Proportional-Integral observer design for nonlinear uncertain systems modelled by a multiple model approach

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47th IEEE Conference on Decision and Control
December 9-11, 2008, Cancun, Mexico
Motivations

Goal
State estimation of a nonlinear system with parameter uncertainties and subject to disturbances

Context
- To take into consideration the complexity of the system in the whole operating range (nonlinear models are needed)
- Observer design problem for generic nonlinear models is very delicate

Proposed strategy
- Multiple model representation of the nonlinear system
- Robust Proportional-integral observer design based on this representation
- Convergence conditions are obtained using the Lyapunov method
- Conditions are given under a LMI form
Outline

1. **Multiple model approach**
   - Basis of Multiple model approach
   - On the decoupled multiple model

2. **State estimation**
   - Proportional-integral observer structure
   - Proportional-integral observer design
   - Proportional-integral observer existence conditions: main result

3. **Simulation example**

4. **Conclusions**
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Multiple model Approach
Basis of Multiple model approach

- Decomposition of the operating space into operating zones
- Modelling each zone by a single submodel
- The contribution of each submodel is quantified by a weighting function

Multiple model = an association of a set of submodels blended by an interpolation mechanism
Multiple model approach

Why using a multiple model?

- Appropriate tool for modelling complex systems (e.g. black box modelling)
- Tools for linear systems can partially be extended to nonlinear systems
- Specific analysis of the system nonlinearity is avoided
Multiple model approach

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### How the submodels can be interconnected?

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Orjuela, Marx, Ragot, Maquin (CRAN) Decoupled multiple model CDC 2008 6 / 20
Decoupled multiple model: multiple model with local state vectors

\[
\begin{align*}
\dot{x}_i(t) &= (A_i + \Delta A_i(t))x_i(t) + (B_i + \Delta B_i(t))u(t) + D_i w(t) \\
y_i(t) &= C_i x_i(t) \\
y(t) &= \sum_{i=1}^{L} \mu_i(\xi(t)) y_i(t) + W w(t)
\end{align*}
\]

Model uncertainties

Uncertainties of each submodel are taken into consideration according to the validity degree of each submodel given by \(\mu_i(\xi(t))\)

\[
\begin{align*}
\Delta A_i(t) &= \mu_i(\xi(t)) M_i F_i(t) N_i \\
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\(F_i(t)\) and \(S_i(t)\) are unknown terms satisfying: \(F_i^T(t)F_i(t) \leq I\) and \(S_i^T(t)S_i(t) \leq I\) \(\forall t\)
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- The multiple model output is given by a weighted sum of the submodel outputs

\[
\sum_{i=1}^{L} \mu_i(\xi(t)) = 1 \text{ and } 0 \leq \mu_i(\xi(t)) \leq 1, \forall t, \forall i \in \{1, \ldots, L\}
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- This multiple model offers a good flexibility and generality for black box modelling

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State estimation using a PI observer
Preliminaries and notations

Augmented form of the multiple model

\[ \dot{x}(t) = (\tilde{A} + \Delta \tilde{A}(t))x(t) + (\tilde{B} + \Delta \tilde{B}(t))u(t) + \tilde{D}w(t) \]

\[ \dot{z}(t) = \tilde{C}(t)x(t) + \mathcal{W}w(t) \]

\[ y(t) = \tilde{C}(t)x(t) + \mathcal{W}w(t) \quad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i \]

\[ x(t) = [x_1^T(t) \ldots x_i^T(t) \ldots x_L^T(t)]^T \]

\[ \tilde{A} = \text{diag}\{A_1 \ldots A_i \ldots A_L\} \]

\[ \tilde{B} = [B_1^T \ldots B_i^T \ldots B_L^T]^T \]

\[ \tilde{D} = [D_1^T \ldots D_i^T \ldots D_L^T]^T \]

\[ \tilde{C}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{C}_i \]

\[ \tilde{C}_i = [0 \ldots C_i \ldots 0] \]

\[ \Delta \tilde{A}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{M}_i F_i(t)\tilde{N}_i \]

\[ \Delta \tilde{B}(t) = \sum_{i=1}^{L} \mu_i(t)\tilde{H}_i S_i(t) E_i \]

\[ \tilde{M}_i = [0 \ldots M_i^T \ldots 0]^T \]

\[ \tilde{N}_i = [0 \ldots N_i \ldots 0] \]

\[ \tilde{H}_i = [0 \ldots H_i^T \ldots 0]^T \]
Augmented form of the multiple model

