

PARAMETER UNCERTAINTIES CHARACTERISATION FOR LINEAR MODELS.

José Ragot^{*}, Didier Maquin^{*}, Olivier Adrot^{**}

^{*} *Centre de Recherche en Automatique de Nancy, CNRS
UMR 7039, INPL. 2, Avenue de la forêt de Haye, F 54516
Vandœuvre-les-Nancy Cedex, {José.Ragot,*

Didier.Maquin}@ensem.inpl-nancy.fr

^{**} *Laboratoire d'Automatique de Grenoble, CNRS UMR
5528, INPG, 961, rue de la Houille Blanche, BP 46, F 38
402 Saint Martin d'Hères Cedex,*

Olivier.Adrot@lag.ensieg.inpg.fr

Abstract: Parameter estimation mainly consists in characterising a parameter set consistent with measurements, the model and the equation error description. The problem to be solved is that of finding the set of admissible parameter values corresponding to an admissible error. The uncertainties must be treated by a global analysis of the problem: both the equation error and the parameter set are considered unknown. Then, a solution is given as a domain of time-variant parameters and a bounded set of the error. This procedure consists in explaining the measurements performed at all time by optimising a precision criterion based on the polytope theory. *Copyright © 2006 IFAC.*

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1. INTRODUCTION

1.1 Historical point of view

The parameter estimation using the set-membership approach started in the eighties, where the strategy initially consisted in circumscribing the domain describing model uncertainties by a simple form. This approach was originally designed to deal with a model linear in uncertain parameters and characterised by a bounded error. The problem of parameter estimation amounts to the determination of the set of constant parameter values called the Feasible Parameter Set (F.P.S.). Each value of this set explains all the available observations which are consistent with the bounds of the error and the model structure. For models linear in their parameters, the F.P.S. is a convex polytope which can be approximated by ellipsoids Fogel and Huang (1982), or orthotopes Milanese and Belforte (1982) containing it. The work in Walter and Piet-Lahanier (1989) on the one hand, and Mo and Norton (1990) on the other, used polytopic domains. The main results are presented in the book

published by Milanese et al. (Milanese et al. (1996)). For models nonlinear in their parameters, various methods exist for determining an approximation of the F.P.S., linear techniques have been extended to the nonlinear case using multiple linearisation of the model Belforte et al. (1990). In Reinelt et al. (2002) a robust identification approach is proposed taking into account unmodeled dynamics and noise affecting data; as uncertainty is evaluated in terms of frequency response, so that it can be handled by \mathcal{H}_∞ technics. In ElGhaoui and Calafiore (2000), the authors explain that the set of possible models is unfalsified by the observed data if that data could actually have been produced by one member of the model set. Results have been obtained (Jaulin (1993)) in order to solve the problem of nonlinear bounded-error estimation using set inversion techniques and based on interval analysis, there make it possible to characterise the F.P.S. by enclosing it between internal and external unions of boxes. The paper Jaulin (2001) deals with a minimax parameter estimation of nonlinear parametric models from experimental data. For specific

model structures, it is possible to obtain sets of linear inequalities describing a domain approximating the F.P.S. Clement and Gentil (1988). Despite the resemblance, the problem considered in Ploix et al. (1999) is noticeably different in the sense that uncertain parameters depend on time; more exactly, they are defined by random variables with bounded realisations; moreover, this paper only deals with MISO representation. The proposed method is a no probability technique for determining the inaccuracy with which each model parameter is known. Only a class of structured and models linear in uncertain parameters is considered. The error is bounded while parameters fluctuate inside a time-invariant bounded domain represented by a zonotope which explains the measurement. Thus, the paper deals with parameters estimation in a bounded-error context for models which are linear in the parameters; parameters could vary with a limited range and measurement errors are bounded but both domains are not a priori known. The objective of this method is to determine the characteristics of these domains (centre, uncertainty range).

