Advances in observer design for Takagi-Sugeno systems with unmeasurable premise variables

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Observer design for nonlinear systems is a challenging problem which is intensively studied in control and diagnosis fields.

Takagi-Sugeno modelling, introduced in 1985, offered an interesting tool for studying nonlinear systems.

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\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\
y(t) &= Cx(t)
\end{align*}
\]

In about 90% of the (very numerous) proposed papers that can be found in the literature, the so-called premise variables or decision variables, \(\xi(t)\) are assumed to be known or accessible to the measurement.

Even if the model is obtained by sector nonlinearity transformations, many authors maintain that hypothesis although very often it does not make sense.

Indeed, the rewriting of a nonlinear model with bounded nonlinearities, using the sector nonlinearity approach, relies specifically on the identification of nonlinearities between state variables, which, on a general point of view, are not all measured!
Motivations

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Indeed, the rewriting of a nonlinear model with bounded nonlinearities, using the sector nonlinearity approach, relies specifically on the identification of nonlinearities between state variables, which, on a general point of view, are not all measured!
Proposition

- To consider more realistic T-S model with unmeasurable premise variables

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- To propose observer design techniques for that kind of model assuming different hypotheses.

- Previous works:
  Ichalal D., Marx B., Ragot J., and Maquin D. (2009-2012)

- To guarantee a given performance level in presence of modeling uncertainties and noise (structural and measurement noises)
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To guarantee a given performance level in presence of modeling uncertainties and noise (structural and measurement noises)
1 Introduction

2 Asymptotic observer design

3 Relaxation of LMI conditions

4 Observer with guaranteed bounded estimation error – Input-to-state stability

5 Extensions – modeling uncertainties, noise

6 Results on academic examples

7 Conclusions and perspectives
Outline

1. Introduction

2. Asymptotic observer design

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Asymptotic observer design

Takagi-Sugeno model

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- Convex sum property: \( \sum_{i=1}^{r} \mu_i(x(t)) = 1 \) and \( 0 \leq \mu_i(x(t)) \leq 1, \forall t, \forall i \in \{1, \ldots, r\} \)
Asymptotic observer design

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Notations

\[
X_\mu = \sum_{i=1}^{r} \mu_i(x(t))X_i, \quad X_{\mu\mu} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(x(t))\mu_j(x(t))X_{ij}
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X_\hat{\mu} = \sum_{i=1}^{r} \mu_i(\hat{x}(t)) X_i, \quad X_{\hat{\mu}\hat{\mu}} = \sum_{i=1}^{r} \sum_{j=1}^{r} \mu_i(\hat{x}(t)) \mu_j(\hat{x}(t)) X_{ij}
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Proposed observer

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Asymptotic observer design

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Unknowns to be determined: \( L_i \) and symmetric positive definite \( P_i, i = 1, \ldots, r \)
Asymptotic observer design

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### State estimation error

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\begin{aligned}
    \dot{e}(t) &= \Phi_{\hat{\mu}} e(t) + \delta(x, \hat{x}, u) \\
    \Phi_{\hat{\mu}} &= A_\hat{\mu} - P^{-1}_\hat{\mu} L_\hat{\mu} C \\
    \delta(x, \hat{x}, u) &= f(\hat{x}, x, u) - f(x, x, u) \\
    f(\hat{x}, x, u) &= A_\hat{\mu} x(t) + B_\hat{\mu} u(t)
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Asymptotic observer design

Assumptions

- **A1.** The function $f$ is Lipschitz with respect to its first variable. Then, there exists a positive scalar $\eta$ such that $\delta^T(x, \hat{x}, u)\delta(x, \hat{x}, u) \leq \eta^2 e^T(t)e(t)$.
- **A2.** There exists positive scalars $\rho_i$ such that the weighting functions satisfy $|\dot{\mu}_i(\hat{x}(t))| \leq \rho_i$.

