MULTIQUADRATIC STABILITY AND STABILISATION OF CONTINUOUS-TIME MULTIPLE MODEL

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Abstract: This paper deals with the stability analysis of a system represented by a multiple model. Based on a multiquadratic Lyapunov function candidate ($\mathcal{MQLF}$), new asymptotic stability conditions for continuous case are presented in linear matrix inequalities ($\mathcal{LMI}$) form. These stability conditions, extended to the controller synthesis, are formulated in bilinear matrix inequalities ($\mathcal{BMI}$). Examples will be given in the final version.

Keywords: Multiple model, nonlinear system, stability analysis, stabilisation, Lyapunov method, linear matrix inequalities.

1. INTRODUCTION

Analysis and synthesis studies of multiple model (Murray-Smith *et al.*, 1997) based on quadratic Lyapunov functions lead to result which are often conservative (see for example (Chadli *et al.*, 2003b), (Blanco *et al.*, 2001a) and (Tanaka *et al.*, 1998)). To overcome these conservatism non quadratic Lyapunov functions may be used. Among these functions, we can quote the piecewise quadratic function. The stability analysis using this type of function was studied these last years by using the incertain system techniques (Zhang *et al.*, 2001), (Cao *et al.*, 1999), (Cao *et al.*, 1996). This approach allows to reduce the conservatism of the quadratic method by taking into account the partition of the state space induced by activation functions with limited local support of the variables (for example trapezoidal or triangular activation functions) (Johansen *et al.*, 1999).

In (Chadli *et al.*, 2002a) another class of non quadratic Lyapunov functions of the form $V(x(t)) = \max(V_i(x(t)))$ with $V_i(x(t)) = x(t)^T P_i x(t)$, $P_i > 0, i \in \{1, 2, .., n\}$ where n is the number of local model, was also considered. The obtained results in the continuous and discrete domains are rather satisfactory in comparison with those obtained with quadratic method. The proposed stability condition in the continuous domain leads also to overcome the pessimism of those obtained by picewise quadratic Lyapunov functions (Johansen *et al.*, 1999) when the activation functions have an infinite support (for example Gaussian activation functions).

Some works also propose another type of non quadratic Lyapunov function called multiquadratic Lyapunov functions ($\mathcal{MQLF}$) of the form $V(x(t)) = x(t)^T \sum_{i=1}^{n} \mu_i(z(t)) P_i x(t)$, $P_i > 0$ (Chadli, 2002b), (Blanco *et al.*, 2001b), (Tanaka *et al.*, 2001), (Daafouz *et al.*, 2001), (Morère *et al.*, 2000), (Johansen, 2000), (Jadbabaie, 1999). The obtained results make it possible to also reduce the conservatism of the quadratic approach. However, it is interesting to notice the great difference between the results of the continuous domain and the discrete domain. If the results obtained for the discrete case are global and may be formulated into a $\mathcal{LMI}$ form (Daafouz *et al.*, 2001), the results in continuous case are often local and in nonconvex form (Chadli, 2002b), (Blanco *et al.*, 2001b), (Tanaka *et al.*, 2001).

The objective of the paper is to formulate stability conditions in term of $\mathcal{LMI}$ form (Boyd *et al.*, 1994) using a $\mathcal{MQLF}$ approach. In the case when these conditions are nonlinear in the synthesis variables, a bilinear form easy to linearise by existing algorithm is given. In the stabilisation part, only the case when the input matrices are positively linearly dependant i.e. $B_i = \beta_i B, \beta_i > 0$ will be considered.

