Design of a sliding mode unknown input observer for uncertain Takagi-Sugeno model

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- To identify the model (out of scope)
- To be able to express convergence conditions in terms of LMI

Linear dynamic model

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) + R\bar{u}(t) \\ y(t) = Cx(t) \end{cases} \quad \text{with} \quad \|\bar{u}(t)\| \leq \rho$$

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Walcott & Zak (1986)

Brief recall of the linear case (1)

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- G such that $(A - GC)$ has stable eigenvalues
- (P, Q) symmetric, positive definite matrices
- F satisfying the following constraints:

$$\begin{cases} (A-GC)^T P + P(A-GC) & = & -Q \\ C^T F^T & = & PR \end{cases}$$

Brief recall of the linear case (2)

Walcott & Zak observer

$$\begin{cases} \dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) - G(C\hat{x}(t) - y(t)) + \nu(t) \\ \hat{y}(t) = C\hat{x}(t) \end{cases}$$

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with

$$\begin{cases} \nu(t) = \begin{cases} -\rho \dfrac{P^{-1}C^T F^T F C}{\|FCe(t)\|} e(t) & \text{if } FCe(t) \neq 0 \\ 0 & \text{otherwise} \end{cases} \\ e(t) = \hat{x}(t) - x(t) \end{cases}$$

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$\hat{y}(t) - y(t)$: output estimation error

Brief recall of the linear case (3)

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Difficulty : $FCe(t) = 0$ whereas $Ce(t) \neq 0$

Structure of the model (simpler form)

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) \left(A_i x(t) + B_i u(t)\right) \\ y(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) C_i x(t) \end{cases}$$

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with
$$\begin{cases} \xi(t) = \{u(t), x(t), y(t)\} \\ \sum_{i=1}^{M} \mu_i(\xi(t)) = 1, \quad 0 \leq \mu_i(\xi(t)) \leq 1 \end{cases}$$

Structure of the model

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) \left((A_i + \Delta A_i(t)) x(t) + B_i u(t) + R_i \bar{u}(t) \right) \\ y(t) = \sum_{i=1}^{M} \mu_i(\xi(t)) C_i x(t) \end{cases}$$

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$\Delta A_i(t)$: unknown time-varying matrices (unmatched uncertainties)

$\bar{u}(t)$: unknown input vector

Uncertain model & unknown inputs

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Hypotheses

$$\|\Delta A_i(t)\| \leq \delta_i \quad \text{(induced two-norm)}$$
$$\|\bar{u}(t)\| \leq \rho \quad \text{(Euclidian norm)}$$

Structure of the model

$$\begin{cases} \dot{x} = \sum_{i=1}^{M} \mu_i \left((A_i + \Delta A_i)x + B_i u + R_i \bar{u} \right) \\ y = \sum_{i=1}^{M} \mu_i C_i x \end{cases}$$

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Structure of the proposed observer

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{M} \mu_i \left(A_i \hat{x} + B_i u + G_i (y - C\hat{x}) + \nu_i + \alpha_i \right) \\ \hat{y} = \sum_{i=1}^{M} \mu_i C_i \hat{x} \end{cases}$$

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Goal: determine G_i, $\nu_i(t)$ and $\alpha_i(t)$

State and ouput estimation errors

$$\begin{cases} e = x - \hat{x} \\ r = y - \hat{y} = \sum_{i=1}^{M} \mu_i C_i e \end{cases}$$

Asymptotic convergence

State and ouput estimation errors

$$\begin{cases} e = x - \hat{x} \\ r = y - \hat{y} = \sum_{i=1}^{M} \mu_i C_i e \end{cases}$$

Dynamic of the state estimation error

$$\dot{e} = \sum_{i=1}^{M} \sum_{j=1}^{M} \mu_i \Big(\bar{A}_{ij} e + \Delta A_i x + R_i \bar{u} - \nu_i - \alpha_i \Big)$$

with:

$$\bar{A}_{ij} = A_i - G_i C_j$$

Theorem 1

The state of the observer converges asymptotically to the state of the Takagi-Sugeno model, if there exists a symmetric positive definite matrix $P \in \mathbf{R}^{n \times n}$, matrices $W_i \in \mathbf{R}^{n \times p}$ and positive scalars β_1, β_2 and β_3 checking the following conditions for all $i, j \in \{1, 2, ..., M\}$:

$$\begin{bmatrix} A_i^T P + P A_i - C_j^T W_i^T - W_i C_j + (\beta_2 \delta_i^2 + \beta_3) I & P \\ P & -\beta_1 I \end{bmatrix} < 0$$