1.2 Model structure

In order to present the principle of the proposed method, let us consider the following MISO model:

$$y_m(k) = x^T(k)\theta(k) \quad k = 1..N \quad (1a)$$

$$y(k) = y_m(k) + e(k) \quad (1b)$$

where $y_m(k)$ is the model output, $x(k) \in \mathcal{R}^p$ is the regressor vector at the instant k , $y(k)$ is the output measurement, whereas $\theta(k) \in \mathcal{R}^p$ defines the uncertain parameter vector. The error $e(k)$ is assumed to be bounded, the bounds being supposed invariant along the time:

$$e(k) \in [-\delta, \delta] \quad (2)$$

Thus, taking (2) into account, (1b) leads to:

$$y(k) - \delta \leq x^T(k)\theta(k) \quad k = 1..N \quad (3a)$$

$$x^T(k)\theta(k) \leq y(k) + \delta \quad k = 1..N \quad (3b)$$

Thus, at each instant k , the known measurement $y(k)$ belongs to an interval defined by (3) and the width of this interval depends both on the bound δ and the value set of $\theta(k)$. In the following, considering the MIMO case, our objective is to search if there exists, at each instant k , at least one value of $(\theta(k), \delta)$ satisfying constraints (3).

1.3 The parameters characterization problem

Let us now formulate the preceding remark for any linear system with bounded time-varying parameters.

In this context, the method proposed by (Ploix et al. (1999)) consists in finding a convex zonotope (its mathematical description is explained thereafter) D_N centred on θ_c and defined by:

$$D_N = \{\theta(k) \in \mathcal{R}^p / \theta(k) = \theta_c + M(\lambda)\nu(k)\} \quad (4)$$

$$M(0) = 0, \quad \|\nu(k)\|_\infty \leq 1$$

such that it contains, at each instant k , at least a value of the time-variant parameter vector $\theta(k)$ which is fully compatible with the measurement $y(k)$. The matrix $M(\lambda)$ characterises the shape of the domain D_N , λ being parameters for adjusting the dimension of that domain. In this way, $\theta(k)$ fluctuates around its central value θ_c inside D_N for satisfying all constraints (2), θ_c being considered as the nominal value of the parameter $\theta(k)$. In order to increase the model precision, D_N must be the smaller domain centred on θ_c with respect to the form imposed by (4).

The problem treated herein is the computation of the central parameter value θ_c , the parameter uncertainties and an appropriate characterisation of the error domain for MIMO systems. Thus, the characterisation procedure consists in determining the bounds of model uncertainties (λ, δ) and the center θ_c which are totally compatible with the set of available measurements.

The paper is organised as follows. In the next section, the formalisation of the problem is detailed. In section 3, a precision criterion is defined in order to identify the model having the minimum uncertainty. The principle of parameter estimation while optimising the given criterion is presented in section 4. In section 5, an example illustrates the proposed method.

2. PROBLEM FORMULATION

We describe in subsections 2.1 and 2.2 the structure of an uncertain system and the uncertainties. In subsection 2.3, we define a time-invariantly zonotope in the parameter space such that, at each instant k , it contains at least one value of the time-varying parameter $\theta(k)$ consistent with the observations. Then, it is shown that the measurement equation maps the zonotope \mathcal{P}_θ in a new zonotope \mathcal{P}_Y in the measurement space.¹

2.1 Modelling of an uncertain system

In order to generalise the representation given by (1), let us consider an uncertain model of a system with several outputs, linear in parameters and observations, and represented by the following structure:

$$Y(k) = X(k)\theta(k) + E(k) \quad k = 1..N \quad (5)$$

where $Y(k) \in \mathcal{R}^n$, $X(k) \in \mathcal{R}^{n \cdot p}$ are the known variables at the time k and $\theta(k) \in \mathcal{R}^p$ defines model parameters. The bounded vector $E(k) \in \mathcal{R}^n$ defines the error taking into account the uncertainties due to the measuring process and to modelling errors at the same time. This type of model includes the particular case of MISO systems and that of MIMO systems. In the MIMO case, according to the presence of uncertain parameters in $\theta(k)$, the outputs $Y(k)$ can be coupled by some of the uncertain parameters $\nu(k)$

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and that can lead to some difficulties in the estimation problem.

