State estimation of nonlinear systems based on heterogeneous multiple models: some recent theoretical results

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Motivations

Context

- State and unknown input estimation is an important topic in automatic control and systems engineering
  - control
  - fault diagnosis
  - encryption/decryption
- Nonlinear models are often unavoidable for modelling complex systems (global modelling)
- Observer design problem for generic nonlinear models is not straightforward

Proposed strategy

- Nonlinear system modelling based on a multiple model representation
- Several realisations (architectures) of multiple models can be considered
- One among them take into account heterogeneous local models
- State and unknown input estimation of nonlinear system based on this multiple model representation is poorly investigated

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Outline

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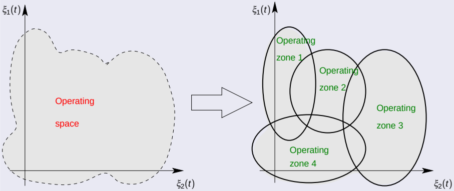
1. Multiple model approach
2. Proportional Observer
3. PI Observer
4. PI Unknown Input Observer
5. Conclusions

Introduction – philosophy

Basis of multiple model approach: divide and conquer



Nonlinear system

Multiple model representation

Multiple model = interpolation of a set of linear submodels

- Appropriate tool for modelling complex systems
- Specific analysis of the system non-linearity is avoided
- Tools for linear systems can partially be extended to nonlinear systems

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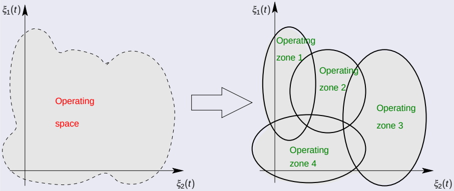

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Classic structure
Submodel parameters interpolation

- Common state vector for all submodels
- Dimension of the submodels must be identical (homogeneous), e.g. Takagi-Sugeno multiple model

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- Less investigated!!

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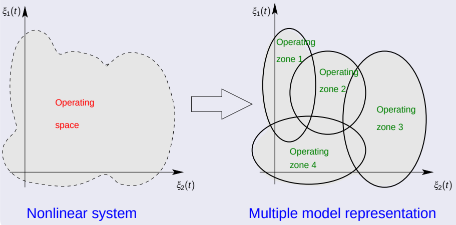

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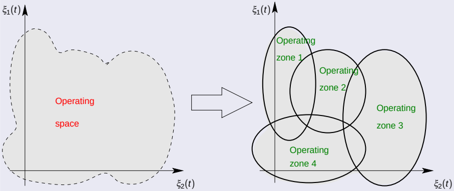

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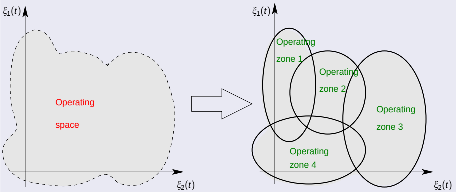

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Heterogeneous multiple model

Heterogeneous multiple model: multiple model with decoupled state vectors

A collection of submodels $\Leftrightarrow$ $\begin{cases} x_i(k+1) = A_i x_i(k) + B_i u(k) + D_i w(t) \\ y_i(k) = C_i x_i(k) \end{cases}$

Disturbances : noise...

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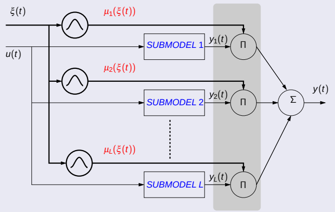
An interpolation mechanism $\Leftrightarrow$ $$y(k) = \sum_{i=1}^{L} \mu_i(\xi(k)) y_i(k) + W w(t)$$

$\xi(k)$: decision variable $\qquad$ $\mu_i(\xi(k))$: weighting functions

$$\sum_{i=1}^{L} \mu_i(\xi(k)) = 1 \quad \text{et} \quad 0 \leq \mu_i(\xi(k)) \leq 1 \quad \forall i \in 1,\dots,L \quad \forall k$$



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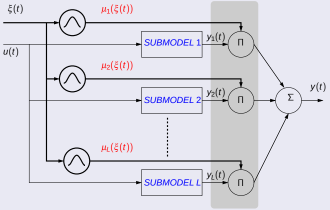

Comments

- The multiple model output is given by a weighted sum of the submodel outputs
- Dimension of the submodels can be different !
- Good flexibility and generality in the modelling stage
- Modelling systems with variable structure