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Linear Matrix Inequalities in P, W_i, β_1, β_2 and β_3

Main result (2)

The gains G_i and the terms $\nu_i(t)$ and $\alpha_i(t)$ of the observer are given by the following equations:

$$\text{If } r \neq 0 \begin{cases} \nu_i = \rho^2 {\beta_3}^{-1} \dfrac{\|P R_i\|^2}{2\, r^T r} P^{-1} \displaystyle\sum_{j=1}^{M} \mu_j C_j^T r \\ \alpha_i = \beta_1 \, (1 + \beta_4) \, \delta_i^2 \dfrac{\hat{x}^T \hat{x}}{2\, r^T r} P^{-1} \displaystyle\sum_{j=1}^{M} \mu_j C_j^T r \end{cases}$$

$$\text{If } r = 0 \begin{cases} \nu_i = 0 \\ \alpha_i = 0 \end{cases}$$

with: $\beta_4 = \frac{\beta_1}{\beta_2 - \beta_1}$

$$G_i = P^{-1} W_i$$

Lemma

For any matrices X and Y with appropriate dimensions, the following property holds for any positive scalar β:

$$X^T Y + Y^T X \leq \beta X^T X + \beta^{-1} Y^T Y$$

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Some elements of the proof

Lyapunov quadratic function: $V = e^T P e$

$$\dot{V} = \sum_{i=1}^{M} \sum_{j=1}^{M} \mu_i \mu_j \Big(e^T (\bar{A}_{ij}^T P + P \bar{A}_{ij}) e + x^T \Delta A_i^T P e + e^T P \Delta A_i x - 2\alpha_i^T P e + 2e^T P R_i \bar{u} - 2e^T P \nu_i \Big)$$

Proof of the theorem (1)

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$$x^T \Delta A_i^T P e + e^T P \Delta A_i x \leq \beta_1^{-1} e^T P^2 e + \beta_1 x^T \Delta A_i^T \Delta A_i x$$

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$$\beta_1 x^T \Delta A_i^T \Delta A_i x \leq \beta_1 \delta_i^2 (\hat{x} + e)^T (\hat{x} + e)$$

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$$\hat{x}^T e + e^T \hat{x} \leq \beta_4^{-1} e^T e + \beta_4 \hat{x}^T \hat{x}$$

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$$\hat{x}^T e + e^T \hat{x} \leq \beta_4^{-1} e^T e + \beta_4 \hat{x}^T \hat{x}$$

$$x^T \Delta A_i^T P e + e^T P \Delta A_i x \leq e^T \left(\beta_1^{-1} P^2 + \beta_2 \delta_i^2 I\right) e + \beta_1 (1 + \beta_4) \delta_i^2 \hat{x}^T \hat{x}$$

Derivative of the Lyapunov function

$$\dot{V} = \sum_{i=1}^{M}\sum_{j=1}^{M}\mu_i\mu_j\Big(e^T(\bar{A}_{ij}^T P + P\bar{A}_{ij})e + x^T\Delta A_i^T Pe + e^T P\Delta A_i x - 2\alpha_i^T Pe + 2e^T PR_i\bar{u} - 2e^T P\nu_i\Big)$$

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Proof of the theorem (3)

Derivative of the Lyapunov function

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$\Rightarrow$ use of lemma

$$2e^T P R_i \bar{u} \leq \beta_3 e^T e + \beta_3^{-1} \| P R_i \bar{u} \|^2$$

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$$2e^T P R_i \bar{u} \leq \beta_3 e^T e + \beta_3^{-1} \|P R_i \bar{u}\|^2$$

$\Rightarrow$ $\bar{u}$ is bounded

$$2e^T P R_i \bar{u} \leq \beta_3 e^T e + \rho^2 \beta_3^{-1} \|P R_i\|^2$$

Proof of the theorem (5)

Derivative of the Lyapunov function

$$\dot{V} \leq \sum_{i=1}^{M}\sum_{j=1}^{M}\mu_i\mu_j\Big(e^T\big(\bar{A}_{ij}^T P + P\bar{A}_{ij} + \beta_1^{-1}P^2 + \beta_2\delta_i^2 I + \beta_3 I\big)e +$$