Let us consider the variables, $X(k)$ and $Y(k)$ of which the measurements are noted respectively $\tilde{X}(k)$ and $\tilde{Y}(k)$. The problem involved with parameter estimation is to determine the parameter values consistent with data

2.2 Description of the uncertainties

Uncertainties affecting a system are classified into two categories. On the one hand, those acting directly on the output are additive uncertainties $E(k)$, and on the other hand, the uncertainties describing the parameter $\theta(k)$ occur in a multiplicative way.

Additive uncertainties are represented by the vector $E(k) \in \mathcal{R}^n$ assumed to belong to the domain $\mathcal{P}_E(\delta)$:

$$\mathcal{P}_E(\delta) = \{Z(\delta)u, \|u\|_\infty \leq 1\} \quad (6)$$

with $\delta = (\delta_1 \dots \delta_n)^T$, $u = (u_1 \dots u_n)^T$ and $Z(\delta) \in \mathcal{R}^{n \times n}$. When these uncertainties affect independently each output, $Z(\delta)$ has the following structure:

$$Z(\delta) = \text{Diag}(\delta)$$

The vector δ defines the magnitude of additive uncertainties which are considered centered and bounded.

Multiplicative uncertainties are represented by the parameter vector $\theta(k) \in \mathcal{R}^p$ which fluctuates in an invariant domain denoted $\mathcal{P}_\theta(\lambda, \theta_c)$, defined by:

$$\mathcal{P}_\theta(\lambda, \theta_c) = \{\theta(k) = \theta_c + M(\lambda)\nu(k), \|\nu(k)\|_\infty \leq 1\} \quad (7)$$

These uncertainties are distributed on the various components of the vector θ via a full row rank matrix $M(\lambda) \in \mathcal{R}^{p \times q}$ depending on the vector $\lambda = (\lambda_1 \dots \lambda_q)^T$. In the rest of the paper, the matrix $M(\lambda)$ is supposed having the following structure:

$$M(\lambda) = M \text{Diag}(\lambda) \quad (8)$$

2.3 Principle of parameter estimation

The parameter estimation problem consists in finding the values of the vectors θ_c , λ and δ which define the parameters domain $\mathcal{P}_\theta(\lambda, \theta_c)$ (7) and the measurement errors domain $\mathcal{P}_E(\delta)$ (6), so that the model explains the available measurements in the most precise way:

$$\tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c) \quad k = 1..N \quad (9)$$

with:

$$\begin{aligned} \mathcal{P}_Y(\lambda, \delta, \theta_c) = \{Y(k) \in / Y(k) = \tilde{X}(k)\theta_c + \\ \tilde{X}(k)M(\lambda)\nu(k) + Z(\delta)u(k), \\ \|u(k)\|_\infty \leq 1, \|\nu(k)\|_\infty \leq 1\} \end{aligned} \quad (10)$$

$\mathcal{P}_Y(\lambda, \delta, \theta_c)$ defines all possible values of the output variables $Y(k)$ consistent with variables $X(k)$, the model parameters θ_c and the model uncertainties

description given by the vectors λ and δ . Indeed, considering (10), if $\tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c)$ then

$$\exists w(k) \in \mathcal{H}_{q+n} / \tilde{Y}(k) = \tilde{Y}_c(\theta_c, k) + T(k, \lambda, \delta)w(k) \quad (11)$$

$$\begin{aligned} T(k, \lambda, \delta) &= (\tilde{X}(k)M(\lambda) \quad Z(\delta)) \\ \tilde{Y}_c(\theta_c, k) &= \tilde{X}(k)\theta_c \quad w(k) = \begin{pmatrix} \nu(k) \\ u(k) \end{pmatrix} \end{aligned} \quad (12)$$

In this paper, the shape of the domain is partially determined by the matrix M which structure is fixed a priori; whatever the choice of this shape, all parameters parameters that are compatible with measurements, error bounds and model structure will be enclosed in the domain.

If we take $\lambda_i = 0$, the scalars δ_i can be chosen as large as we want for a given value of θ_c , since that consists in increasing the volume of the domain of uncertainties occurring in the model, until being compatible with all measurements.

