Advances in observer design for Takagi-Sugeno systems with unmeasurable premise variables

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Motivations

- Observer design for nonlinear systems is a challenging problem which is intensively studied in control and diagnosis fields
- Takagi-Sugeno modelling, introduced in 1985, offered an interesting tool for studying nonlinear systems.

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases}$$

- In about 90 % of the (very numerous) proposed papers that can be found in the literature, the so-called *premise variables* or *decision variables*, $\xi(t)$ are assumed to be known or accessible to the measurement.
- Even if the model is obtained by sector nonlinearity transformations, many authors maintain that hypothesis although very often it does not make sense.
- Indeed, the rewritting of a nonlinear model with bounded nonlinearities, using the sector nonlinearity approach, relies specifically on the identification of nonlinearities between state variables, which, on a general point of view, are not all measured !

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Proposition

- To consider more realistic T-S model with unmeasurable premise variables

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))\,(A_i x(t) + B_i u(t)) \\ y(t) = Cx(t) \end{cases}$$

- To propose observer design techniques for that kind of model assuming different hypotheses.
- Previous works :
  Bergsten P., Palm R., and Driankov D. (2000-2002)
  Yoneyama, J. (2008-2009)
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- To guarantee a given performance level in presence of modeling uncertainties and noise (structural and measurement noises)

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1. Introduction
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3. Relaxation of LMI conditions
4. Observer with guaranteed bounded estimation error – Input-to-state stability
5. Extensions – modeling uncertainties, noise
6. Results on academic examples
7. Conclusions and perspectives

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Takagi-Sugeno model

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- Convex sum property : $\sum_{i=1}^{r} \mu_i(x(t)) = 1$ and $0 \leq \mu_i(x(t)) \leq 1, \forall t, \forall i \in \{1, ..., r\}$

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Notations

$$X_\mu = \sum_{i=1}^{r} \mu_i(x(t)) X_i \; , \quad X_{\mu\mu} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x(t)) \mu_j(x(t)) X_{ij}$$

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Unknowns to be determined : L_i and symmetric positive definite P_i, $i = 1, \ldots, r$

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State estimation error

$$\begin{aligned} \dot{e}(t) &= \Phi_{\hat{\mu}\hat{\mu}} e(t) + \delta(x, \hat{x}, u) \\ \Phi_{\hat{\mu}\hat{\mu}} &= A_{\hat{\mu}} - P_{\hat{\mu}}^{-1} L_{\hat{\mu}} C \\ \delta(x, \hat{x}, u) &= f(\hat{x}, x, u) - f(x, x, u) \\ f(\hat{x}, x, u) &= A_{\hat{\mu}} x(t) + B_{\hat{\mu}} u(t) \end{aligned}$$

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Assumptions

- **A1.** The function f is Lipschitz with respect to its first variable. Then, there exists a positive scalar η such that $\delta^T(x,\hat{x},u)\delta(x,\hat{x},u) \leq \eta^2 e^T(t)e(t)$.
- **A2.** There exists positive scalars ρ_i such that the weighting functions satisfy $|\dot{\mu}_i(\hat{x}(t)| \leq \rho_i$.

Lyapunov based approach for stability analysis

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Its time derivative is given by

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Substituting the expression of $e(t)$ leads to

$$\dot{V}(e(t)) = e^T(t)\left(\Phi^T_{\hat{\mu}\hat{\mu}}P_{\hat{\mu}} + P_{\hat{\mu}}\Phi_{\hat{\mu}\hat{\mu}} + \dot{P}_{\hat{\mu}}\right)e(t) + 2e^T(t)P_{\hat{\mu}}\delta(x,\hat{x},u)$$

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Derivative of the Lyapunov function

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- For any positive λ

$$2e^T(t)P_{\hat{\mu}}\delta(x,\hat{x},u) \leq \lambda\delta^T(x,\hat{x},u)\delta(x,\hat{x},u) + \lambda^{-1}e^T(t)P_{\hat{\mu}}P_{\hat{\mu}}e(t)$$

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- We have

$$\dot{P}_{\hat{\mu}} = \sum_{i=1}^{r}\dot{\mu}_i(\hat{x})P_i, \qquad \sum_{i=1}^{r}\dot{\mu}_i(\hat{x}) = 0 \Rightarrow \sum_{i=1}^{r}\dot{\mu}_i(\hat{x})P_0 = 0$$