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Proof of the theorem (5)

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Appropriate choice of the functions $\alpha_i(t)$ and $\nu_i(t)$

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Stability conditions (1)

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Convergence conditions of the state estimation error

$$(A_i - G_iC_j)^T P + P(A_i - G_iC_j) + \beta_1^{-1}P^2 + \beta_2\delta_i^2 I + \beta_3 I < 0, \quad \forall i,j$$

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Nonlinear matrix inequalities in G_i, P, β_1^{-1}, β_2 and β_3

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Stability conditions (2)

Convergence conditions of the state estimation error

$$(A_i - G_i C_j)^T P + P(A_i - G_i C_j) + \beta_1^{-1} P^2 + \beta_2 \delta_i^2 I + \beta_3 I < 0, \quad \forall i, j$$

Change of variables $W_i = PG_i$ and Schur complement

$$\begin{bmatrix} A_i^T P + P A_i - C_j^T W_i^T - W_i C_j + (\beta_2 \delta_i^2 + \beta_3) I & P \\ P & -\beta_1 I \end{bmatrix} < 0$$

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Linear Matrix Inequalities in P, W_i, β_1, β_2 and β_3

Relaxed stability conditions

L. Xiaodong & Z. Qingling, *Automatica*, 2003

L. Xiaodong & Z. Qingling, *Automatica*, 2003

Theorem 2

The state estimation error converges asymptotically towards zero, if the following inequalities hold for all $i, j \in \{1, 2, ..., M\}$:

$$\left\{ \begin{array}{l} \begin{bmatrix} A_i^T P + PA_i - C_i^T W_i^T - W_i C_i + Q_{ii} + (\beta_2 \delta_i^2 + \beta_3) I & P \\ P & -\beta_1 I \end{bmatrix} < 0 \\ \begin{bmatrix} T + (\beta_2 \delta_i^2 + \beta_3) I & P \\ P & -\beta_1 I \end{bmatrix} < 0 \\ \begin{pmatrix} Q_{11} & Q_{12} & \cdots & Q_{1M} \\ Q_{12}^T & Q_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ Q_{1M}^T & \cdots & \cdots & Q_{MM} \end{pmatrix} > 0 \qquad P, Q_{ii} \text{ symmetric, positive definite} \end{array} \right.$$

Nonlinear model made up of two local models

$$\begin{cases} \dot{x} = \sum_{i=1}^{2} \mu_i(u)\Big((A_i + \Delta A_i)x + B_i u + R_i \bar{u}\Big) & x \in \mathbf{R}^3 \\ y = \sum_{i=1}^{2} \mu_i(u) C_i x & y \in \mathbf{R}^2 \end{cases}$$

Simulation example (1)

Nonlinear model made up of two local models

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Model uncertainties

$$\Delta A_{i,(j,k)}(t) = 0.2 A_{i,(j,k)} \eta(t) \quad j,k \in \{1,3\} \text{ and } i \in \{1,2\}$$

