Generation of Analytical Redundancy Relations for FDI purposes

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Abstract: All the methods of fault detection and isolation (FDI) are based, either explicitly or implicitly, on the use of redundancy, i.e. relations among the measured variables. Since the founder work of Potter and Suman [22], the problem of generation of redundancy relations has been widely addressed. This paper presents the fundamental results obtained in this area.

Keywords: redundancy relations, fault detection and isolation, model-based diagnosis.

I. INTRODUCTION

Modern control systems are often large and complex. If faults occur, consequences can be extremely serious in terms of human lives, environmental impact and economic loss. Higher performances and more rigorous security requirements have invoked an ever increasing demand to develop real time fault detection and isolation systems. The problem of fault diagnosis using analytical redundancy (model-based) methods has received increasing attention during recent years due to the rapid growth in available computer power. All the methods, in one way or another, involve generation and evaluation of signals that are accentuated by faults that have actually occurred. The procedures for generating such signals, called residuals, are based on two main distinct approaches. The first one (direct approach) consists in the elimination of all the unknown variables (states, unknown inputs, ...) keeping input-output relations involving only observable variables. The second one (indirect approach) estimates states, outputs or parameters in order to generate discrepancy signals obtained by the difference between the actual variables and their estimates.

However, the main problem obstructing the progress and improvement in reliability of fault diagnosis is the robustness with respect to uncertainties which arises, for example, due to process noise, parameter variations and modelling errors. For methods which use the direct approach, when perfect de-coupling cannot be achieved, a performance index which measures the sensitivity with respect to faults and the insensitivity with respect to uncertainties must be defined and optimised. For the indirect approaches, the estimation procedure must be as insensitive as possible with respect to unknown inputs.

This paper is focused on the presentation of methods for generating input-output relations (the direct approach). Its organisation is the following: in the second section, the main concepts of model-based monitoring are reminded. The section III is dedicated to the structural analysis which allows the study of the monitoring ability of large scale system through a structural canonical decomposition. The results of this analysis provide a way to generate residuals. The basic residual generators allowing a perfect de-coupling with respect to unknown variables and perturbations are presented in section IV in the case of polynomial models, which often provide a good approximation of systems non linearities. Some results issued from Elimination Theory and especially the concept of Gröbner basis are briefly presented in this section. Linear systems are the simplest case of polynomial ones. Variable elimination, in this case, is equivalent to projections into specific subspaces. The Parity Space approach is presented in section V, for static as well as for dynamic models. When perfect de-coupling is not possible, approximate solutions may be searched. Section VI presents some optimisation approaches which can be used. The last section illustrates the residual generation method for an induction machine.

II. MODEL-BASED MONITORING

A. System modelling

The behavioural model of a system gives some information about the variables the plant involves. It can indicate the values that some variables should have in their simpler expression or express some knowledge about the generating process of these variables.

The analytical model gives an explicit formulation of the behavioural model. It is generally made up of two parts:

- the first one describes the operation of the plant, including the actuators and the process dynamics. It expresses the way in which the controls are transformed into states. The state trajectories depend on the initial state for dynamic models.

- the second one describes the measurements which are available. It expresses the way in which the sensors transform some states of the process into output signals which can be used for control or FDI purposes.

Both parts of the analytical model may depend on some parameters.

B. Residual generation

Let us consider a dynamical system observed on a temporal window. The analytical model of the system first expresses the relations between the internal variables $\mathcal{X}$ (as the state variables) and the control variables $\mathcal{U}$ and secondly the measurement $\mathcal{Y}$ as functions of the internal variables.

$$F(\mathcal{X}(t),\mathcal{U}(t)) = 0 \quad (1a)$$
$$\mathcal{Y}(t) = G(\mathcal{X}(t)) \quad (1b)$$

where $\mathcal{Z}(t) = \begin{pmatrix} z(t) & z(t-1) & \cdots & z(t-p) \end{pmatrix}$ is the vector of values of a vector z on a temporal window of size p. However, notice that (1) also stands for static systems.

In order to perform the FDI algorithms in real time, those algorithms must only make use of known variables, namely the values of $\mathcal{U}$ and $\mathcal{Y}$. The unknown variables $\mathcal{X}$ have thus to be eliminated in the system (1). Parity equations or Analytical Redundancy Relations (A.R.R.) may be obtained by re-writing the plant and measurement models in which only known variables intervene. It leads to an input-output model which expresses some invariance property of the form:

$$\Omega\big(\mathcal{U}(t),\mathcal{Y}(t)\big) = 0 \quad (2)$$

Because of measurement uncertainties and modelling errors, the equality (2) is never exactly verified. That leads to a residual vector:

$$r(t) = \Omega\big(\mathcal{U}(t),\mathcal{Y}(t)\big) \quad (3)$$

The design of residual based FDI algorithms rises the two following questions:

- Is it possible for a given system to obtain equations like (2) ? Structural analysis is aimed at answering this question.

