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Optimality conditions for the truncated network of the generalized discrete orthonormal basis having real poles

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Abstract

The main contribution of this paper is the synthesis of optimality conditions for the truncated network of the generalized orthonormal basis in the case where all the poles belong to the set of real numbers. These conditions are brought to a very simple form, but their solutions are not trivial. They generalize the optimality conditions for the truncated Laguerre network and are very attractive in system identification, model representation, and model reduction frameworks.

Key words: System representation, Laguerre filters, Generalized orthonormal basis, and optimization

1. Introduction

The output $Y(z)$ of any linear time invariant (LTI) system may be represented by a sum of an infinite series of orthonormal functions $\{\mathcal{G}_n(z)\}_{n=0}^{\infty}$ weighted by their respective Fourier coefficients g_n, provided they constitute a complete basis in $H^2(D^c)$ where D^c is the complement of the unit disk in the complex Hardy plane H^2.

$$Y(z)=\sum_{n=0}^{\infty} g_n \mathcal{G}_n(z)U(z) \qquad (1)$$

In practice, a truncation at $N+1$ terms is often performed and the usefulness of system representation on a desired basis is limited by the rate of convergence of the Fourier coefficients, g_n's, or the rate by which the error term $\sum_{n=N+1}^{\infty}|g_n|^2$ tends to zero as N tends to infinity.

The most popular of these bases is definitely the basis of FIR filters in which case each function $\mathcal{G}_n(z)$ is simply replaced by the n^{th} term of the sequence $\{z^{-k}\}_{k=0}^{\infty}$. However, the major problem with FIR filters is the low convergence rate of their Fourier coefficients, especially for discretized systems with high sampling frequency compared to the dominant time constant. An extended discussion of this problem is presented in [Linskog, 1996].

The advantage of Laguerre functions is that they overcome this problem by introducing an extra parameter 'a' known in the discrete case as *time scale factor*.

$$L_n(z,a)=\sqrt{1-a^2}\,\frac{(1-za)^n}{(z-a)^{n+1}} \qquad n=0,\ldots\infty \qquad (2)$$

All the authors among whom, [Wahlberg, 1991], [Wang, 1994], and [Linskog, 1996] agree that the time scale factor 'a' must be chosen near the dominating time constant of the system in order to ensure a fast convergence rate of Fourier coefficients.

[Clowes, 1965] was the first to raise the question of the optimal choice of the Laguerre time scale factor. He established the well known optimality condition for the truncated Laguerre network with a real pole

$$g_N(a)g_{N+1}(a)=0 \qquad (3)$$

Though this equation seems simple, its solution is not trivial because 'a' intervenes in a nonlinear way in the Fourier coefficients. Due to the complexity of the solutions of equation (3) many researchers have investigated the choice of optimal time scale factor. First proposed algorithms [Parks, 1971], [Fu *et al*, 1993] and [Wang *et al*, 1994] were restrained to the use of input signals which power spectrum is white ensuring the orthogonality of the output of Laguerre filters. Other algorithms followed, [e Silva, 1995] and [Malti *et al*, 1998], dropping this constraint away.

Though the use of Laguerre filters improves considerably system's representation with orthonormal functions, their major inconvenience is that they accept only one real pole. Hence, they are not suited to approximate under-damped systems which have complex conjugate poles. To overcome this problem, [Kautz, 1954] proposes to use another decomposition basis which took his name. It is mostly known in the case of two-parameters Kautz filters and has two complex conjugate poles p and p^*.

After generalizing the optimality condition (3), for the truncated Laguerre network with a complex pole, [e Silva, 1995] has established the optimality conditions for the truncated Kautz network with complex conjugate poles. The two are similar since the latter is

$$g_N^{odd}(p,p^*)=g_N^{odd}(p,p^*)=0 \qquad (4)$$

or $$g_{N+1}^{even}(p,p^*)=g_{N+1}^{even}(p,p^*)=0$$

Recently, [Ninness and Gustafsson, 1997] have proposed, to utilize a generalized orthonormal basis which is not limited to the use of either one real pole or two complex conjugate ones. They are presented either as causal filters when $d = 1$ or strictly causal when $d = 0$ in the following definition

