Proportional-Integral observer design for nonlinear uncertain systems modelled by a multiple model approach

Rodolfo Orjuela, Benoît Marx, José Ragot and Didier Maquin

Centre de Recherche en Automatique de Nancy (CRAN)
Nancy-Université, CNRS

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Goal

State estimation of a nonlinear system with parameter uncertainties and subject to disturbances

Context

- To take into consideration the complexity of the system in the whole operating range (nonlinear models are needed)
- Observer design problem for generic nonlinear models is very delicate

Proposed strategy

- Multiple model representation of the nonlinear system
- Robust Proportional-integral observer design based on this representation
- Convergence conditions are obtained using the Lyapunov method
- Conditions are given under a LMI form

Outline

1. Multiple model approach
   - Basis of Multiple model approach
   - On the decoupled multiple model
2. State estimation
   - Proportional-integral observer structure
   - Proportional-integral observer design
   - Proportional-integral observer existence conditions: main result
3. Simulation example
4. Conclusions

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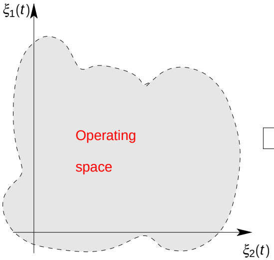
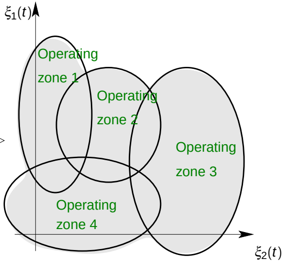
Multiple model Approach

- Decomposition of the operating space into operating zones
- Modelling each zone by a single submodel
- The contribution of each submodel is quantified by a weighting function



Nonlinear system



Multiple model representation

Multiple model = an association of a set of submodels blended by an interpolation mechanism

Why using a multiple model?

- Appropriate tool for modelling complex systems (e.g. black box modelling)
- Tools for linear systems can partially be extended to nonlinear systems
- Specific analysis of the system nonlinearity is avoided

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How the submodels can be interconnected?

Classic structure
Takagi-Sugeno multiple model

- Common state vector for all submodels
- Dimension of the submodels must be identical

Proposed structure
Decoupled multiple model

- A different state vector for each submodel
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Decoupled multiple model: multiple model with local state vectors

$$\begin{aligned}\dot{x}_i(t) &= (A_i + \Delta A_i(t))x_i(t) + (B_i + \Delta B_i(t))u(t) + D_i w(t)\\ y_i(t) &= C_i x_i(t)\\ y(t) &= \sum_{i=1}^{L} \mu_i(\xi(t)) y_i(t) + W w(t)\end{aligned}$$

Model uncertainties

Uncertainties of each submodel are taken into consideration according to the validity degree of each submodel given by $\mu_i(\xi(t))$

$$\Delta A_i(t) = \mu_i(\xi(t)) M_i F_i(t) N_i \qquad \Delta B_i(t) = \mu_i(\xi(t)) H_i S_i(t) E_i$$

$F_i(t)$ and $S_i(t)$ are unknown terms satisfying: $F_i^T(t)F_i(t) \leq \mathrm{I}$ and $S_i^T(t)S_i(t) \leq \mathrm{I} \quad \forall t$

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- The multiple model output is given by a weighted sum of the submodel outputs

$$\sum_{i=1}^{L} \mu_i(\xi(t)) = 1 \text{ and } 0 \leq \mu_i(\xi(t)) \leq 1, \forall t, \forall i \in \{1, \ldots, L\}$$

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State estimation using a PI observer

Augmented form of the multiple model

$$\begin{aligned}\dot{x}(t) &= (\tilde{A} + \Delta\tilde{A}(t))x(t) + (\tilde{B} + \Delta\tilde{B}(t))u(t) + \tilde{D}w(t)\\ \dot{z}(t) &= \tilde{C}(t)x(t) + Ww(t)\\ y(t) &= \tilde{C}(t)x(t) + Ww(t) \qquad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i\end{aligned}$$

$$x(t) = \left[x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)\right]^T \qquad \tilde{A} = diag\left\{A_1 \cdots A_i \cdots A_L\right\}$$

$$\tilde{B} = \left[{B_1}^T \cdots {B_i}^T \cdots {B_L}^T\right]^T \qquad \tilde{D} = \left[{D_1}^T \cdots {D_i}^T \cdots {D_L}^T\right]^T$$

