State estimation of nonlinear systems based on heterogeneous multiple models: some recent theoretical results

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Abstract: State estimation of nonlinear systems plays an important role in several control engineering problems. Multiple model approach is an interesting way to cope with this relevant problem. Indeed, multiple models are recognized as a powerful modelling tool for nonlinear dynamic systems. In this framework, several realisations of multiple models can be considered for submodel interconnections. In contrast to the most popular results found in the multiple model literature, we consider here *heterogeneous multiple models* which allow to use submodels of different state space dimensions. Thanks to this fact, flexibility and generality can be introduced in the modelling stage. This paper provides survey of recent results in state estimation strategies based on heterogeneous multiple models. Different kinds of observers are investigated in order to improve the state estimation with respect to disturbance as well as unknown inputs. Theoretical results on the observers design and the state estimation error convergence are presented. Discussion and criticisms of the suggested approaches are also proposed and further research are pointed out.

Keywords: multiple models; state estimation; unknown input estimation; observers design

1. INTRODUCTION

State estimation of dynamic systems is a central topic in system theory and control engineering. Indeed, in many real world situations, the state vector of a system is completely or partially unavailable from the measured output vector. This in turn motivates intensively efforts for years to replace the direct measurement of the system state via a sensor by an estimation obtained with the help of a state estimator (i.e. a software sensor).

For systems described by linear time-invariant (LTI) models, the state estimation can be successfully accomplished from their available inputs and outputs thanks to the well known Luenberger observer or Kalman filter. However, it turns out that LTI models fail for accurately describe the dynamic behaviour of relatively complex systems in a large operating range (*global modelling*). Indeed, LTI models only provide a good representation of such systems around an operating point (*local modelling*) and consequently the generated state estimation is not valid far away from this point. Hence, nonlinear models become unavoidable in order to cope with the global modelling task. Unfortunately, these relatively complex models cannot be easily used for observer design because no generic method for observer design based on any nonlinear model is available.

From the observer design viewpoint, this makes more difficult the modelling task because the accurate and global description of the input-output behaviour of the system must be accomplished with the help of a mathematical model as simple as possible for observer design. For such a purpose, a multiple model representation of the system under investigation can be used.

In this modelling approach, the dynamic behaviour of the system is accurately captured by the judicious interconnection of a set of linear submodels, each of them being valid in a particular operating zone of the system (Murray-Smith and Johansen, 1997; Leith and Leithead, 1999). It should be noted that various realisations of multiple models can be employed in order to generate the global output of the multiple model (Filev, 1991; Gregorcic and Lightbody, 2008). Two essential realisations of multiple models can be distinguished whether *homogeneous* or *heterogeneous* submodels are taken into account for modelling the system behaviour. In the first case, an interpolation of the submodel parameters is considered and the submodels share the same state space (e.g. the so-called *Takagi-Sugeno multiple model*). In the second one, only the submodel outputs are interpolated and consequently each submodel have its own state space which can be of different dimension (e.g. the so-called *decoupled multiple model*).

Some of the approaches often used to obtain a multiple model representation of a nonlinear system are:

(1) linearisation of the nonlinear model around several operating points;
(2) parameter estimation of the submodels using input and output data and appropriate identification techniques, see for instance (Babuska, 1998; Boukhris et al., 1999; Venkat et al., 2003).

Nowadays, multiple models are recognized as a powerful modelling tool because a large class of nonlinear dynamic behaviours can be captured using this representation. Multiple model representation makes it possible the partial extension of some results obtained in the linear framework to nonlinear systems avoiding specific analysis of the nonlinearity of the system. However, most relevant studies only concern the observer design based on the homogeneous multiple model (Murray-Smith and Johansen, 1997; Tanaka and Wang, 2001; Marx et al., 2007), much less are devoted to observer design using heterogeneous multiple models. To the best of the authors' knowledge, this problem is only dealt in few recent works (Uppal et al., 2006).

This fact strongly motivates us to propose the present contribution. Indeed, the state estimation problem of heterogeneous multiple models has drawn our attention in the recent years and encouraging results have been obtained in this direction. The aim of this contribution is to present our ideas and our recent theoretical results concerning the observer design and particularly conditions for ensuring state estimation error convergence. Different observer structures are proposed and the observer existence conditions, under LMI, are presented as a slight extension of the ideas previously developed in (Orjuela et al., 2008a,b,c, 2009). Consequently, discussions and related references are provided instead of complete mathematical proofs, here voluntary omitted. It should be noted that some of the suggested observers seem poorly investigated so far in the multiple model framework (homogeneous or heterogeneous realisations). In this way, the presented results provide a relevant contribution to the state estimation methods based on multiple models.