$$\mathcal{B}_n\left(z,\underline{\xi_n}\right) = z^d \frac{\sqrt{1-\left|\xi_n\right|^2}}{z-\xi_n} \prod_{k=0}^{n-1} \frac{1-\overline{\xi_k} z}{z-\xi_k} \quad \boldsymbol{n = 0, \ldots \infty} \qquad \textbf{(5a)}$$

where all the poles ξ_n are assumed inside the unit circle and $\underline{\xi_n}$ is defined as $\underline{\xi_n} = \left[\xi_0\ \xi_1\ \cdots\ \xi_n\right]^T$. All the vectors are underlined throughout this paper.
From the definition (5a), one may establish the following recursive formula

$$\mathcal{B}_n\left(z,\underline{\xi_n}\right) = D_{n,n-1} \frac{1-\overline{\xi_{n-1}} z}{z-\xi_n} \mathcal{B}_{n-1}\left(z,\underline{\xi_{n-1}}\right) \qquad \textbf{(5b)}$$

with $D_{n,n-1} = \dfrac{\sqrt{1-\left|\xi_n\right|^2}}{\sqrt{1-\left|\xi_{n-1}\right|^2}}$

which eases the network representation of the functions of the generalized basis, q defining the forward shift operator

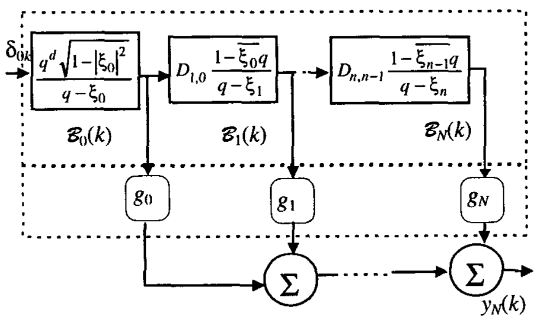

Figure 1 - Network representation

All the discussed functions are orthonormal and form a complete basis with respect to the scalar product which is defined as

$$\left(\mathcal{G}_i(k),\mathcal{G}_j(k)\right) = \sum_{k=0}^{\infty} \mathcal{G}_i(k)\overline{\mathcal{G}_j(k)} = \delta_{i,j}, \qquad \textbf{(6a)}$$

and which reciprocal in the frequency domain is established using Parseval's formula

$$\left(\mathcal{G}_i(z),\mathcal{G}_j(z)\right) = \frac{1}{2\pi j}\oint_T \mathcal{G}_i(z)\mathcal{Z}\left\{\overline{\mathcal{G}_j(k)}\right\}_{z\to z^{-1}} z^{-1}dz = \delta_{i,j} \qquad \textbf{(6b)}$$

where T is the unit circle and $\mathcal{Z}\{.\}$ is the z-transform of the function between brackets.

2. *Optimality conditions for the truncated network of the generalized basis*

Though the generalized basis is defined for any number of complex poles of any multiplicity, we will restrict ourselves to the determination of optimality conditions in the case where all the poles belong to the set of real numbers. Hence, the conjugate part will be dropped in the definitions (5a&b) and (6a&b).
The estimated output which depends on the truncation order and the set of poles is written as

$$Y_N\left(z,\underline{\xi_N}\right) = \sum_{n=0}^{N} g_n\left(\underline{\xi_n}\right)\mathcal{B}_n\left(z,\underline{\xi_n}\right)$$

The modeling error, $E_N\left(k,\underline{\xi_N}\right)$, due to the truncation, is presented as

$$E_N\left(z,\underline{\xi_N}\right) = Y(z) - \sum_{n=0}^{N} g_n\left(\underline{\xi_n}\right)\mathcal{B}_n\left(z,\underline{\xi_n}\right) \qquad \textbf{(7a)}$$

or as $E_N\left(z,\underline{\xi_N}\right) = \displaystyle\sum_{n=N+1}^{\infty} g_n\left(\underline{\xi_n}\right)\mathcal{B}_n\left(z,\underline{\xi_n}\right) \qquad \textbf{(7b)}$

The quadratic error, which is defined as half the scalar product of the error by itself, is formulated in the time domain as

$$I_N\left(\underline{\xi_N}\right) = \frac{1}{2}\sum_{k=0}^{\infty}\left(y(k) - \sum_{n=0}^{N} g_n\left(\underline{\xi_n}\right)\mathcal{B}_n\left(k,\underline{\xi_n}\right)\right)^2$$

or in the frequency domain as

$$I_N\left(\underline{\xi_N}\right) = \frac{1}{4\pi j}\oint_T E_N\left(z,\underline{\xi_N}\right)E_N\left(z^{-1},\underline{\xi_N}\right)z^{-1}dz$$

