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State Equations and Decomposition into Laguerre Functions

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ABSTRACT: *For a linear system, one establishes a connection between the coefficients of the decomposition into Laguerre functions of the inputs and the outputs of the system. When the system is modelled by a state equation, it is showed that the two series are related through a state equation whose matrices are depending of those of the state equations of the system. The new state representation may be used for analysing linear dynamic systems.* Copyright © 1996 Published by Elsevier Science Ltd

I. Introduction

An important problem in control and communication theory is the expansion of a signal in a truncated orthonormal expansion. The Laguerre functions, a complete orthonormal set on $L^2[0, \infty[$, have been widely used often both for their convenient network realisation (**1**) and for their similarity to transient signal, for the approximation of time series (**2**) and identification of linear system (**3, 4**) and recently for the approximation of delay systems. In the last case, one can use *a priori* information about the time constants of the system to improve the modelling performance related to Laguerre expansion (**5**).

Nevertheless, only a few works based on systems state space representation with orthonormal series approach are published (**6, 7**). The first work on this subject is due to Nurges and Yaaksoo (**6**) who suggest a relation involving Laguerre polynomials. Our presentation may be considered as an extension of this work.

II. Definitions

2.1 *System equations*

Consider a linear, time-invariant, asymptotically-stable, dynamic multivariable system with n state variables, m input and q output variables described by the state equations

$$\frac{d\mathbf{X}(t)}{dt} = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t) \tag{1a}$$

$$\mathbf{Y}(t) = \mathbf{C}\mathbf{X}(t) \tag{1b}$$

where **X** is the state vector, **U** the input and **Y** the output vectors and where A, B and C are matrices of appropriate dimensions.

2.2 *Laguerre functions*

The continuous Laguerre functions with order n may be defined by

$$\mathrm{L}_n(t) = \frac{\mathrm{e}^{(\gamma-\alpha)t}}{n!}\frac{\mathrm{d}^n}{\mathrm{d}t^n}[t^n\,\mathrm{e}^{-\gamma t}] \tag{2}$$

where α and γ are two scaling parameters. Sometimes, a simplification is given by considering Laguerre functions depending on only one parameter by taking $\gamma = 2\alpha$ (**4**). It is clear that α is a key parameter which determines the decreasing rate of the Laguerre functions during time. The choice of the parameter α is, however, difficult and only a few number of papers have suggest solutions to this problem (**5**, **8**).

Their derivative in respect to the time may be computed from the relation

$$\frac{\mathrm{dL}_n(t)}{\mathrm{d}t} = -\alpha\mathrm{L}_n(t) - \gamma\sum_{i=0}^{n-1}\mathrm{L}_i(t). \tag{3}$$

This identity may be obtained by direct computation in the time domain. A convenient representation of the Laguerre functions in the s domain is given by

$$\mathrm{L}_n(s) = \frac{(s-\gamma+\alpha)^n}{(s+\alpha)^{n+1}} \tag{4}$$

which also allows the recurrence (3) to be established. The s-domain Laguerre functions form an orthonormal basis in H_2, the Hardy space of all analytical functions on the open right half complex plane that are square integrable.

Note that laguerre functions also exist with a discrete representation (**9**) and have similar properties.

2.3 *Decomposition of a time function into Laguerre functions*

According to the previous sections, any linear time stable signal $f(t)$ can be expressed by using Laguerre expansion as

$$f(t) = \sum_{n=0}^{\infty} a_n\mathrm{L}_n(t) \tag{5}$$

where a_n are constants.

From Eqns (4) and (5), in the s-domain, we have the expansion

$$F(s) = \sum_{n=0}^{\infty} a_n\mathrm{L}_n(s) = \sum_{n=0}^{\infty} a_n\frac{(s-\gamma+\alpha)^n}{(s+\alpha)^{n+1}}. \tag{6}$$

Usually, for practical reasons, the summation is limited to a finite number N and it can easily be shown (**9**) that, if $F(s) \in \mathrm{H}_2$ and $F(s)$ is uniformly continuous over the imaginary axis, for any $\varepsilon > 0$ there exist $N(\varepsilon)$ such that

$$|F(s)-F_{N(\varepsilon)}(s)| < \varepsilon \quad \forall s \in C_+ = \{\sigma + j\omega | \sigma \geqslant 0, \quad \omega \in R\} \tag{7}$$

where $F_{N(\varepsilon)}$ is the summation (6) limited to $N(\varepsilon)$ terms.

For determining the coefficients of the expansion of $f(t)$ one generally uses the minimization of the square norm

$$\Phi = \frac{1}{2}\int_0^\infty \left(f(t) - \sum_{n=0}^{N} a_n \mathrm{L}_n(t)\right)^2 \gamma\, \mathrm{e}^{-\beta t}\, \mathrm{d}t. \tag{8}$$

The coefficient $\beta = \gamma - 2\alpha$ defines a weighting function allowing to emphasize a region of interest in the interval $[0, +\infty[$. Setting the first derivatives of Φ with respect to a_n $(n = 0, \ldots, N)$ equal to 0 and taking into account the orthogonality property of the Laguerre functions in $\mathrm{L}^2[0, \infty[$, we obtain

$$a_n = \int_0^\infty f(t)\mathrm{L}_n(t)\gamma\, \mathrm{e}^{-\beta t}\, \mathrm{d}t. \tag{9}$$

If the s transform of $f(t)$, denoted $F(s) \in \mathrm{H}_2$ is explicitly known, the coefficients a_n can be computed by integration in the complex plane by virtue of the Parseval formula; for other time signals, the integration in Eq. (9) has to be performed numerically.