Notation: In this paper, X^T denotes the transpose of the matrix X, $X > 0$ $(X \geq 0)$ means that X is a symmetric positive definite (semidefinite) matrix, $\langle\cdot\rangle$ denotes the scalar product, $|\cdot|$ denotes the absolute value, $I_s = \{1, 2, .., s\}$ and

$\sum_{i\neq j:1}^{n} \mu_i(z)\mu_j(z) = \sum_{i:1}^{n}\sum_{j\neq i:1}^{n} \mu_i(z)\mu_j(z)$

$\sum_{i<j:1}^{n} \mu_i(z)\mu_j(z) = \sum_{i<j:1}^{n}\sum_{j:1}^{n} \mu_i(z)\mu_j(z)$

2. CONTINUOUS MULTIPLE MODEL

The continuous multiple model is represented as follows:

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t))(A_i x(t) + B_i u(t)) \tag{1a}$$

$$y(t) = \sum_{i=1}^{n} \mu_i(z(t)) C_i x(t) \tag{1b}$$

where n is the number of local models, $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^l$ is the output vector, $z(t) \in \mathbb{R}^q$ is the vector of the so-called decision variables, $A_i \in \mathbb{R}^{p.p}$, $B_i \in \mathbb{R}^{p.m}$ et $C_i \in \mathbb{R}^{l.p}$. The activation functions $\mu_i(.)$ are such that :

$$\begin{cases} \sum_{i=1}^{n} \mu_i(z(t)) = 1 \\ \mu_i(z(t)) \geq 0, \quad \forall\, i \in I_n \end{cases} \tag{2}$$

The choice of the variable $z(t)$ leads to different classes of models. It can depend on the measurable state variables, be a function of the measurable outputs of the system and possibly on the input. In this case, the system (1a) describes a nonlinear system. It can also be an unknown constant value, system (1a) then represents a PLDI.

3. STABILITY ANALYSIS

The considered $\mathcal{MQLF}$ depends only on the system state and has the form :

$$V(x(t)) = x(t)^T P(x(t))\, x(t) \tag{3}$$

with

$$P(x(t)) = \sum_{i=1}^{n} \mu_i(x(t))\, P_i, P_i > 0 \tag{4}$$

Taking into account the properties of the matrices P_i and those of the activation functions, the function (3) is a Lyapunov function candidate since :

$$c_1 \|x(t)\|^2 \leq V(x(t)) \leq c_2 \|x(t)\|^2 \tag{5}$$

where c_1 and c_2 are positive scalars :

$$c_1 = \max_{i\in I_n}(\lambda_{\min}(P_i)),\ c_2 = \min_{i\in I_n}(\lambda_{\max}(P_i)) \tag{6}$$

The proposed method using this type of function needs a priori bound on the state variation. This hypothesis implies a bounded derived activation function $\frac{d\mu_i(x(t))}{dt}$.

Assumption 1 : the activation functions $\mu_i(x(t))$ are continuously differentiable.

Considering the derivative time of the $\mathcal{MQLF}$ (3):

$$\begin{aligned}\dot{V}(x(t)) = \dot{x}(t)^T P(x(t))\, x(t) + x(t)^T P(x(t)) \cdot \\ \dot{x}(t) + x(t)^T \dot{P}(x(t))\, x(t)\end{aligned} \tag{7}$$

Taking definition (4) into account, the last term in the right member of (7) could be bounded as follows

$$\begin{aligned} & x(t)^T \dot{P}(x(t))\, x(t) = \\ & x(t)^T \sum_{i=1}^{n} \left\langle \frac{\partial \mu_i(x(t))}{\partial x(t)}, \frac{\partial x(t)}{\partial t} \right\rangle P_i x(t) \\ & \leq x(t)^T \sum_{i=1}^{n} \left| \left(\frac{\partial \mu_i(x(t))}{\partial x(t)} \right)^T \dot{x}(t) \right| P_i x(t) \end{aligned} \tag{8}$$

Let us now formulate the following assumption

Assumption 2 : there exists a scalar $\nu > 0$ such that $\left|\left(\frac{\partial\mu_i(x(t))}{\partial x(t)}\right)^T \dot{x}(t)\right| \leq \nu, \forall x(t) \in \mathbb{R}^p, i \in I_n$