The outline of this paper is as follows. A brief description of the decoupled multiple model is presented in section 2. Section 3 deals with the state estimation problem based on heterogeneous multiple models. Three observer structures are proposed and sufficient conditions, on the basis of the Lyapunov method, are given in order to ensure the convergence of the state estimation error provided by these observers. Discussion on the presented approach and directions for further researches are given in the last section.

2. HETEROGENEOUS MULTIPLE MODEL

Heterogeneous multiple model appears in the literature under some quite divers names such as: *quasilinear fuzzy model based on the second interpretation* (Filev, 1991), *local-state local model network* (Gawthrop, 1995), *heterogeneous multiple model* (Murray-Smith and Johansen, 1997), *multiple local model* (Gatzke and Doyle III, 1999; Venkat et al., 2003), *local model networks blending the outputs* (Gregorcic and Lightbody, 2008), *multiple mod-*

els with no common state (Kanev and Verhaegen, 2006), *decoupled multiple model* (Orjuela et al., 2008c). In this section, the heterogeneous multiple model is presented.

2.1 Heterogeneous multiple model structure

The structure of the proposed heterogeneous multiple model can be viewed as a parallel interconnection of L submodels via a weighted sum of their outputs (Gatzke and Doyle III, 1999). By considering a state space representation, this structure takes the following form (Orjuela et al., 2008c):

$$\dot{x}_i(t) = A_i x_i(t) + B_i u(t) + D_i w(t) \ , \tag{1a}$$

$$y_i(t) = C_i x_i(t) \ , \tag{1b}$$

$$y(t) = \sum_{i=1}^{L} \mu_i(\xi(t)) y_i(t) + W w(t) \ , \tag{1c}$$

where $x_i \in \mathbb{R}^{n_i}$ and $y_i \in \mathbb{R}^p$ are respectively the state vector and the output of the ith submodel; $u \in \mathbb{R}^m$ is the multiple model input, $y \in \mathbb{R}^p$ is the multiple model output and $w \in \mathbb{R}^r$ a perturbation e.g. a noise. The matrices $A_i \in \mathbb{R}^{n_i \times n_i}$, $B_i \in \mathbb{R}^{n_i \times m}$, $D_i \in \mathbb{R}^{n_i \times r}$, $C_i \in \mathbb{R}^{p \times n_i}$ and $W \in \mathbb{R}^{p \times r}$ are known and constant.
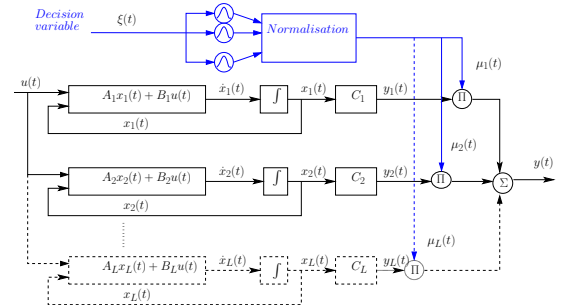

Figure 1. Heterogeneous multiple model structure

The current operating point of the system is taken into account by means of the so-called *decision variable* $\xi(t)$ which is a real-time accessible variable (e.g. inputs, outputs and/or other exogenous measured signal). The relative contribution of each submodel to the global model according to the current operating point of the system is quantified by the *weighting functions* $\mu_i(\xi(t))$ (i.e. the interpolation mechanism). They are associated to each operating zone and satisfy the following convex sum properties:

$$\sum_{i=1}^{L} \mu_i(\xi(t)) = 1 \ , \tag{2a}$$

$$0 \le \mu_i(\xi(t)) \le 1 \ , \quad \forall i = 1, \dots, L \text{ and } \forall t \ . \tag{2b}$$

In this multiple model the contribution of each submodel is taken into account via a weighted sum of the submodel outputs. Consequently, *heterogeneous submodels*, i.e. submodels of different vector dimensions, can be considered in the modelling stage. Hence, this multiple model structure:

- is well suited for modelling nonlinear systems which structure varies with the operating regime, for example when the complexity of the dynamic behaviour

is not uniform across the operating space (Gregorcic and Lightbody, 2008),

- offers an interesting degree of flexibility to cope with the *curse of dimensionality* problem whereby the number of parameters in the multiple model increases extremely rapidly as the complexity of the nonlinear system increases (Leith and Leithead, 1999).