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Quadratic form in $e(t)$

The negativity of $\dot{V}(e(t))$ is ensured if (sufficient condition)

$$A_{\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}}A_{\hat{\mu}} - C^T L_{\hat{\mu}}^T - L_{\hat{\mu}}C + \sum_{i=1}^{r}\rho_i(P_i - P_0) + \lambda\eta^2 I + \lambda^{-1}P_{\hat{\mu}}P_{\hat{\mu}} < 0$$

Theorem 1

Under the assumptions **A1** and **A2**, if there exists a symmetric matrix P_0, symmetric and positive definite matrices P_i, gain matrices L_i and a positive scalar λ satisfying the following LMI

$$M_{ij} < 0, \quad i, j = 1, \ldots, r$$
$$P_i - P_0 \geq 0, \quad i = 1, \ldots, r$$

where

$$M_{ij} = \begin{pmatrix} A_i^T P_j + P_j A_i - C^T L_i^T - L_i C + \sum_{i=1}^{r} \rho_i (P_i - P_0) + \lambda \eta^2 I & P_j \\ P_j & -\lambda I \end{pmatrix}$$

then the state estimation error asymptotically converges towards zero.

There exists many approaches for relaxing the previous conditions (expressed as LMI with double summation indexes) [1], [2]

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$$\begin{cases} M_{ii} < 0, \; i = 1, ..., r \\ M_{ii} + M_{ij} + M_{ji} < 0, \; j \neq i \\ M_{ij} + M_{ji} + M_{ik} + M_{ki} + M_{jk} + M_{kj} < 0, \quad i \neq j, \; i \neq k, \; j \neq k \end{cases}$$

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- **A3.** The input $u(t)$ is bounded
- **A4.** The system is input-to-state stable (ISS), *i.e.* the system state $x(t)$ is bounded for bounded input $u(t)$
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$$\dot{e}(t) = \Phi_{\hat{\mu}\hat{\mu}} e(t) + \delta(t)$$

$\delta(t)$ is a bounded perturbation term

Bounded estimation error

Definition – Input-to-state stability [Sontag, 1985]

The system describing the state estimation error is said to be ISS if there exists a function $\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for each input $\delta(t)$ satisfying $\|\delta(t)\|_\infty < \infty$ and each initial condition $e(0) \in \mathbb{R}^n$, the trajectory of $e(t)$ associated with $e(0)$ and $\delta(t)$ satisfies

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Convergence into a ball

Starting from the same nonlinear Lyapunov function

$$\dot{V}(e(t)) = e^T(t)\left(\Phi_{\hat{\mu}\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}}\Phi_{\hat{\mu}\hat{\mu}} + \dot{P}_{\hat{\mu}}\right)e(t) + 2e^T(t)P_{\hat{\mu}}\delta(t)$$

$$\dot{V}(e(t)) \leq e^T(t)\left(\Phi_{\hat{\mu}\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}}\Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r}\rho_i(P_i - P_0)\right)e(t) + 2e^T(t)P_{\hat{\mu}}\delta(t)$$

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$$2e^T(t)P_{\hat{\mu}}\delta(t) \geq c\delta^T(t)\delta(t) + c^{-1}P_{\hat{\mu}}P_{\hat{\mu}}$$

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Defining an augmented state $e_a(t) = [e^T(t) \;\; \delta^T(t)]^T$ and using a Schur complement

$$\dot{V}(e(t)) \leq e_a^T(t)\Xi_{\hat{\mu}\hat{\mu}}e_a(t) - \alpha e^T(t)P_{\hat{\mu}}e(t) + c\delta^T(t)\delta(t)$$

where

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If $\Xi_{\hat{\mu}\hat{\mu}} < 0$ holds, then

$$\dot{V}(e(t)) \leq -\alpha e^T(t)P_{\hat{\mu}}e(t) + c\delta^T(t)\delta(t)$$

Bounded estimation error

Convergence into a ball

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Multplying both sides by $e^{\alpha t}$ and integrating from 0 to t, one obtains

$$V(e(t)) \leq V(0)e^{-\alpha t} + c\int_0^t e^{-\alpha(t-s)} \|\delta(s)\|_2^2 \, ds$$

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Moreover, due to the convex sum property, we have

$$\alpha_1 \|e(t)\|_2^2 \leq V(e(t)) = e^T(t)P_{\hat{\mu}}e(t) \leq \alpha_2 \|e(t)\|_2^2$$

where

$$\alpha_1 = \min_{1\leq i\leq r} \lambda_{\min}(P_i) \qquad \alpha_2 = \max_{1\leq i\leq r} \lambda_{\max}(P_i)$$

$\lambda_{\min}(M)$ and $\lambda_{\max}(M)$: minimal and maximal eigenvalues of M.