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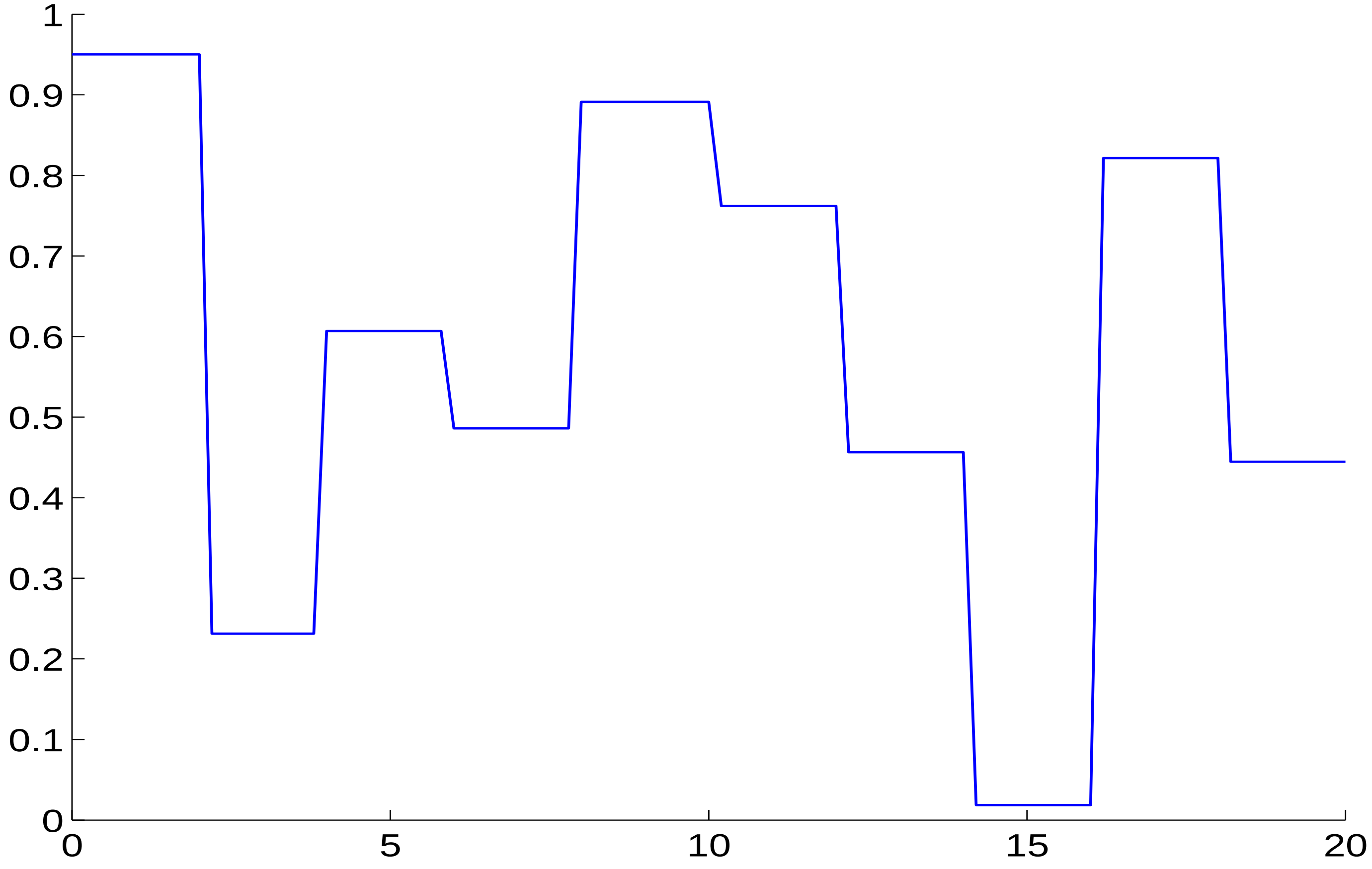
$\eta(t)$: piece-wise constant function which magnitude is uniformly distributed on the interval $[0 \;\; 1]$

Uncertainties



The piece-wise constant function $\eta(t)$

Sliding mode observer

$$\begin{cases} \dot{\hat{x}} = \sum_{i=1}^{2} \mu_i \Big(A_i \hat{x} + B_i u + G_i \left(y - \hat{y} \right) + \nu_i + \alpha_i \Big) \\ \hat{y} = \sum_{i=1}^{2} \mu_i C_i \hat{x} \end{cases}$$

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Practical implementation

If $\|r\| \geq \varepsilon = 10^{-3}$ $\begin{cases} \nu_i = \rho^2 \beta_3^{-1} \dfrac{\|P R_i\|^2}{2 r^T r} P^{-1} \sum_{j=1}^{2} \mu_j C_j^T r \\ \alpha_i = \beta_1 \left(1 + \beta_4\right) \delta_i^2 \dfrac{\hat{x}^T \hat{x}}{2 \; r^T r} P^{-1} \sum_{j=1}^{2} \mu_j C_j^T r \end{cases}$

If $\|r\| < \varepsilon = 10^{-3}$ $\begin{cases} \nu_i = 0 \\ \alpha_i = 0 \end{cases}$

Model matrices

$$A_1 = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 1 & -6 \end{bmatrix} \quad A_2 = \begin{bmatrix} -3 & 2 & 2 \\ 5 & -8 & 0 \\ 0.5 & 0.5 & -4 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} 1 \\ 0.5 \\ 0.5 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0.5 \\ 1 \\ 0.25 \end{bmatrix} \quad R_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad R_2 = \begin{bmatrix} 1 \\ 0.5 \\ 2 \end{bmatrix}$$

$$C_1 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \quad C_2 = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

