Actuator Fault diagnosis: $H_\infty$ framework with relative degree notion

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Consider the linear system

\[ \dot{x} = Ax +Ef \]
\[ y =Cx \]

where \( x \in \mathbb{R}^n \) is the state vector, \( f \in \mathbb{R} \) is the fault signal and \( y \in \mathbb{R} \) is the output signal.

**Objective**

The objective is to detect or estimate the fault \( f \) from the measurement \( y \).
Problem statement

When the system is affected by a perturbation $d$, the so-called $H_-/H_\infty$ which consists in computing a residual generator

$$\begin{cases} 
    \dot{\hat{x}}(t) = A\hat{x}(t) + L(y(t) - \hat{y}(t)) \\
    \hat{y}(t) = C\hat{x}(t) \\
    r(t) = M(y(t) - \hat{y}(t)) 
\end{cases}$$

with gains $L$ and $M$ in order to satisfy

- Stability of $(A - LC)$
- Minimization of the effect of $d$ on $r$
- Maximization of the sensitivity of $f$ on $r$

This is a min/max problem
Problem statement

In order to transform the min/max problem on a simple min problem, a virtual residual generator is defined as follows

\[ r_e = r - f = MCe - f \]

where \( e = x - \hat{x} \). Then the system generating the state estimation error is given by

\[
\begin{align*}
\dot{e}(t) &= (A - LC) e(t) + Ef(t) \\
r_e(t) &= MCe(t) - f(t)
\end{align*}
\]

In standard \( H_\infty \) framework, the matrices \( L \) and \( M \) should be determined in such a way to satisfy the following constraints

\[
\begin{align*}
\lim_{t \to +\infty} r_e(t) &= 0 & \text{if } f(t) = 0 \\
\|r_e(t)\|_2 &< \gamma \|f(t)\|_2 & \text{if } f(t) \neq 0
\end{align*}
\]
Problem statement

A solution can be found by solving an optimization problem under LMI constraints (SISO case)

\[
\begin{align*}
\min_{P,K,M} \gamma \\
\text{s.t.} \quad \left( \begin{array}{ccc}
A^T P + PA - C^T K^T - KC & PE & C^T M^T \\
E^T P & -\gamma & -1 \\
MC & -1 & -\gamma
\end{array} \right) < 0 \quad (1)
\end{align*}
\]

where \( P = P^T > 0 \). After solving the optimization problem, the matrices of the residual generator are obtained by \( L = P^{-1} K \) and \( M \) is obtained directly. The attenuation level is given by \( \gamma \).
Problem statement

If the previous optimization problem is solved, then one has:

\[
\begin{pmatrix}
-\gamma & -1 \\
-1 & -\gamma
\end{pmatrix} \prec 0
\] (2)

which leads to \( \gamma > 1 \).

Then the best value for \( \gamma \) is \( 1 + \epsilon \) where \( \epsilon \) is a positive small number.
$H_\infty$ Residual Generator with relative degree consideration

Let us consider the system

$$\dot{x} = Ax + Ef$$
$$y = Cx$$

where the relative degree is $r$. This means that

$$y^{(r)}(t) = CA^r x(t) + CA^{r-1} Ef(t)$$

Now, let us consider the new output $\tilde{y}(t)$ defined by

$$\tilde{y}(t) = \begin{pmatrix} y(t) \\ \dot{y}(t) \\ \vdots \\ y^{(r)}(t) \end{pmatrix} = \begin{pmatrix} C \\ CA \\ \vdots \\ CA^r \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ CA^{r-1} E \end{pmatrix} f(t)$$
The system with the new generated output becomes

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + Ef(t) \\
\hat{y}(t) &= \hat{C}\hat{x}(t) + Rf(t)
\end{align*}
\]

The proposed residual generator is

\[
\begin{align*}
\dot{\hat{x}}(t) &= A\hat{x}(t) + L(\hat{y}(t) - \hat{\hat{y}}(t)) \\
\hat{\hat{y}}(t) &= \hat{\hat{C}}\hat{x}(t) \\
r(t) &= M(\hat{y}(t) - \hat{\hat{y}}(t))
\end{align*}
\]
If there exist a symmetric and positive definite matrix $P$, gain matrices $K$ and $M$ and a positive scalar $\gamma$ solution to the following optimization problem

$$\min_{P,K,M} \gamma$$

s.t.

$$\begin{pmatrix}
A^T P + PA - \tilde{C}^T K^T - K \tilde{C} & PE - K \tilde{C} & \tilde{C}^T M^T \\
E^T P - \tilde{C}^T K^T & \tilde{C}^T M^T - 1 & -\gamma \\
M \tilde{C} & MR - 1 & -\gamma 
\end{pmatrix} < 0$$

The gain $L$ of the residual generator is obtained from the equation $L = P^{-1}K$. The attenuation level $\gamma$ describes the sensitivity of $r(t)$ with respect to $f(t)$. The smallest is $\gamma$ the greatest is the sensitivity.
**H∞ Residual Generator with relative degree consideration**