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Bounded estimation error

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Since $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $\forall a, b \in \mathbb{R}^+$

$$\|e(t)\|_2 \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|e(0)\|_2 e^{-\frac{\alpha}{2} t} + \sqrt{\frac{c}{\alpha \alpha_1}} \|\delta(t)\|_\infty$$

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The radius of the convergence region D is upper bounded by $\sqrt{\frac{c}{\alpha\alpha_1}} \|\delta(t)\|_\infty$

Bounded estimation error

Convergence into a ball

- The radius of the convergence region D is upper bounded by $\sqrt{\frac{c}{\alpha\alpha_1}}\|\delta(t)\|_\infty$
- This bound depends on the selected matrices P_i and the parameters α and c.
- The set D should be made as small as possible to ensure a good accuracy of convergence.
- The choice of α, c and P_i providing a small set of convergence is not obvious because the problem is nonlinear.

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Proposed solution

$$\sqrt{\frac{c}{\alpha\alpha_1}} \leq \sqrt{\gamma} \quad \text{where } \gamma \text{ is a positive scalar to minimize.}$$

If we impose $\alpha_1 \geq 1$, the minimization of γ for a given $\alpha > 0$ can be done under the constraint

$$c - \alpha\gamma \leq 0$$

Theorem 2

Under the assumptions **A3**, **A4** and **A5**, given a scalar $\alpha > 0$, if there exists a symmetric matrix P_0, symmetric matrices P_i, gain matrices L_i and positive scalars γ and c solution to the following optimization problem

$$\min_{P_0, P_i, L_i, c} \gamma$$

$$s.t. \quad \begin{cases} P_0 \geq I \\ P_i - P_0 \geq 0, \; i = 1, \ldots, r \\ \Xi_{ii} < 0, \quad i = 1, ..., r \\ \Xi_{ii} + \Xi_{ij} + \Xi_{ji} < 0, \quad j \neq i \\ \Xi_{ij} + \Xi_{ji} + \Xi_{ik} + \Xi_{ki} + \Xi_{jk} + \Xi_{kj} < 0, \quad i \neq j, \; i \neq k, \; j \neq k \\ c - \alpha\gamma \leq 0 \end{cases}$$

then the error dynamics is ISS with respect to $\delta(t)$ and satisfy

$$\|e(t)\|_2 \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|e(0)\|_2 \, e^{-\frac{\alpha}{2}t} + \sqrt{\frac{c}{\alpha\alpha_1}} \|\delta(t)\|_\infty$$

The gains L_i of the observer are obtained directly and the attenuation level of the transfer from $\delta(t)$ to $e(t)$ is $\sqrt{\frac{c}{\alpha\alpha_1}}$.

Robustness with respect to modeling uncertainties

Consider the uncertain system

$$\begin{cases} \dot{x}(t) = (A_\mu + \Delta A_\mu)x(t) + (B_\mu + \Delta B_\mu)u(t) \\ y(t) = (C + \Delta C)x(t) \end{cases}$$

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with its corresponding observer (the same as before)

$$\begin{cases} \dot{\hat{x}}(t) = A_{\hat{\mu}}\hat{x}(t) + B_{\hat{\mu}}u(t) + P_{\hat{\mu}}^{-1}L_{\hat{\mu}}(y(t) - \hat{y}(t)) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

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The state estimation error obeys the differential equation

$$\dot{e}(t) = \Phi_{\hat{\mu}\hat{\mu}}e(t) + \delta(x, \hat{x}, u)$$

where $\Phi_{\hat{\mu}\hat{\mu}} = A_{\hat{\mu}} - P_{\hat{\mu}}^{-1}L_{\hat{\mu}}C$ and

$$\delta(x, \hat{x}, u) = \left(A_\mu - A_{\hat{\mu}} + \Delta A_\mu - P_{\hat{\mu}}^{-1}L_{\hat{\mu}}\Delta C\right)x(t) + \left(B_\mu - B_{\hat{\mu}} + \Delta B_\mu\right)u(t)$$

All the uncertain terms are included in the disturbance-like term $\delta(x, \hat{x}, u)$

Extensions – modeling uncertainties, noise

Robustness with respect to modeling uncertainties

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Noise consideration

Consider the noised system

$$\begin{cases} \dot{x}(t) = A_\mu x(t) + B_\mu u(t) + \omega(t) \\ y(t) = Cx(t) + \nu(t) \end{cases}$$