If the measurements are not affected by errors, then the model may be compatible with the measurements by increasing the magnitude of λ_i .

In the other cases, it will be possible to define a criterion representative of the precision, the latter being related to the domain extent: indeed increasing "arbitrarily" the values of λ_i and δ_i in order to explain measurements is not satisfactory. Therefore, it is necessary to find a quantity which is sensitive to the difference between real measurements and their estimates. Ploix et al. (1999), defined a criterion based on interval arithmetic (Moore (1979), Neumaier (1990)) for a model with only one output. An obvious and intuitive choice that one can make, is to consider the volume of the domain. It is easy to show that its volume is proportional to the components of λ . Then, the solution is the smallest λ which explains all measurements.

The application of this procedure, when $\mathcal{P}_Y(\lambda, \delta, \theta_c)$ has an complicated form, leads to some calculation problems. One needs to find a criterion which is at once representative of the model precision and which does not lead to computation difficulties.

3. CHOICE OF THE CRITERION

The aim of this section is to define a criterion which provides a solution $(\lambda_s, \delta_s, \theta_{c,s})$ in such a way that the domain $\mathcal{P}_Y(\lambda_s, \delta_s, \theta_{c,s})$, corresponding to the estimation of $Y(k)$, contains all the measurements $\tilde{Y}(k)$ while having a minimal size.

3.1 Definition of a vertex

Let us consider a bounded polytope $\Delta \subset \mathcal{R}^n$ defined by r linear inequalities ($r > n$) which can be written as: $Ay \leq b$, $\forall y \in \Delta$, with $A \in \mathcal{R}^{r \times n}$ and $b \in \mathcal{R}^r$. S is a vertex of Δ if the two following conditions hold:

$$AS - b \text{ contains at least } n \text{ nul elements.} \quad (13a)$$

$$AS \leq b. \quad (13b)$$

Since the rows of A and the elements of b define all the hyperplanes which constitute the faces of Δ ², the first condition (13a) means that a vertex S is the intersection of at least n hyperplanes limiting the hull of Δ . Hence, there are n indexes $i_j, 1 \leq i_1 \dots i_n \leq r$

such that $\Gamma S = t$ with $\Gamma = \begin{pmatrix} a_{i_1}^T \\ \dots \\ a_{i_n}^T \end{pmatrix}$, $t = (b_{i_1} \dots b_{i_n})^T$,

a_j is the j th row of A and b_j the j th element of b . The second condition (13b) is that the vertex S belongs to Δ : $A\Gamma^{-1}t \leq b$. If this does not hold, S is called a pseudo-vertex.

3.2 Data zonotope characterisation

In the previous section, the definition of a polytope has been recalled; this definition may be directly applied for representing either the parameter domain of an uncertain system or the domain of the measurements of the system. Then, the distances between the centre of $\mathcal{P}_Y(\lambda, \delta, \theta_c)$ and its vertices can also describe this shape. So, it is then possible to consider these distances as a criterion of the model precision.

Principle for the polytope generation

The expression which generates the domain $\mathcal{P}_Y(\lambda, \delta, \theta_c)$, parametrized by λ , δ and θ_c is given in (10) in which the matrix $T(k, \lambda, \delta)$, defined in (12), has also the form:

$$T(k, \lambda, \delta) = (\lambda_1 t_1(k) \dots \lambda_q t_q(k) \delta_1 e_1 \dots \delta_n e_n) \quad (14)$$

with $t_i(k) = \tilde{X}(k)m_i$, for $i = 1..n$, and $I_n = (e_1 \dots e_n)$ being the identity matrix in $\mathcal{R}^{n \times n}$. For a such form of the matrix $T(k, \lambda, \delta)$, it is possible to generate, by combination, systematically all linear inequalities describing $\mathcal{P}_Y(\lambda, \delta, \theta_c)$ as:

$$\tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c) \Leftrightarrow R(k)\tilde{Y}(k) \leq d(k, \lambda, \delta, \theta_c) \quad (15)$$

with $R(k) \in \mathcal{R}^{\bar{r} \times n}$ (notice that \bar{r} is the number of inequalities defining the domain \mathcal{P}_Y). This is justified by the fact that $\tilde{Y}(k)$ is linear in respect to $w(k)$ which is itself bounded; therefore $\tilde{Y}(k)$ is also bounded and $d(k, \lambda, \delta, \theta_c)$ is linear in λ , θ_c and δ . The determination of $d(k, \lambda, \delta, \theta_c)$ and $R(k)$ is presented in the remainder of this section.

