Process input estimation with a multiple model. Application to communication

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Goals
Introduction

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- To design a nonlinear observer with unknown inputs
- To be able to estimate the unknown inputs
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- To design a nonlinear observer with unknown inputs
- To be able to estimate the unknown inputs

Means
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• To design a nonlinear observer with unknown inputs
• To be able to estimate the unknown inputs

Means
• To describe the system behaviour with the help of a multiple model
• To cancel or compensate the unknown inputs
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• To design a nonlinear observer with unknown inputs

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• To describe the system behaviour with the help of a multiple model

• To cancel or compensate the unknown inputs

Difficulties
Goals

- To design a nonlinear observer with unknown inputs
- To be able to estimate the unknown inputs

Means

- To describe the system behaviour with the help of a multiple model
- To cancel or compensate the unknown inputs

Difficulties

- Design of the multiple model (out of purpose)
- Satisfaction of structural constraints
- Resolution of bilinear matrix inequalities
Brief recall of the linear case (1)

Linear dynamical model

\[
\begin{align*}
  x(t + 1) &= A x(t) + B u(t) + R\tilde{u}(t) \\
  y(t) &= C x(t)
\end{align*}
\]
Brief recall of the linear case (1)

Linear dynamical model

\[
\begin{align*}
    x(t + 1) &= Ax(t) + Bu(t) + R\ddot{u}(t) \\
    y(t) &= Cx(t)
\end{align*}
\]

Unknown input observer

\[
\begin{align*}
    z(t + 1) &= Nz(t) + Gu(t) + Ly(t) \\
    \hat{x}(t) &= z(t) - Ey(t)
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State estimation error

\[
\begin{align*}
    e(t) &= x(t) - \hat{x}(t) \\
    e(t) &= (I + EC')x(t) - z(t)
\end{align*}
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State estimation error dynamics

\[ e(t + 1) = Ne(t) + (PA - NP - LC)x(t) + (PB - G)u(t) + \\
(PR - KF)\bar{u}(t) + EF\bar{u}(t + 1) \]
State estimation error dynamics

\[ e(t + 1) = Ne(t) + (PA - NP - LC)x(t) + (PB - G)u(t) + \\
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Autonomous conditions

\[ PA - NP - LC = 0 \]
State estimation error dynamics

\[ e(t + 1) = Ne(t) + (PA - NP - LC)x(t) + (PB - G)u(t) + \]
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Autonomous conditions

\[ PA - NP - LC = 0 \]
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Autonomous conditions

\[ PA - NP - LC = 0 \]
\[ PB - G = 0 \]
\[ PR - KF = 0 \]
\[ EF = 0 \]
Brief recall of the linear case (2)

State estimation error dynamics

\[ e(t + 1) = Ne(t) + (PA - NP - LC)x(t) + (PB - G)u(t) + \\
(PR - KF)\bar{u}(t)) + EF\bar{u}(t + 1) \]

Autonomous conditions

\[ PA - NP - LC = 0 \]
\[ PB - G = 0 \]
\[ PR - KF = 0 \]
\[ EF = 0 \]

Asymptotic convergence

\[ N \text{ stable} \]
Structure description

\[
\begin{aligned}
x(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + R_i \bar{u}(t)) \\
y(t) &= Cx(t) + F\bar{u}(t)
\end{aligned}
\]
Multiple model case

Structure description

\[
\begin{align*}
\begin{cases}
x(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + R_i \bar{u}(t)) \\
y(t) &= C x(t) + F \bar{u}(t)
\end{cases}
\end{align*}
\]

with

\[
\begin{align*}
\begin{cases}
\xi(t) &= \{u(t), x(t), y(t)\} \\
\sum_{i=1}^{M} \mu_i(\xi(t)) &= 1, \quad 0 \leq \mu_i(\xi(t)) \leq 1
\end{cases}
\end{align*}
\]
Multiple model

\[
\begin{align*}
\begin{cases}
  x(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + R_i \bar{u}(t)) \\
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Design of a multiple observer (1)

Multiple model

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\begin{align*}
    x(t+1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) \left( A_i x(t) + B_i u(t) + R_i \bar{u}(t) \right) \\
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\end{align*}
\]

Multiple observer

\[
\begin{align*}
    z(t+1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) \left( N_i z(t) + G_i u(t) + L_i y(t) \right) \\
    \hat{x}(t) &= z(t) - E y(t)
\end{align*}
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Design of a multiple observer (1)