Lyapunov based approach for stability analysis
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Let us consider the nonlinear Lyapunov function

$$V(e(t)) = e^T(t)P_{\hat{x}}e(t)$$
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$$P_{\dot{\mu}} = \sum_{i=1}^{r} \mu_i(\hat{x}(t)) P_i$$
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$$V(e(t)) = e^T(t)P_{\hat{\mu}}e(t)$$

Its time derivative is given by

$$\dot{V}(e(t)) = \dot{e}^T(t)P_{\hat{\mu}}e(t) + e^T(t)P_{\hat{\mu}}\dot{e}(t) + e^T(t)\dot{P}_{\hat{\mu}}e(t)$$
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Substituting the expression of $e(t)$ leads to

$$\dot{V}(e(t)) = e^T(t) \left( \Phi_{\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}} + \dot{P}_{\hat{\mu}} \right) e(t) + 2e^T(t) P_{\hat{\mu}} \delta(x, \hat{x}, u)$$
Lyapunov based approach for stability analysis

Derivative of the Lyapunov function

\[ \dot{V}(e(t)) = e^T(t) \left( \Phi_{\mu\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\mu\hat{\mu}} + \dot{P}_{\hat{\mu}} \right) e(t) + 2e^T(t)P_{\hat{\mu}} \delta(x, \hat{x}, u) \]

- For any positive \( \lambda \)

\[ 2e^T(t)P_{\hat{\mu}} \delta(x, \hat{x}, u) \leq \lambda \delta^T(x, \hat{x}, u) \delta(x, \hat{x}, u) + \lambda^{-1}e^T(t)P_{\hat{\mu}} P_{\hat{\mu}} e(t) \]

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- We have

\[ \dot{P}_{\hat{\mu}} = \sum_{i=1}^{r} \dot{\mu}_i(\hat{x}) P_i, \quad \sum_{i=1}^{r} \dot{\mu}_i(\hat{x}) = 0 \Rightarrow \sum_{i=1}^{r} \dot{\mu}_i(\hat{x}) P_0 = 0 \]

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for any \( P_0 \) such that \( P_i - P_0 \geq 0 \).
Asymptotic observer design

Lyapunov based approach for stability analysis

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Lyapunov based approach for stability analysis

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\dot{V}(e(t)) \leq e^T(t) (\Phi_{\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) + \lambda \eta^2 I + \lambda^{-1} P_{\hat{\mu}} P_{\hat{\mu}}) e(t)
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Derivative of the Lyapunov function

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Quadratic form in \( e(t) \)
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Derivative of the Lyapunov function

\[
\dot{V}(e(t)) = e^T(t) \left( \Phi_{\hat{\mu} \hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu} \hat{\mu}} + \dot{P}_{\hat{\mu}} \right) e(t) + 2e^T(t) P_{\hat{\mu}} \delta(x, \hat{x}, u)
\]

\[
\dot{V}(e(t)) \leq e^T(t) (\Phi_{\hat{\mu} \hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu} \hat{\mu}} + \sum_{i=1}^{r} \rho_i(P_i - P_0) + \lambda \eta^2 I + \lambda^{-1} P_{\hat{\mu}} P_{\hat{\mu}}) e(t)
\]

Quadratic form in \(e(t)\)

The negativity of \(\dot{V}(e(t))\) is ensured if (sufficient condition)

\[
A_{\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} A_{\hat{\mu}} - C^T L_{\hat{\mu}}^T L_{\hat{\mu}}^T - C_{\hat{\mu}}^T L_{\hat{\mu}} + \sum_{i=1}^{r} \rho_i(P_i - P_0) + \lambda \eta^2 I + \lambda^{-1} P_{\hat{\mu}} P_{\hat{\mu}} < 0
\]
Asymptotic observer design

Theorem 1

Under the assumptions $\textbf{A1}$ and $\textbf{A2}$, if there exists a symmetric matrix $P_0$, symmetric and positive definite matrices $P_i$, gain matrices $L_i$ and a positive scalar $\lambda$ satisfying the following LMI

$$M_{ij} < 0, \quad i, j = 1, \ldots, r$$
$$P_i - P_0 \geq 0, \quad i = 1, \ldots, r$$

where

$$M_{ij} = \begin{pmatrix}
A_i^T P_j + P_j A_i - C^T L_i^T - L_i C + \sum_{i=1}^{r} \rho_i (P_i - P_0) + \lambda \eta^2 I
P_j
P_j
\end{pmatrix}$$

then the state estimation error asymptotically converges towards zero.
Relaxation of LMI conditions