Augmented state vector \( \Rightarrow \)

\[
\begin{align*}
\dot{x}(t) &= (\tilde{A} + \Delta \tilde{A}(t))x(t) + (\tilde{B} + \Delta \tilde{B}(t))u(t) + \tilde{D}w(t) \\
\dot{z}(t) &= \tilde{C}(t)x(t) + \mathcal{W}w(t) \\
y(t) &= \tilde{C}(t)x(t) + \mathcal{W}w(t) \quad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i
\end{align*}
\]

\[
x(t) = [x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)]^T
\]

\[
\tilde{B} = [B_1^T \cdots B_i^T \cdots B_L^T]^T
\]

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Preliminaries and notations

Augmented form of the multiple model

\[ \dot{x}(t) = (\tilde{A} + \Delta \tilde{A}(t))x(t) + (\tilde{B} + \Delta \tilde{B}(t))u(t) + \tilde{D}w(t) \]

Supplementary variable

\[ \dot{z}(t) = \tilde{C}(t)x(t) + \tilde{W}w(t) \Rightarrow z(t) = \int_0^t y(\xi)d\xi \]

Integral term

\[ y(t) = \tilde{C}(t)x(t) + \tilde{W}w(t) \quad x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i \]

Supplementary variable

\[ x(t) = [x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)]^T \]

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\[ \tilde{A} = \text{diag}\{A_1 \cdots A_i \cdots A_L\} \]

\[ \tilde{D} = [D_1^T \cdots D_i^T \cdots D_L^T]^T \]

\[ \tilde{C}_i = [0 \cdots C_i \cdots 0] \]

\[ \tilde{M}_i = [0 \cdots M_i^T \cdots 0]^T \]

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\[ \tilde{H}_i = [0 \cdots H_i^T \cdots 0]^T \]
Augmented form of the multiple model

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\begin{align*}
\dot{x}(t) &= (\tilde{A} + \Delta \tilde{A}(t))x(t) + (\tilde{B} + \Delta \tilde{B}(t))u(t) + \tilde{D}w(t) \\
\dot{z}(t) &= \tilde{C}(t)x(t) + \mathcal{W}w(t) \quad \Rightarrow z(t) = \int_{0}^{t} y(\xi) d\xi \\
\end{align*}
\]

Nonlinear form: blending outputs

\[
\begin{align*}
y(t) &= \tilde{C}(t)x(t) + \mathcal{W}w(t) \\
x \in \mathbb{R}^n \quad n = \sum_{i=1}^{L} n_i 
\end{align*}
\]

\[
\begin{align*}
x(t) &= [x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)]^T \\
\tilde{A} &= \text{diag} \{A_1 \cdots A_i \cdots A_L\} \\
\tilde{B} &= [B_1^T \cdots B_i^T \cdots B_L^T]^T \\
\tilde{C} &= \sum_{i=1}^{L} \mu_i(t) \tilde{C}_i \\
\tilde{C}_i &= [0 \cdots C_i \cdots 0] \\
\Delta \tilde{A}(t) &= \sum_{i=1}^{L} \mu_i(t) \tilde{M}_i F_i(t) \tilde{N}_i \\
\tilde{M}_i &= [0 \cdots M_i^T \cdots 0]^T \\
\tilde{N}_i &= [0 \cdots N_i \cdots 0] \\
\Delta \tilde{B}(t) &= \sum_{i=1}^{L} \mu_i(t) \tilde{H}_i S_i(t) E_i \\
\tilde{H}_i &= [0 \cdots H_i^T \cdots 0]^T 
\end{align*}
\]
Proportional-integral observer structure

Decoupled multiple model

\[
\dot{x}_a(t) = (\tilde{A}_a(t) + \bar{C}_1 \Delta \tilde{A}(t) \bar{C}_1^T) x_a(t) + \bar{C}_1 (\tilde{B} + \Delta \tilde{B}(t)) u(t) + \tilde{D}_a w(t)
\]
\[
y(t) = \tilde{C}(t) \bar{C}_1^T x_a(t) + Ww(t)
\]
\[
z(t) = \bar{C}_2^T x_a(t)
\]