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\end{align*}
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State estimation error

\[
e(t) = x(t) - \hat{x}(t)
\]
Asymptotic convergence of the estimation error

\[ e(t + 1) = \sum_{i=1}^{M} \mu_i(\xi(t)) N_i e(t) \]
Asymptotic convergence of the estimation error

\[ e(t + 1) = \sum_{i=1}^{M} \mu_i(\xi(t)) N_i e(t) \]

if

\[
\begin{align*}
P &= I + EC \\
N_i P &= PA_i - L_i C \\
K_i &= N_i E + L_i \\
PR_i &= KiF \\
G_i &= PB_i \\
EF &= 0
\end{align*}
\]

and

\[ N = \sum_{i=1}^{M} \mu_i(\xi(t)) N_i \quad \text{stable} \]
Asymptotic convergence of the estimation error

\((A_i, C)\) observable pairs, \(F\) with full column rank and \(\forall i, \in \{1, ..., M\}\) :

\[
N_i^T X N_i - X < 0
\]

\[
N_i = PA_i - K_i C
\]

\[
P = I + EC
\]

\[
PR_i = K_i F
\]

\[
EF = 0
\]

\[
L_i = K_i - N_i E
\]

\[
G_i = PB_i
\]

where \(X \in \mathbb{R}^{n \times n}\) is symmetric and positive definite.
Asymptotic convergence of the estimation error

\((A_i, C)\) observable pairs, \(F\) with full column rank and
\(\forall i, \in \{1, \ldots, M\}\):

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where \(X \in IR^{n \times n}\) is symmetric and positive definite.
Solution of the system of equation (1)

System

\[ N_i^T X N_i - X < 0 \]
\[ N_i = PA_i - K_i C \]
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Solution of the system of equation (1)

System

\[ N_i^T X N_i - X < 0 \]
\[ N_i = PA_i - K_i C \]
\[ P = I + EC \]
\[ PR_i = K_i F \]
\[ EF = 0 \]

Solution

\[ E = I - FF^- \]

\[ F^- \] generalized inverse of \( F \)
Solution of the system of equation (2)

System

\[ N_i^T X N_i - X < 0 \]

\[ N_i = P A_i - K_i C \]

\[ P = I + EC \]

\[ PR_i = K_i F \]

\[ EF = 0 \]
Solution of the system of equation (2)

System

\[ N_i^T X N_i - X < 0 \]

\[ N_i = PA_i - K_i C \]

\[ P = I + EC \]

\[ PR_i = K_i F \]

\[ EF = 0 \]

Solution

\[ N_i^T X N_i - X = (PA_i - K_i C)^T X (PA_i - K_i C) < 0 \]

\[ PR_i = K_i F \]

Bilinear matrix inequalities w.r.t. to \( K_i \) and \( X \) s.t. equality constraints.
Solution of the system of equation (3)

Matrix inequalities

\[
\begin{align*}
(P A_i - K_i)^T X (P A_i - K_i) &< 0 \\
PR_i &= K_i F
\end{align*}
\]
Solution of the system of equation (3)

Matrix inequalities

\[
\begin{align*}
(PA_i - K_i)^TX(PA_i - K_i) &< 0 \\
PR_i &= K_iF
\end{align*}
\]

Change of variables

\[W_i = XK_i\]
Solution of the system of equation (3)

Matrix inequalities

\[
\begin{cases}
(PA_i - K_i)^T X (PA_i - K_i) < 0 \\
PR_i = K_i F
\end{cases}
\]

Change of variables

\[W_i = XK_i\]

Schur complement

\[
\begin{cases}
X \\
XP_Ai - WiC \\
PR_i = Wi F
\end{cases}
\begin{pmatrix}
X \\
A_i^T PX - C^TW_i \\
X
\end{pmatrix} > 0
\]
Solution of the system of equation (3)

Matrix inequalities

\[
\begin{align*}
(PA_i - K_i)^T X (PA_i - K_i) &< 0 \\
PR_i &= K_i F
\end{align*}
\]

Change of variables

\[W_i = X K_i\]

Schur complement

\[
\begin{align*}
\begin{pmatrix}
X & A_i^T PX - C^T W_i \\
X PA_i - W_i C & X
\end{pmatrix} > 0 \\
X PR_i &= W_i F
\end{align*}
\]

Linear matrix inequalities w.r.t. \(X\) and \(W_i\)
Solution of the system of equation (4)

System

\[ N_i^T X N_i - X < 0 \]

\[ N_i = PA_i - K_i C \]

\[ P = I + EC \]

\[ PR_i = K_i F \]

\[ EF = 0 \]

\[ L_i = K_i - N_i E \]

\[ G_i = PB_i \]
Solution of the system of equation (4)

System

\[ N_i^T X N_i - X < 0 \]
\[ N_i = P A_i - K_i C \]
\[ P = I + E C \]
\[ P R_i = K_i F \]
\[ E F = 0 \]
\[ L_i = K_i - N_i E \]
\[ G_i = P B_i \]