Let us remark that in some particular situations, a discontinuity in the multiple model output $y(t)$ appears when a submodel that is away of the current operating zone is suddenly taken into consideration for building the multiple model output. Of course, this phenomenon does not appear in a systematic way and it depends on the dynamics of the submodels and/or on the choice of the decision variable. When the decision variable is the input of the system, this undesirable phenomenon can effectively be overcome by using a filtered value of the decision variable instead of its direct value. Usually, a low-pass filter with unit gain is employed from a priori knowledge of the system as proposed by Gatzke and Doyle III (1999). Note also that the outputs $y_i(t)$ of the submodels can be considered as *artificial modelling signals* only used in order to provide an approximation of the output of the real system. Therefore the outputs $y_i(t)$ cannot be employed as accessible signals for driving an observer.

2.2 Notations

The following standard notations will be used throughout the paper: $P > 0$ ($P < 0$) denotes a positive (negative) definite matrix P; X^T denotes the transpose of the matrix X; I is the identity matrix of appropriate dimension and $diag\{\}$ is a block diagonal matrix of appropriate dimension. The $\mathcal{L}_2-$norm of a signal, quantifying its energy, is denoted and defined by $\|e\|_2^2 = \int_0^\infty e^T(t)e(t)dt$ and we shall simply write $\mu_i(\xi(t)) = \mu_i(t)$.

2.3 Preliminaries

For the simplicity of manipulations, the heterogeneous multiple model (1) is rewritten in the following compact form (Orjuela et al., 2008c):

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}u(t) + \tilde{D}w(t) \ , \tag{3a}$$

$$y(t) = \tilde{C}(t)x(t) + Ww(t) \tag{3b}$$

where:

$$x(t) = \left[x_1^T(t) \cdots x_i^T(t) \cdots x_L^T(t)\right]^T \in \mathbb{R}^n, \ n = \sum_{i=1}^L n_i \ , \tag{4a}$$

$$\tilde{A} = diag\left\{A_1 \ \cdots \ A_i \ \cdots \ A_L\right\} \in \mathbb{R}^{(n \times n)} \ , \tag{4b}$$

$$\tilde{B} = \left[B_1{}^T \ \cdots \ B_i{}^T \ \cdots \ B_L{}^T\right]^T \in \mathbb{R}^{(n \times m)} \ , \tag{4c}$$

$$\tilde{D} = \left[D_1{}^T \ \cdots \ D_i{}^T \ \cdots \ D_L{}^T\right]^T \in \mathbb{R}^{(n \times r)} \ , \tag{4d}$$

$$\tilde{C}(t) = [\mu_1(t)C_1 \ \cdots \ \mu_i(t)C_i \ \cdots \ \mu_L(t)C_L] \in \mathbb{R}^{(p \times n)} \ . \tag{4e}$$

The matrix $\tilde{C}(t)$ is time-varying because the weighting functions are taken into consideration in this matrix. It should be noted that this matrix can be rewritten as a weighted sum of constant matrices as follows:

$$\tilde{C}(t) = \sum_{i=1}^L \mu_i(t)\tilde{C}_i \tag{5}$$

where $\tilde{C}_i$ is a constant block matrix given by:

$$\tilde{C}_i = [0 \ \cdots \ C_i \ \cdots \ 0] \tag{6}$$

such that the term C_i is found on the ith block column of $\tilde{C}_i$.

2.4 Comments on the stability of the multiple model

The stability of the multiple model (1) can be easily established by investigating the eigenvalues of the constant matrix $\tilde{A}$ given by (4b). This matrix is a block diagonal matrix and therefore its eigenvalues are in the open left-half complex plane if and only if all eigenvalues of all matrices A_i are also in the left-half complex plane. Consequently, the stability of the multiple model (1) is equivalent to the stability of all the submodels.