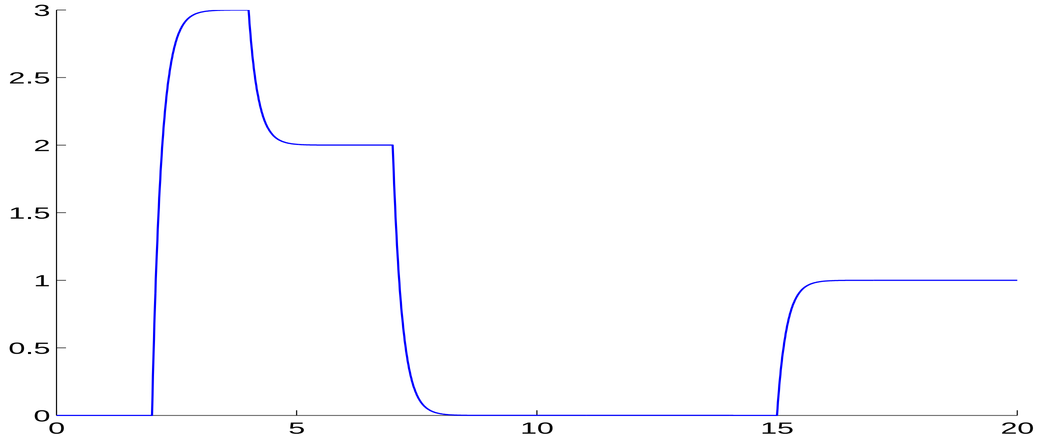
Results from theorem 2

$$G_1 = \begin{bmatrix} 0.55 & 2.18 \\ 1.58 & -0.67 \\ 0.18 & -0.93 \end{bmatrix} \quad G_2 = \begin{bmatrix} 2.62 & 1.04 \\ -1.34 & 1.29 \\ 2.22 & -2.19 \end{bmatrix}$$

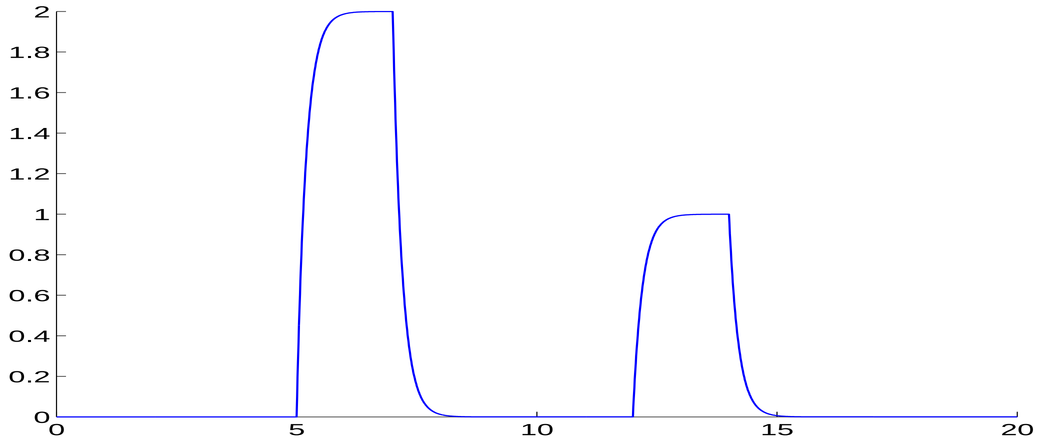
$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad Q_{11} = \begin{bmatrix} 1.66 & -0.13 & -0.44 \\ -0.13 & 1.44 & -0.12 \\ -0.44 & -0.12 & 2.16 \end{bmatrix}$$

$$Q_{22} = \begin{bmatrix} 1.87 & -0.27 & 0.18 \\ -0.27 & 1.66 & 0.12 \\ 0.18 & 0.12 & 1.39 \end{bmatrix} \quad Q_{12} = \begin{bmatrix} 1.16 & 0.14 & 0.22 \\ 0.34 & 0.17 & 0.66 \\ -0.33 & 0.87 & -0.69 \end{bmatrix}$$