There exists many approaches for relaxing the previous conditions (expressed as LMI with double summation indexes) \(^1,^2\)

---


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### Tuan’s lemma

For example, with the use of Tuan’s lemma

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\begin{align*}
M_{ii} &< 0, \quad i = 1, ..., r \\
\frac{2}{r-1} M_{ii} + M_{ij} + M_{ji} &< 0, \quad j \neq i
\end{align*}
\]

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\end{aligned}
\]

### Polya’s theorem

\[
\begin{aligned}
M_{ij} &< 0, \quad i = 1, \ldots, r \\
M_{ii} + M_{ij} + M_{ji} &< 0, \quad j \neq i \\
M_{ij} + M_{ji} + M_{ik} + M_{ki} + M_{jk} + M_{kj} &< 0, \quad i \neq j, \quad i \neq k, \quad j \neq k
\end{aligned}
\]

---

The assumption **A1.** about the Lipschitz property is, on a general point of view, difficult to justify and to prove. Another observer design is proposed to relax that assumption.
Relaxation of the Lipschitz constraint

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\begin{itemize}
  \item **A3.** The input \( u(t) \) is bounded
  \item **A4.** The system is input-to-state stable (ISS), \textit{i.e.} the system state \( x(t) \) is bounded for bounded input \( u(t) \)
  \item **A5.** There exists positive scalars \( \rho_i \) such that the weighting functions satisfy \( |\dot{\mu}_i(\hat{x}(t))| \leq \rho_i \).
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**State estimation error**

$$\dot{e}(t) = \Phi_{\hat{\mu}\hat{\mu}} e(t) + \delta(x, \hat{x}, u)$$
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### State estimation error

\[
\dot{e}(t) = \Phi_{\hat{\mu}\hat{\mu}} e(t) + \delta(t)
\]

\( \delta(t) \) is a bounded perturbation term.
**Definition – Input-to-state stability [Sontag, 1985]**

The system describing the state estimation error is said to be ISS if there exists a function $\beta : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ and a function $\alpha : \mathbb{R} \to \mathbb{R}$ such that, for each input $\delta(t)$ satisfying $\|\delta(t)\|_\infty < \infty$ and each initial condition $e(0) \in \mathbb{R}^n$, the trajectory of $e(t)$ associated with $e(0)$ and $\delta(t)$ satisfies

$$\|e(t)\|_2 \leq \beta (\|e(0)\|_2, t) + \alpha (\|\delta(t)\|_\infty), \forall t$$
Bounded estimation error

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**Convergence into a ball**

Starting from the same nonlinear Lyapunov function

$$\dot{V}(e(t)) = e^T(t) \left( \Phi^T \hat{\mu} \hat{\mu} P \hat{\mu} + P \hat{\mu} \Phi \hat{\mu} \hat{\mu} + \dot{P} \hat{\mu} \right) e(t) + 2e^T(t)P \hat{\mu} \delta(t)$$

$$\dot{V}(e(t)) \leq e^T(t) \left( \Phi^T \hat{\mu} \hat{\mu} P \hat{\mu} + P \hat{\mu} \Phi \hat{\mu} \hat{\mu} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t)P \hat{\mu} \delta(t)$$
Convergence into a ball

\[
\dot{V}(e(t)) \leq e^T(t) \left( \Phi_{\hat{\mu}\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t) P_{\hat{\mu}} \delta(t)
\]
Bounded estimation error

Convergence into a ball

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\dot{V}(e(t)) \leq e^T(t) \left( \Phi_{\hat{\mu}\hat{\mu}}^T P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t)P_{\hat{\mu}} \delta(t)
\]

We have

\[
2e^T(t)P_{\hat{\mu}} \delta(t) \geq c\delta^T(t)\delta(t) + c^{-1}P_{\hat{\mu}} P_{\hat{\mu}}
\]
Bounded estimation error