Notations

\[
x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}
\]
\[
\tilde{A}_a(t) = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C}(t) & 0 \end{bmatrix}
\]
\[
\tilde{D}_a = \begin{bmatrix} \tilde{D} \\ W \end{bmatrix}
\]
\[
\bar{C}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]
\[
\bar{C}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Proportional-integral Observer

\[
\dot{\hat{x}}_a(t) = \tilde{A}_a(t) \hat{x}_a(t) + \bar{C}_1 \tilde{B} u(t) + K_P (y(t) - \hat{y}(t)) + K_I (z(t) - \hat{z}(t))
\]
\[
\hat{y}(t) = \tilde{C}(t) \bar{C}_1^T \hat{x}_a(t)
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**Decoupled multiple model**

\[
\dot{x}_a(t) = (\tilde{A}_a(t) + \tilde{C}_1 \Delta \tilde{A}(t) \tilde{C}_1^T)x_a(t) + \tilde{C}_1(\tilde{B} + \Delta \tilde{B}(t))u(t) + \tilde{D}_aw(t)
\]

\[
y(t) = \tilde{C}(t)\tilde{C}_1^T x_a(t) + \tilde{w}w(t)
\]

\[
z(t) = \tilde{C}_2^T x_a(t)
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**Notations**

\[
x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \quad \tilde{A}_a(t) = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C}(t) & 0 \end{bmatrix} \quad \tilde{D}_a = \begin{bmatrix} \tilde{D} \\ \tilde{w} \end{bmatrix} \quad \tilde{C}_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \tilde{C}_2 = \begin{bmatrix} 0 \\ I \end{bmatrix}
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**Proportional-integral Observer**

\[
\dot{x}_a(t) = \tilde{A}_a(t)\hat{x}_a(t) + \tilde{C}_1 \tilde{B}u(t) + K_P(y(t) - \hat{y}(t)) + K_I(z(t) - \hat{z}(t))
\]

\[
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Proportional-integral observer structure

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\[
\dot{x}_a(t) = (\tilde{A}_a(t) + C_1 \Delta \tilde{A}(t) \bar{C}_1^T) x_a(t) + C_1 (\tilde{B} + \Delta \tilde{B}(t)) u(t) + \tilde{D}_a w(t)
\]

\[
y(t) = \tilde{C}(t) \bar{C}_1^T x_a(t) + W w(t)
\]

\[
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\bar{C}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
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\dot{\hat{x}}_a(t) = \tilde{A}_a(t) \hat{x}_a(t) + \bar{C}_1 \tilde{B} u(t) + K_P (y(t) - \hat{y}(t)) + K_I (z(t) - \hat{z}(t))
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y(t) &= \tilde{C}(t)\tilde{C}_1^T x_a(t) + Ww(t) \\
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Proportional-integral Observer

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\dot{x}_a(t) &= \tilde{A}_a(t)x_a(t) + \tilde{C}_1\tilde{B}u(t) + K_P(y(t) - \hat{y}(t)) + K_I(z(t) - \hat{z}(t)) \\
\hat{y}(t) &= \tilde{C}(t)\tilde{C}_1^T \hat{x}_a(t) \quad \text{Proportional action} \\
\hat{z}(t) &= \tilde{C}_2^T \hat{x}_a(t)
\end{align*}
\]
Proportional-integral observer structure

Decoupled multiple model

\[
\dot{x}_a(t) = (\tilde{A}_a(t) + \tilde{C}_1 \Delta \tilde{A}(t) \tilde{C}_1^T) x_a(t) + \tilde{C}_1 (\tilde{B} + \Delta \tilde{B}(t)) u(t) + \tilde{D}_a w(t)
\]
\[
y(t) = \tilde{C}(t) \tilde{C}_1^T x_a(t) + Ww(t)
\]
\[
z(t) = \tilde{C}_2^T x_a(t) \quad \Rightarrow \text{Integral term: supplementary variable}
\]

Notations

\[
x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \quad \tilde{A}_a(t) = \begin{bmatrix} \tilde{A} \\ \tilde{C}(t) \\ 0 \\ 0 \end{bmatrix} \quad \tilde{D}_a = \begin{bmatrix} \tilde{D} \\ W \end{bmatrix} \quad \tilde{C}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \tilde{C}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
\]

