State Equations and Decomposition into Laguerre Functions

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ABSTRACT: For a linear system, one establishes a connection between the coefficients of the decomposition into Laguerre functions of the inputs and the outputs of the system. When the system is modelled by a state equation, it is showed that the two series are related through a state equation whose matrices are depending of those of the state equations of the system. The new state representation may be used for analysing linear dynamic systems. Copyright © 1996 Published by Elsevier Science Ltd

I. Introduction

An important problem in control and communication theory is the expansion of a signal in a truncated orthonormal expansion. The Laguerre functions, a complete orthonormal set on \( L^2[0, \infty[ \), have been widely used often both for their convenient network realisation (1) and for their similarity to transient signal, for the approximation of time series (2) and identification of linear system (3, 4) and recently for the approximation of delay systems. In the last case, one can use a priori information about the time constants of the system to improve the modelling performance related to Laguerre expansion (5).

Nevertheless, only a few works based on systems state space representation with orthonormal series approach are published (6, 7). The first work on this subject is due to Nurges and Yaaksoo (6) who suggest a relation involving Laguerre polynomials. Our presentation may be considered as an extension of this work.

II. Definitions

2.1 System equations

Consider a linear, time-invariant, asymptotically-stable, dynamic multivariable system with \( n \) state variables, \( m \) input and \( q \) output variables described by the state equations

\[
\frac{dX(t)}{dt} = AX(t) + BU(t) \quad (1a)
\]

\[
Y(t) = CX(t) \quad (1b)
\]
where \( \mathbf{X} \) is the state vector, \( \mathbf{U} \) the input and \( \mathbf{Y} \) the output vectors and where \( \mathbf{A}, \mathbf{B} \) and \( \mathbf{C} \) are matrices of appropriate dimensions.

2.2 Laguerre functions

The continuous Laguerre functions with order \( n \) may be defined by

\[
L_n(t) = \frac{e^{(\gamma-a)t}}{n!} \frac{d^n}{dt^n} \left[ t^n e^{-\gamma t} \right]
\]

(2)

where \( \alpha \) and \( \gamma \) are two scaling parameters. Sometimes, a simplification is given by considering Laguerre functions depending on only one parameter by taking \( \gamma = 2\alpha \) (4). It is clear that \( \alpha \) is a key parameter which determines the decreasing rate of the Laguerre functions during time. The choice of the parameter \( \alpha \) is, however, difficult and only a few number of papers have suggest solutions to this problem (5, 8).

Their derivative in respect to the time may be computed from the relation

\[
\frac{dL_n(t)}{dt} = -\alpha L_n(t) - \gamma \sum_{i=0}^{n-1} L_i(t).
\]

(3)

This identity may be obtained by direct computation in the time domain. A convenient representation of the Laguerre functions in the \( s \) domain is given by

\[
L_n(s) = \frac{(s - \gamma + \alpha)^n}{(s + \alpha)^{n+1}}
\]

(4)

which also allows the recurrence (3) to be established. The \( s \)-domain Laguerre functions form an orthonormal basis in \( H_2 \), the Hardy space of all analytical functions on the open right half complex plane that are square integrable.

Note that laguerre functions also exist with a discrete representation (9) and have similar properties.

2.3 Decomposition of a time function into Laguerre functions

According to the previous sections, any linear time stable signal \( f(t) \) can be expressed by using Laguerre expansion as

\[
f(t) = \sum_{n=0}^{\infty} a_n L_n(t)
\]

(5)

where \( a_n \) are constants.

From Eqns (4) and (5), in the \( s \)-domain, we have the expansion

\[
F(s) = \sum_{n=0}^{\infty} a_n L_n(s) = \sum_{n=0}^{\infty} a_n \frac{(s - \gamma + \alpha)^n}{(s + \alpha)^{n+1}}.
\]

(6)

Usually, for practical reasons, the summation is limited to a finite number \( N \) and it can easily be shown (9) that, if \( F(s) \in H_2 \) and \( F(s) \) is uniformly continuous over the imaginary axis, for any \( \varepsilon > 0 \) there exist \( N(\varepsilon) \) such that
|F(s) - F_{N(e)}(s)| < \varepsilon \quad \forall s \in C_+ = \{\sigma + j\omega|\sigma \geq 0, \quad \omega \in R\} \quad (7)

where $F_{N(e)}$ is the summation (6) limited to $N(e)$ terms.