Sequence of calculus

\[ E \rightarrow P \rightarrow G_i \text{ and } (X, W_i) \rightarrow K_i = X^{-1} W_i \rightarrow N_i \rightarrow L_i \]
Model of the system

\[
\begin{align*}
x(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + R_i \bar{u}(t)) \\
y(t) &= C x(t) + F \bar{u}(t)
\end{align*}
\]
Model of the system

\[
\begin{align*}
x(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i x(t) + B_i u(t) + R_i \bar{u}(t)) \\
y(t) &= C x(t) + F \bar{u}(t)
\end{align*}
\]

Model applied on signal estimations

\[
\begin{align*}
\hat{x}(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) (A_i \hat{x}(t) + B_i \hat{u}(t) + R_i \hat{u}(t)) \\
\hat{y}(t) &= C \hat{x}(t) + F \hat{u}(t)
\end{align*}
\]

where \( \hat{u}(t) \) is an estimation of the unknown input \( \bar{u}(t) \)
Model applied on signal estimations

\[
\begin{align*}
\hat{x}(t + 1) &= \sum_{i=1}^{M} \mu_i(\xi(t)) \left( A_i \hat{x}(t) + B_i u(t) + R_i \hat{u}(t) \right) \\
\hat{y}(t) &= C \hat{x}(t) + F \hat{u}(t)
\end{align*}
\]
Unknown input estimation (2)

Model applied on signal estimations

\[
\begin{cases}
\hat{x}(t + 1) = \sum_{i=1}^{M} \mu_i(\xi(t)) \left( A_i \hat{x}(t) + B_i u(t) + R_i \hat{\bar{u}}(t) \right) \\
\hat{y}(t) = C \hat{x}(t) + F \hat{\bar{u}}(t)
\end{cases}
\]

Unknown input estimation

\[
\hat{\bar{u}}(t) = (W^T W)^{-1} W^T \left( \hat{x}(t + 1) - \sum_{i=1}^{M} \mu_i(\xi(t)) \left( A_i \hat{x}(t) + B_i u(t) \right) \right)
\]

\[
\hat{x}(t + 1) - \sum_{i=1}^{M} \mu_i(\xi(t)) \left( A_i \hat{x}(t) + B_i u(t) \right) \\
y(t) - C \hat{x}(t)
\]

with

\[
W = \left( \sum_{i=1}^{M} \mu_i(\xi(t)) R_i \right) F
\]
Multiple model

\[
\begin{aligned}
x(t + 1) &= \sum_{i=1}^{2} \mu_i(\xi(t))(A_i x(t) + R_i \bar{u}(t)) \\
y(t) &= C x(t) + F \bar{u}(t)
\end{aligned}
\]
Multiple model

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\begin{align*}
    x(t + 1) &= \sum_{i=1}^{2} \mu_i(\xi(t))(A_i x(t) + R_i \bar{u}(t)) \\
    y(t) &= C x(t) + F \bar{u}(t)
\end{align*}
\]

with

\[
\begin{align*}
    \xi(t) &= y(t) \\
    \mu_1(\xi(t)) &= \frac{1}{2}(1 - \tanh(\xi(t))) \\
    \mu_2(\xi(t)) &= 1 - \mu_1(\xi(t))
\end{align*}
\]
Multiple model

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\begin{align*}
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  \mu_2(\xi(t)) &= 1 - \mu_1(\xi(t))
\end{align*}
\]

\[
A_1 = \begin{bmatrix} -1.1 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.8 & -0.1 \\ 1 & 1.1 \end{bmatrix}
\]

\[
C = \begin{bmatrix} 0.5 & 0.5 \end{bmatrix}, \quad F = 5
\]
Encoding – decoding

\[ \bar{u}(t) \xrightarrow{\text{Multiple model}} y(t) \xrightarrow{\text{Transmission}} \Delta u(t) \]

Encoding

Decoding
Transmitted and original signals
Actual and estimated states and decoded message
Activation functions

Output and message
• Multiple model representation is well adapted for nonlinear system modelling
• Multiple model representation is well adapted for nonlinear system modelling

• Capacity to transpose the classical “results” (linear case) to nonlinear systems
Conclusion

- Multiple model representation is well adapted for nonlinear system modelling
- Capacity to transpose the classical “results” (linear case) to nonlinear systems
- General range of the method suggested related to the capacity of representation of the multiple model
Conclusion

- Multiple model representation is well adapted for nonlinear system modelling
- Capacity to transpose the classical “results” (linear case) to nonlinear systems
- General range of the method suggested related to the capacity of representation of the multiple model
- Promising application in the context of secure communications