Polytope generation

Now, we are interested in the computation of all vertices of $\mathcal{P}_Y(\lambda, \delta, \theta_c)$; according to (15), these vertices are defined by a set of inequalities $R(k)\tilde{Y}(k) \leq d(k, \lambda, \delta, \theta_c)$. This procedure is performed by the following two steps. The first one concerns the condition (13a) and consists in finding all matrices

$$\Gamma_i(k) = \begin{pmatrix} a_{i_1}^T(k) \\ \dots \\ a_{i_n}^T(k) \end{pmatrix} \quad (i = 1..n_k, n_k \leq C_{\bar{r}}^n, 1 \leq i_j \leq \bar{r})$$

containing n linearly independent rows of $R(k)$ and the corresponding vector $\gamma_i(k, \lambda, \delta, \theta_c) =$

$(d_{i_1}(k, \lambda, \delta, \theta_c) \dots d_{i_n}(k, \lambda, \delta, \theta_c))^T$. Then we have to determine the points $S_i(k)$ which are the intersections of the n considered hyperplanes. This leads to the expression of $S_i(k)$:

$$S_i(k) = \Gamma_i^{-1}(k)\gamma_i(k, \lambda, \delta, \theta_c) \quad (16)$$

The second step concerns condition (13b) and checks whether the point $S_i(k)$ is a vertex or a pseudo-vertex of $\mathcal{P}_Y(\lambda, \delta, \theta_c)$; this consists in testing whether $S_i(k)$ belongs $\mathcal{P}_Y(\lambda, \delta, \theta_c)$. Using (15) and (16):

$$S_i(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c) \Leftrightarrow R(k)\Gamma_i^{-1}(k)\gamma_i(k, \lambda, \delta, \theta_c) \leq d(k, \lambda, \delta, \theta_c) \quad (17)$$

Unfortunately, the last inequality cannot be easily tested because it is parameterised by λ , δ and θ_c which are unknown. Consequently, in the following, all the points $S_i(k)$ checking only the first condition (13a) are considered (thus without any distinction between vertices and pseudo-vertices).

Vertices generation

The determination of each point $S_i(k)$ requires initially the knowledge of its associated matrix $\Gamma_i(k)$ and vector $d_i(k, \lambda, \delta, \theta_c)$ which are based on the knowledge of $R(k)$ and $d(k, \lambda, \delta, \theta_c)$ corresponding to the linear inequalities describing $\mathcal{P}_Y(\lambda, \delta, \theta_c)$. Then, the problem is to find $R(k)$ and $d(k, \lambda, \delta, \theta_c)$ such that:

$$\begin{aligned} \tilde{Y}(k) \in \mathcal{P}_Y(\lambda, \delta, \theta_c) &\Leftrightarrow R(k)\tilde{Y}(k) \leq d(k, \lambda, \delta, \theta_c) \\ &\Leftrightarrow \exists w \in \mathcal{H}_{q+n} / \tilde{Y}(k) = \tilde{Y}_c(\theta_c, k) + T(k, \lambda, \delta)w(k) \end{aligned} \quad (18)$$

Considering the relation (18), the idea is to analyse the influence of the bounded variable $w(k)$ on each component of $\tilde{Y}(k)$.