Convergence into a ball

\[ \dot{V}(e(t)) \leq e^T(t) \left( \Phi_{\hat{\mu}\hat{\mu}} P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t)P_{\hat{\mu}} \delta(t) \]

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Defining an augmented state \( e_a(t) = [e^T(t) \quad \delta^T(t)]^T \) and using a Schur complement

\[ \dot{V}(e(t)) \leq e_{a}^T(t)\Xi_{\hat{\mu}\hat{\mu}}e_a(t) - \alpha e^T(t)P_{\hat{\mu}} e(t) + c\delta^T(t)\delta(t) \]

where

\[ \Xi_{\hat{\mu}\hat{\mu}} = \begin{pmatrix} \Phi_{\hat{\mu}\hat{\mu}} P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) + \alpha P_{\hat{\mu}} & P_{\hat{\mu}} \\ P_{\hat{\mu}} & -cl \end{pmatrix} \]
Bounded estimation error

Convergence into a ball

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Bounded estimation error

Convergence into a ball

\[ \dot{V}(e(t)) \leq e^T(t) \left( \Phi_{\hat{\mu}\hat{\mu}} P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu}\hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t)P_{\hat{\mu}} \delta(t) \]

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Bounded estimation error

Convergence into a ball

\[
\dot{V}(e(t)) \leq e^T(t) \left( \Phi_{\hat{\mu} \hat{\mu}} P_{\hat{\mu}} + P_{\hat{\mu}} \Phi_{\hat{\mu} \hat{\mu}} + \sum_{i=1}^{r} \rho_i (P_i - P_0) \right) e(t) + 2e^T(t)P_{\hat{\mu}} \delta(t)
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\[
\Xi_{\hat{\mu} \hat{\mu}} = 
\begin{pmatrix}
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\sum_{i=1}^{r} \rho_i (P_i - P_0) + \alpha P_{\hat{\mu}} & -cI
\end{pmatrix}
\]

If \(\Xi_{\hat{\mu} \hat{\mu}} < 0\) holds, then

\[
\dot{V}(e(t)) \leq -\alpha e^T(t)P_{\hat{\mu}} e(t) + c\delta^T(t)\delta(t)
\]
Convergence into a ball

\[ \dot{V}(e(t)) \leq -\alpha V(e(t)) + c\delta^T(t)\delta(t) \]
Bounded estimation error

Convergence into a ball

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Multiplying both sides by \( e^{\alpha t} \) and integrating from 0 to \( t \), one obtains

\[ V(e(t)) \leq V(0)e^{-\alpha t} + c \int_0^t e^{-\alpha(t-s)} \| \delta(s) \|^2_2 ds \]
Bounded estimation error

Convergence into a ball

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Multiplying both sides by \( e^{\alpha t} \) and integrating from 0 to \( t \), one obtains

\[ V(e(t)) \leq V(0)e^{-\alpha t} + c \int_{0}^{t} e^{-\alpha(t-s)} \|\delta(s)\|_2^2 ds \]

Moreover, due to the convex sum property, we have

\[ \alpha_1 \|e(t)\|_2^2 \leq V(e(t)) = e^T(t)P\hat{\mu}e(t) \leq \alpha_2 \|e(t)\|_2^2 \]

where

\[ \alpha_1 = \min_{1 \leq i \leq r} \lambda_{\min}(P_i) \quad \alpha_2 = \max_{1 \leq i \leq r} \lambda_{\max}(P_i) \]

\( \lambda_{\min}(M) \) and \( \lambda_{\max}(M) \) : minimal and maximal eigenvalues of \( M \).
Bounded estimation error

Convergence into a ball

\[ \dot{V}(e(t)) \leq -\alpha V(e(t)) + c\delta^T(t)\delta(t) \]

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Bounded estimation error

Convergence into a ball

\[ \alpha_1 \| e(t) \|^2_2 \leq \alpha_2 \| e(0) \|^2_2 e^{-\alpha t} + \frac{C}{\alpha} \| \delta(t) \|^2_{\infty} \]
Bounded estimation error