Preliminaries and notations

Augmented form of the multiple model

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t) + D_i w(t) \quad \Leftrightarrow$$

$$y(t) = \sum_{i=1}^{L} \mu_i(\xi(t)) C_i x_i(t) \quad \Leftrightarrow$$

Stability condition

The multiple model stability is ensured by the stability of all submodels ($\tilde{A}$ is block diagonal)

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State estimation

Strategy

- Let us consider the heterogeneous multiple model representation of a nonlinear system
- Extension of some LTI results to heterogeneous multiple models
- Robustness properties of the state estimation with respect to disturbances and unknown inputs (UI) are investigated
- Sufficient conditions for observer design are established on the basis of the Lyapunov method

Observer structures

- Survey of recent results in state estimation strategies based on heterogeneous multiple models
- Different kinds of observers are investigated:
    1. Proportional gain observer
    2. Proportional-Integral observer for disturbance attenuation (first case)
    3. Proportional-Integral observer for unknown input estimation (second case)

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Proportional gain observer design 1/2

Proportional gain observer structure

Proportional gain observer : model of the system with a correction action $\tilde{K}$

$$\begin{aligned} \dot{\hat{x}}(t) &= \tilde{A}\hat{x}(t) + \tilde{B}u(t) - \tilde{K}(y(t) - \hat{y}(t)) \\ \hat{y}(t) &= \tilde{C}(t)\hat{x}(t) \end{aligned}$$

Goal

1. Consider the state estimation error: $e(t) = x(t) - \hat{x}(t)$
2. Consider its time-derivative:

$$\dot{e}(t) = (\tilde{A} - \tilde{K}\tilde{C}(t))e(t) + (\tilde{D} - \tilde{K}W)w(t) \text{ where } \tilde{K} = \left[K_1^T \cdots K_i^T \cdots K_L^T\right]^T \in \mathbb{R}^{n \times p}$$

3. Goal: $\tilde{K}$ to be determined such that :

$$\lim_{t \to \infty} e(t) = 0 \text{ for } \omega(t) = 0 \quad \text{and} \quad \|e\|_2^2 \leq \gamma^2 \|w\|_2^2 \text{ for } w(t) \neq 0 \text{ and } e(0) = 0$$

- γ is the $\mathcal{L}_2$ gain from $w(t)$ to $e(t)$ to be minimised
- The convergence of the estimation error in the disturbance-free case is ensured
- The robust state estimation in presence of a disturbance is also ensured

Proportional gain observer design 2/2

Theorem: observer existence conditions

The robust proportional observer is obtained if there exists a matrix $P = P^T = 0$ and a matrix G solution of the constrained optimisation problem for a given scalar $\alpha \geq 0$:

$$\min \bar{\gamma} \quad \text{subject to}$$

$$\begin{bmatrix} \mathcal{A}_i + \mathcal{A}_i^T + \mathrm{I} & \mathcal{B} \\ \mathcal{B}^T & -\bar{\gamma}\mathrm{I} \end{bmatrix} < 0 \ , \quad i = 1, \ldots, L$$

where $\mathcal{A}_i = P(\tilde{A} + \alpha \mathrm{I}) - G\tilde{C}_i$ and $\mathcal{B} = P\tilde{D} - GW$
The observer gain is $\tilde{K} = P^{-1}G$ and α is the decay rate for exponential convergence of $e(t)$

Sketch of the proof

1. Consider the following quadratic Lyapunov function:

$$V(t) = e^T(t)Pe(t) \quad P > 0 \quad P = P^T$$

2. Robust constraint are satisfied if:

$$\dot{V}(t) + 2\alpha V(t) < \gamma^2 w^T(t)w(t) - e^T(t)e(t)$$

3. See the proceedings for a detailed proof

Proportional-Integral observer design: first case 1/3

Proportional-Integral observer: justification

- Only one degree of freedom $\tilde{K}$ is available to reject disturbances and to improve the dynamic performances (two antagonist design goals)
- The gain $\tilde{K}$ is replaced by two correction actions: proportional and integral
- A supplementary integral variable $z(t)$ is introduced:

$$z(t) = \int_0^t y(\xi)d\xi \quad \Rightarrow \quad \dot{z}(t) = y(t)$$

- The multiple model in now given

$$\begin{aligned} \dot{x}_a(t) &= \tilde{A}_1(t)x_a(t) + \bar{C}_1\tilde{B}u(t) + (\bar{C}_1\tilde{D} + \bar{C}_2W)w(t) \ , \\ y(t) &= \tilde{C}(t)\bar{C}_1^T x_a(t) + Ww(t) \ , \\ z(t) &= \bar{C}_2^T x_a(t) \end{aligned}$$