Thus in this case the term $\left|\left(\frac{\partial\mu_i(x(t))}{\partial x(t)}\right)^T \dot{x}(t)\right|$ is bounded independently from the state:

$$x(t)^T \dot{P}(x(t))\, x(t) \leq x(t)^T \nu \sum_{i=1}^{n} P_i x(t) \qquad (9)$$

Consequently the time derivative of the $\mathcal{MQLF}$ (7) becomes :

$$\dot{V}(x(t)) \leq \dot{x}(t)^T P(x(t))\, x(t) + x(t)^T P(x(t)) \cdot \dot{x}(t) + x(t)^T \nu \sum_{i=1}^{n} P_i x(t) \qquad (10)$$

Based on the work of (Chadli, 2002b), (Blanco *et al.*, 2001b), (Tanaka *et al.*, 2001) and (Jadbabaie, 1999) the results, which will be presented thereafter in $\mathcal{LMI}$ formulation, use the bounded time derivative of the $\mathcal{MQLF}$ candidate (10). In order to improve the analysis result and then to obtain conditions less constraining as possible, the bound should be determined as precisely as possible. To achieve this goal, the following proposition a priori supposes certain knowledge on the coupling of the activation functions, i.e. the maximum number (r) of local models simultaneously activated at each time.

Proposition 1 : taking into account the properties of the activation functions (2), the following inequalities hold, $\forall\, r \in \{2, .., n\}$:

$$\sum_{i \neq j:1}^{n} \mu_i(z)\, \mu_j(z) \leq 1 - \frac{1}{r} \qquad (11)$$

$$\sum_{i=1}^{n} \mu_i^2(z) \geq \frac{1}{r} \qquad (12)$$

where r is the maximum number of local models simultaneously activated at each time.

Proof. : from the properties (2) of the activation functions one deduces :

$$1 = \left(\sum_{i=1}^{n} \mu_i(z)\right)^2 = \sum_{i=1}^{n} \mu_i(z)^2 + \sum_{i \neq j:1}^{n} \mu_i(z)\, \mu_j(z) \qquad (13)$$

and then using (Tanaka *et al.*, 1998):

$$\sum_{i=1}^{n} \mu_i(z)^2 \geq \frac{1}{r-1} \sum_{i \neq j:1}^{n} \mu_i(z)\, \mu_j(z) \qquad (14)$$

we obtain the property (11). In the same way the inequality (12) is deduced directly from the inequality (11) and from the equality (13). □

The following result supposes a priori bound (ν) on the state variation (assumption 2).

Theorem 1 : suppose that there exists symmetric matrices $Q > 0$, $P_i > 0, i \in I_n$, M and N which verify the following $\mathcal{LMI}$:

$$P_i > P_{j+r},\ i \in I_r,\ j \in I_{n-r} \qquad (15)$$

$$A_i^T P_i + P_i A_i \leq M, \forall\ i \in I_n \qquad (16)$$

$$A_i^T P_j + P_j A_i + A_j^T P_i + P_i A_j \leq 2N \quad \forall\ (i,j) \in I_n^2, i < j \qquad (17)$$

$$M - N \leq 0 \qquad (18)$$

$$N + r^{-1}(M - N) + \nu \sum_{i=1}^{r} P_i < -Q \qquad (19)$$

with $\mu_i(z(t))\, \mu_j(z(t)) \neq 0$, r is the maximum number of local models simultaneously activated at each time and ν is a bound related to the state variation (assumption 2). Then the equilibrium point of the unforced multiple model (1a) is globally exponentially stable.