The optimal values of the coefficients are obtained by applying the orthogonality property of Hilbert spaces. The result is the following set of normal equations.

$$\left(E_N\left(z,\underline{\xi_N}\right),\mathcal{B}_n\left(z,\underline{\xi_n}\right)\right) = 0 \quad \boldsymbol{n = 0, \ldots, N} \qquad \textbf{(8)}$$

Furthermore, the stationary points of the set of the poles must satisfy the following vector equation

$$\frac{\partial I_N\left(\underline{\xi_N}\right)}{\partial\underline{\xi_N}} = \left(\frac{\partial E_N\left(z,\underline{\xi_N}\right)}{\partial\underline{\xi_N}}, E_N\left(z,\underline{\xi_N}\right)\right) = 0_{(N+1)\times 1} \qquad \textbf{(9)}$$

Note that I_N and E_N are differentiated w.r.t. the vector $\underline{\xi_N}$ and that the scalar product in (9) is performed between each 'function' element of that vector and the scalar 'function' which is on the RHS of the parentheses. This notation is kept through equations (10) and (11).
Combining (9) and (7a) and then taking into account (8) and (7b) gives the following

$$\sum_{n=0}^{N} -g_n\left(\underline{\xi_n}\right)\left(\frac{\partial\mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial\underline{\xi_N}}, \sum_{k=N+1}^{\infty} g_k\left(\underline{\xi_k}\right)\mathcal{B}_k\left(z,\underline{\xi_k}\right)\right) = 0_{(N+1)\times 1} \qquad \textbf{(10)}$$

Now, a relation needs to be established between the derivatives of $\mathcal{B}_n$'s w.r.t. the poles and $\mathcal{B}_n$'s. It is presented in lemma 1.

Lemma 1 - Expansion of $\dfrac{\partial\mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial\xi_i}$ **on** $\mathcal{B}_n$**'s**

Let $\mathcal{B}_n(z,\underline{\xi_n})$ be the generalized basis defined by (5a). Then, the following decomposition holds for every $i \le n$

$$\frac{\partial \mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial \xi_i}=\sum_{j=0}^{\infty} C_{n,i,j}\left(\underline{\xi_j}\right)\mathcal{B}_j\left(z,\underline{\xi_j}\right) \tag{11}$$

where $C_{n,i,j}$'s are defined in table 1. ⊠

Convention It will be assumed throughout this paper that $\prod_{i=m}^{n} f_i = 1$ if the lower index m is greater than the upper index n, f_i being any non-zero expression. ⊠

Remark 1 One should keep in mind that $C_{n,i,j}=0$ if $i>n$, because every function $\mathcal{B}_n$ does not depend on the pole which index is greater than n. Otherwise, the values of $C_{n,i,j}$ are to be taken from table 1.⊠

The basic idea to prove lemma 1 is first to establish, by introducing the logarithm on both sides of (5a) and differentiating them w.r.t. the poles, a relation between $\frac{\partial \mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial \xi_i}$, $\mathcal{B}_n\left(z,\underline{\xi_n}\right)$ and other terms in z.

After noticing that each expression is a linear transfer function, the property of completeness of the generalized basis is used to decompose each derivative in terms of the functions of the generalized basis by performing its scalar product with $\mathcal{B}_j\left(z,\underline{\xi_j}\right)$ when j varies from 0 to ∞.

The substitution of (11) in (10) and the use of the orthonormality property of the functions of the generalized basis, yields

$$\frac{\partial I_N\left(\underline{\xi_N}\right)}{\partial \underline{\xi_N}}=-\sum_{n=0}^{N} g_n\left(\underline{\xi_n}\right)\sum_{j=N+1}^{\infty} g_j\left(\underline{\xi_j}\right)\underline{C_{n,j}\left(\underline{\xi_j}\right)}=0_{(N+1)\times 1} \tag{12}$$

where $\underline{C_{n,j}}\left(\underline{\xi_j}\right)=\left[C_{n,0,j}\left(\underline{\xi_j}\right)\; C_{n,1,j}\left(\underline{\xi_j}\right)\cdots C_{n,N,j}\left(\underline{\xi_j}\right)\right]^T$