Convergence into a ball

\[ \alpha_1 \| e(t) \|_2^2 \leq \alpha_2 \| e(0) \|_2^2 e^{-\alpha t} + \frac{c}{\alpha} \| \delta(t) \|_\infty^2 \]

Since \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \), \( \forall a, b \in \mathbb{R}^+ \)

\[ \| e(t) \|_2 \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \| e(0) \|_2 e^{-\frac{\alpha}{2} t} + \sqrt{\frac{c}{\alpha \alpha_1}} \| \delta(t) \|_\infty \]
Bounded estimation error

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- If \( \| \delta(t) \|_\infty = 0 \) then \( \| e(t) \|_2 \rightarrow 0 \) when \( t \rightarrow \infty \)
- In the presence of the perturbation \( \delta(t) \), the error \( \| e(t) \|_2 \) is bounded by \( \sqrt{\frac{c}{\alpha \alpha_1}} \| \delta(t) \|_\infty \)
Bounded estimation error

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The system describing the evolution of the state estimation error is ISS
Bounded estimation error

Convergence into a ball

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The system describing the evolution of the state estimation error is ISS

The radius of the convergence region \( D \) is upper bounded by \( \sqrt{\frac{c}{\alpha_\alpha_1}} \| \delta(t) \|_\infty \)
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This bound depends on the selected matrices \( P_i \) and the parameters \( \alpha \) and \( c \).

The set \( D \) should be made as small as possible to ensure a good accuracy of convergence.

The choice of \( \alpha \), \( c \) and \( P_i \) providing a small set of convergence is not obvious because the problem is nonlinear.
Bounded estimation error

Convergence into a ball

- The radius of the convergence region $D$ is upper bounded by $\sqrt{\frac{c}{\alpha_1}} \| \delta(t) \|_{\infty}$.
- This bound depends on the selected matrices $P_i$ and the parameters $\alpha$ and $c$.
- The set $D$ should be made as small as possible to ensure a good accuracy of convergence.
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Proposed solution

$\sqrt{\frac{c}{\alpha_1}} \leq \sqrt{\gamma}$ where $\gamma$ is a positive scalar to minimize.

If we impose $\alpha_1 \geq 1$, the minimization of $\gamma$ for a given $\alpha > 0$ can be done under the constraint $c - \alpha \gamma \leq 0$. 
Bounded estimation error

Convergence into a ball

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$$c - \alpha \gamma \leq 0$$
Theorem 2

Under the assumptions **A3**, **A4** and **A5**, given a scalar $\alpha > 0$, if there exists a symmetric matrix $P_0$, symmetric matrices $P_i$, gain matrices $L_i$ and positive scalars $\gamma$ and $c$ solution to the following optimization problem

$$
\min_{P_0, P_i, L_i, c} \gamma
$$

s.t.

$$
\begin{align*}
& P_0 \geq I \\
& P_i - P_0 \geq 0, \ i = 1, \ldots, r \\
& \Xi_{ii} < 0, \ i = 1, \ldots, r \\
& \Xi_{ii} + \Xi_{ij} + \Xi_{ji} < 0, \ j \neq i \\
& \Xi_{ij} + \Xi_{ji} + \Xi_{ik} + \Xi_{ki} + \Xi_{jk} + \Xi_{kj} < 0, \ i \neq j, \ i \neq k, \ j \neq k \\
& c - \alpha \gamma \leq 0
\end{align*}
$$

then the error dynamics is ISS with respect to $\delta(t)$ and satisfy

$$
\|e(t)\|_2 \leq \sqrt{\frac{\alpha_2}{\alpha_1}} \|e(0)\|_2 e^{-\frac{\alpha_1}{2} t} + \sqrt{\frac{c}{\alpha \alpha_1}} \|\delta(t)\|_\infty
$$

The gains $L_i$ of the observer are obtained directly and the attenuation level of the transfer from $\delta(t)$ to $e(t)$ is $\sqrt{\frac{c}{\alpha \alpha_1}}$. 
Robustness with respect to modeling uncertainties