Proof. : The derivative (10) of the $\mathcal{MQLF}$, along the trajectory of the unforced multiple model (1a), taking into account the conditions (15), (16) and (17) with the assumption 2 is expressed :

$$\dot{V}(x(t)) \leq x(t)^T \nu \sum_{i=1}^{r} P_i x(t) + x(t)^T \left(\sum_{i=1}^{n} \mu_i^2(x(t)) M + 2 \sum_{i<j:1}^{n} \mu_i \mu_j N\right) x(t) \qquad (20)$$

or in an equivalent form :

$$\dot{V}(x(t)) \leq x(t)^T \nu \sum_{i=1}^{r} P_i x(t) + x(t)^T \left(N + \sum_{i=1}^{n} \mu_i(x(t))^2 (M - N)\right) x(t) \qquad (21)$$

The condition (18) and the property (12) allow to write

$$\dot{V}(x(t)) \leq x(t)^T \left(N + r^{-1}(M-N) + \nu \sum_{i=1}^{r} P_i \right) x(t) \tag{22}$$

Consequently the condition (19) guarantees exponential stability of the unforced multiple model (1a). □

Let us notice that when the activation functions are defined on an infinite support, i.e. $r = n$, the condition (15) is trivial.

4. CONTROLLER SYNTHESIS

In the case when the input matrices are positively linearly dependant i.e. $B_i = \beta_i B, \beta_i > 0$, it is interesting to consider the control law (Guerra *et al.*, 2001) :

$$u(t) = -\frac{\sum_{i=1}^{n} \mu_i(z(t)) \beta_i K_i}{\sum_{i=1}^{n} \mu_i(z(t)) \beta_i} x(t) \tag{23}$$

With this control law, a closed loop continuous multiple model is then written as follows :

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t)) G_{ii} x(t) \tag{24}$$

with

$$G_{ii} = A_i - B_i K_i \tag{25}$$

Notice that in this case the closed loop continuous multiple model depend only on the dominant terms $G_{ii} = A_i - B_i K_i$. The coupled terms $G_{ij} = A_i - B_i K_j$ with $i \neq j$ are ignored.

The derivative of the $\mathcal{MQLF}$ (10) along the trajectory of the multiple model (24) is expressed:

$$\dot{V}(x(t)) = x(t)^T R(x) x(t) + x(t)^T \dot{P}(x(t)) x(t) \tag{26}$$

with

$$R(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} \mu_i(x) \mu_j(x) S_{ij} \tag{27}$$

$$S_{ij} = G_{ii}^T P_j + P_j G_{ii} \tag{28}$$

The term $x(t)^T \dot{P}(x(t)) x(t)$ depends on the local gain $K_i, i \in I_n$:

$$x^T \dot{P}(x) x \leq x^T \sum_{i=1}^{n} \left| \left(\frac{\partial \mu_i(x)}{\partial x} \right)^T \dot{x} \right| P_i x \tag{29}$$

The following bound may be used :

$$\begin{aligned} \left| \left(\frac{\partial \mu_i(x(t))}{\partial x(t)} \right)^T \dot{x}(t) \right| &\leq \left\| \frac{\partial \mu_i(x(t))}{\partial x(t)} \right\| \|\dot{x}(t)\| \\ &\leq \left\| \frac{\partial \mu_i(x(t))}{\partial x(t)} \right\| \|x(t)\| \alpha \end{aligned} \tag{30}$$

where

$$\alpha = \max_{i \in I_n} (\|G_{ii}\|) \tag{31}$$

Using the definition (25), the condition (31) is verified if the following $\mathcal{LMI}$ depending on α and K_i are feasible:

$$\begin{pmatrix} \alpha I & (A_i - B_i K_i)^T \\ A_i - B_i K_i & \alpha I \end{pmatrix} \geq 0, \; i \in I_n \tag{32}$$

Assumption 3 : there exits α, K_i verifying (32) and $\eta > 0$ such that :

$$\left\| \frac{\partial \mu_i(x(t))}{\partial x(t)} \right\| \|x(t)\| \leq \eta, \forall x(t) \in \mathbb{R}^p \tag{33}$$

In this case the expression (29) can be bounded as follows:

$$x(t)^T \dot{P}(x(t)) x(t) \leq \alpha\eta x(t)^T \sum_{i=1}^{n} P_i x(t), \forall x \in \mathbb{R}^p \tag{34}$$