- How should one proceed for effective calculation of equations (2) ? Elimination theory (and parity space approach which traduces it in the linear case) answers the second question.

III. STRUCTURAL ANALYSIS

On a structural point of view, the system is modelled as a network of elementary activities, each of them processing a subset of variables. Among the set of all the variables, only some of them are known (computed by elementary activities) or measured. For a given instrumentation scheme, the canonical decomposition [11, 27] of the system structure exhibits a subsystem on which failure detection and identification procedures can be designed. Note that since only structural information are used, this approach applies to large scale systems described by a great number of variables, even when their analytical models are not precisely known.

A. Structure of the model

From the very general point of view which is that of structural analysis, the model of the system is only considered as a set of constraints which apply to a set of variables.

Let $\mathbf{F} = \{f_1, f_2, \ldots, f_m\}$ be the set of the constraints which represent the system model and $\mathbf{Z} = \{z_1, z_2, \ldots, z_n\}$ be the set of variables. The set $\mathbf{Z}$ contains two subsets $\mathbf{K}$ and $\mathbf{X}$ where $\mathbf{K}$ is the set of known variables: the control variables set U and the measured variables set Y. The set $\mathbf{X}$ is the subset of unknown variables. The structure of the model is a digraph $(\mathbf{F}, \mathbf{Z}, \mathbf{A_Z})$ which associates the two sets $\mathbf{F}$ and $\mathbf{Z}$ through the set of links between their elements $\mathbf{A_Z}$.

$$\mathbf{F} \times \mathbf{Z} \quad \mathbf{A_Z}$$
$$(f_i, z_j) \in \mathbf{A_Z} \Leftrightarrow \text{ constraint } f_i \text{ applies to variable } x_j$$

Let a belong to $\mathbf{A_Z}$, $v(a)$ denotes the extremity of a in $\mathbf{Z}$ and $c(a)$ the extremity of a in $\mathbf{F}$, so a can be written: $a = \big(c(a), v(a)\big)$.

B. Monitorable subsystems

Let $P(\mathbf{E})$ be the set of the subsets of a given set $\mathbf{E}$. Constraint and variable structure are defined using the following application:

$$Q : P(\mathbf{F}) \to P(\mathbf{Z})$$
$$\mathsf{F} \mapsto Q(\mathsf{F}) = \{z_j \mid \big(\exists f_i \in \mathsf{F}\big) \wedge (f_i, z_j) \in \mathbf{A_Z}\}$$

A subsystem is a pair $(\mathsf{F}, Q(\mathsf{F}))$ where F is a subset of $\mathbf{F}$

Let $Q(\mathsf{F}) = Q_k(\mathsf{F}) \cup Q_x(\mathsf{F})$ where $Q_k(\mathsf{F})$ is the subset of known variables while $Q_x(\mathsf{F})$ is the subset of unknown ones. The constraints which define the subsystem may be written as:

$$\mathsf{F}\big(Q_k(\mathsf{F}), Q_x(\mathsf{F})\big) = 0 \quad (4)$$

Then a subsystem is monitorable if it is equivalent to analytical redundancy relations of form (2). This property can be expressed as follows:

The system $(F, Q(F))$ is monitorable if and only if a transformation T can be found such that $T[(F, Q(F))] = (F', Q(F'))$ with $\mathbf{K} \subseteq Q(F')$. The analytical redundancy relations are then expressed as:

$$F'(Q(F')) = 0 \tag{5}$$

C. Canonical decomposition

According to the previous definitions, the problem of finding monitorable subsystems is equivalent to the finding of subsystems in which $Q_x(F)$ can be eliminated. The analysis of the system structure with regard to the unknown variables set $\mathbf{X}$ can be a guideline for researching these subsystems.

Let us consider the graph $G(\mathbf{F_X}, \mathbf{X}, \mathbf{A_X})$ which is the restriction of the system structural graph to the set of vertices $\mathbf{X}$. The subset $\mathbf{A_X}$ only contains the arcs of $\mathbf{A}$ which link $\mathbf{F}$ to $\mathbf{X}$:

$$\mathbf{F_X} = \{f_i \mid (\exists x_i \in \mathbf{Z}) \wedge (f_i, x_j) \in \mathbf{A_X}\}$$

Definitions:

- $G(\mathbf{F}, \mathbf{X}, \mathbf{A})$ is a *matching* on $G(\mathbf{F_X}, \mathbf{X}, \mathbf{A_X})$ if and only if:

  i) $\mathbf{A} \subseteq \mathbf{A_X}$
  ii) $\forall a_1, a_2 \in \mathbf{A}$ with $a_1 \neq a_2$,
  $c(a_1) \neq c(a_2)$ and $v(a_1) \neq v(a_2)$

- A *maximal matching* on $G(\mathbf{F_X}, \mathbf{X}, \mathbf{A_X})$ is a matching $G(\mathbf{F}, \mathbf{X}, \mathbf{A})$ such that:

  $\forall \mathbf{A}' \supseteq \mathbf{A}$, $\mathbf{A} \neq \mathbf{A}'$, $G(\mathbf{F}, \mathbf{X}, \mathbf{A}')$ is not a matching.

- A matching on $G(\mathbf{F_X}, \mathbf{X}, \mathbf{A_X})$ is complete with regard to $\mathbf{F}$ (respectively with regard to $\mathbf{X}$) if and only if:

  $\forall f \in \mathbf{F},\ \exists a \in \mathbf{A} \mid c(a) = f$
  $(\text{resp. } \forall x \in \mathbf{X},\ \exists a \in \mathbf{A} \mid v(a) = x)$

The problem of finding a maximal matching has been intensively addressed [2, 20, 26] in order to propose algorithms whose complexity is only polynomial instead of exponential.

It has been demonstrated [11] that a system can be decomposed according to a canonical form using a maximal matching. The fig. 1 exhibits this decomposition of the incidence matrix of the structure. The oblique straight line symbolises a maximal matching $G(\mathbf{F}, \mathbf{X}, \mathbf{A})$. Some other results concerning digraph decomposition and algorithms in order to find canonical components can be found in [12, 21].

The different subsets of $\mathbf{X}$ and $\mathbf{F}$ are defined as follows:

The matching $G(\mathbf{F}, \mathbf{X}, \mathbf{A_X})$ is complete with regard to $\mathbf{X}^*$.

The matching $G(\mathbf{F}, \mathbf{X}, \mathbf{A_X})$ is complete with regard to $\overline{\mathbf{F}}^*$.

The matching $G_*(\mathbf{F}_*, \mathbf{X}_*, \mathbf{A}_{*\mathbf{X}})$ is complete with regard to $\mathbf{X}_*$ (but $\text{card}(\mathbf{F}_*) > \text{card}(\mathbf{X}_*)$)

Let us now give an interpretation of the notion of complete matching. Let us consider a subsystem $(\mathbf{F}', Q(\mathbf{F}'))$. In the case of numerical analytical model, the set of constraints applied to $(\mathbf{F}', Q(\mathbf{F}'))$ can be processed as a set of equations to be solved with regard to $\mathbf{X}' = Q_u(\mathbf{F}')$:

$$\mathbf{F}'(Q_k(\mathbf{F}'), \mathbf{X}') = 0 \tag{6}$$

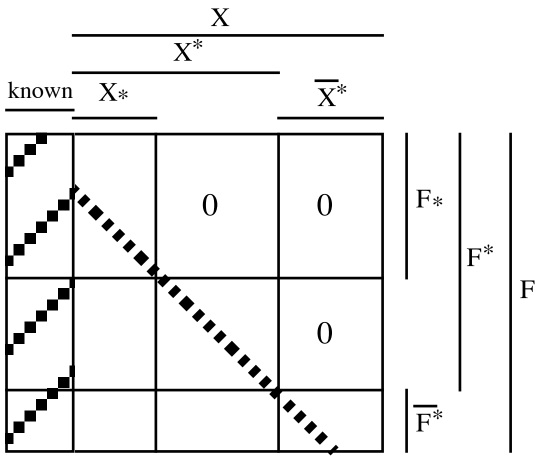

Fig. 1: Canonical decomposition

The existence of a complete matching with regard to $\mathbf{X}'$ and $\mathbf{F}'$ is a necessary condition for the system (6) to be solved in $\mathbf{X}'$ [11].

So, analysing the canonical decomposition of Fig. 1, the subsystem $(\mathbf{F}_*, \mathbf{X}_*)$ is monitorable. Moreover this subsystem may be decomposed according Fig. 2:

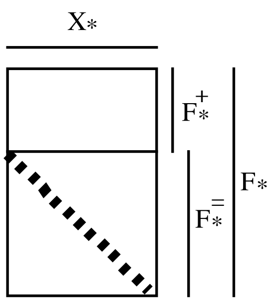

Fig.2: Monitorable subsystem

Let $\mathbf{X_S}(Q_k(\mathbf{F}_*^-))$ be the solution of the corresponding system of equations for given values of $Q_k(\mathbf{F}_*^-)$. Using the remaining relations $\mathbf{F}_*^+$ leads to:

$$\mathbf{F}_*^+(Q_k(\mathbf{F}_*^+), \mathbf{X_S}(Q_k(\mathbf{F}_*^-))) = 0 \tag{7}$$

which constitutes a set of analytical redundancy relations.