Simulation example (6)

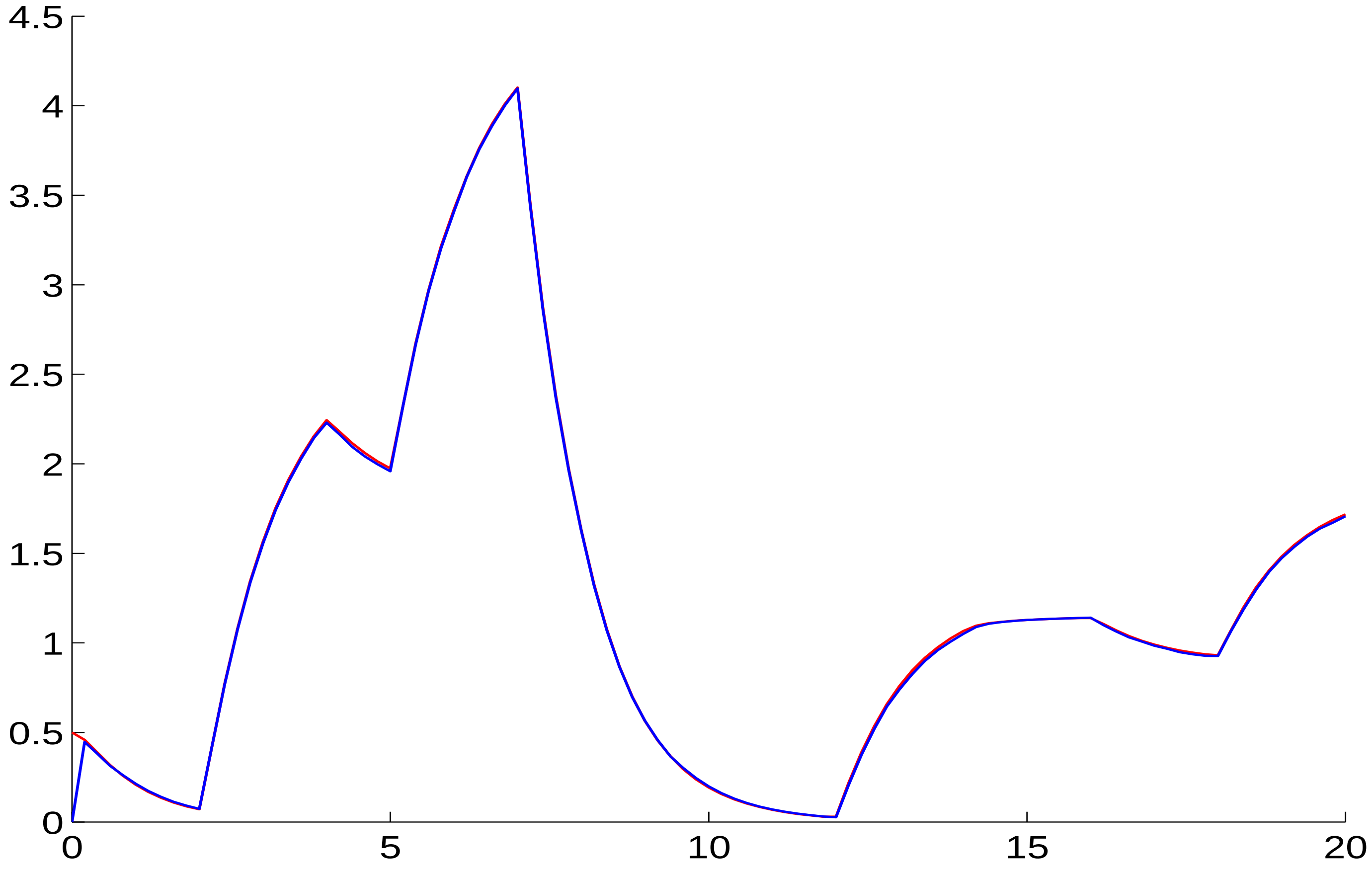


Known input $u(t)$



Unknown input $\bar{u}(t)$

State estimation



Real and estimated $x_1(t)$ state

Conclusion

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- Whether the activation functions μ_i of the T-S model depend on $u(t)$ or $y(t)$, the method is able to diagnose sensor or actuator faults