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Nonquadratic stability analysis of Takagi-Sugeno models

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Abstract

This paper deals with the stability analysis of Takagi-Sugeno models (T-S). Based on a piecewise quadratic Lyapunov functions and the use of the so-called S-procedure, new asymptotic stability conditions for both continuous and discrete T-S models are presented. The stability conditions are formulated in linear matrix inequalities (LMIs). Examples are given to illustrate the advantage of the proposed method.

Keywords: T-S model, nonlinear models, stability analysis, Lyapunov methods, S-procedure, LMIs.

1 Introduction

In the last decade, the issue of stability analysis for nonlinear models described by Takagi-Sugeno models [5][15] has been considered actively. Having the property of universal approximation [8][9], this approach includes the multiple model approach [17] and can be seen also as Polytopic Linear Differential Inclusions (PLDI) [10]. Indeed, there is significant amount of research on quadratic stability. These studies utilize the recently developed interior-point convex optimization methods for solving Linear Matrix Inequalities (LMIs) [10]. It is well known that the stability depends on the existence of a common positive definite matrix satisfying a set of LMIs [2][4][10][12][16]. Nevertheless, restriction to the class of quadratic Lyapunov function candidate may lead to significant conservativeness. In [19] stability conditions are derived in terms of M-matrices and vector Lyapunov functions by regarding the T-S models as an interconnection of LTI subsystems while in [24] they are treated as a linear system having modeling uncertainty. To overcome this limitation, stability conditions relaxing previous constraints have been established using a piecewise quadratic Lyapunov function and the S-procedure [13][14]. While in [13] the continuity of Lyapunov function is carried out by requiring additional constraints, in [14] the function of Lyapunov can be discontinuous. A polyquadratic Lyapunov function which is built on the same basis as the T-S model itself is studied for continuous case [1][3][7][11][18]. Using convex optimization, this type of Lyapunov function is also computed for discrete systems with time varying uncertainties [6]. In Linear Parameter Varying (LPV) systems, to reduce the conservativeness, quadratic parameter dependant Lyapunov functions are used [20]-[23]. The LPV systems may also be represented by T-S models. However, the way which consist to embed nonlinear systems into LPV framework, i.e. when the states are viewed as time varying parameter, will lead obviously to conservative results [1].

In this paper, the stability analysis of T-S models are considered. New sufficient conditions for global asymptotic stability are obtained using a nonquadratic Lyapunov function and the so-called S-procedure. The stability conditions are derived in a set of LMIs with additional scalars parameters which can be numerically computed. The proposed method is proved to be less conservative compared to those derived via quadratic stability analysis.

The organization of the paper is as follows. The section 2 is dedicated to the description of the continuous T-S model. In section 3, a basic stability condition, helping to precise the motivation of the paper, is described, while the main result is established in section 4. The proposed analysis is then extended to discrete T-S models and compared to previous nonquadratic stability conditions. Numerical examples for both continuous and discrete cases are presented.

2 T-S continuous model

A continuous T-S model is based on the interpolation between several LTI local models as follows:

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t))(A_i x(t) + B_i u(t)) \tag{1}$$

where n is the number of submodels, $x(t) \in \mathbb{R}^p$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $A_i \in \mathbb{R}^{p.p}, B_i \in \mathbb{R}^{p.m}$ and $z(t) \in \mathbb{R}^q$ is the decision variable vector.

The choice of the variable $z(t)$ leads to different class of models. It can depend on the measurable state variables, be a function of the measurable outputs of the system and possibly on the input. In this case, the system (1) describes a nonlinear system. It can also be an unknown constant value, system (1) then represents a PLDI.

The normalized activation function $\mu_i(z(t))$ in relation with the i^{th} submodel is such that:

$$\begin{cases} \sum_{i=1}^{n} \mu_i(z(t)) = 1 \\ \mu_i(z(t)) \geq 0 \quad \forall\, i \in \{1,..,n\} \end{cases} \tag{2}$$

The global output of T-S model is interpolated as follows:

$$y(t) = \sum_{i=1}^{n} \mu_i(z(t)) C_i x(t) \tag{3}$$

where $y(t) \in \mathbb{R}^l$ is the output vector and $C_i \in \mathbb{R}^{l.p}$. More detail about this type of representation can be found in [2][4].

In the rest of the paper, the following useful notation is used: X^T denote the transpose of the matrix X and $X > 0$ ($X \geq 0$) denote symmetric positive definite (semidefinite) matrix.

3 Motivation of the paper

Consider the unforced continuous T-S model of (1):

$$\dot{x}(t) = \sum_{i=1}^{n} \mu_i(z(t)) A_i x(t) \tag{4}$$

Sometimes, it is possible to prove the stability of a T-S model (4) using a quadratic Lyapunov function $V(x(t)) = x^T(t) P x(t)$, $P > 0$. The method is based on the following sufficient conditions:

If there exists a common symmetric positive definite matrix P such that [2]

$$A_i^T P + P A_i < 0 \qquad \forall i \in \{1, \ldots, n\} \tag{5}$$

Then the T-S model (4) is globally asymptotically stable.

The existence of such a common positive definite matrix P is a key issue to check the stability of a T-S model. Inequalities (5) give a sufficient condition for ensuring stability of (4). However, it is well known that in a lot of cases, a common positive definite matrix P does not exist whereas the T-S model is stable. The following lemma gives a sufficient conditions for the non existence of a such matrix P.

Lemma 1 [10]: If there exists matrices $X_i, \forall\, i \in \{1,..,n\}$ not all zero such that

$$X_i \geq 0 \text{ and } \sum_{i=1}^{n} A_i^T X_i + X_i A_i \geq 0$$

then the inequalities (5) do not admit a solution $P > 0$.■

The example in section 4.2 illustrates this fact. In some cases, one way to reduce the conservativeness of the quadratic analysis is the use of the so-called S-procedure and the piecewise quadratic Lyapunov function [13][14]. It should be pointed out that the Lyapunov function used in [14] is allowed to be discontinuous since the continuity is not used in the proof.