The structural analysis of a system constitutes a good way to exhibits redundancy. Indeed this approach makes no hypothesis about the kind of model which will be used and can then be applied to various process models. From this point of view, it can be considered as a very powerful pre-processing of any classical residual generation method applied to large scale systems.

Notice that structural analysis is not limited to the finding of the monitorable subsystems of a given system. Indeed, when the set $\mathbf{X}$ of the unknown variables only contains the unknown states, structural analysis gives the possibility or not to obtain some ARR like equation (2). Moreover, if we include in the set $\mathbf{X}$ some unknown parameters, then the resulting ARR, when they exist, will be robust with respect to them [7]. Further, if $\mathbf{X}$ contains a subset of known parameters or outputs (which are considered as if they were unknown), then the resulting ARR, when they exist, are not sensitive to the faults on those parameters and outputs. In this way, structured residuals by perfect de-coupling may be achieved.

From a computational point of view, a complete matching with respect to some variables $\mathbf{X}$ leads to solve the system of equations (6) in order to obtain the ARR given by (7). One easily sees that the whole procedure is equivalent to eliminate the variables $\mathbf{X}$ from the system of equations (6)-(7).

IV. THEORY OF ELIMINATION

Basically, perfect de-coupling consists in the elimination, from a set of equations, of all the unknown variables. Some very interesting results issued from commutative algebra and algebraic geometry may be used for this purpose. Knowledge about this kind of mathematics is not very wide-spread among engineers but there is not place enough here to recall the basics of this branch [13]. We refer the reader to [10]. Roughly speaking, let us consider a system modelled by a set of constraints $\mathbf{F} = \{f_1, \cdots, f_m\}$ involving a set of unknown variables $\mathbf{X} = \{x_1, \cdots, x_p\}$ and known variables $\mathbf{K} = \{k_1, \cdots, k_q\}$. Each f_i is described by $f_i(x_1, \cdots, x_p, k_1, \cdots, k_q) = 0$. The following presentation is limited to polynomial functions of the form:

$$f_i = \sum_j c_j x_1^{a_{j,1}} \cdots x_p^{a_{j,p}} k_1^{b_{j,1}} \cdots k_p^{b_{j,q}} \tag{8}$$

where $c_j \in \mathbf{R}$, exponents $a_{j,k}, b_{j,k} \in \mathbf{N}$.

The set (or more precisely, the ring) of polynomials in the variables $x_1, \cdots, x_p, k_1, \cdots, k_q$ with coefficients from $\mathbf{R}$ is denoted $\mathbf{R}[x_1, \cdots, x_p, k_1, \cdots, k_q]$.

Let $f_1, \cdots, f_m \in \mathbf{R}[x_1, \cdots, x_p, k_1, \cdots, k_q]$. The *ideal* generated by the f_i is the set of all polynomials f that can be written $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n$ for some $\alpha_i \in \mathbf{R}[x_1, \cdots, x_p, k_1, \cdots, k_q]$. It is denoted $\langle f_1, \cdots, f_n \rangle$.

The theory of ideals and their cousins is called *commutative algebra* or, if the main interest is in zero-sets of the polynomials, *algebraic geometry*.

The objective is to eliminate elements of $\mathbf{X}$, leading to a non trivial polynomial relation containing only variables of $\mathbf{K}$. The theory of elimination provides mathematical tools to find these polynomials relations.

A first approach is based on direct elimination of variables using Euclidean and generalised Euclidean polynomial division [9]. The theory of Gröbner bases, invented by Bruno Buchberger [3] which generalises the previous ideas, is a special generating set of an ideal. It has the following property:

Consider the two sets of variables, $\mathbf{X}$ and $\mathbf{K}$. If G is a Gröbner basis for the ideal $\mathbf{I} \subset \mathbf{R}[\mathbf{X}, \mathbf{K}]$ with regard to the ranking $\mathbf{X} \succ \mathbf{K}$ then:

$$\mathbf{I} \cap \mathbf{R}[\mathbf{K}] = (\mathrm{G} \cap \mathbf{R}[\mathbf{K}]) \tag{9}$$

The ranking of the variables determines which variables to eliminate first. So $\mathbf{I} \cap \mathbf{R}[\mathbf{K}]$ constitutes a set of polynomials which are a consequence of the original ones in which any variables belonging to $\mathbf{X}$ has been eliminated, so this set is composed of Analytical Redundancy Relations [14].