Proportional-integral Observer

\[
\hat{x}_a(t) = \tilde{A}_a(t) \hat{x}_a(t) + \tilde{C}_1 \tilde{B} u(t) + K_P (y(t) - \hat{y}(t)) + K_I (z(t) - \hat{z}(t))
\]
\[
\hat{y}(t) = \tilde{C}(t) \tilde{C}_1^T \hat{x}_a(t) \quad \text{Proportional action}
\]
\[
\hat{z}(t) = \tilde{C}_2^T \hat{x}_a(t) \quad \text{Integral action}
\]
Proportional-integral observer design

State estimation error

\[ e_a(t) = x_a(t) - \hat{x}_a(t) \]
\[ \dot{e}_a(t) = (\dot{A}_a(t) - K_P C(t) \bar{C}_1 - K_I \bar{C}_2) e_a(t) + \bar{C}_1 \Delta \ddot{A} x(t) + \bar{C}_1 \Delta \ddot{B} u(t) + (\bar{D}_a - K_P W) w(t) \]

Main advantages of the PI observer

Two degrees of freedom for the observer design:

(i) \( K_P \) can be used to reduce the impact of \( w(t) \) on \( e_a(t) \)

(ii) \( K_I \) can be used to improve the observer dynamics

Analysis of the state estimation error

\[ \dot{e}(t) = A_{obs}(t) e(t) + \Phi \bar{w}(t) \Rightarrow \text{Compact form} \]
\[ e(t) = \begin{bmatrix} e_a^T(t) \\ x^T(t) \end{bmatrix} \]
\[ A_{obs}(t) = \begin{bmatrix} \ddot{A}_a(t) - K_P C(t) \bar{C}_1^T & -K_I \bar{C}_2^T \\ 0 & \bar{C}_1 \Delta \ddot{A} \end{bmatrix} \]
\[ \bar{w}(t) = \begin{bmatrix} w^T(t) \\ u^T(t) \end{bmatrix} \]
\[ \Phi = \begin{bmatrix} \bar{D}_a - K_P W & \bar{C}_1 \Delta \ddot{B} \\ \bar{D} & \bar{B} + \Delta \bar{B} \end{bmatrix} \]
Proportional-integral observer design

State estimation error

\[
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\]

\[
\dot{e}_a(t) = (\bar{A}_a(t) - K_P C(t)\bar{C}_1 - K_I \bar{C}_2) e_a(t) + \bar{C}_1 \Delta \bar{A} x(t) + \bar{C}_1 \Delta \bar{B} u(t) + (\bar{D}_a - K_P W) w(t)
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Analysis of the state estimation error

\[
\dot{\epsilon}(t) = A_{obs}(t) \epsilon(t) + \Phi \tilde{w}(t) \Rightarrow \text{Compact form}
\]

\[
\epsilon(t) = [e^T_a(t) \quad x^T(t)]^T
\]

\[
A_{obs}(t) = \begin{bmatrix}
\bar{A}_a(t) - K_P C(t)\bar{C}_1 - K_I \bar{C}_2 & \bar{C}_1 \Delta \bar{A} \\
0 & \bar{A} + \Delta \bar{A}
\end{bmatrix}
\]

\[
\tilde{w}(t) = [w^T(t) \quad u^T(t)]^T
\]

\[
\Phi = \begin{bmatrix}
\bar{D}_a - K_P W & \bar{C}_1 \Delta \bar{B} \\
\bar{D} & \bar{B} + \Delta \bar{B}
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Proportional-integral observer design

State estimation error

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\[ \tilde{w}(t) = \begin{bmatrix} w^T(t) & u^T(t) \end{bmatrix} \]
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Proportional-integral observer design

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Analysis of the state estimation error