Theorem 2 : suppose that there exists symmetric matrices $Q > 0$, $P_i > 0, i \in I_n$, M and N, matrices $K_i, i \in I_n$ and a scalar α which verify the following constraints :

$$P_i > P_{j+r}, \; i \in I_r, \; j \in I_{n-r} \tag{35}$$

$$S_{ii} \leq M, \; i \in I_n \tag{36}$$

$$S_{ij} + S_{ji} \leq 2N, \; (i,j) \in I_n^2, i < j \tag{37}$$

$$M - N \leq 0 \tag{38}$$

$$N + r^{-1}(M-N) + \alpha\eta \sum_{i=1}^{r} P_i < -Q \tag{39}$$

$$\begin{pmatrix} \alpha I & (A_i - B_i K_i)^T \\ A_i - B_i K_i & \alpha I \end{pmatrix} \geq 0, \; i \in I_n \tag{40}$$

with $\mu_i(z(t)) \mu_j(z(t)) \neq 0, S_{ij} = (A_i - B_i K_i)^T P_j + P_j (A_i - B_i K_i)$ and η a bound respecting (33).

Then the multiple model (24) is globally exponentially stable.

Proof. : The equality (27) can be written

$$R(x) = x^{T}\left(\sum_{i=1}^{n}\mu_i^2 S_{ii} + \sum_{i<j:1}^{n}\mu_i\mu_j(S_{ij}+S_{ji})\right)x$$

With conditions (36) and (37), we obtain

$$R(x) \leq x^{T}\left(N + \sum_{i=1}^{n}\mu_i\left(x(t)\right)^2(M-N)\right)x \tag{41}$$

By using the proposition 1 with the assumption (34) and the constraints (35) and (40), the expression (26) is expressed as follows

$$\dot{V}(x) \leq x^{T}\left(N + r^{-1}(M-N) + \alpha\eta\sum_{i=1}^{r}P_i\right)x \tag{42}$$

finally, with the result (39), we verify $\dot{V}(x(t)) < -x(t)^{T}Qx(t)$, which ends the proof. □

Remarks :

- In order to avoid obtaining non exploitable stabilisation conditions by numerical tools, the case of only two local models simultaneously activated (without any conditions on the input matrices B_i) is considered in (**?**). This case gives interesting results easy to linearise by existing technics.
- *Linearisation* - The result obtained in theorem 2 are in $\mathcal{BMI}$ form in P_i and K_i. We know that $\mathcal{BMI}$ problems are not convex and may have multiple local solutions. For solving this problem, we can use, for example, the path-following method, developed in (Hassibi *et al.*, 1999) (see (Chadli *et al.*, 2003a) for more detail). For that purpose, let P_{i0} and K_{i0} be initial values and let $P_i = P_{i0} + \delta P_i$ and $K_i = K_{i0} + \delta K_i$. The $\mathcal{LMI}$ formulation corresponds to substitue S_{ij} by

$$S_{ij} = A_i^T P_{j0} + P_{j0}A_i - K_{i0}^T B_i^T P_{j0} - P_{j0}B_iK_{i0} + A_i^T\delta P_j + \delta P_j A_i - K_{i0}^T B_i^T \delta P_j - \delta P_j B_i K_{i0} - \delta K_i^T B_i^T P_{j0} - P_{j0}B_i\delta K_i \tag{43}$$

with the supplementary constraints

$$\begin{pmatrix} \zeta P_{i0} & \delta P_i \\ \delta P_i & \zeta P_{i0} \end{pmatrix} > 0, \begin{pmatrix} (\zeta\,\|K_{i0}\|)^2 & \delta K_i^T \\ \delta K_i & \mathcal{I} \end{pmatrix} > 0 \tag{44}$$

where $0 < \zeta \ll 1$.

5. CONCLUSION

In this paper, the stability analysis of a nonlinear system described by a multiple model and also the controller synthesis are considered. Using the $\mathcal{MQLF}$, sufficient conditions for global asymptotic stability are given using a $\mathcal{LMI}$ formulation. The results of stabilisation are given under $\mathcal{BMI}$ form. This set of inequality may be solved using a linearisation technique.

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