Another interesting thing in commutative algebra is the concept of algebraic dependence. We say that $p_1, \cdots, p_m \in \mathbf{R}[x_1, \cdots, x_n]$ are algebraically dependent over $\mathbf{R}$ if there is a nonzero polynomial f such that $f(p_1, \cdots, p_m) \equiv 0$. A nice way to retrieve the dependency relation f using the Gröbner bases is simply to form the ideal:

$$\mathbf{I} = (z_1 - p_1(x), z_2 - p_2(x), \cdots, z_m - p_m(x)) \tag{10}$$

and then compute one of its Gröbner basis with regard to some ranking that eliminates the x_i. The z_i are called *tag variables.*

V. THE CASE OF LINEAR SYSTEMS

A. Static parity space

Let us consider the following linear static model at time *t*:

$$y(t) = Cx(t) + e(t) + Fd(t) \tag{11}$$

where $y(t)$ is the measurement vector of dimension m, $x(t)$ the state vector of dimension n, $d(t)$ the vector of faults of dimension p and $e(t)$ the vector of measurement noise which is often considered to be normally distributed with zero mean and known covariance matrix Σ. The matrix C characterises the measurement system and F is the distribution matrix of faults. In the following, we only consider the case where m, the number of measurement is greater than n the number of variables, in order to be able to obtain redundancies (this condition is necessary but not sufficient). For such system, the parity vector may be written as:

$$p(t) = Wy(t) \tag{12}$$

where W is a projection matrix which is orthogonal to C [22]. So we have:

$$WC = 0 \tag{13}$$

That implies:

$$p(t) = We(t) + WFd(t) \tag{14}$$

Equation (12) is the computational form of the parity vector and (14) is the internal form (containing the influence of faults). Under ideal circumstances, the parity vector is statistically equal to zero. Its covariance matrix is equal to $W\Sigma W^T$. If the matrix WF is regular, (14) provides a mean to detect and isolate faults. It clearly shows that the rank of this matrix, which depends, of course, on W must be studied. Particularly, a "bad" choice of W could conceal some fault directions (when the matrix WF contains a column of zeros).

More generally, a parity vector sensitive to certain faults and insensitive to the others may be designed. The measurement vector may be written as:

$$y(t) = Cx(t) + e(t) + F^+ d^+(t) + F^- d^-(t) \tag{15}$$

where $d^+(t)$ and $d^-(t)$ denotes the subvectors of faults with regard to which sensitivity and insensitivity is required. The main principle of parity vector generation is kept. A matrix W orthogonal to the subspace spanned by the columns of the matrix $(C \quad F^-)$ has to be found.

$$\begin{cases} WC = 0 \\ WF^- = 0 \\ WF^+ \neq 0 \end{cases} \tag{16}$$

The previous formalism may be easily extended when considering constraints on state variables. The measurement equation is completed by a constraint equation:

$$\begin{cases} y(t) = Cx(t) + e(t) + Fd(t) \\ Ax(t) = Ed(t) \end{cases} \tag{17}$$

The structure of the previous case may be find again in writing this system under the following form:

$$\begin{bmatrix} I \\ 0 \end{bmatrix} y(t) = \begin{bmatrix} C \\ A \end{bmatrix} x(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} F \\ E \end{bmatrix} d(t) \tag{18}$$

The parity vector is then obtained by elimination of the state vector:

$$W \begin{bmatrix} C \\ A \end{bmatrix} = 0 \tag{19}$$

and its computational and internal forms are:

$$p(t) = W \begin{bmatrix} I \\ 0 \end{bmatrix} y(t) \tag{20a}$$

$$p(t) = W \left(\begin{bmatrix} I \\ 0 \end{bmatrix} e(t) + \begin{bmatrix} F \\ E \end{bmatrix} d(t) \right) \tag{20b}$$

As previously, the vector $d(t)$ may be split into components with regards to which sensitivity and insensitivity are required.

B. Dynamic parity space

Let us now consider a dynamic system modelled by a linear time invariant system described by the following state space equations:

$$\dot{x}(t) = Ax(t) + Bu(t) + Ed(t) \tag{21a}$$

$$y(t) = Cx(t) + Du(t) + Fd(t) + e(t) \tag{21b}$$

The same principle as for static systems applies. It consists to eliminate the unknown variables in order to obtain input-output relations. Using symbolic calculus, the system may be re-written:

$$\begin{bmatrix} B & 0 \\ -D & I \end{bmatrix} \begin{bmatrix} u(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} sI - A \\ C \end{bmatrix} x(s) + \begin{bmatrix} 0 \\ I \end{bmatrix} e(s) + \begin{bmatrix} -E \\ F \end{bmatrix} d(s) \tag{22}$$