In order to take into account these dependancies, the method consists in considering $(n-1)$ elements $s_j = \{w_{j_1}(k) \dots w_{j_{n-1}}(k)\}$ of $w(k)$ among $(q+n)$ and corresponding to independent columns of $T(k)$, then looking for a linear combination of the components of \tilde{Y} ($\tilde{Y}_i, i = 1 \dots n$), noted $C_j = g_j^T \tilde{Y}$ which is independent of $w_{j_1}(k) \dots w_{j_{n-1}}(k)$. Then, g_j is the vector orthogonal to the $(n-1)$ columns $\tilde{t}_{j_1}(k) \dots \tilde{t}_{j_{n-1}}(k)$ of the matrix $\tilde{T}(k)$ (\tilde{t}_i is the i th column of $\tilde{T}(k)$). Therefore, C_j depends only on the $(q+1)$ components of $w(k)$ which do not belong to the set s_j . As these components vary in \mathcal{H}_{q+1} , then it is possible to determine a lower and an upper bounds of C_j as:

$$g_j^T \tilde{Y}(k) \leq g_j^T \tilde{Y}_c(\theta_c, k) + |g_j^T \tilde{T}(k)| \alpha \quad (19a)$$

$$g_j^T \tilde{Y}(k) \geq g_j^T \tilde{Y}_c(\theta_c, k) - |g_j^T \tilde{T}(k)| \alpha \quad (19b)$$

By iterating this procedure for all sets $s_j = \{w_{j_1}(k) \dots w_{j_{n-1}}(k)\}$ of bounded variables to eliminate ($j = 1..n_y, n_y = C_{q+n}^{n-1}$) and aggregating the pairs of inequalities (19), one obtains:

² if a_i^T is the i^{th} row of A and b_i the i^{th} element of b , then the i^{th} face of Δ is $F_i = \{y \in \mathcal{R}^n / a_i^T y = b_i \text{ and } Ay \leq b\}$.

$$\begin{pmatrix} g_1^T \\ \dots \\ g_{n_y}^T \\ -g_1^T \\ \dots \\ -g_{n_y}^T \end{pmatrix} (\tilde{Y}(k) - \tilde{Y}_c(\theta_c, k)) \leq \begin{pmatrix} |g_1^T \tilde{T}(k)| \\ \dots \\ |g_{n_y}^T \tilde{T}(k)| \\ |g_1^T \tilde{T}(k)| \\ \dots \\ |g_{n_y}^T \tilde{T}(k)| \end{pmatrix} \alpha \quad (20)$$

Finally, the parallelotope $\mathcal{P}_Y(\alpha, \theta_c)$ is defined as:

$$\tilde{Y}(k) \in \mathcal{P}_Y(\alpha, \theta_c) \Leftrightarrow R(k)\tilde{Y}(k) \leq d(k, \alpha, \theta_c) \quad (21)$$

$$R(k) = (g_1 \dots g_{n_y} - g_1 \dots - g_{n_y})$$

$$d(k, \alpha, \theta_c) = R(k)\tilde{Y}_c(\theta_c, k) + |R(k)\tilde{T}(k)| \alpha$$

where $R(k) \in \mathcal{R}^{2(n_y+n).n}$ and $d(k, \alpha, \theta) \in \mathcal{R}^{2(n_y+n)}$.

3.3 Precision criterion

The main result of section 3.2 provides the bounds of a domain to which the measurements $\tilde{Y}(k)$ belong. This domain is characterized by several parameters, i.e. the center θ_c of the parameter domain, the shape of the domain described by the λ parameter and the bound δ of the error. It is clear that the "best" parameter vector is that which can explain all the measurements with the smaller fluctuations of its parameters, these fluctuations depending on λ and δ . For that purpose, we have to compute the distances between the centre of $\mathcal{P}_Y(\alpha, \theta_c)$ and its vertices. For that, the following consists in finding all matrices $\Gamma_i(k) = (a_{i_1}^T(k) \dots a_{i_n}^T(k))^T$ ($i = 1..n_k, n_k \leq C_r$) containing n linearly independent rows of $R(k)$ and the corresponding vector $d_i(k, \alpha, \theta_c) = (d_{i_1}(k, \alpha, \theta_c) \dots d_{i_n}(k, \alpha, \theta_c))^T$, and then determine the points $S_i(k)$ such that:

$$\Gamma_i(k)(S_i(k) - \tilde{Y}_c(\theta_c, k)) = |\Gamma_i(k)\tilde{T}(k)| \alpha$$