3. OBSERVER DESIGN PROBLEM

The observer design problem for systems represented with the help of heterogeneous multiple models is poorly investigated in the literature. Recently, the heterogeneous multiple model, namely *Neuro-Fuzzy Decoupling Fault Detection Scheme* NFDFDS, is successfully used for state estimation in a fault detection and isolation perspective by Uppal et al. (2006). In this paper, an independent local observer design based on each submodel is considered and only a note on the estimation error convergence is proposed. This approach cannot be guaranteed, in a general way, the convergence towards zero of the estimation error because the interpolation of the submodel is not taken into consideration in the observer design stage. Indeed, the weighting sum of a set of stable systems is not a stable system. On the other hand, the robustness properties of the state estimation with respect to disturbances and unknown inputs (UI) are not investigated in this work.

These interesting problems are tackled in our recent works (Orjuela et al., 2008a,c). Sufficient conditions for ensuring the exponential convergence towards zero of the estimation errors $e_i(t) = x_i(t) - \hat{x}_i(t)$ have been established on the basis of the well-known Lyapunov method. Different observer structures have been proposed in order to improve the state estimation with respect to perturbations and UI acting on the system. These observers are presented in the next sections and the conditions for ensuring their stability are summarised. They are expressed in the form of a set of linear matrix inequalities (LMIs) (Boyd et al., 1994).

3.1 Proportional gain observer

A state estimation of a nonlinear system characterised by a heterogeneous multiple model (1) can be generated on the basis of a proportional observer defined by:

$$\dot{\hat{x}}_i(t) = A_i\hat{x}_i(t) + B_iu(t) + K_i(y(t) - \hat{y}(t)) \ ,$$

$$\hat{y}(t) = \sum_{i=1}^L \mu_i(\xi(t))C_i\hat{x}_i(t)$$

where $K_i \in \mathbb{R}^{n_i \times p}$ is the gain associated to the ith observer.

Assumption 1. The perturbation is a bounded energy signal, i.e. $\|w\|_2^2 < \infty$.

By using the augmented form (3) of the multiple model (1), the state estimation error can be defined as:

$$e(t) = x(t) - \hat{x}(t) \tag{8}$$

and its time-derivative by:

$$\dot{e}(t) = (\tilde{A} - \tilde{K}\tilde{C}(t))e(t) + (\tilde{D} - \tilde{K}W)w(t) \tag{9}$$

where $\tilde{K} = \begin{bmatrix} K_1^T & \cdots & K_i^T & \cdots & K_L^T \end{bmatrix}^T \in \mathbb{R}^{n \times p}$ is the augmented gain of the observer to be determined such that the following robust performances are respected:

$$\lim_{t \to \infty} e(t) = 0 \quad \text{for} \quad \omega(t) = 0 \ , \tag{10a}$$

$$\|e\|_2^2 \le \gamma^2 \|w\|_2^2 \quad \text{for} \quad w(t) \ne 0 \text{ and } e(0) = 0 \tag{10b}$$

where γ is the $\mathcal{L}_2$ gain from $w(t)$ to $e(t)$ to be minimised. Notice that the convergence of the estimation error in the disturbance-free case is ensured by (10a) and robust state estimation in presence of a disturbance is ensured by (10b). In other words, the influence of $w(t)$ on the estimation error (8) must be attenuated and the state estimation errors must remain globally bounded.

The following theorem provides sufficient conditions for ensuring the proportional observer existence under constraints (10).

Theorem 1. **(Orjuela et al., 2008a)** The proportional observer (7) for the multiple model (1), under constraints (10), is obtained if there exists a symmetric, positive definite matrix P and a matrix G solution of the constrained optimisation problem for a given scalar $\alpha \ge 0$:

$$\min \overline{\gamma} \quad \text{subject to}$$
$$\begin{bmatrix} \mathcal{A}_i + \mathcal{A}_i^T + \mathrm{I} & \mathcal{B} \\ \mathcal{B}^T & -\overline{\gamma}\mathrm{I} \end{bmatrix} < 0 \ , \quad i = 1, \dots, L$$

where

$$\mathcal{A}_i = P(\tilde{A} + \alpha\mathrm{I}) - G\tilde{C}_i \ ,$$
$$\mathcal{B} = P\tilde{D} - GW$$

and $\tilde{C}_i$ is given by (6).

The observer gain is given by $\tilde{K} = P^{-1}G$, the $\mathcal{L}_2$ gain from $\omega(t)$ to $e(t)$ is given by $\gamma = \sqrt{\overline{\gamma}}$ and α is the so-called *decay rate* for exponential convergence of $e(t)$.