The structure of this equation is the same as (11). The generation of the parity vector may be done in the same manner. It leads to a "projection" matrix $W(s) = (W_1(s) \quad W_2(s))$ such that:

$$W(s) \begin{bmatrix} sI - A \\ C \end{bmatrix} = 0 \tag{23}$$

The two forms of the parity vector may be written as:

$$p(s) = \left(-W_1(s)E + W_2(s)F \right) d(s) + W_2(s)e(s) \tag{24a}$$

$$p(s) = \left(W_1(s)B - W_2(s)D \right) u(s) + W_2(s)y(s) \tag{24b}$$

As the static case, the vector $d(s)$ may be split into subvectors of faults with regard to which sensitivity and insensitivity is required. With obvious notation, the problem leads to search a polynomial matrix $W(s)$ such that:

$$W(s)\begin{bmatrix} sI - A & E^{-} \\ C & F^{-} \end{bmatrix} = 0 \qquad (25)$$

Equations (23) or (25) are not only mathematical writing. With the help of symbolic calculus software such as Maple or Mathematica for example, finding such polynomial matrix is very easy. Moreover, these softwares not only give a solution but all the parametrised solutions to the problem.

This type of approach may be easily extended to the case of dynamical singular systems [17] or, equivalently, to systems with unknown inputs [24]. It unifies the case of static and dynamic systems. However, numerically efficient methods for generating redundancy relations of dynamical systems have also been developed [6, 25]

VI. ENHANCEMENT OF ROBUSTNESS PARTIAL DE-COUPLING

All previous methods present some limitations and lead to consider that exact de-coupling may be a too strong constraint. Relaxing this constraint allows to search a trade-off which can give better results than the previous approaches can do. For sake of simplicity, the static case is only presented. The dynamical case only differs by definition of the influence of the dynamic of the failure d on the residual vector.

Let us consider again equation (15). The perfect de-coupling is not always attainable because of the rank of the matrix $(C \quad F^{-})$. When the matrix W does not exist, it is interesting to search an approximate solution. The following objectives may be pursued: to generate a parity vector totally de-coupled from the state $x(t)$, "very" sensitive to faults to be detected $d^{+}(t)$ and "very" insensitive to the perturbations $d^{-}(t)$. For example, the vector w could be the solution of the optimisation problem:

$$\begin{cases} w^T C = 0 \\ \max\limits_{w} f^{+}\left(w^T F^{+}\right) \\ \min\limits_{w} f^{-}\left(w^T F^{-}\right) \end{cases} \qquad (26)$$

where the functions $f^{-}(\bullet)$ and $f^{+}(\bullet)$ has to be defined by the user. It consists in a multicriteria problem [24] and the solution may be obtained only if the relative weight of the different objectives are given. Taking into account the statistical aspect of the residual leads to an additional constraint $w^T \Sigma w = 1$ (see the reference [4]).

A classical transformation of this problem consists in modifying (26) in a single criterion problem. If the statistical constraint is relaxed, that leads to find the solution of:

$$\begin{cases} w^T C = 0 \\ \max\limits_{w} \dfrac{f^{+}\left(w^T F^{+}\right)}{f^{-}\left(w^T F^{-}\right)} \end{cases} \qquad (27)$$

When the function to be optimised is a ratio of quadratic ones, the problem may be written as:

$$\begin{cases} w^T C = 0 \\ \max\limits_{w} \dfrac{\left\| w^T F^{+} \right\|^2}{\left\| w^T F^{-} \right\|^2} \end{cases} \qquad (28)$$

It is well known that the solution of this type of problem may be expressed as a problem of finding generalised eigenvectors and eigenvalues. The vector w is solution of:

$$(QA - \lambda QB)w = 0 \qquad (29)$$

with

$$A = F^{+}(F^{+})^T \quad B = F^{-}(F^{-})^T \quad Q = I - C^T (C C^T)^{-1} C$$

The structure of (29) shows that w is a generalised eigenvector of the pair $(QA \quad QB)$. Moreover, it can be shown that the vector λ is the least eigenvalue associated to this vector.