III. Representation of a System with Laguerre Functions

Following the preceding definition, we expand the input, the state and the output of a process as

$$\mathbf{U}(t) = \sum_{n=0}^{\infty} \mathbf{U}_n \mathrm{L}_n(t) \tag{10a}$$

$$\mathbf{X}(t) = \sum_{n=0}^{\infty} \mathbf{X}_n \mathrm{L}_n(t) \tag{10b}$$

$$\mathbf{Y}(t) = \sum_{n=0}^{\infty} \mathbf{Y}_n \mathrm{L}_n(t). \tag{10c}$$

Obviously, the vectors $\mathbf{U}_n$ and $\mathbf{Y}_n$ have the same dimension as those of $\mathbf{U}$ and $\mathbf{Y}$.

3.1 *Relation between the input and output decompositions*

Substituting (10a) and (10b) in (1a) gives

$$\sum_{n=0}^{\infty} \mathbf{X}_n \frac{\mathrm{d}\mathrm{L}_n(t)}{\mathrm{d}t} = \mathrm{A} \sum_{n=0}^{\infty} \mathbf{X}_n \mathrm{L}_n(t) + \mathbf{B} \sum_{n=0}^{\infty} \mathbf{U}_n \mathrm{L}_n(t)$$

and taking into account the relation (3) we obtain

$$-\left[\alpha \mathbf{X}_0 \mathrm{L}_0(t) + \sum_{n=1}^{\infty} \mathbf{X}_n \left(\alpha \mathrm{L}_n(t) + \gamma \sum_{i=0}^{n-1} \mathrm{L}_i(t)\right)\right] = \sum_{n=0}^{\infty} (\mathbf{A}\mathbf{X}_n + \mathbf{B}\mathbf{U}_n)\mathrm{L}_n(t) \tag{11}$$

or by expanding the summations

$$-(\mathbf{X}_0\alpha+\mathbf{X}_1\gamma+\mathbf{X}_2\gamma\ldots)\mathrm{L}_0(t)-(\mathbf{X}_1\alpha+\mathbf{X}_2\gamma+\mathbf{X}_3\gamma\ldots)\mathrm{L}_1(t)-\cdots$$
$$=(\mathbf{AX}_0+\mathbf{BU}_0)\mathrm{L}_0(t)+(\mathbf{AX}_1+\mathbf{BU}_1)\mathrm{L}_1(t)+(\mathbf{AX}_2+\mathbf{BU}_2)\mathrm{L}_2(t)+\cdots.$$

Since functions $\mathrm{L}_n(t)$ are independent, by identifying the $\mathrm{L}_n(t)$th coefficient in the both hands of the previous equation one obtains

$$-(\mathbf{X}_n\alpha+\mathbf{X}_{n+1}\gamma+\mathbf{X}_{n+2}\gamma+\cdots+)=\mathbf{AX}_n+\mathbf{BU}_n. \tag{12}$$

Subtracting two consecutive equations from rank n and $n+1$, one obtains a relation between the decomposition of the state and the input:

$$-\mathbf{X}_n\alpha-\mathbf{X}_{n+1}(\gamma-\alpha)=\mathbf{AX}_n+\mathbf{BU}_n-\mathbf{AX}_{n+1}-\mathbf{BU}_{n+1}. \tag{13}$$

Or when reordering

$$[\mathrm{A}-(\gamma-\alpha)I]\mathbf{X}_{n+1}=(\mathrm{A}+\alpha I)\mathbf{X}_n+\mathbf{BU}_n-\mathbf{BU}_{n+1} \tag{14}$$

which looks like a state equation. A standard form of the state equation may be obtained with the change of variable

$$\mathbf{Z}_n=[\mathrm{A}-(\gamma-\alpha)I]\mathbf{X}_n+\mathbf{BU}_n. \tag{15}$$

Substituting (15) into (14) yields

$$\mathbf{Z}_{n+1}=(\mathrm{A}+\alpha I)[\mathrm{A}-(\gamma-\alpha)I]^{-1}\mathbf{Z}_n+\gamma[\mathrm{A}-(\gamma-\alpha)I]^{-1}\mathbf{BU}_n. \tag{16a}$$

According to the same transformation, using (1b), the measurement equation is expressed

$$\mathbf{Y}_n=\mathrm{C}[\mathrm{A}-(\gamma-\alpha)I]^{-1}Z_n-\mathrm{C}[\mathrm{A}-(\gamma-\alpha)I]^{-1}\mathbf{BU}_n. \tag{16b}$$