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$$\delta(x, \hat{x}, u) = \left(A_\mu - A_{\hat{\mu}}\right) x(t) + \left(B_\mu - B_{\hat{\mu}}\right) u(t) + \omega(t) + P_{\hat{\mu}}^{-1} L_{\hat{\mu}} \nu(t)$$

Noise consideration

Consider the noised system

$$\begin{cases} \dot{x}(t) = A_\mu x(t) + B_\mu u(t) + \omega(t) \\ y(t) = Cx(t) + \nu(t) \end{cases}$$

The state estimation error obeys the differential equation

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Rossler chaotic system with 2 local models

Rossler chaotic system with 2 local models

$$A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\max} \\ 0 & x_{1\max} & -0.37 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\min} \\ 0 & x_{1\min} & -0.37 \end{pmatrix},$$

$$B_1 = B_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$x_{1\min} = -9.8693 \quad \text{and} \quad x_{1\max} = 13.8164$$

$$\mu_1(x(t)) = \frac{x_1(t) - x_{1\min}}{x_{1\max} - x_{1\min}} \qquad \mu_2(x(t)) = \frac{x_{1\max} - x_1(t)}{x_{1\max} - x_{1\min}}$$

$$|\dot{\mu}_i(\hat{x}(t)| \leq \rho_i, \quad \rho_1 = \rho_2 = 4.5, \quad \text{Lipschitz constant } \eta = 173.35$$

Rossler chaotic system with 2 local models

- Using the results presented in P. Bergsten and R. Palm (2000) $\Rightarrow$
  no solution (maximal admissible Lipschitz constant : 29.73)
- Asymptotic observer (Theorem 1) $\Rightarrow$
  no solution (LMI are not feasible)
- Observer with guaranteed bounded estimation error (Theorem 2) with $\alpha = 10$
  Obtained attenuation level : $\sqrt{\gamma} = 0.0549$

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Results on academic examples

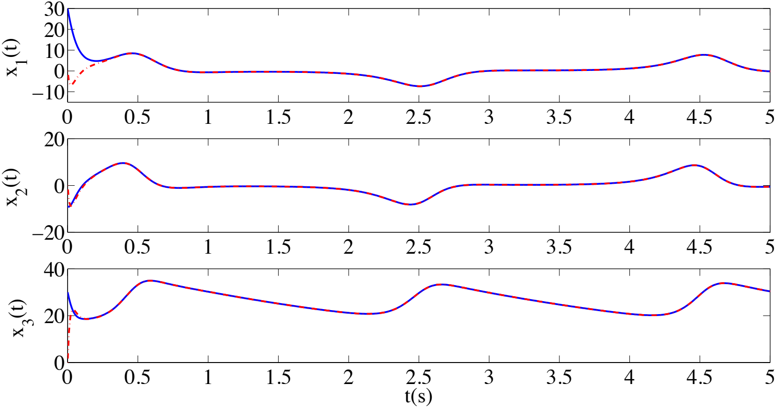

FIGURE: State variables (blue line) and their estimates (red dashed lines)

Second example : flexible link robot

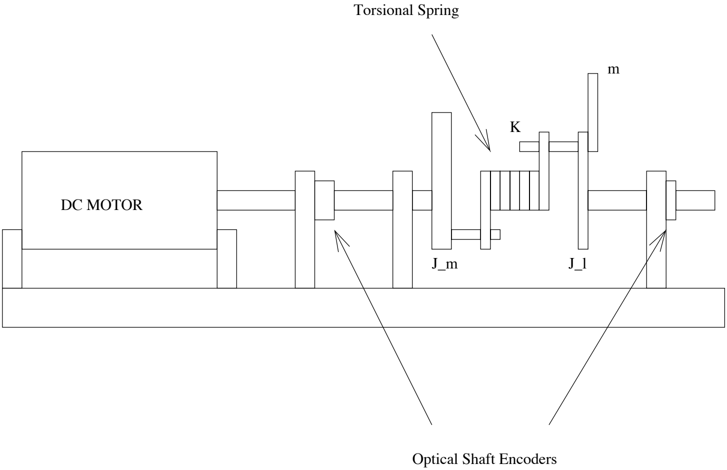

FIGURE: Flexible Joint Robot

Example : Flexible link robot

The model of the flexible link robot is given by :

$$\dot{x}(t) = Ax(t) + Bu(t) + \phi(x(t))$$

$$y(t) = Cx(t)$$

where :

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} \theta_m(t) \\ \omega_m(t) \\ \theta_l(t) \\ \omega_l(t) \end{pmatrix}, \quad \phi(x(t)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3.33\sin(x_3(t)) \end{pmatrix}$$