Transforming the above vector equation to N+1 scalar equations, by substituting $C_{n,j}\left(\underline{\xi_j}\right)$, obtained from the last column of table 1 and keeping in mind remark 1yields a set of N+1 equations which general term is given by

$$\frac{\partial I_N\left(\underline{\xi_N}\right)}{\partial \xi_i}=-\sum_{n=i}^{N} g_n\left(\underline{\xi_n}\right)\sum_{j=N+1}^{\infty} m_i A_{n,j} g_j\left(\underline{\xi_j}\right)\frac{\prod_{k=n+1}^{j-1}\left(\xi_i-\xi_k\right)}{\prod_{k=n}^{j}\left(1-\xi_i\xi_k\right)}=0$$

$$i=0,\ldots,N \tag{13}$$

Note that each equation is of infinite dimension and has infinite degrees of freedom, because no constraint is imposed on ξ_j when j goes from N+1 to ∞, while looking for the optimal ξ_j when j varies from 0 to N.

Simplifying m_i from (13), substituting $A_{n,j}$ by its value as defined in table 1, inverting the order of the summations, and factoring yields

$$\left[\sum_{n=i}^{N}\left(\frac{\sqrt{1-\xi_n^2}\,g_n\left(\underline{\xi_n}\right)}{\left(1-\xi_i\xi_n\right)}\prod_{k=n+1}^{N}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right]\left[\sum_{j=N+1}^{\infty}\frac{\sqrt{1-\xi_j^2}\,g_j\left(\underline{\xi_j}\right)}{\left(1-\xi_i\xi_j\right)}\prod_{k=N+1}^{j-1}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right]=0 \tag{14}$$

N degrees of freedom will be used in order to compute the optimality conditions of the N^{th} order network.

To see how the next steps will be done, one should rewrite the first terms of the infinite summation in (14) for $i=N$. He will see, then, that (ξ_N - ξ_{N+1}) can be factored out of all the terms starting at $j=N+2$ to ∞ which are all set to zero by imposing

$$\xi_{N+1}=\xi_N. \tag{15}$$

Repeating these steps which general formulation is presented in (16) for every i, allows to obtain all the constraints which contribute to simplify (14).

(see at the bottom of this page) **(16)**

By imposing the constraint

$$\xi_{2N-i+1}=\xi_i,\quad \text{for every } i=0,\ldots N \tag{17}$$

j	$j<i$	$i\le j<n$	$j=n$	$j>n$
$C_{n,i,j}=\left(\frac{\partial \mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial \xi_i},\mathcal{B}_j\left(z,\underline{\xi_j}\right)\right)$	0	$-A_{n,j}m_i\frac{\prod_{k=j+1}^{n-1}\left(\xi_i-\xi_k\right)}{\prod_{k=j}^{n}\left(1-\xi_i\xi_k\right)}$	0	$A_{n,j}m_i\frac{\prod_{k=n+1}^{j-1}\left(\xi_i-\xi_k\right)}{\prod_{k=n}^{j}\left(1-\xi_i\xi_k\right)}$
with $A_{n,j}=\sqrt{1-\xi_n^2}\sqrt{1-\xi_j^2}$ and m_i is the multiplicity of the pole ξ_i				

Table 1 - Coefficients of the decomposition of $\frac{\partial \mathcal{B}_n\left(z,\underline{\xi_n}\right)}{\partial \xi_i}$ on $\mathcal{B}_j\left(z,\underline{\xi_j}\right)$

$$\left[\sum_{n=i}^{N}\left(\frac{\sqrt{1-\xi_n^2}\,g_n\left(\underline{\xi_n}\right)}{\left(1-\xi_i\xi_n\right)}\prod_{k=n+1}^{N}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right]\left\{\sum_{j=N+1}^{2N-i+1}\left[\frac{\sqrt{1-\xi_j^2}\,g_j\left(\underline{\xi_j}\right)}{\left(1-\xi_i\xi_j\right)}\left(\prod_{k=N+1}^{j-1}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right]+\underbrace{\sum_{j=2N-i+2}^{\infty}\left[\frac{\sqrt{1-\xi_j^2}\,g_j\left(\underline{\xi_j}\right)}{\left(1-\xi_i\xi_j\right)}\left(\prod_{k=N+1}^{j-1}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right]}_{\frac{\left(\xi_i-\xi_{2N-i+1}\right)}{\left(1-\xi_i\xi_{2N-i+1}\right)}\text{(other terms)}}\right\}=0 \tag{16}$$