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\dot{\varepsilon}(t) = A_{obs}(t) \varepsilon(t) + \Phi \tilde{w}(t) \Rightarrow \text{Compact form}
\]

\[
\varepsilon(t) = \begin{bmatrix} e_a^T(t) & x^T(t) \end{bmatrix}^T
\]

\[
A_{obs}(t) = \begin{bmatrix} \tilde{A}_a(t) - K_P C(t) \tilde{C}_1 & -K_I \tilde{C}_2 \\ 0 & \tilde{A} + \Delta \tilde{A} \end{bmatrix}
\]

\[
\tilde{w}(t) = \begin{bmatrix} w^T(t) & u^T(t) \end{bmatrix}^T
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\Phi = \begin{bmatrix} \tilde{D}_a - K_P W & \tilde{C}_1 \Delta \tilde{B} \\ \tilde{D} & \tilde{B} + \Delta \tilde{B} \end{bmatrix}
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Proportional-integral observer design

### State estimation error

\[
e_a(t) = x_a(t) - \hat{x}_a(t)
\]

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\dot{e}_a(t) = (\tilde{A}_a(t) - K_P C(t) \bar{C}_1^T - K_I \bar{C}_2^T) e_a(t) + \bar{C}_1 \Delta \tilde{A} x(t) + \bar{C}_1 \Delta \tilde{B} u(t) + (\tilde{D}_a - K_P W) w(t)
\]

### Main advantages of the PI observer

Two degrees of freedom for the observer design:

(i) \(K_P\) can be used to reduce the impact of \(w(t)\) on \(e_a(t)\)

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### Analysis of the state estimation error

\[
\dot{\epsilon}(t) = A_{obs}(t) \epsilon(t) + \Phi \tilde{w}(t) \quad \Rightarrow \quad \text{Compact form}
\]

\[
\epsilon(t) = \begin{bmatrix} e_a^T(t) & x^T(t) \end{bmatrix}^T
\]

\[
A_{obs}(t) = \begin{bmatrix} \tilde{A}_a(t) - K_P C(t) \bar{C}_1^T - K_I \bar{C}_2^T & \bar{C}_1 \Delta \tilde{A} \\ 0 & \tilde{A} + \Delta \tilde{A} \end{bmatrix}
\]

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\tilde{w}(t) = \begin{bmatrix} w^T(t) & u^T(t) \end{bmatrix}^T
\]

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\Phi = \begin{bmatrix} \tilde{D}_a - K_P W & \bar{C}_1 \Delta \tilde{B} \\ \tilde{D} & \tilde{B} + \Delta \tilde{B} \end{bmatrix}
\]

(i) \(\epsilon(t)\) is stable if the decoupled multiple model is stable and
Proportional-integral observer design

State estimation error

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\[ \dot{e}_a(t) = (\dot{A}_a(t) - K_P C(t)\bar{C}_1^T - K_I \bar{C}_2^T) e_a(t) + \bar{C}_1 \Delta \tilde{A}x(t) + \bar{C}_1 \Delta \tilde{B}u(t) + (\bar{D}_a - K_P W)w(t) \]

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\[ \dot{\epsilon}(t) = A_{obs}(t)\epsilon(t) + \Phi \tilde{w}(t) \Rightarrow \text{Compact form} \]
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\[ \Phi = \begin{bmatrix} \tilde{D}_a - K_P W & \bar{C}_1 \Delta \tilde{B} \end{bmatrix} \]

(i) \( \epsilon(t) \) is stable if the decoupled multiple model is stable and

(ii) \( K_P \) and \( K_I \) are chosen so that \( \dot{A}_a(t) - K_P C(t)\bar{C}_1^T - K_I \bar{C}_2^T \) is also stable
Observer design

Goal

- Ensuring the stability of $\varepsilon(t)$ for any $\bar{w}(t)$
- Finding the matrices $K_P$ and $K_I$ such that the influence of $\bar{w}(t)$ on $e_a(t)$ is attenuated

Performances of the PI observer

\[
\lim_{t \to \infty} e_a(t) = 0 \quad \text{for } w(t) = 0, F_i(t) = 0, S_i(t) = 0 \quad \Rightarrow \text{Convergence toward zero}
\]
\[
\|\nu(t)\|_2^2 \leq \gamma^2 \|\bar{w}(t)\|_2^2 \quad \text{for } \bar{w}(t) \neq 0 \text{ and } \nu(0) = 0 \quad \Rightarrow \text{Disturbance attenuation}
\]
\[
\nu(t) = Y e_a(t) \quad \text{and } \gamma \text{ is the } L_2 \text{ gain from } \bar{w}(t) \text{ to } \nu(t) \text{ to be minimized.}
\]