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$$x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \tilde{A}_1(t) = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C}(t) & 0 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} I_n \\ 0 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} 0 \\ I_p \end{bmatrix}.$$

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Proportional-Integral observer design: first case 2/3

Proportional-Integral observer structure

- Proportional-Integral observer

$$\begin{aligned}
\dot{\hat{x}}_a(t) &= \tilde{A}_1(t)\hat{x}_a(t)+\bar{C}_1\tilde{B}u(t)+K_P(y(t)-\hat{y}(t))+K_I(z(t)-\hat{z}(t)) \\
\hat{y}(t) &= \tilde{C}(t)\bar{C}_1^T\hat{x}_a(t) \qquad \tilde{K}_P \text{Proportional action} \quad \tilde{K}_I \text{Integral action} \\
\hat{z}(t) &= \bar{C}_2^T\hat{x}_a(t)
\end{aligned}$$

- Consider the state estimation error: $e_a(t)=x_a(t)-\hat{x}_a(t)$
- Consider its time-derivative:

$$\dot{e}_a(t) = (\tilde{A}_i(t)-\tilde{K}_P\tilde{C}(t)\bar{C}_1^T-\tilde{K}_I\bar{C}_2^T)e_a(t)+(\bar{C}_1\tilde{D}+\bar{C}_2W-\tilde{K}_PW)w(t)$$

Comments

The PI observer offers two degrees of freedom $\tilde{K}_P$ and $\tilde{K}_I$

1. $\tilde{K}_P$ can be used to reduce the impact of the disturbance $w(t)$ on $e_a(t)$
2. $\tilde{K}_I$ can be used to improve the observer dynamics performances
3. PI observer existence conditions are obtained as in the previous case

Proportional-Integral observer design: first case 2/3

Proportional-Integral observer structure

- Proportional-Integral observer

$$\begin{aligned}
\dot{\hat{x}}_a(t) &= \tilde{A}_1(t)\hat{x}_a(t)+\bar{C}_1\tilde{B}u(t)+K_P(y(t)-\hat{y}(t))+K_I(z(t)-\hat{z}(t)) \\
\hat{y}(t) &= \tilde{C}(t)\bar{C}_1^T\hat{x}_a(t) \\
\hat{z}(t) &= \bar{C}_2^T\hat{x}_a(t)
\end{aligned}$$

$\tilde{K}_P$ Proportional action $\tilde{K}_I$ Integral action

- Consider the state estimation error: $e_a(t)=x_a(t)-\hat{x}_a(t)$
- Consider its time-derivative:

$$\dot{e}_a(t) = (\tilde{A}_i(t)-\tilde{K}_P\tilde{C}(t)\bar{C}_1^T-\tilde{K}_I\bar{C}_2^T)e_a(t)+(\bar{C}_1\tilde{D}+\bar{C}_2W-\tilde{K}_PW)w(t)$$

Comments

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Proportional-Integral observer design: first case 3/3

Theorem: Observer existence conditions

The robust proportional-integral observer is obtained if there exists a matrix $P = P^T = 0$ and a matrices G_P and G_I solution of the constrained optimisation problem for a given scalar $\alpha \geq 0$:

$$\min \bar{\gamma} \quad \text{subject to}$$

$$\begin{bmatrix} \mathcal{A}_i + \mathcal{A}_i^T + \mathrm{I} & \mathcal{B} \\ \mathcal{B}^T & -\bar{\gamma}\mathrm{I} \end{bmatrix} < 0 \ , \quad i = 1, \ldots, L$$

where

$$\mathcal{A}_i = P(\tilde{A}_1 + \alpha \mathrm{I}) - G_P \tilde{C}_i \bar{C}_1^T - G_I \bar{C}_2^T \quad \text{and} \quad \mathcal{B} = P\bar{C}_1\tilde{D} + P\bar{C}_2 W - G_P W$$

The gains are $\tilde{K}_P = P^{-1} G_P$ and $\tilde{K}_I = P^{-1} G_I$

Comments

- These conditions are similar to the previous ones (the LMI structure is quite similar)
- An additional parameter must be found (the integral gain)
- Unknown input estimation is not taken into consideration!