Sketch of the proof. This theorem is obtained by considering a quadratic Lyapunov function:

$$V(t) = e^T(t)Pe(t) \quad P > 0 \quad P = P^T \ . \tag{11}$$

The following well-known inequality (Boyd et al., 1994) must be ensured in order to satisfying the robust state estimation problem (10):

$$\dot{V}(t) + 2\alpha V(t) < \gamma^2 w^T(t)w(t) - e^T(t)e(t) \tag{12}$$

where α is the decay rate and γ is the attenuation level from $w(t)$ to e(t) for robust estimation. The LMI in theorem 1 is obtained by using (8), (9) and (11) in (12) on the one hand and the convex sum properties (2) of the weighting functions $\mu_i(t)$ on the other hand. It can

be noted that for $w(t) = 0$ the exponential convergence toward zero of the state estimation is guaranteed thanks to the decay rate. Asymptotic convergence is obtained by considering $\alpha = 0$.$\square$

Exponential convergence of the estimation error is carried out by assigning the eigenvalues of the matrix $\tilde{A} - \tilde{K}\tilde{C}(t)$, in (9), in the shifted left-half complex plane. However, in this approach, only the real parts of the eigenvalues are assigned and consequently, dynamics with strong oscillations can appear because the imaginary part of the eigenvalues is unbounded. In order to avoid this phenomenon, the eigenvalues can be assigned in a specific region S of the complex plane by using the general characterization for eigenvalues clustering in subregions of the complex plane in terms of LMIs proposed by Chilali and Gahinet (1996).

Note that an arbitrary decay rate α cannot be obtained from the LMIs in the theorem 1 due to the particular form of matrices $\tilde{A}$ and $\tilde{C}_i$ used in these LMIs. Indeed, all columns of matrix $\tilde{C}_i$ are equal to zero except the ith block column, see (6). Hence, eigenvalues assignment of the matrix $\tilde{A} - \tilde{K}\tilde{C}(t)$ may be limited because the pairs $(\tilde{A}, \tilde{C}_i)$, respectively given by (4b) and (6), are not observable and therefore the decay rate choice can be limited. However, this problem has been partially solved in (Orjuela et al., 2009) by considering the weighting functions in the state estimation error (9)

Notice finally, from equation (9), that the proportional observer (7) offers only one degree of freedom $\tilde{K}$ for reducing the influence of the disturbance and providing at the same time good dynamic performances (two antagonist design goals). In the next section another observer structure is proposed to avoid this problem.

3.2 Proportional-Integral observer, first case

A Proportional-Integral observer is suggested in this section in order to introduce a supplementary degree of freedom in the observer design stage, robustness can be improved in this way. This observer is characterised by the use of two corrective injection terms, proportional and integral, instead of the only one proportional correction previously employed. *Proportional-integral observers* are well-known by their interesting robustness properties with respect to perturbations (Busawon and Kabore, 2001).

In order to introduce the supplementary integral variable $z(t) = \int_0^t y(\xi)d\xi$, the multiple model (3) is rewritten as follows:

$$\dot{x}_a(t) = \tilde{A}_1(t)x_a(t) + \bar{C}_1\tilde{B}u(t)$$
$$\qquad\qquad + (\bar{C}_1\tilde{D} + \bar{C}_2W)w(t) \ , \tag{13a}$$
$$y(t) = \tilde{C}(t)\bar{C}_1^T x_a(t) + Ww(t) \ , \tag{13b}$$
$$z(t) = \bar{C}_2^T x_a(t) \tag{13c}$$

where

$$x_a(t) = \begin{bmatrix} x(t) \\ z(t) \end{bmatrix}, \tilde{A}_1(t) = \begin{bmatrix} \tilde{A} & 0 \\ \tilde{C}(t) & 0 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} \mathrm{I}_n \\ 0 \end{bmatrix}, \bar{C}_2 = \begin{bmatrix} 0 \\ \mathrm{I}_p \end{bmatrix}.$$

The proposed proportional-integral observer, based on the multiple model (13), is given by:

$$\dot{\hat{x}}_a(t) = \tilde{A}_1(t)\hat{x}_a(t) + \bar{C}_1\tilde{B}u(t) + K_P(y(t) - \hat{y}(t))$$
$$+ K_I(z(t) - \hat{z}(t)) \ , \quad (14a)$$
$$\hat{y}(t) = \tilde{C}(t)\bar{C}_1^T \hat{x}_a(t) \ , \quad (14b)$$
$$\hat{z}(t) = \bar{C}_2^T \hat{x}_a(t) \ . \quad (14c)$$

Notice that the use of the two correction terms K_P and K_I is at the origin of the designation proportional-integral observer.