one gets rid of the last term of the summation and is left only with

$$\left[\sum_{n=i}^{N}\left(\frac{\sqrt{1-\xi_n^2}\,g_n\left(\underline{\xi_n}\right)}{\left(1-\xi_i\xi_n\right)}\prod_{k=n+1}^{N}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right] \left[\sum_{j=N+1}^{2N-i+1}\left(\frac{\sqrt{1-\xi_j^2}\,g_j\left(\underline{\xi_j}\right)}{\left(1-\xi_i\xi_j\right)}\prod_{k=N+1}^{j-1}\frac{\left(\xi_i-\xi_k\right)}{\left(1-\xi_i\xi_k\right)}\right)\right]=0 \quad \textbf{(18)}$$

The obtained result is summarized in the following theorem.

Theorem 1 Let $\mathcal{B}_n(z, \underline{\xi_n})$ be the n[th] function of the generalized basis defined by (5a) and $g_n\left(\underline{\xi_n}\right)$ the Fourier coefficients associated to $\mathcal{B}_n(z, \underline{\xi_n})$. If all the poles belong to the set of real numbers *i.e.* $\xi_i \in \Re$ ($\forall\ i = 0 \ldots N$), then, the optimality conditions of the truncated generalized basis network of order N+1, n varying from 0 to N, is the solution of the set of N+1 equations which general term is (18), under the N+1 constraints which general term is (17), both resulting from varying i from 0 to N.☒

Further simplifications of theorem 1

The aim of this section is to simplify further theorem 1, by rewriting equation (18) in a simpler and more convenient manner. For that purpose, an equivalence relation needs to be established between two N^{th} order generalized networks having the same set of poles ordered in different manner. Further details are given in the following lemma.

Lemma 2 The two sets of functions, $\left\{\mathcal{B}_N\left(z,\underline{\xi_N}\right)\right\}$ and $\left\{\mathcal{B}_N\left(z,\underline{\xi_N^{[i,N]}}\right)\right\}$, defined by (5a), where

- $\underline{\xi_N}=\left[\xi_0\ \xi_1\ \cdots\ \xi_N\right]^T$
- and $^{[i,N]}$ defines a permutation between any element ξ_i, i = 0, ...(N - 1), and the element ξ_N, from the vector $\underline{\xi_N}$, which means that

$$\underline{\xi_N^{[i,N]}}=\left[\xi_0\ \cdots\ \xi_{i-1}\ \xi_N\ \xi_{i+1}\ \cdots\ \xi_{N-1}\ \xi_i\right]^T, \quad \textbf{(19)}$$

span the same vector space.☒

Lemma 2 indicates that the decomposition of any LTI, stable transfer function, $G(z)$, on either of the truncated networks $\left\{\mathcal{B}_N\left(z,\underline{\xi_N}\right)\right\}$ or $\left\{\mathcal{B}_N\left(z,\underline{\xi_N^{[i,N]}}\right)\right\}$ is equivalent. Assuming that $\overline{g}_n\left(\underline{\xi_N^{[i,N]}}\right)$ are the Fourier coefficients associated with $\left\{\mathcal{B}_N\left(z,\underline{\xi_N^{[i,N]}}\right)\right\}$, this is mathematically written as

$$\sum_{n=0}^{N} g_n\left(\underline{\xi_N}\right)\mathcal{B}_n\left(z,\underline{\xi_N}\right)=\sum_{n=0}^{N}\overline{g}_n\left(\underline{\xi_N^{[i,N]}}\right)\mathcal{B}_n\left(z,\underline{\xi_N^{[i,N]}}\right)\forall\ z \quad \textbf{(20)}$$

Replacing all the $\mathcal{B}_n$'s by their expression obtained from (5a), and equating the coefficients of all the powers of z to zero gives rise to (N - i +1) equations which are sufficient to establish a relation between the coefficients $\overline{g}_n\left(\underline{\xi_N^{[i,N]}}\right)$ and $g_n\left(\underline{\xi_N}\right)$ for every $n = i, \ldots N$.