The negativity of (10) implies that

$$\begin{pmatrix} -\gamma & R^T M^T - 1 \\ MR - 1 & -\gamma \end{pmatrix} < 0$$

which is equivalent to

$$\gamma^2 > \left(R^T M^T - 1\right) \left(MR - 1\right)$$

Since this paper considers only systems with single fault and single output, the term \(MR - 1\) is just a scalar, it follows

$$\gamma > MR - 1 \quad (3)$$

Since \(R\) has full column rank due to the relative degree, it is then possible to chose \(M\) such that \((MR - 1) \to 0\). Thus, the parameter \(\gamma > 0\) may takes values small than 1 which enhance the residual sensitivity with respect to the fault compared to the classical approach where \(\gamma > 1\).
$H_\infty$ Residual Generator with relative degree consideration (MIMO case)

Under the observability of the pair $(C, A)$ and the relative degree vector $\{r_1, ..., r_{ny}\}$, the residual generator exists if there exist a symmetric and positive definite matrix $P$, a gain matrix $K$ and a positive scalar $\gamma$ solution to the following optimization problem

$$\min_{P, K, M} \gamma$$

s.t.

$$\begin{pmatrix}
A^T P + PA - \tilde{C}^T \tilde{K}^T - \tilde{K} \tilde{C} & PE - \tilde{K} \tilde{C} & \tilde{C}^T \tilde{M}^T \\
E^T P - \tilde{C}^T \tilde{K}^T & -\gamma I_{nf} & R^T \tilde{M}^T - I_{nf} \\
\tilde{M} \tilde{C} & \tilde{M} R - I_{nf} & -\gamma I_{nf}
\end{pmatrix} < 0$$

The gain $L$ of the residual generator is obtained from the equation $L = P^{-1} K$. The attenuation level $\gamma$ describes the sensitivity of $r(t)$ with respect to $f(t)$. The smallest is $\gamma$ the greatest is the sensitivity.
Notice that the theorem considers the worst case. However, with a simple analysis on the system error dynamics, it can be concluded that: If the condition

\[
\text{rank}\left(\begin{bmatrix} \tilde{C} & \tilde{R} \\ 0 & I_{nf} \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} \tilde{C} & \tilde{R} \end{bmatrix}\right)
\]

Then there exists a matrix \( M \) such that

\[
\begin{cases}
M\tilde{C} = 0 \\
M\tilde{R} = I_{nf}
\end{cases}
\]

Consequently, the error dynamics becomes

\[
\begin{cases}
\dot{e}(t) = (A - \tilde{L}\tilde{C})e(t) + (E - L\tilde{R})f(t) \\
\dot{r}(t) = 0
\end{cases}
\]

Then \( r = f \).
If the rank condition above is not satisfied but if
\[ \text{rank} \left( \begin{bmatrix} \tilde{R} \\ I_{nf} \end{bmatrix} \right) = \text{rank} \left( \tilde{R} \right), \text{rank} \left( \begin{bmatrix} E \\ \tilde{R} \end{bmatrix} \right) = \text{rank} \left( E \right) \]

Then there exist matrices $M$ and $L$ such that
\[
\begin{align*}
L\tilde{R} &= E \\
M\tilde{R} &= I_{nf}
\end{align*}
\]

and in addition, the matrix $L$ stabilizes the matrix $A - LC$, the error dynamics becomes
\[
\begin{align*}
\dot{e}(t) &= \left( A - \tilde{L}\tilde{C} \right) e(t) \\
r_e(t) &= M\tilde{C}e(t)
\end{align*}
\]

Then $r$ converges asymptotically to $f$. 
Simulation example

Consider the system with the matrices

\[ A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \]

The system is observable and the output \( y(t) \) have a relative degree 2 with respect to the fault \( f(t) \).

For the classical approach, solving the optimization problem under the LMI constraint (1) leads to the following solution

\[ P = 10^4 \times \begin{pmatrix} 2.082 & -0.0009 \\ -0.0009 & 0.0000 \end{pmatrix}, \quad L = 10^6 \times \begin{pmatrix} 0.0009 \\ 1.9286 \end{pmatrix}, \quad M = -9.2522, \gamma = 1.001 \]
Simulation example

Figure: Fault and residual signal (classical approach)
Simulation example

With the proposed approach, one has

\[ P = \begin{pmatrix} 1.1417 & 0 \\ 0 & 1.1417 \end{pmatrix}, \]

\[ L = 10^3 \times \begin{pmatrix} 0.0005 & 0 & 0 \\ 0.0010 & 1.0005 & 0.0010 \end{pmatrix}, \quad M = \begin{pmatrix} 1 & 2 & 1 \end{pmatrix} \]

and \( \gamma \) is around \( 10^{-11} \) (the rank conditions are satisfied.)
Simulation example

Figure: Fault and residual signal (proposed approach)
Simulation example

Figure: Residual signals (Comparison)
Conclusion and perspectives

Conclusions

- $H_\infty$ Residual Generator with relative degree consideration
- Rank conditions for exact, asymptotic and bounded fault estimation error convergence

Perspectives

- Including the perturbation affecting the system (Use of Sobolev space and norms)
- Extension to LPV systems
Thank you for your attention