As previously, by using the augmented forms (13) and (14), an augmented state estimation error can be defined as:

$$e_a(t) = x_a(t) - \hat{x}_a(t) \quad (15)$$

and its time-derivative by:

$$\dot{e}_a(t) = (\tilde{A}_1 - \tilde{K}_P\tilde{C}(t)\bar{C}_1^T - \tilde{K}_I\bar{C}_2^T)e_a(t) \quad (16)$$
$$+ (\bar{C}_1\tilde{D} + \bar{C}_2W - \tilde{K}_PW)w(t) \ .$$

It can be seen, from equation (16), that:

(1) the proportional gain $\tilde{K}_P$ can be used to reduce the impact of the disturbance $w(t)$ on $e_a(t)$,
(2) the integral gain $\tilde{K}_I$ can be used to improve the observer dynamics performances.

Hence, the proposed observer (14) offers two degrees of freedom $\tilde{K}_P$ and $\tilde{K}_I$. This fact constitutes one of the most important interest of the proportional-integral observer (14) with respect to the proportional observer (7).

The following theorem provides sufficient conditions for ensuring the existence of the proportional-integral observer (14) under constraints (10) slightly modified to take into account $e_a(t)$ instead of $e(t)$.

Theorem 2. **(Orjuela et al., 2008a)** The proportional-integral observer (14) for the multiple model (13), under constraints (10), is obtained if there exists a symmetric, positive definite matrix P and matrices G_P and G_I solution of the constrained optimisation problem for a given scalar decay rate $\alpha \geq 0$:

$$\min \overline{\gamma} \quad \text{subject to}$$
$$\begin{bmatrix} \mathcal{A}_i + \mathcal{A}_i^T + \text{I} & \mathcal{B} \\ \mathcal{B}^T & -\overline{\gamma}\text{I} \end{bmatrix} < 0 \ , \quad i = 1, \dots, L$$

where

$$\mathcal{A}_i = P(\tilde{A}_1 + \alpha\text{I}) - G_P\tilde{C}_i\bar{C}_1^T - G_I\bar{C}_2^T \ ,$$
$$\mathcal{B} = P\bar{C}_1\tilde{D} + P\bar{C}_2W - G_PW$$

and $\tilde{C}_i$ is given by (6).

The observer gains are given by $\tilde{K}_P = P^{-1}G_P$ and $\tilde{K}_I = P^{-1}G_I$, the $\mathcal{L}_2$ gain from $\omega(t)$ to $e_a(t)$ is given by $\gamma = \sqrt{\overline{\gamma}}$.

Sketch of the proof. As in the previous section using the following quadratic Lyapunov function: $V(t) = e_a^T(t)Pe_a(t)$, $P > 0$, $P = P^T$.□

3.3 Proportional-Integral observer, second case

In this section, the multiple model (3) is considered but the disturbance $w(t)$ is now considered as an interesting

unknown input (UI) to be estimated instead of a disturbance to be attenuated. This UI acting on the system can be employed in order to characterize an actuator failure and/or an abnormal behaviour of an internal component of the system.

Assumption 2. The unknown input signal $w(t)$ is supposed to be a constant signal, i.e. $\dot{w}(t) = 0$.

The goal is to generate both state and UI estimations of nonlinear systems modelled by the multiple model (1). To this end, the following proportional-integral unknown input observer is employed (Orjuela et al., 2008b):

$$\dot{\hat{x}}(t) = \tilde{A}\hat{x}(t) + \tilde{B}u(t) + \tilde{D}\hat{w}(t) + \tilde{K}_P(y(t) - \hat{y}(t)) \ (17a)$$
$$\dot{\hat{w}}(t) = \tilde{K}_I(y(t) - \hat{y}(t)) \ , \quad (17b)$$
$$\hat{y}(t) = \tilde{C}(t)\hat{x}(t) + W\hat{w}(t) \ . \quad (17c)$$

where $\hat{w}(t)$ provides an estimation of the UI $w(t)$. The use of an integral action given by (17b) is at the origin of the designation proportional-integral observer.

Remark 1. The two observers, (14) and (17), previously proposed are proportional-integral observers. However the integral action is used in two different ways. In the first case, the observer defined by (14), the integral action introduces an additional degree of freedom for disturbance attenuation. In the second one, the observer defined by (17), the integral action allows the UI estimation.