For determining the coefficients of the expansion of $f(t)$ one generally uses the minimization of the square norm

$$\Phi = \frac{1}{2} \int_0^\infty \left(f(t) - \sum_{n=0}^N a_n L_n(t)\right)^2 e^{-\beta t} dt. \quad (8)$$

The coefficient $\beta = \gamma - 2 \alpha$ defines a weighting function allowing to emphasize a region of interest in the interval $[0, +\infty[$. Setting the first derivatives of $\Phi$ with respect to $a_n$ ($n = 0, \ldots, N$) equal to 0 and taking into account the orthogonality property of the Laguerre functions in $L^2[0, \infty[$, we obtain

$$a_n = \int_0^\infty f(t) L_n(t) e^{-\beta t} dt. \quad (9)$$

If the $s$ transform of $f(t)$, denoted $F(s) \in H_2$ is explicitly known, the coefficients $a_n$ can be computed by integration in the complex plane by virtue of the Parseval formula; for other time signals, the integration in Eq. (9) has to be performed numerically.

### III. Representation of a System with Laguerre Functions

Following the preceding definition, we expand the input, the state and the output of a process as

$$U(t) = \sum_{n=0}^\infty U_n L_n(t) \quad (10a)$$

$$X(t) = \sum_{n=0}^\infty X_n L_n(t) \quad (10b)$$

$$Y(t) = \sum_{n=0}^\infty Y_n L_n(t). \quad (10c)$$

Obviously, the vectors $U_n$ and $Y_n$ have the same dimension as those of $U$ and $Y$.

#### 3.1 Relation between the input and output decompositions

Substituting (10a) and (10b) in (1a) gives

$$\sum_{n=0}^\infty X_n \frac{dL_n(t)}{dt} = A \sum_{n=0}^\infty X_n L_n(t) + B \sum_{n=0}^\infty U_n L_n(t)$$

and taking into account the relation (3) we obtain

$$-\alpha X_0 L_0(t) + \sum_{n=1}^\infty X_n (\alpha L_n(t) + \gamma \sum_{i=0}^{n-1} L_i(t)) = \sum_{n=0}^\infty (AX_n + BU_n)L_n(t) \quad (11)$$

or by expanding the summations
\[-(X_0 \gamma + X_1 \gamma + X_2 \gamma + \cdots) L_0(t) - (X_1 \gamma + X_2 \gamma + X_3 \gamma + \cdots) L_1(t) - \cdots = (AX_0 + BU_0) L_0(t) + (AX_1 + BU_1) L_1(t) + (AX_2 + BU_2) L_2(t) + \cdots.\]

Since functions \(L_n(t)\) are independent, by identifying the \(L_n(t)\)th coefficient in the both hands of the previous equation one obtains
\[-(X_n \gamma + X_{n+1} \gamma + X_{n+2} \gamma + \cdots + ) = AX_n + BU_n. \quad (12)\]

Subtracting two consecutive equations from rank \(n\) and \(n+1\), one obtains a relation between the decomposition of the state and the input:
\[-X_n \gamma - X_{n+1} (\gamma - \alpha) = AX_n + BU_n - AX_{n+1} - BU_{n+1}. \quad (13)\]

Or when reordering
\[[A - (\gamma - \alpha) I] X_{n+1} = (A + \alpha I) X_n + BU_n - BU_{n+1} \quad (14)\]

which looks like a state equation. A standard form of the state equation may be obtained with the change of variable
\[Z_n = [A - (\gamma - \alpha) I] X_n + BU_n. \quad (15)\]

Substituting (15) into (14) yields
\[Z_{n+1} = (A + \alpha I) [A - (\gamma - \alpha) I]^{-1} Z_n + \gamma [A - (\gamma - \alpha) I]^{-1} BU_n. \quad (16a)\]

According to the same transformation, using (1b), the measurement equation is expressed
\[Y_n = C [A - (\gamma - \alpha) I]^{-1} Z_n - C [A - (\gamma - \alpha) I]^{-1} BU_n. \quad (16b)\]