Another way to solve the problem consists in using penalty function. The problem (26) is then replaced by:

$$\begin{cases} w^T C = 0 \\ w^T \Sigma w = 1 \\ \max\limits_{w} \left(f^{+}\left(w^T F^{+}\right) - \beta^2 f^{-}\left(w^T F^{-}\right) \right) \end{cases} \qquad (30)$$

Where the coefficient β^2 weights the two objectives of sensitivity and insensibility. When quadratic functions are used, this problem is easily solved using the following Lagrangian function [8]:

$$L = w^T \left(A - \beta^2 B \right) w + w^T C \mu + \lambda \left(1 - w^T \Sigma w \right) \qquad (31)$$

The optimal value λ^{*} of the criteria is the maximum zero of the quadripole $[(A - \beta^2 B),\ -\Sigma,\ C,\ 0]$ and the vector $\left(w' \quad \mu' \right)$ belongs to the kernel of the matrix:

$$R = \begin{bmatrix} A - \beta^2 B - \lambda^{*} \Sigma & C \\ C^T & 0 \end{bmatrix} \qquad (32)$$

When perfect de-coupling cannot be attained or when the quality of the partial de-coupling is not satisfactory, the instrument scheme must be modified or completed. Some works have been developed in this direction [1, 16, 19].

The problem of de-coupling may also be viewed with regard to small variations of the parameter of the system model. Let us consider the model (11) where the matrix C

belongs to a finite set $C \in \{C_0, C_1, \cdots, C_q\}$. Practically, C_0 may be considered as the nominal value of C and the C_i are "distributed" around C_0. The matrix W must satisfy the relations:

$$WC_i = 0 \qquad i = 1, \ldots, q \tag{33}$$

However, in general this problem has no solution and it is interesting to search an approximate one. A solution may be obtained by searching the minimum of the following criteria [15]:

$$J = \| W\mathbf{C} \|_F \tag{34}$$

where $\mathbf{C} = \begin{pmatrix} C_0 & C_1 & \cdots & C_q \end{pmatrix}$, and $\|M\|_F$ denotes the Frobenius norm of the matrix M, i.e. $\|M\|_F = \left(\mathrm{Tr}(MM^T)\right)^{1/2}$ where Tr( ) stands for the trace operator.

The solution of this problem may be easily obtained. Considering the singular value decomposition of matrix $\mathbf{C}$:

$$\mathbf{C} = U\Sigma V \text{ and } \Sigma = \left[\begin{array}{cccc|ccc} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \ddots & \vdots & \vdots & & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & \sigma_m & 0 & \cdots & 0 \end{array}\right] \tag{35}$$

where U and V are orthogonal matrices, and $\sigma_1 \geq \cdots \geq \sigma_m$ are the singular values of $\mathbf{C}$, ordered by magnitude. The matrix W minimising (34) is given by:

$$W = \begin{pmatrix} u_1 & u_2 & \cdots & u_p \end{pmatrix} \tag{36}$$

where the u_i, $i = 1, .., p$ are the first p left singular vectors of $\mathbf{C}$, i.e. the first p columns of U. In this case, the residual criterion, which represents a robustness index is $J_r = \sum_{i=1}^{r} \sigma_i^2$.

This technique may be extended to systems which parameters belong to a given interval, $C = C(\theta)$ with $\theta \in [\theta_{\min}, \theta_{\max}]$. The problem reduces to find parity vectors as orthogonal as possible to all possible values of C. Let us consider a uniform distribution of the parameters in their intervals, the minimisation of the following criteria is intuitive:

$$J = \int_{\theta_{\min}}^{\theta_{\max}} \| WC(\theta) \|_F \, d\theta \tag{37}$$

(In this notation, the integration is done with regard to all the entries of the parameter vector θ). According to the previous result, the optimal matrix W is given by (36) where the u_i, $i = 1, .., p$ are now the p eigenvectors associated to the first p least eigenvalues λ_i of the matrix [23]:

$$\mathbf{C} = \int_{\theta_{\min}}^{\theta_{\max}} C(\theta) C^T(\theta) \, d\theta \tag{38}$$

VII. EXAMPLE

In this section, the elimination theory is applied in order to generate a set of residuals. Let us consider a simplified dynamical electrical model of an induction machine. The chosen state variables are the stator and rotor currents (I), all expressed in the Park basis linked to the stator frame; the input is the supply voltage (V). The model may then be written as a classical linear dynamical model:

$$\begin{aligned} \dot{X}(t) &= A(t)X(t) + BU(t) \\ Y(t) &= CX(t) \end{aligned}$$

with

$$A(t) = \frac{1}{\sigma} \begin{bmatrix} -l_r r_s & m_{sr}^2 \omega(t) & m_{sr} r_r & l_r m_{sr} \omega(t) \\ -m_{sr}^2 \omega(t) & -l_r r_s & -l_r m_{sr} \omega(t) & m_{sr} r_r \\ m_{sr} r_s & -l_s m_{sr} \omega(t) & -l_s r_r & -l_s l_r \omega(t) \\ l_s m_{sr} \omega(t) & m_{sr} r_s & l_s l_r \omega(t) & -l_s r_r \end{bmatrix}$$