$$\tilde{C}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{C}_i \qquad \tilde{C}_i = \left[0 \cdots C_i \cdots 0\right]$$

$$\Delta\tilde{A}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{M}_i F_i(t)\tilde{N}_i \qquad \Delta\tilde{B}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{H}_i S_i(t) E_i$$

$$\tilde{M}_i = \left[0 \cdots M_i^T \cdots 0\right]^T \qquad \tilde{H}_i = \left[0 \cdots H_i^T \cdots 0\right]^T$$

$$\tilde{N}_i = \left[0 \cdots N_i \cdots 0\right]$$

Augmented form of the multiple model

Augmented state vector ⇒

$$\begin{aligned}\dot{x}(t) &= (\tilde{A} + \Delta\tilde{A}(t))x(t) + (\tilde{B} + \Delta\tilde{B}(t))u(t) + \tilde{D}w(t)\\ \dot{z}(t) &= \tilde{C}(t)x(t) + Ww(t)\\ y(t) &= \tilde{C}(t)x(t) + Ww(t) \qquad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i\end{aligned}$$

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Augmented form of the multiple model

$$\dot{x}(t) = (\tilde{A} + \Delta\tilde{A}(t))x(t) + (\tilde{B} + \Delta\tilde{B}(t))u(t) + \tilde{D}w(t)$$

Supplementary variable

Integral term ⇒

$$\dot{z}(t) = \tilde{C}(t)x(t) + Ww(t) \quad \Rightarrow z(t) = \int_0^t y(\xi)d\xi$$

$$y(t) = \tilde{C}(t)x(t) + Ww(t) \qquad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i$$

$$x(t) = \left[x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)\right]^T \qquad \tilde{A} = diag\left\{A_1 \cdots A_i \cdots A_L\right\}$$

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Nonlinear form: blending outputs

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$$\tilde{N}_i = \left[0 \cdots N_i \cdots 0\right]$$

Decoupled multiple model

$$\begin{aligned}
\dot{x}_a(t) &= (\tilde{A}_a(t) + \overline{C}_1 \Delta\tilde{A}(t)\overline{C}_1^T) x_a(t) + \overline{C}_1(\tilde{B} + \Delta\tilde{B}(t))u(t) + \tilde{D}_a w(t) \\
y(t) &= \tilde{C}(t)\overline{C}_1^T x_a(t) + W w(t) \\
z(t) &= \overline{C}_2^T x_a(t)
\end{aligned}$$

Notations

$$x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \quad \tilde{A}_a(t) = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C}(t) & 0 \end{bmatrix} \quad \tilde{D}_a = \begin{bmatrix} \tilde{D} \\ W \end{bmatrix} \quad \overline{C}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \overline{C}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Proportional-integral Observer

$$\begin{aligned}
\dot{\hat{x}}_a(t) &= \tilde{A}_a(t)\hat{x}_a(t) + \overline{C}_1\tilde{B}u(t) + K_P(y(t) - \hat{y}(t)) + K_I(z(t) - \hat{z}(t)) \\
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Proportional action

Decoupled multiple model

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\end{aligned}$$

$\Rightarrow$ Integral term: supplementary variable

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Proportional action   Integral action

State estimation error

$$e_a(t) = x_a(t) - \hat{x}_a(t)$$

$$\dot{e}_a(t) = (\tilde{A}_a(t) - K_P C(t)\overline{C}_1^T - K_I \overline{C}_2^T)e_a(t) + \overline{C}_1 \Delta\tilde{A}x(t) + \overline{C}_1 \Delta\tilde{B}u(t) + (\tilde{D}_a - K_P W)w(t)$$

Main advantages of the PI observer

Two degrees of freedom for the observer design :

(i) K_P can be used to reduce the impact of $w(t)$ on $e_a(t)$

(ii) K_I can be used to improve the observer dynamics

Analysis of the state estimation error

$$\dot{\varepsilon}(t) = A_{obs}(t)\varepsilon(t) + \Phi\bar{w}(t) \Rightarrow \quad \text{Compact} \quad \text{form}$$

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State estimation error

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(i) $\varepsilon(t)$ is stable if the decoupled multiple model is stable and