Consider the uncertain system

\[
\begin{align*}
\dot{x}(t) &= (A_\mu + \Delta A_\mu)x(t) + (B_\mu + \Delta B_\mu)u(t) \\
y(t) &= (C + \Delta C)x(t)
\end{align*}
\]
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\end{align*}
\]

with its corresponding observer (the same as before)

\[
\begin{align*}
\dot{\hat{x}}(t) &= A_\hat{\mu}\hat{x}(t) + B_\hat{\mu}u(t) + P_\hat{\mu}^{-1}L_\hat{\mu}(y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C\hat{x}(t)
\end{align*}
\]
Robustness with respect to modeling uncertainties

Consider the uncertain system

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y(t) &= (C + \Delta C)x(t)
\end{align*}
\]

with its corresponding observer (the same as before)

\[
\begin{align*}
\dot{\hat{x}}(t) &= A_\hat{\mu} \hat{x}(t) + B_\hat{\mu} u(t) + P^{-1}_\hat{\mu} L_\hat{\mu} (y(t) - \hat{y}(t)) \\
\hat{y}(t) &= C \hat{x}(t)
\end{align*}
\]

The state estimation error obeys the differential equation

\[
\dot{e}(t) = \Phi_{\hat{\mu} \hat{\mu}} e(t) + \delta(x, \hat{x}, u)
\]

where \(\Phi_{\hat{\mu} \hat{\mu}} = A_\hat{\mu} - P^{-1}_\hat{\mu} L_\hat{\mu} C\) and

\[
\delta(x, \hat{x}, u) = \left( A_\mu - A_\hat{\mu} + \Delta A_\mu - P^{-1}_\hat{\mu} L_\hat{\mu} \Delta C \right) x(t) + (B_\mu - B_\hat{\mu} + \Delta B_\mu) u(t)
\]

All the uncertain terms are included in the disturbance-like term \(\delta(x, \hat{x}, u)\)
Robustness with respect to modeling uncertainties

Consider the uncertain system

\[
\begin{aligned}
\dot{x}(t) &= (A_\mu + \Delta A_\mu)x(t) + (B_\mu + \Delta B_\mu)u(t) \\
y(t) &= (C + \Delta C)x(t)
\end{aligned}
\]

with its corresponding observer (the same as before)

\[
\begin{aligned}
\dot{\hat{x}}(t) &= A_\hat{\mu} \hat{x}(t) + B_\hat{\mu} u(t) + P_{\hat{\mu}}^{-1} L_{\hat{\mu}} (y(t) - \hat{y}(t)) \\
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The state estimation error obeys the differential equation

\[
\dot{e}(t) = \Phi_{\hat{\mu}} e(t) + \delta(x, \hat{x}, u)
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where \( \Phi_{\hat{\mu}} = A_{\hat{\mu}} - P_{\hat{\mu}}^{-1} L_{\hat{\mu}} C \) and

\[
\delta(x, \hat{x}, u) = \left( A_\mu - A_{\hat{\mu}} + \Delta A_\mu - P_{\hat{\mu}}^{-1} L_{\hat{\mu}} \Delta C \right) x(t) + (B_\mu - B_{\hat{\mu}} + \Delta B_\mu) u(t)
\]

All the uncertain terms are included in the disturbance-like term \( \delta(x, \hat{x}, u) \)
Consider the noised system

\[
\begin{aligned}
\dot{x}(t) &= A_\mu x(t) + B_\mu u(t) + \omega(t) \\
y(t) &= Cx(t) + \nu(t)
\end{aligned}
\]
Extensions – modeling uncertainties, noise

**Noise consideration**

Consider the noised system

\[
\begin{align*}
\dot{x}(t) &= A_{\mu} x(t) + B_{\mu} u(t) + \omega(t) \\
y(t) &= C x(t) + v(t)
\end{align*}
\]