The expression of the last coefficient of the permuted basis *i.e.* $\overline{g}_N\left(\underline{\xi_N^{[i,N]}}\right)$, in terms of the coefficients of the non-permuted basis $g_n\left(\underline{\xi_N}\right)$ $n = i, \ldots N$, is obtained after tedious calculations

$$\frac{\overline{g}_N\left(\underline{\xi_N^{[i,N]}}\right)}{\sqrt{1-\xi_i^2}}=\sum_{n=i}^{N}\left(\frac{\sqrt{1-\xi_n^2}\,g_n\left(\underline{\xi_n}\right)}{\left(1-\xi_i\xi_n\right)}\prod_{k=n+1}^{N}\frac{\xi_i-\xi_k}{1-\xi_i\xi_k}\right) \quad \textbf{(21)}$$

The same reasoning holds on the approximation of a system on the basis of ($2N$ + i - 1) filters of the generalized basis by imposing the set of constraints defined by (17) on the elements of the permuted vector *i.e.* by imposing

$$\left[\xi_{2N+1}\ \cdots\ \xi_{2N-i}\ \xi_{N+1}\ \xi_{2N-i+2}\ \cdots\ \xi_{N+2}\ \xi_{2N-i+1}\right] =\left[\xi_0\ \cdots\ \xi_{i-1}\ \xi_N\ \xi_{i+1}\ \cdots\ \xi_{N-1}\ \xi_i\right].$$

It yields the following result, after tedious calculations

$$\frac{\overline{g}_{N+1}\left(\underline{\xi_N^{[i,N]}}\right)}{\sqrt{1-\xi_i^2}}=\sum_{j=N+1}^{2N-i+1}\left(\frac{\sqrt{1-\xi_j^2}\,g_j\left(\underline{\xi_j}\right)}{\left(1-\xi_i\xi_j\right)}\prod_{k=N+1}^{j-1}\frac{\xi_i-\xi_k}{1-\xi_i\xi_k}\right) \quad \textbf{(22)}$$

Replacing equations (21) and (22) in (18) simplifies theorem 1 which is now presented as

Theorem 1 bis Let

- $\mathcal{B}_n(z,\underline{\xi_N})$ be defined by (5a),
- $g_n\left(\underline{\xi_N}\right)$ be the Fourier coefficients associated with $\mathcal{B}_n(z,\underline{\xi_N})$
- $\underline{\xi_N}=\left[\xi_0\ \cdots\ \xi_{i-1}\ \xi_i\ \xi_{i+1}\ \cdots\ \xi_{N-1}\ \xi_N\right]^T$ be the vector of poles associated with $\mathcal{B}_n(z,\underline{\xi_N})$ where $\xi_i \in \Re$ ($\forall\ i = 0 \ldots N$),
- and $\underline{\xi_N^{[i,N]}}=\left[\xi_0\ \cdots\ \xi_{i-1}\ \xi_N\ \xi_{i+1}\ \cdots\ \xi_{N-1}\ \xi_i\right]^T$ be the permuted vector of poles. The permutation is done between the element i and the element N in the vector $\underline{\xi_N}$.

Then, the optimality conditions for the truncated generalized basis network of order N+1 (from 0 to N) is the solution of the set of **N + 1 equations (23) resulting** from all the permutations of the element ξ_i, i = 0, 1, ... N, with the element ξ_N in the vector of poles $\underline{\xi_N}$.

$$g_N\left(\underline{\xi_N^{[i,N]}}\right)g_{N+1}\left(\underline{\xi_N^{[i,N]}}\right)=0 \qquad i=0,1,\ldots N \quad \textbf{(23)}$$

The N^{th} and the (N^{th}+1) Fourier coefficients are recalculated after each permutation. The vector of poles used to calculate $g_{N+1}\left(\underline{\xi_N^{[i,N]}}\right)$ is the following

$$\underline{\xi}_N^{[i,N]} = \Bigg[\underbrace{\xi_0 \ \cdots \ \xi_{i-1} \ \xi_N \ \xi_{i+1} \ \cdots \ \xi_{N-1} \ \xi_i \ \xi_i}_{N+1 \text{ elements}}\Bigg]^T . \boxtimes$$

A straightforward implication of *theorem 1 bis*, is the derivation of the well-known optimality conditions for the truncated Laguerre network which is due to [Clowes, 1965], presented in equation (3).