(ii) K_P and K_I are chosen so that $\tilde{A}_a(t) - K_P C(t)\overline{C}_1^T - K_I \overline{C}_2^T$ is also stable

Goal

- Ensuring the stability of $\varepsilon(t)$ for any $\bar{w}(t)$
- Finding the matrices K_P and K_I such that the influence of $\bar{w}(t)$ on $e_a(t)$ is attenuated

Performances of the PI observer

$$\lim_{t\to\infty} e_a(t) = 0 \quad \text{for } w(t)=0, F_i(t)=0, S_i(t)=0 \Rightarrow \text{Convergence toward zero}$$

$$\|v(t)\|_2^2 \leq \gamma^2\|\overline{w}(t)\|_2^2 \quad \text{for } \overline{w}(t)\neq 0 \text{ and } v(0)=0 \Rightarrow \text{Disturbance attenuation}$$

$v(t) = Y\, e_a(t)$ and γ is the $\mathcal{L}_2$ gain from $\bar{w}(t)$ to $v(t)$ to be minimized.

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Proportional-integral observer existence conditions

Theorem

There exists a PIO ensuring the robust objectives if there exists symmetric positive definite matrices P_1 and P_2, matrices L_P and L_I and positive scalars $\bar{\gamma}$, τ_1^i and τ_2^i such that the following condition holds for $i = 1...L$

$$\min \bar{\gamma} \quad \text{subject to}$$

$$\begin{bmatrix} \Gamma_i + \Gamma_i^T + Y^T Y & 0 & \Psi & 0 & P_1 \overline{C}_1 \tilde{M}_i & P_1 \overline{C}_1 \tilde{H}_i \\ 0 & \Lambda_i & P_2 \tilde{D} & P_2 \tilde{B} & P_2 \tilde{M}_i & P_2 \tilde{H}_i \\ (*) & (*) & -\bar{\gamma} I & 0 & 0 & 0 \\ 0 & (*) & 0 & \phi_i & 0 & 0 \\ (*) & (*) & 0 & 0 & -\tau_1^i I & 0 \\ (*) & (*) & 0 & 0 & 0 & -\tau_2^i I \end{bmatrix} < 0$$

where

$$\begin{aligned} \Gamma_i &= P_1 \overline{A}_i - L_P \tilde{C}_i \overline{C}_1^T - L_I \overline{C}_2^T \\ \Psi &= P_1 \tilde{D}_a - L_P W \\ \Lambda_i &= P_2 \tilde{A} + \tilde{A}^T P_2 + \tau_1^i \tilde{N}_i^T \tilde{N}_i \\ \phi_i &= -\bar{\gamma} I + \tau_2^i E_i^T E_i \end{aligned}$$

for a prescribed matrix Y.
$K_P = P_1{}^{-1} L_P$ and $K_I = P_1{}^{-1} L_I$; the $\mathcal{L}_2$ gain from $\bar{w}(t)$ to $v(t)$ is given by $\gamma = \sqrt{\bar{\gamma}}$.

Idea

(i) Consider the following quadratic Lyapunov function:

$$V(t) = e_a^T(t)P_1e_a(t) + x^T(t)P_2x(t)$$

(ii) Robust performance ($\|v(t)\|_2^2 \leq \gamma^2\|\overline{w}(t)\|_2^2$) is guaranteed if

$$\dot{V}(t) < -v^T(t)v(t) + \gamma^2\overline{w}^T(t)\overline{w}(t) \quad \text{where} \quad v(t) = Y e_a(t)$$

(iii) The unknown bounded-norm terms (i.e. uncertainties) can be avoided using the well known inequality

$$XF(t)Y + Y^TF^T(t)X^T \leq XQ^{-1}X^T + Y^TQY$$

(iv) Using the estimation error equation and some algebraic manipulations...