Summarizing, the coefficients of the expansion into Laguerre functions of the inputs and the outputs of a process are linked by the state equations

$$\mathbf{Z}_{n+1}=\bar{\mathrm{A}}\mathbf{Z}_n+\bar{\mathrm{B}}\mathbf{U}_n;\quad \mathbf{Z}_0=0 \tag{17a}$$

$$\mathbf{Y}_n=\bar{\mathrm{C}}\mathbf{Z}_n+\bar{\mathrm{D}}\mathbf{U}_n. \tag{17b}$$

The matrices $\bar{\mathrm{A}}$, $\bar{\mathrm{B}}$, $\bar{\mathrm{C}}$ and $\bar{\mathrm{D}}$ may be deduced from those of the system with the transformations

$$\bar{\mathrm{A}}=(\mathrm{A}+\alpha I)[\mathrm{A}-(\gamma-\alpha)I]^{-1} \tag{18a}$$

$$\bar{\mathrm{B}}=\gamma[\mathrm{A}-(\gamma-\alpha)I]^{-1}\mathrm{B} \tag{18b}$$

$$\bar{\mathrm{C}}=\mathrm{C}[\mathrm{A}-(\gamma-\alpha)I]^{-1} \tag{18c}$$

$$\bar{\mathrm{D}}=-\mathrm{C}[\mathrm{A}-(\gamma-\alpha)I]^{-1}\mathrm{B}. \tag{18d}$$

Remark 1: the preceding formulation needs the regularity of the matrix $[\mathrm{A}-(\gamma-\alpha)I]$. This condition is fulfilled if $(\gamma-\alpha)$ is not an eigenvalue of A.

Remark 2: instead of using the recursive formula (17) for computing the Laguerre coefficients of the output, direct integration gives the result

$$\mathbf{Y}_n = \bar{\mathbf{C}} \sum_{i=0}^{n-1} \bar{\mathbf{A}}^{n-1-i} \bar{\mathbf{B}} \mathbf{U}_i + \bar{\mathbf{D}} \mathbf{U}_n$$

which may be useful for the control design of the system.

3.2 *Laguerre state equations dynamic behaviour*

From definition (18a), one obtains the relation between the eigenvalues of A and $\bar{\mathrm{A}}$

$$\lambda(\bar{\mathrm{A}}) = \frac{\lambda(\mathrm{A}) + \alpha}{\lambda(\mathrm{A}) + \alpha - \gamma}. \tag{19}$$

The stability of the Laguerre expansion is guaranteed if

$$|\lambda(\bar{\mathrm{A}})| < 1.$$

Taking into account (19), it is needed that

$$\lambda(\mathrm{A}) < \frac{\gamma}{2} - \alpha. \tag{20}$$

In other words, in the complex plane the real part of the eigenvalues of the matrix A must be at the left side of $\gamma/2 - \alpha$; this guarantees the stability of the series defined by Eq. (17).

3.3 *Application to control*

We only consider here the case of SISO systems for which, in order to stabilize the model, we introduce the following quadratic cost function

$$\Phi = \frac{1}{2} \int_0^{\tau} [(y(t) - r(t))^2 + u(t)^2 \alpha^2] \gamma \, \mathrm{e}^{-\beta t} \, \mathrm{d}t$$

where α^2 is a weight between the regulation error and the energy consumption. The solution $u(t)$ minimizing Φ for the trajectories of the model (1) is now derived from the Laguerre series expansions. From Eq. (10), let us express the process variables as

$$u(t) = U^T \mathbf{L}(t)$$

$$r(t) = R^T \mathbf{L}(t)$$

$$y(t) = Y^T \mathbf{L}(t)$$

where U, R and Y contain the coefficients of the Laguerre series expansion of $u(t)$, $r(t)$ and $y(t)$ and where $\mathbf{L}$ is the vector of the Laguerre functions. As it has been proved (remark 2), Y may be written

$$Y = GU$$

and consequently the cost function takes the form

$$\Phi = \frac{1}{2}\int_0^{\tau} [(GU-R)^T \mathbf{L}(t)\mathbf{L}^T(t)(GU-R) + U^T\mathbf{L}(t)\mathbf{L}^T(t)U\alpha^2]\gamma\, \mathrm{e}^{-\beta t}\, \mathrm{d}t.$$

The minimum of Φ, in respect to $u(t)$ or equivalently in respect to U is obtained when

$$U = (G^T PG + \alpha^2 P)^{-1} G^T PR$$

with the definition

$$P = \int_0^{\tau} \mathbf{L}(t)\mathbf{L}^T(t)\gamma\, \mathrm{e}^{-\beta t}\, \mathrm{d}t.$$

Summarizing, we have derived the optimal control of a LTI system from the series expansion of the reference trajectory. It is clear that recurrence expression may be obtained in order to provide a more convenient form of this control.

IV. Conclusion

We have given a new formulation of the representation of a system by establishing an original link between the conventional state equations and the decomposition of the input and output into Laguerre functions. Indeed, we have pointed out a simple transformation allowing the computation of the new representation. A simple application has been suggested in the paper but the reader should imagine how to design more complex regulator ensuring stability or optimal control by using the proposed representation.

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