3. Example

Determine the optimal second order truncated network of the generalized basis for the system which unit-sample response is

$$h(k) = \sum_{i=1}^{3} A_i b_i^{k-1} + A_4 (k-1) b_4^{k-1}, \ k = 1, 2, \ldots \infty \qquad \textbf{(24)}$$

where $A_1 = 0.5$, $A_2 = 2$, $A_3 = -1$, $A_4 = 0.156$,
and $b_1 = 0.6$, $b_2 = -0.7$, $b_3 = -0.5$, $b_4 = 0.8$.
Its quadratic norm is $\|H\|^2 = 4.799$.

Independently from the proposed method and for the sake of verifying the results, we have plotted the quadratic criterion and isocriteria curves versus the poles. Indeed, using equation (7a) and the property of orthonormality, it is straightforward to show that

$$I_1(z, \underline{\xi_1}) = \frac{1}{2}\left[\|H(z)\|^2 - g_0(\underline{\xi_0})^2 - g_1(\underline{\xi_1})^2\right].$$

Where the coefficients are expressed with respect to the poles by projecting $H(z)$ on each function of the basis

$$g_n(\underline{\xi_n}) = \left(H(z), \mathcal{B}_n(z, \underline{\xi_n})\right) = \frac{1}{2\pi j}\oint_T H(z^{-1}) \mathcal{B}_n(z, \underline{\xi_n}) z^{-1} dz \qquad \textbf{(25)}$$

Figure (2) shows the obtained iso-criteria curves. Note that there exists several extrema points. They are all symmetrical with respect to the axis $\xi_0 = \xi_1$. The purpose of solving the optimality equations is essentially to determine these stationary points.

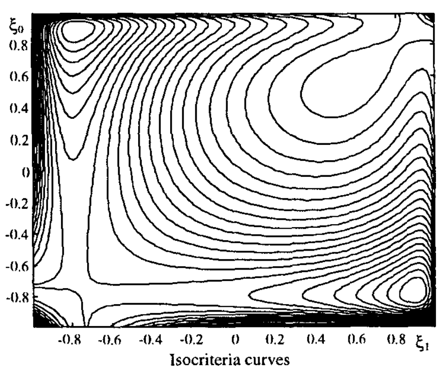

Figure 2 - Quadratic error versus the poles

For that purpose, the use of theorem 1 shows that the optimality conditions for a second order generalized basis is the solution of the two equations

$$f_0(\xi_0, \xi_1) = \left[g_0(\underline{\xi_0})(\xi_1 - \xi_0) - g_1(\underline{\xi_1}) A_{0,1}\right] \left[g_3(\underline{\xi_1})(\xi_1 - \xi_0) - g_2(\underline{\xi_1}) A_{0,1}\right] = 0 \qquad \textbf{(26)}$$

$$f_1(\xi_0, \xi_1) = g_1(\underline{\xi_1}) g_2(\underline{\xi_1}) = 0$$

where g_2 and g_3 are computed after imposing the constraints $\xi_2 = \xi_1$ and $\xi_3 = \xi_0$.

Due to the complexity of $f_0(\xi_0,\xi_1)$ and $f_1(\xi_0,\xi_1)$, calculated using the software Maple V, the computation of their zeros is not trivial and cannot be expressed analytically. However, their isolevel at the plane defined by (26) is plotted in figure (3). Each intersection point of the function $f_0(\xi_0,\xi_1)$ with $f_1(\xi_0,\xi_1)$ in that plane gives a stationary point of the criterion with respect to the poles.

Note that $f_0(\xi_0,\xi_1)$ and $f_1(\xi_0,\xi_1)$ are symmetrical w.r.t. the line $\xi_0 = \xi_1$. Hence, the optimality conditions can also be written as

$$f_1(\xi_1, \xi_0) = g_1([\xi_1 \ \xi_0]) g_2([\xi_1 \ \xi_0 \ \xi_0]) = 0$$

$$f_1(\xi_0, \xi_1) = g_1([\xi_0 \ \xi_1]) g_2([\xi_0 \ \xi_1 \ \xi_1]) = 0$$

which corresponds to, nothing else but, the result announced in *theorem 1 bis*.

Note that if $\xi_0 = \xi_1$, then $f_0(\xi_0, \xi_0)$ simplifies to $f_1(\xi_0, \xi_0)$ in (26), which represents the optimality conditions for the truncated Laguerre network (see figure 3).