Main difficulties

- Interaction between submodels must be taken into consideration
- Ensuring the observer stability for any combination between the submodels and for any initial conditions ($\forall e_a(0)$)
Observer design

Goal

- Ensuring the stability of $\varepsilon(t)$ for any $\bar{w}(t)$
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Performances of the PI observer

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\[
\|v(t)\|_2^2 \leq \gamma^2 \|\bar{w}(t)\|_2^2 \quad \text{for } \bar{w}(t) \neq 0 \text{ and } v(0) = 0 \quad \Rightarrow \text{Disturbance attenuation}
\]

\[v(t) = Y e_a(t)\] and $\gamma$ is the $\mathcal{L}_2$ gain from $\bar{w}(t)$ to $v(t)$ to be minimized.

Main difficulties

- Interaction between submodels must be taken into consideration
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### Observer design

#### Goal

- Ensuring the stability of $\varepsilon(t)$ for any $\tilde{w}(t)$
- Finding the matrices $K_P$ and $K_I$ such that the influence of $\tilde{w}(t)$ on $e_a(t)$ is attenuated

#### Performances of the PI observer

\[
\lim_{t \to \infty} e_a(t) = 0 \quad \text{for } w(t) = 0, F_i(t) = 0, S_i(t) = 0 \quad \Rightarrow \text{Convergence toward zero}
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\|v(t)\|_2^2 \leq \gamma^2 \|\tilde{w}(t)\|_2^2 \quad \text{for } \tilde{w}(t) \neq 0 \text{ and } v(0) = 0 \quad \Rightarrow \text{Disturbance attenuation}
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\[v(t) = Y e_a(t)\] and $\gamma$ is the $L_2$ gain from $\tilde{w}(t)$ to $v(t)$ to be minimized.

#### Main difficulties

- Interaction between submodels must be taken into consideration
- Ensuring the observer stability for any combination between the submodels and for any initial conditions ($\forall e_a(0)$)
Theorem

There exists a PIO ensuring the robust objectives if there exists symmetric positive definite matrices $P_1$ and $P_2$, matrices $L_P$ and $L_I$ and positive scalars $\bar{\gamma}$, $\tau_1^i$ and $\tau_2^i$ such that the following condition holds for $i = 1...L$

$$\min \bar{\gamma} \quad \text{subject to}$$

$$\begin{bmatrix}
\Gamma_i + \Gamma_i^T + Y^T Y & 0 & \psi & 0 & P_1 \bar{C}_1 \tilde{M}_i & P_1 \bar{C}_1 \tilde{H}_i \\
0 & \Lambda_i & P_2 \tilde{D} & P_2 \tilde{B} & P_2 \tilde{M}_i & P_2 \tilde{H}_i \\
(*) & (*) & -\bar{\gamma} I & 0 & 0 & 0 \\
0 & (*) & 0 & \phi_i & 0 & 0 \\
(*) & (*) & 0 & 0 & -\tau_1^i I & 0 \\
(*) & (*) & 0 & 0 & 0 & -\tau_2^i I
\end{bmatrix} < 0$$

where

$$\Gamma_i = P_1 \bar{A}_i - L_P \bar{C}_i \bar{C}_1^T - L_I \bar{C}_2^T$$

$$\psi = P_1 \tilde{D}_a - L_P W$$

$$\Lambda_i = P_2 \tilde{A} + \tilde{A}^T P_2 + \tau_1^i \tilde{N}_i^T \tilde{N}_i$$

$$\phi_i = -\bar{\gamma} I + \tau_2^i E_i^T E_i$$

for a prescribed matrix $Y$.

$K_P = P_1^{-1} L_P$ and $K_I = P_1^{-1} L_I$; the $L_2$ gain from $\bar{w}(t)$ to $v(t)$ is given by $\gamma = \sqrt{\bar{\gamma}}$. 