We have $S_i(k) = \tilde{Y}_c(\theta_c, k) + \Gamma_i^{-1}(k) |\Gamma_i(k)\tilde{T}(k)| \alpha$ and then, the distance between the point $S_i(k)$ and the centre $\tilde{Y}_c(\theta_c, k)$ of the parallelotope $\mathcal{P}_Y(\alpha, \theta_c)$ is:

$$\delta_i(k) = \|S_i(k) - \tilde{Y}_c(\theta_c, k)\| = \sqrt{\alpha^T Q_i(k) \alpha}$$

$$Q_i(k) = |\Gamma_i(k)\tilde{T}(k)|^T \Gamma_i^{-T}(k) \Gamma_i^{-1}(k) |\Gamma_i(k)\tilde{T}(k)| \quad (22)$$

The number of the points $S_i(k)$ being equal to n_k , the quadratic mean of $\delta_i(k)$ at a time k is:

$$\bar{\delta}(k) = \alpha^T \left(\frac{1}{n_k} \sum_{i=1}^{n_k} Q_i(k) \right) \alpha. \quad (23)$$

Then, taking into account (23) and all the available data ($k = 1..N$), the final expression of the criterion of precision may be written:

$$J(\alpha) = \alpha^T \sum_{k=1}^N \left(\frac{1}{n_k} \sum_{i=1}^{n_k} Q_i(k) \right) \alpha \quad (24)$$

4. EXAMPLE

In order to illustrate this procedure, let us consider a system linear in parameters and measurements, described by the following model:

$$Y(k) = X(k)\theta(k) \quad (25)$$

with $Y(k) \in \mathcal{R}^2$, $X(k) \in \mathcal{R}^{2 \times 2}$ and $\theta(k) \in \mathcal{R}^2$. For sake of simplicity, $X(k), k = 1..500$ are constant and equal to :

$$X = \begin{pmatrix} -1.5 & 0.5 \\ -1.0 & 3.0 \end{pmatrix}$$

and only the values of $\tilde{Y}(k)$ change due to the measurement noise. The domain described by the uncertain parameters is generated by the following equation:

$$\theta(k) = \theta_c + M(\lambda)\nu(k) \quad (26)$$

$$\theta_c = \begin{pmatrix} 5 \\ 5 \end{pmatrix} M = 0.1 \begin{pmatrix} -3 & 1 & -3 \\ -2 & 5 & 1 \end{pmatrix} \quad \lambda = (1 \ 1.25 \ 2)^T$$

The measurement noise has been generated by using an uniform *pdf* taking values between -1 and $+1$. In this example, the centre θ_c of the parameter domain and the matrix M are considered known and only the size of uncertainties remains unknown in order to observe the efficiency of the chosen criterion for uncertainty characterization. The matrix $\tilde{T}(k) = \tilde{X}(k)M$ is then constant (equal to \tilde{T}) and therefore all the matrices $Q(k)$ take the common value:

$$Q = \frac{1}{n_k} \sum_{i=1}^{n_k} (\Gamma_i^{-1} |\Gamma_i \tilde{T}|)^T (\Gamma_i^{-1} |\Gamma_i \tilde{T}|) \quad (27)$$

corresponding to the set of the points $S_i(k)$ at the time k . For this particular example where θ_c is given ($\beta = \alpha$) and the measurement are noise free ($\alpha = \lambda$), the precision criterion only depends on the λ parameter:

$$J(\lambda) = \sum_{k=1}^N \lambda^T Q \lambda \quad (28)$$

and the problem is thus reduced to the minimisation of $\bar{J}(\beta) = \beta^T Q \beta$. In this example the matrix Q has the value:

$$Q = \begin{pmatrix} 640.2 & 32.4 & 1043 \\ 32.4 & 59.94 & 62.6 \\ 1043 & 62.6 & 1745.9 \end{pmatrix} \quad (29)$$

and the corresponding constraints imposed by measurements $A_N \lambda \geq b_N$ are such that:

$$A_N = \begin{pmatrix} 1.48 & 1.02 & 0 & 1.15 & .22 \\ 0.98 & 0.08 & 0.63 & 0 & .54 \\ 3.12 & 1.64 & 0.42 & 1.91 & 0 \end{pmatrix} \quad (30)$$

and $b_N = (8.9 \ 4.10 \ 2.67 \ 4.80 \ 1.79)$.