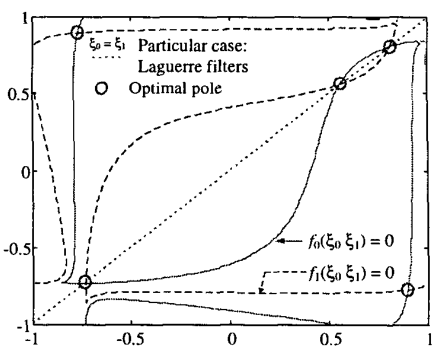

Figure 3 - Intersection of the two curves is the solution of optimality conditions

The solutions of (26), computed numerically, are presented in table 2 with their respective coefficients and quadratic error.

	ξ_0	ξ_1	g_0	g_1	$\dfrac{I_1(\xi_0,\xi_1)}{\|H\|^2/2}$
1st line	-0.774	0.898	1.924	1.034	0.58 %
2nd line	0.898	-0.774	1.337	-1.727	0.58 %
3rd line	-0.733	-0.733	1.932	0	22.25 %
4th line	0.5580	0.558	1.357	0	61.62 %
5th line	0.8096	0.810	1.381	0	60.28 %

Table 2 - The solutions of (29), their respective coefficients, and quadratic error

By comparing all the values of the last column of table 2, one may notice that the choice of the first two lines correspond to the global minimum of the generalized basis' functions. The third line corresponds to the global minimum of Laguerre basis' functions.

Applying the results of either line one or two of table 2 to equation (5a), the optimal second order generalized basis' network was computed.

Then, applying the results of line three to equation (2), the optimal second order Laguerre basis' network was computed. Both of them were excited with a dirac delta and a step signal. Their outputs are compared to the response of the original unit sample response (24) in the following

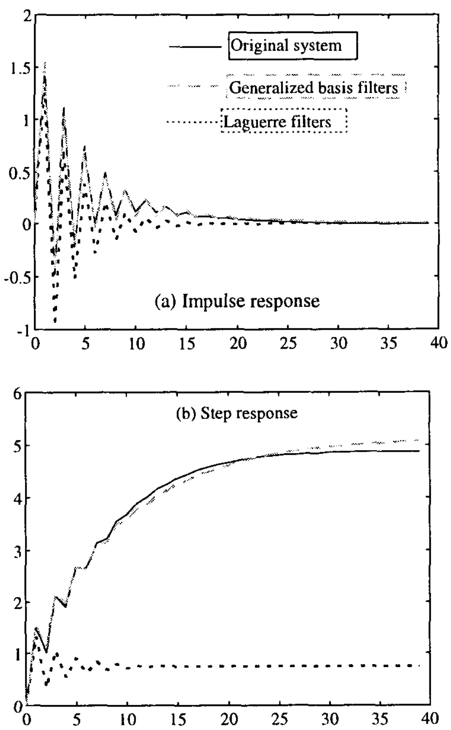

Figure 4 - Responses to particular input signals

Figure (4) shows that a second order optimal Laguerre network fails to approach the original system, especially in the steady state (low frequencies). This remark is confirmed through Bode diagrams plotted in figure (5). Higher order Laguerre networks are, definitely, necessary. However, a second order generalized basis network with optimally chosen poles is much more suited to approximate systems with scattered poles.

4. Conclusions

The optimality conditions for the choice of real poles of the generalized basis were established in this paper. They generalize the well known results concerning the choice of the Laguerre pole. The algorithms discussed herein are suited to model simplification and model reduction. However, based on the results established in *theorems 1* and *1 bis*, nonlinear optimization algorithms can be realized for the choice of optimal poles in the scope of system identification. We are currently working on the generalization of the results obtained herein to the case of complex poles.

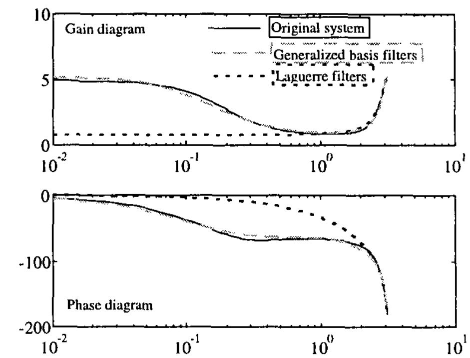

Figure 5 - Bode plots

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