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Example

Multiple model parameters

$L = 2$ submodels with different dimensions ($n_1 = 3$ and $n_2 = 2$), given by:

$$A_1 = \begin{bmatrix} -0.1 & -0.3 & 0.6 \\ -0.5 & -0.4 & 0.1 \\ -0.3 & -0.2 & -0.6 \end{bmatrix} \qquad A_2 = \begin{bmatrix} -0.3 & -0.1 \\ 0.4 & -0.2 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 0.3 & 0.5 & 0.6 \end{bmatrix}^T \qquad B_2 = \begin{bmatrix} 0.4 & 0.3 \end{bmatrix}^T$$

$$D_1 = \begin{bmatrix} 0.1 & -0.1 & 0.1 \end{bmatrix}^T \qquad D_2 = \begin{bmatrix} -0.1 & -0.1 \end{bmatrix}^T$$

$$C_1 = \begin{bmatrix} -0.4 & 0.3 & 0.5 \\ 0.5 & 0.3 & 0.4 \end{bmatrix} \qquad C_2 = \begin{bmatrix} 0.4 & -0.2 \\ 0.3 & 0.2 \end{bmatrix}$$

$$M_1 = \begin{bmatrix} -0.1 & 0.2 & -0.1 \end{bmatrix}^T \qquad M_2 = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}^T$$

$$N_1 = \begin{bmatrix} 0.1 & -0.2 & 0.3 \end{bmatrix} \qquad N_2 = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}$$

$$H_1 = \begin{bmatrix} 0.3 & -0.1 & 0.2 \end{bmatrix}^T \qquad H_2 = \begin{bmatrix} -0.1 & -0.2 \end{bmatrix}^T$$

$$E_1 = -0.2 \qquad E_2 = -0.3$$

$$W = \begin{bmatrix} 0.1 & -0.1 \end{bmatrix} \qquad Y = \mathrm{I}_{(7\times 7)}$$

The weighting functions are

$$\mu_i(\xi(t)) = \eta_i(\xi(t)) / \sum_{j=1}^{L} \eta_j(\xi(t)) \quad \text{where} \quad \eta_i(\xi(t)) = \exp\left(-(\xi(t) - c_i)^2/\sigma^2\right),$$

with $\sigma = 0.6$ and $c_1 = -0.3$ and $c_2 = 0.3$, $\xi(t)$ is the input signal $u(t) \in [-1, 1]$.

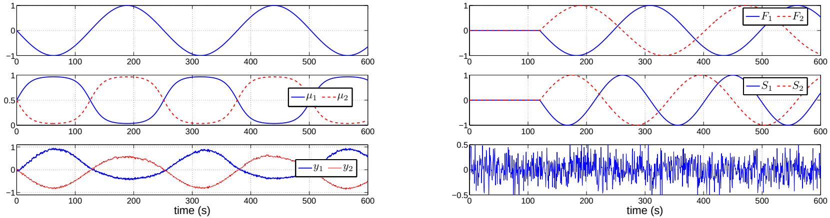

Figure: Input, weighting functions and outputs (left) $F_i(t)$, $S_i(t)$ and $w(t)$ (right)

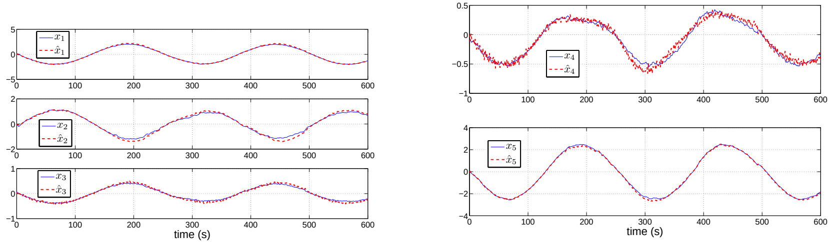

Figure: States of submodels and their estimates

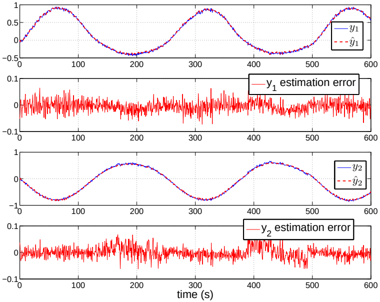

Figure: Output, its estimates and the output estimation errors

Comments

- The minimal attenuation level is $\gamma = 0.8654$
- The state estimation of each submodel is not always close to zero
- Interaction between submodels is at the origin of some compensation phenomenons in the state estimation
- The overall output estimation of the multiple model is not truly affected

Conclusions

- Robust state estimation based on a multiple model representation of an uncertain nonlinear system is investigated
- Originality: the dimension of each submodel may be different (flexibility in a black box modelling stage can be provided)
- Conception of a Proportional-Integral observer is proposed using the Lyapunov theory
- The Proportional-Integral observer offers more degrees of freedom with respect to a classic proportional (Luenberger) observer

Thank you!