The state estimation error obeys the differential equation

\[
\dot{e}(t) = \Phi_{\hat{\mu} \hat{\mu}} e(t) + \delta(x, \hat{x}, u)
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where \( \Phi_{\hat{\mu} \hat{\mu}} = A_{\hat{\mu}} - P_{\hat{\mu}}^{-1} L_{\hat{\mu}} C \) and

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Noise consideration

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\[
\delta(x, \hat{x}, u) = (A_\mu - A_{\hat{\mu}}) x(t) + (B_\mu - B_{\hat{\mu}}) u(t) + \omega(t) + P^{-1}_{\hat{\mu}} L_{\hat{\mu}} \nu(t)
\]
Rossler chaotic system with 2 local models

\[ A_1 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\text{max}} \\ 0 & x_{1\text{max}} - 0.37 \end{bmatrix}, \]
\[ A_2 = \begin{bmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\text{min}} \\ 0 & x_{1\text{min}} - 0.37 \end{bmatrix}, \]
\[ B_1 = B_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \]
\[ C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

\[ x_{1\text{min}} = -9.8693 \] and \[ x_{1\text{max}} = 13.8164. \]
Results on academic examples

Rossler chaotic system with 2 local models

\[
A_1 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\text{max}} \\ 0 & x_{1\text{max}} & -0.37 \end{pmatrix}, \quad A_2 = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & -x_{1\text{min}} \\ 0 & x_{1\text{min}} & -0.37 \end{pmatrix},
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\]

\[x_{1\text{min}} = -9.8693 \text{ and } x_{1\text{max}} = 13.8164\]

\[
\mu_1(x(t)) = \frac{x_1(t) - x_{1\text{min}}}{x_{1\text{max}} - x_{1\text{min}}} \quad \mu_2(x(t)) = \frac{x_{1\text{max}} - x_1(t)}{x_{1\text{max}} - x_{1\text{min}}}
\]

\[|\dot{\mu}_i(\hat{x}(t))| \leq \rho_i, \quad \rho_1 = \rho_2 = 4.5, \quad \text{Lipschitz constant } \eta = 173.35\]
Results on academic examples

Rossler chaotic system with 2 local models

- Using the results presented in P. Bergsten and R. Palm (2000) ⇒ no solution (maximal admissible Lipschitz constant : 29.73)
- Asymptotic observer (Theorem 1) ⇒ no solution (LMI are not feasible)
- Observer with guaranteed bounded estimation error (Theorem 2) with $\alpha = 10$
  Obtained attenuation level : $\sqrt{\gamma} = 0.0549$
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Results on academic examples

**Figure:** State variables (blue line) and their estimates (red dashed lines)
Second example: flexible link robot

**FIGURE**: Flexible Joint Robot
Example: Flexible link robot

The model of the flexible link robot is given by:

\[
\dot{x}(t) = Ax(t) + Bu(t) + \phi(x(t))
\]

\[
y(t) = Cx(t)
\]

where:

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = \begin{pmatrix} \theta_m(t) \\ \omega_m(t) \\ \theta_l(t) \\ \omega_l(t) \end{pmatrix}, \quad \phi(x(t)) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -3.33 \sin(x_3(t)) \end{pmatrix}
\]

\[
A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -48.6 & -1.25 & 48.6 & 0 \\ 0 & 0 & 0 & 1 \\ 19.5 & 0 & -19.5 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 21.6 \\ 0 \\ 0 \end{pmatrix}
\]

\[
C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
\]

\[
u(t) = \sin(t), \quad x_0 = [0.5 \quad 0.75 \quad 1 \quad 1]^T
\]
Example: Flexible link robot

The obtained T-S model is:

\[
\dot{x}(t) = \sum_{i=1}^{2} \mu_i(x_3(t))A_i x(t) + Bu(t), \quad y(t) = Cx(t)
\]

**Figure**: Nonlinear vs Takagi-Sugeno
Example: Flexible link robot

The observer takes the form:

\[
\begin{align*}
\dot{\hat{x}}(t) &= \sum_{i=1}^{2} \mu_i(\hat{x}_3(t))(A_i\hat{x}(t) + \left(\sum_{j=1}^{2} \mu_j(\hat{x}_3(t))P_j\right)^{-1}L_i(y(t) - \hat{y}(t))) + Bu(t) \\
\hat{y}(t) &= C\hat{x}(t)
\end{align*}
\]

where the gains \( L_i \) and the matrices \( P_j \) are computed from solving the LMIs given in the Theorem 2 with the parameters \( \alpha = 4.7 \) and \( \rho_1 = \rho_2 = 1 \).