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$u(t) = \sin(t), \quad x_0 = [0.5 \quad 0.75 \quad 1 \quad 1]^T$$

The obtained T-S model is :

$$\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_3(t))A_i x(t) + Bu(t), \quad y(t) = Cx(t)$$

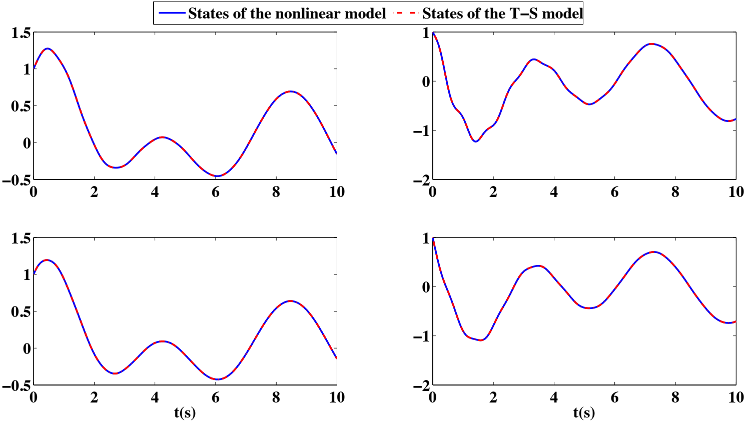

FIGURE: Nonlinear vs Takagi-Sugeno

Example : Flexible link robot

The observer takes the form :

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{2} \mu_i(\hat{x}_3(t))(A_i\hat{x}(t) + (\sum_{j=1}^{2} \mu_j(\hat{x}_3(t))P_j)^{-1}L_i(y(t) - \hat{y}(t))) + Bu(t) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

where the gains L_i and the matrices P_j are computed from solving the LMIs given in the Theorem 2 with the parameters $\alpha = 4.7$ and $\rho_1 = \rho_2 = 1$.

The radius of the convergence ball is :

$$\sqrt{\frac{c}{\alpha\alpha_1}}\,\|\delta(t)\|_\infty = 1.095\,\|\delta(t)\|_\infty$$

For the following simulations, a centered random noise with maximal magnitude 0.2 is added to the output.

Example : Flexible link robot

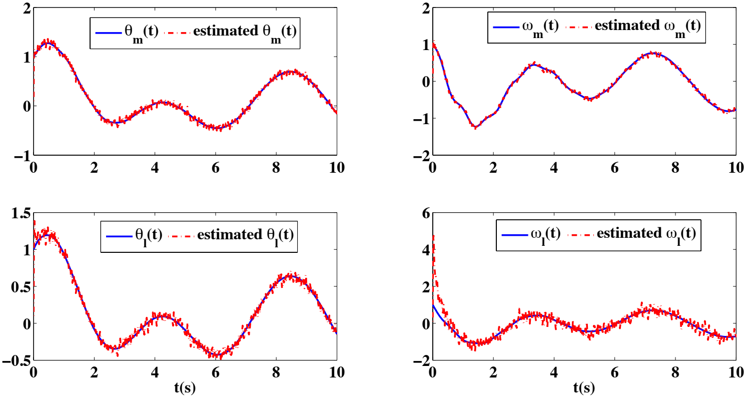

FIGURE: State estimation

Conclusions

- Two propositions of observer design for T-S model with unmeasurable premise variables have been proposed

Perspectives

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- Overcome the hypothesis about the boundedness of the derivatives of the weighting function (assumptions **A2** or **A5**)
- Application on a real system

Get in touch

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Additive material

Construction of TS models – 3 main approaches

- Transformation of a nonlinear model into a multiple model
  - Linearization around some "well-chosen" points

    Identification of the weighting function parameter to minimize the ouput error

  - Sector nonlinearity transformation

    Rewriting of the model in a compact subspace of the state space

$$\left\{\begin{array}{l}\dot{x}(t) = f(x(t), u(t)) \\ y(t) = h(x(t), u(t))\end{array}\right. \Rightarrow \left\{\begin{array}{l}\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\ y(t) = \sum_{i=1}^{r} \mu_i(\xi(t))(C_i x(t) + D_i u(t))\end{array}\right.$$

Takagi-Sugeno model

Construction of TS models – 3 main approaches

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  - Choice of the number of modalities of each premise variable
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Sector nonlinearity transformation

Example

$$\begin{cases} \dot{x}_1(t) = -x_1(t)\cos(x_1(t)) + u(t) \\ \dot{x}_2(t) = x_1^2(t) - x_2(t) \end{cases}$$

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