Orjuela, Marx, Ragot, Maquin (CRAN)
(i) Consider the following quadratic Lyapunov function:

\[ V(t) = e_{a}^{T}(t)P_{1}e_{a}(t) + x^{T}(t)P_{2}x(t) \]

(ii) Robust performance \( (\|v(t)\|_{2}^{2} \leq \gamma^{2}\|\tilde{w}(t)\|_{2}^{2}) \) is guaranteed if

\[ \dot{V}(t) < -v^{T}(t)v(t) + \gamma^{2}\tilde{w}^{T}(t)\tilde{w}(t) \quad \text{where} \quad v(t) = Ye_{a}(t) \]

(iii) The unknown bounded-norm terms (i.e. uncertainties) can be avoided using the well known inequality

\[ XF(t)Y + Y^{T}F^{T}(t)X^{T} \leq XQ^{-1}X^{T} + Y^{T}QY \]

(iv) Using the estimation error equation and some algebraic manipulations...

(v) See the proceedings for a detailed proof
PIO existence conditions: idea

**Idea**

(i) Consider the following quadratic Lyapunov function:

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Example
Simulation example

Multiple model parameters

$L = 2$ submodels with different dimensions ($n_1 = 3$ and $n_2 = 2$), given by:

\[
A_1 = \begin{bmatrix}
-0.1 & -0.3 & 0.6 \\
-0.5 & -0.4 & 0.1 \\
-0.3 & -0.2 & -0.6
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
-0.3 & -0.1 \\
0.4 & -0.2
\end{bmatrix}
\]

\[
B_1 = \begin{bmatrix}
0.3 & 0.5 & 0.6
\end{bmatrix}^T, \quad B_2 = \begin{bmatrix}
0.4 & 0.3
\end{bmatrix}^T
\]

\[
D_1 = \begin{bmatrix}
0.1 & -0.1 & 0.1
\end{bmatrix}^T, \quad D_2 = \begin{bmatrix}
-0.1 & -0.1
\end{bmatrix}^T
\]

\[
C_1 = \begin{bmatrix}
-0.4 & 0.3 & 0.5 \\
0.5 & 0.3 & 0.4
\end{bmatrix}, \quad C_2 = \begin{bmatrix}
0.4 & -0.2 \\
0.3 & 0.2
\end{bmatrix}
\]

\[
M_1 = \begin{bmatrix}
-0.1 & 0.2 & -0.1
\end{bmatrix}^T, \quad M_2 = \begin{bmatrix}
-0.2 & 0.1
\end{bmatrix}^T
\]

\[
N_1 = \begin{bmatrix}
0.1 & -0.2 & 0.3
\end{bmatrix}, \quad N_2 = \begin{bmatrix}
0.1 & 0.2
\end{bmatrix}
\]

\[
H_1 = \begin{bmatrix}
0.3 & -0.1 & 0.2
\end{bmatrix}^T, \quad H_2 = \begin{bmatrix}
-0.1 & -0.2
\end{bmatrix}^T
\]

\[E_1 = -0.2, \quad E_2 = -0.3\]

\[
W = \begin{bmatrix}
0.1 & -0.1
\end{bmatrix}
\]

\[
Y = I_{(7 \times 7)}
\]

The weighting functions are

\[
\mu_i(\xi(t)) = \eta_i(\xi(t)) / \sum_{j=1}^L \eta_j(\xi(t)) \quad \text{where} \quad \eta_i(\xi(t)) = \exp \left(-\frac{(\xi(t) - c_i)^2}{\sigma^2}\right),
\]

with $\sigma = 0.6$ and $c_1 = -0.3$ and $c_2 = 0.3$, $\xi(t)$ is the input signal $u(t) \in [-1, 1]$. 

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Simulation example

Figure: Input, weighting functions and outputs (left) $F_i(t)$, $S_i(t)$ and $w(t)$ (right)

Figure: States of submodels and their estimates
Simulation example

![Graph showing output, estimates, and estimation errors](image-url)

**Figure:** Output, its estimates and the output estimation errors

**Comments**

- The minimal attenuation level is $\gamma = 0.8654$
- The state estimation of each submodel is not always close to zero
- Interaction between submodels is at the origin of some compensation phenomena in the state estimation
- The overall output estimation of the multiple model is not truly affected
Conclusions

- Robust state estimation based on a multiple model representation of an uncertain nonlinear system is investigated.

- **Originality**: the dimension of each submodel may be different (flexibility in a black box modelling stage can be provided).

- Conception of a Proportional-Integral observer is proposed using the Lyapunov theory.

- The Proportional-Integral observer offers more degrees of freedom with respect to a classic proportional (Luenberger) observer.
Thank you!