Consider now the following augmented state vector:

$$\Sigma(t) = \begin{bmatrix} e(t) \\ \varepsilon(t) \end{bmatrix} = \begin{bmatrix} x(t) - \hat{x}(t) \\ w(t) - \hat{w}(t) \end{bmatrix} \in \mathbb{R}^{n+r} \ , \quad (18)$$

where $e(t)$ is the state estimation error and $\varepsilon(t)$ the UI estimation error. By considering the assumption 2, the time-derivative of (18) is given by:

$$\begin{bmatrix} \dot{e}(t) \\ \dot{\varepsilon}(t) \end{bmatrix} = \begin{bmatrix} \tilde{A} - \tilde{K}_P\tilde{C}(t) & \tilde{D} - \tilde{K}_PW \\ -\tilde{K}_I\tilde{C}(t) & -\tilde{K}_IW \end{bmatrix} \begin{bmatrix} e(t) \\ \varepsilon(t) \end{bmatrix} \ , \quad (19)$$

which can be rewritten as:

$$\dot{\Sigma}(t) = (A_a - K_aC_a(t))\Sigma(t) \quad (20)$$

where

$$A_a = \begin{bmatrix} \tilde{A} & \tilde{D} \\ 0 & 0 \end{bmatrix}, \ K_a = \begin{bmatrix} \tilde{K}_P \\ \tilde{K}_I \end{bmatrix}, \ C_a(t) = \begin{bmatrix} \tilde{C}(t) \ W \end{bmatrix} \ . \quad (21)$$

The following theorem provides sufficient conditions for ensuring the existence of the proportional-integral unknown input observer (17) under assumption 2.

Theorem 3. **(Orjuela et al., 2008b)** The proportional-integral unknown input observer (17) for the multiple model (3), is obtained if there exists a symmetric, positive definite matrix P and matrix a G_a solution of the constrained optimisation problem for a given scalar decay rate $\alpha \geq 0$:

$$\mathcal{A}_i + \mathcal{A}_i^T < 0 \quad , \quad i = 1, \dots, L$$

where

$$\mathcal{A}_i = P(A_a + \alpha\text{I}) - G_a\bar{C}_i \quad ,$$
$$\bar{C}_i = \begin{bmatrix} \tilde{C}_i \ W \end{bmatrix} \in \mathbb{R}^{p \times (n+r)}$$

and $\tilde{C}_i$ is given by (6). The observer gain is given by $K_a = P^{-1}G_a$.

Sketch of the proof. As in the previous sections, the conditions of theorem 3 are obtained by considering $V(t) = \Sigma^T(t)P\Sigma(t)$, $P > 0$, $P = P^T$ and the following well-known inequality for ensuring exponential error convergence towards zero:

$$\dot{V}(t) + 2\alpha V(t) < 0 \qquad (22)$$

where α is the *decay rate* for convergence velocity. Finally, the LMIs in theorem 3 are obtained by using (18) and (20) in (22). □

It can be noted that here, disturbances acting on the system are not considered. However, the disturbance and UI can be both taken into account by introducing their respective distribution matrices. In this case, robust performance as proposed in (10) must be considered in the state estimation study as proposed in sections 3.1 and 3.2.

Notice that the assumption 2 is needed for theoretical proofs but our approach remains effective in practical cases where the assumption 2 is not truly satisfied, of course the UI must be a low frequency signal e.g. constant or slowly varying-time signal (Orjuela et al., 2008b).

The suggested unknown input observer (17) can be used, as an extension of the classic generalized observer scheme, in the detection and the isolation of sensor and actuator failures of complex systems. Indeed, the UI estimation can be directly employed as a residual signal in a FDI scheme.

4. CONCLUSION AND FURTHER RESEARCHES

Recent theoretical results concerning the state estimation of nonlinear systems represented by heterogeneous multiple models are presented in this contribution. Three novel observer structures based on the proportional gain and the proportional-integral gains are investigated. The observer existence conditions, based on the LMI framework, have been established in order to ensure exponential convergence and robust performances of the estimation error with respect to disturbances and unknown inputs.

Further research will aim at reducing the conservatism of the proposed solution using other Lyapunov functions, such as piecewise Lyapunov functions. Improvements to the proposed proportional-integral unknown input observer, in order to take into consideration a more general class of unknown inputs, provides promising prospects in the future.

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