$$B = \frac{1}{\sigma} \begin{bmatrix} l_r & 0 \\ 0 & l_r \\ -m_{sr} & 0 \\ 0 & -m_{sr} \end{bmatrix} \qquad C = I_{4 \times 4} \qquad \sigma = l_s l_r - m_{sr}^2$$

$$X(t) = \begin{bmatrix} I_{sd}(t) \\ I_{sq}(t) \\ I_{rd}(t) \\ I_{rq}(t) \end{bmatrix} \qquad U(t) = \begin{bmatrix} V_{sd}(t) \\ V_{sq}(t) \end{bmatrix}$$

and where r_s and r_r are respectively the equivalent statoric and rotoric resistance, l_s and l_r the equivalent statoric and rotoric cyclic inductance and m_{sr}, the equivalent cyclic mutual stator/rotor. If the statoric currents are systematically measured, it is not the case for the rotoric currents. However, these variables may be indirectly observed by the measurement of the fluxes provided some expensive devices may be set up. However, for sake of simplicity in this didactic presentation, the observation matrix C will be assigned to identity.

The proposed model is not time invariant because of the presence, in the matrix A, of the angular velocity of the machine $\omega(t)$. However, when the machine is in steady-state, this velocity may be considered as constant.

Considering one of the control signals $V_{sd}(t)$ or $V_{sq}(t)$ as a perturbation, it is possible, using the decoupling methods presented in section V, to generate redundancy relations which do not depend on this signal. The general parametrised solution, which depends on three parameters, is to big for being presented here. For instance, the following relations do not depend on $V_{sq}(t)$:

$$r_r V_{sd} + l_r \dot{V}_{sd} + \sigma \ddot{I}_{sd} + \tau \dot{I}_{sd} + r_r r_s I_{sd} - m_{sr} l_r \omega I_{rq} - m_{sr}^2 \omega I_{sq} = 0$$

with $\tau = r_s l_r + r_r l_s$

$$l_r V_{sd} \square \square I_{sd} + l_r r_s I_{sd} + m_{sr} l_r \square I_{rq} + m_{sr}^{2} \square I_{sq} + I_{rd} = 0$$

The general solution also shows that, for this particular example, it is possible to eliminate all the control signals, leading, for example, to the following relation:

$$m_{sr} \square \left(l_s l_r + m_{sr}^2 \right) I_{sd} \square \square m_{sr} \dot{I}_{sq} + \square l_r \square I_{rd} \square l_r \square \dot{I}_{rq} \square \square r_r I_{rq} = 0$$

Using the theory of elimination, and Gröbner bases, it is also possible to find three redundancy relations where the angular velocity does not intervene:

$$r_s I_{sd} + l_s \dot{I}_{sd} + m_{sr} \dot{I}_{rd} \square V_{sd} = 0$$

$$r_s \square I_{sq} + l_s \square \dot{I}_{sq} + l_s r_r m_{sr} \left(I_{rq} \square I_{rd} \right) + m_{sr} \square \dot{I}_{rq} \square \square V_{sq} = 0$$

$$\begin{aligned} & m_{sr} \left(l_r I_{rq} + m_{sr} I_{sq} \right) V_{sq} + m_{sr} \left(l_r I_{rd} + m_{sr} I_{sd} \right) V_{sd} \\ & \quad + l_r \left(r_r l_s I_{rd} \square r_s m_{sr} I_{sq} + \square \dot{I}_{rq} \right) I_{rq} \\ & \quad + \left(r_r l_s m_{sr} I_{sq} + m_{sr} \left(r_r l_s \square r_s l_r \right) I_{sd} + l_r \square \dot{I}_{rd} \right) I_{rd} \\ & \quad + r_r l_s l_r I_{rd}^2 \square r_s m_{sr}^2 \left(I_{sq}^2 + I_{sd}^2 \right) + m_{sr} \square \left(\dot{I}_{rq} I_{sq} + \dot{I}_{rd} I_{sd} \right) = 0 \end{aligned}$$

VIII. CONCLUDING REMARKS

An attempt has been made to present residual generation methods for fault detection purposes. The presentation has been organised along the concept of unknown or unmeasured variable elimination by using the model of the process which may be either consider as static or dynamic. The first stage of the analysis is independent of the nature of the model because this model is reduced to a set of constraints applied to a set of variables. The resulting structural analysis gives qualitative results with regard to the observability and the redundancy of variables. The second stage of residual generation gives an explicit form of the residuals depending only on the known variables of the process. Perfect elimination of unknown variables being not possible in some situations, partial de-coupling has been presented. These techniques must still be enhanced and deserve to be further developed. Robustness in the face of modelling errors is probably one of the most important extension to these methods.

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