Summarizing, the coefficients of the expansion into Laguerre functions of the inputs and the outputs of a process are linked by the state equations
\[Z_{n+1} = \tilde{A} Z_n + \tilde{B} U_n; \quad Z_0 = 0 \quad (17a)\]
\[Y_n = \tilde{C} Z_n + \tilde{D} U_n. \quad (17b)\]

The matrices \(\tilde{A}, \tilde{B}, \tilde{C}\) and \(\tilde{D}\) may be deduced from those of the system with the transformations
\[\tilde{A} = (A + \alpha I) [A - (\gamma - \alpha) I]^{-1} \quad (18a)\]
\[\tilde{B} = \gamma [A - (\gamma - \alpha) I]^{-1} B \quad (18b)\]
\[\tilde{C} = C [A - (\gamma - \alpha) I]^{-1} \quad (18c)\]
\[\tilde{D} = -C [A - (\gamma - \alpha) I]^{-1} B. \quad (18d)\]

**Remark 1:** the preceding formulation needs the regularity of the matrix \([A - (\gamma - \alpha) I]\). This condition is fulfilled if \((\gamma - \alpha)\) is not an eigenvalue of \(A\).

**Remark 2:** instead of using the recursive formula (17) for computing the Laguerre coefficients of the output, direct integration gives the result
\[ Y_n = \sum_{i=0}^{n-1} A^{n-i} Bu_i + Du_n \]

which may be useful for the control design of the system.

3.2 **Laguerre state equations dynamic behaviour**

From definition (18a), one obtains the relation between the eigenvalues of \( A \) and \( \bar{A} \)

\[
\lambda(\bar{A}) = \frac{\lambda(A) + \alpha}{\lambda(A) + \alpha - \gamma}.
\]  \( \text{(19)} \)

The stability of the Laguerre expansion is guaranteed if

\[ |\lambda(\bar{A})| < 1. \]

Taking into account (19), it is needed that

\[ \lambda(A) < \frac{\gamma}{2} - \alpha. \]  \( \text{(20)} \)

In other words, in the complex plane the real part of the eigenvalues of the matrix \( A \) must be at the left side of \( \gamma/2 - \alpha \); this guarantees the stability of the series defined by Eq. (17).

3.3 **Application to control**

We only consider here the case of SISO systems for which, in order to stabilize the model, we introduce the following quadratic cost function

\[
\Phi = \frac{1}{2} \int_0^t [(y(t) - r(t))^2 + u(t)^2 \alpha^2] e^{-\beta t} dt
\]

where \( \alpha^2 \) is a weight between the regulation error and the energy consumption. The solution \( u(t) \) minimizing \( \Phi \) for the trajectories of the model (1) is now derived from the Laguerre series expansions. From Eq. (10), let us express the process variables as

\[
u(t) = U^T L(t) \\
r(t) = R^T L(t) \\
y(t) = Y^T L(t)
\]

where \( U, R \) and \( Y \) contain the coefficients of the Laguerre series expansion of \( u(t) \), \( r(t) \) and \( y(t) \) and where \( L \) is the vector of the Laguerre functions. As it has been proved (remark 2), \( Y \) may be written

\[ Y = GU \]

and consequently the cost function takes the form
\[ \Phi = \frac{1}{2} \int_0^\infty \left[ (GU - R)^T L(t) L^T(t)(GU - R) + U^T L(t) L^T(t) U \right] e^{-\beta t} \] 

The minimum of \( \Phi \), in respect to \( u(t) \) or equivalently in respect to \( U \) is obtained when

\[ U = (G^T P G + \alpha^2 P)^{-1} G^T P R \]

with the definition

\[ P = \int_0^\infty L(t) L^T(t) \gamma e^{-\beta t} \] 

Summarizing, we have derived the optimal control of a LTI system from the series expansion of the reference trajectory. It is clear that recurrence expression may be obtained in order to provide a more convenient form of this control.

**IV. Conclusion**

We have given a new formulation of the representation of a system by establishing an original link between the conventional state equations and the decomposition of the input and output into Laguerre functions. Indeed, we have pointed out a simple transformation allowing the computation of the new representation. A simple application has been suggested in the paper but the reader should imagine how to design more complex regulator ensuring stability or optimal control by using the proposed representation.

**References**