The radius of the convergence ball is:

\[
\sqrt{\frac{c}{\alpha \alpha_1}} \| \delta(t) \|_\infty = 1.095 \| \delta(t) \|_\infty
\]

For the following simulations, a centered random noise with maximal magnitude 0.2 is added to the output.
Example: Flexible link robot

**FIGURE:** State estimation
Conclusions

- Two propositions of observer design for T-S model with unmeasurable premise variables have been proposed.

Perspectives

- Extension of the work to the simultaneous estimation of state and unknown inputs (fault diagnosis and/or fault tolerant control).
- Overcoming the hypothesis about the boundedness of the derivatives of the weighting function (assumptions A2 or A5).
- Application on a real system.
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Two propositions of observer design for T-S model with unmeasurable premise variables have been proposed.

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Conclusions and perspectives

Conclusions

▶ Two propositions of observer design for T-S model with unmeasurable premise variables have been proposed
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▶ The design reduces to the solution of optimization problems subject to LMI (Linear Matrix Inequality) constraints

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More information?

Research Lab: http://www.cran.uhp-nancy.fr/anglais/
Additive material
Construction of TS models – 3 main approaches

- Transformation of a nonlinear model into a multiple model
  - Linearization around some “well-chosen” points
  - Identification of the weighting function parameter to minimize the output error

- Sector nonlinearity transformation
  - Rewriting of the model in a compact subspace of the state space

\[
\begin{align*}
\dot{x}(t) &= f(x(t), u(t)) \\
y(t) &= h(x(t), u(t))
\end{align*}
\Rightarrow
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{l} \mu_i(\xi(t)) (A_i x(t) + B_i u(t)) \\
y(t) &= \sum_{i=1}^{l} \mu_i(\xi(t)) (C_i x(t) + D_i u(t))
\end{align*}
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- **Identification approach**
  - Choice of premise variables
  - Choice of the number of modalities of each premise variable
  - Choice of the structure of the local models
  - Parameter identification

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Sector nonlinearity transformation

Example

\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t) \cos(x_1(t)) + u(t) \\
\dot{x}_2(t) &= x_1^2(t) - x_2(t)
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\Rightarrow
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\dot{x}(t) &= \begin{pmatrix}
-\cos(x_1(t)) \\
x_1(t)
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1 \\
0
\end{pmatrix} u(t)
\end{align*}
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\]

\[-1 \leq \cos(x_1) \leq 1\]

\[x_1 \cos(x_1) = F_1^1(x_1)x_1 - F_2^1(x_1)x_1, \text{ with } F_1^1(x_1) = \frac{\cos(x_1) + 1}{2} \text{ and } F_2^1(x_1) = \frac{1 - \cos(x_1)}{2}\]
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-\cos(x_1(t)) & 0 \\
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\(A_1 = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}, \ A_2 = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}, \ A_3 = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \ A_4 = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}, \ B_i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\)
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  1 & -1
\end{pmatrix}, \quad A_3 = \begin{pmatrix}
  1 & 0 \\
  1 & -1
\end{pmatrix}, \quad A_4 = \begin{pmatrix}
  1 & 0 \\
  -1 & -1
\end{pmatrix}, \quad B_i = \begin{pmatrix}
  1 \\
  0
\end{pmatrix}\]

\[
\begin{cases}
  \dot{x}_1(t) = -x_1(t)\cos(x_1(t)) + u(t) \\
  \dot{x}_2(t) = -x_1^2(t) - x_2(t)
\end{cases}
\Rightarrow \dot{x}(t) = \sum_{i=1}^{4} \mu_i(x_1(t))(A_ix(t) + B_iu(t))
\]