The vector λ_{opt} which minimises $J_1(\lambda) = \lambda^T Q \lambda$ while checking $A_N \lambda \geq b_N$, is

$$\lambda_{opt} = (0.987 \ 1.246 \ 1.988)^T$$

knowing that simulation was made by taking $\lambda = (1.00 \ 1.25 \ 2.00)^T$. When increasing the number of measurements, a better estimation (in regard to the true values) may be obtained. For, example with $N = 1000$ observations, we get $\lambda_{opt} = (0.989 \ 1.249 \ 1.998)^T$. Figure 1 shows a projection on the space (Y_1, Y_2) of all measurements ($k = 1..N$) which belong to the considered field representing different possible values that can take measurements. The same data are presented on figure 1 on which the identified domain has been drawn. On Figure 2,

the true and the identified data domains have been displayed and can be compared. Figure 3 presents the domain \mathcal{P}_θ . At last, when identifying together λ and θ_c , we obtain :

$$\lambda_{opt} = (0.966 \ 1.152 \ 1.943)^T \quad \theta_c = 4.998 \ 5.002$$

5. CONCLUSION

Parameter estimation of a MIMO model has been studied. This is a well known problem, however when the bounds of the equation error are not admissible, i.e. for the given measurements and an equation-error description, the existence of a solution (parameter set) is not guaranteed if the parameters are supposed time invariant. A method, consisting in explaining all the measurements while optimising a criterion of precision is proposed in the most general case where the parameters are time-varying without considering the notion of parameter variation speed. Moreover, the uncertainties characterisation of a MIMO model highlights dependencies between the outputs of the model, these dependencies being created by the parameters to be estimated. A technique taking into account these dependencies, combined with the calculation of a criterion of precision is proposed. It provides an optimal solution (via the precision criterion) as a parameter set, its central value and the bounds of the equation error. Further, it would be also interesting to use polytopes instead of parallelotopes in order to improve parameter estimation procedure. The idea is to find some linear inequalities defining the parameter set as a polytope in which the time-varying parameter vector varying in time, explain all measurements for a given model structure.

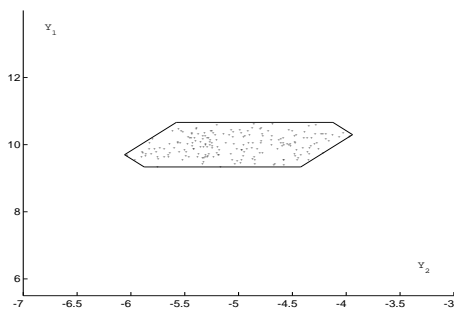


Fig. 1. Data and estimated domain

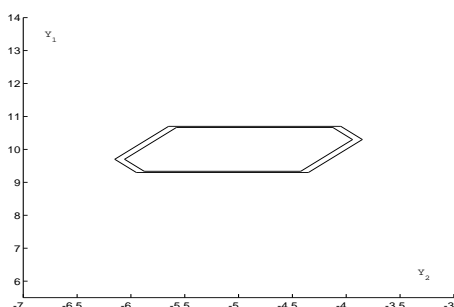


Fig. 2. True and estimated data domains

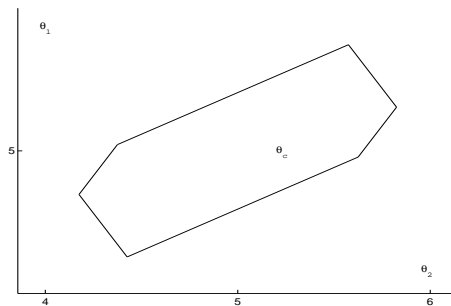


Fig. 3. Parameter domain

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