Proportional-Integral unknown input observer design: second case 1/3

Unknown input assumptions

- $w(t)$ is now an unknown input (UI) to be estimated instead of a disturbance to be attenuated
- This UI can be employed to characterize an actuator failure and/or an abnormal behaviour
- The UI $w(t)$ is supposed to be a constant signal, i.e. $\dot{w}(t) = 0$

Proportional-Integral unknown input observer

- The goal is to generate both state and UI estimations
- The proportional-integral unknown input observer is given

$$\begin{aligned}
\dot{\hat{x}}(t) &= \tilde{A}\hat{x}(t) + \tilde{B}u(t) + \tilde{D}\hat{w}(t) + \tilde{K}_P(y(t) - \hat{y}(t)) \quad \tilde{K}_P \text{ Proportional action} \\
\dot{\hat{w}}(t) &= \tilde{K}_I(y(t) - \hat{y}(t)) \quad \tilde{K}_I \text{ Integral action} \\
\hat{y}(t) &= \tilde{C}(t)\hat{x}(t) + W\hat{w}(t)
\end{aligned}$$

where $\hat{w}(t)$ provides an estimation of the UI $w(t)$

- $\tilde{K}_P$ is a correction injection term
- $\tilde{K}_I$ is used to UI estimation

Proportional-Integral unknown input observer design: second case 1/3

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Proportional-Integral unknown input observer design: second case 2/3

Proportional-Integral unknown input observer

- Consider now the following augmented state vector:

$$\Sigma(t) = \begin{bmatrix} e(t) \\ \varepsilon(t) \end{bmatrix} = \begin{bmatrix} x(t) - \hat{x}(t) \\ w(t) - \hat{w}(t) \end{bmatrix} \in \mathbb{R}^{n+r}$$

- Its time-derivative is given by:

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} - \tilde{K}_P \tilde{C}(t) & \tilde{D} - \tilde{K}_P W \\ -\tilde{K}_I \tilde{C}(t) & -\tilde{K}_I W \end{bmatrix} \begin{bmatrix} e(t) \\ \varepsilon(t) \end{bmatrix}$$

- Consider the compact form

$$\dot{\Sigma}(t) = (A_a - K_a C_a(t)) \Sigma(t)$$

where $A_a = \begin{bmatrix} \tilde{A} & \tilde{D} \\ 0 & 0 \end{bmatrix}$, $K_a = \begin{bmatrix} \tilde{K}_P \\ \tilde{K}_I \end{bmatrix}$, $C_a(t) = \begin{bmatrix} \tilde{C}(t) & W \end{bmatrix}$

- PI observer existence conditions are obtained as in the previous case
- The augmented PI unknown input observer structure is quite similar to the P observer structure!

Proportional-Integral unknown input observer design: second case 2/3

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Proportional-Integral unknown input observer design: second case 3/3

Theorem: observer existence conditions

The robust proportional-integral unknown input observer is obtained if there exists a matrix $P = P^T = 0$ and matrix a G_a solution of the constrained optimisation problem for a given scalar decay rate $\alpha \geq 0$:

$$\mathcal{A}_i + \mathcal{A}_i^T < 0 \quad , \quad i = 1, \ldots, L$$

where

$$\mathcal{A}_i = P(A_a + \alpha I) - G_a \bar{\mathcal{C}}_i \qquad \text{and} \qquad \bar{\mathcal{C}}_i = \begin{bmatrix} \tilde{C}_i & W \end{bmatrix}$$

The observer gain is given by $K_a = P^{-1} G_a$.

Comments

- Disturbance (noise) acting on the system can easily be considered
- The UI can be a low frequency signal e.g. slowly varying-time signal
- This UI observer can be used for detection and isolation of sensor and actuator failures
- UI estimation can be directly employed as a residual signal in a FDI scheme.

Conclusions

Conclusions

- Heterogeneous multiple models are poorly investigated for state estimation and diagnosis
- Recent theoretical results on the state and UI estimation based on heterogeneous multiple models are presented
- New robust observer design conditions under LMI forms are proposed
- The Proportional, Proportional-Integral and Proportional-Integral Unknown Input observer are investigated

Perspectives

- The conservatism of the proposed LMI conditions must be investigated. How restrictive is this condition?
- A more general class of unknown inputs can be considered by improving the proposed proportional-integral UI observer

Thank you!
comments are welcome!