4 Stability analysis of continuous T-S models

4.1 Analysis

In the following, a new sufficient condition for global asymptotic stability of T-S model (4) is established. Before let us recall the following useful lemma.

Lemma 2 (*S-Procedure,* [10]):

Let $F_0(x(t)), \ldots, F_q(x(t))$ be quadratic functions of the variable $x(t) \in \mathbb{R}^p$. Consider the following statements:

$F_0(x(t)) \leq 0$ for all $x(t)$ such that

$$F_i(x(t)) \leq 0, \forall\, i \in \{1,..,q\} \tag{6}$$

If there exists scalars $\tau_1 \geq 0, .., \tau_q \geq 0$ such that

$$F_0(x(t)) - \sum_{i=1}^{q} \tau_i F_i(x(t)) \leq 0 \tag{7}$$

then (6) holds. ■

The following theorem gives sufficient stability conditions by using the S-procedure lemma and nonquadratic Lyapunov function candidate [10] of the form

$$V(x(t)) = \max(V_1(x(t)),..,V_i(x(t)),..,V_n(x(t))) \quad (8)$$

where $V_i(x(t)) = x(t)^T P_i x(t), \quad P_i > 0, \forall\, i \in \{1,..,n\}$

Theorem 1 : Suppose that there exists symmetric matrices $P_i, \forall\, i \in \{1,..,n\}$ and scalars $\tau_{ijk} \geq 0$ such that

$$\begin{pmatrix} A_i^T P_j + P_j A_i + \sum_{k=1,k\neq j}^{n} \tau_{ijk}(P_j - P_k) & 0 \\ 0 & -P_i \end{pmatrix} < 0 \quad (9)$$

$\forall\, i, j \in \{1,..,n\}$. Then the T-S model (4) is globally asymptotically stable.

Proof:

Considering the nonquadratic Lyapunov function candidate (8). It follows that

$$V(x(t)) = V_i(x(t)) \text{ if } V_i(x(t)) \geq V_j(x(t)), \forall\, j \neq i \in \{1,..,n\} \quad (10)$$

Consequently

$$\frac{dV(x(t))}{dt} = \frac{dV_i(x(t))}{dt} \text{ if } V_i(x(t)) \geq V_j(x(t)),\, j \neq i \in \{1,..,n\}.$$

Considering all possible situations, we have

$$\frac{dV(x(t))}{dt} = \begin{cases} \bullet\, x(t)^T \sum_{i=1}^{n} \mu_i(z(t))(A_i^T P_1 + P_1 A_i)x(t) \quad when\, \forall x(t): \\ V_1 \geq V_2, .. V_1 \geq V_j, .., V_1 \geq V_n \\ \vdots \\ \bullet\, x(t)^T \sum_{i=1}^{n} \mu_i(z(t))(A_i^T P_n + P_n A_i)x(t) \quad when\, \forall x(t): \\ V_n \geq V_1, .., V_n \geq V_j, .., V_n \geq V_{n-1} \end{cases} \quad (11)$$

To prove the stability of the system (4), it suffices to check, along the trajectory of (4), that

$$\frac{dV(x(t))}{dt} < 0 \;\; \forall\; x(t) \neq 0 \quad (12)$$

Consequently if $\forall\, i \in \{1,..,n\}$

$$\frac{dV(x(t))}{dt} = \begin{cases} \bullet\, x(t)^T (A_i^T P_1 + P_1 A_i)x(t) < 0 \quad when\, \forall x(t): \\ x(t)^T (P_1 - P_2)x(t) \geq 0, .., x(t)^T (P_1 - P_n)x(t) \geq 0 \\ \vdots \\ \bullet\, x(t)^T (A_i^T P_n + P_n A_i)x(t) < 0 \quad when\, \forall x(t): \\ x(t)^T (P_n - P_1)x(t) \geq 0, .., x(t)^T (P_n - P_{n-1})x(t) \geq 0 \end{cases} \quad (13)$$

then the T-S model (4) is globally asymptotically stable.

Finally, constraints (9) are obtained by applying the S-procedure lemma to (13). ■

Remarks:

1) It should be noted that the quadratic conditions (5) are included in conditions derived in (9). So when $P_i = P\; \forall\, i \in \{1,..,n\}$ we have $P_i - P_j = 0$ and $V(x(t)) = \max_{i:1,..,n}(V_i(x(t))) = x(t)^T P x(t)$. Then conditions (9) become

$$\begin{pmatrix} A_i^T P + P A_i & 0 \\ 0 & -P \end{pmatrix} < 0 \quad (14)$$

which is only those of the quadratic case (5).

2) The same result can be obtained by using the nonquadratic Lyapunov function

$$V(x(t)) = \min_{i:1,..,n}(V_i(x(t))) \quad (15)$$

where $V_i(x(t))$ is defined in (8).

3) The use of the S-procedure lemma and the nonquadratic Lyapunov function (8) leads to a non convex problem (9). However, if we fix τ_{ijk}, the results of theorem 1 are convex in $P_i\; \forall\, i \in \{1,..,n\}$ and lead to n^2 LMIs to satisfy. The following example illustrates this case.

4.2 Numerical example

Consider the T-S model (4) with $n = 3$

$$\dot{x}(t) = \sum_{i=1}^{3} \mu_i(z(t)) A_i x(t) \quad (16)$$

where the state matrices are

$$A_1 = \begin{pmatrix} 0 & 1 \\ -0.06 & -1 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 1 \\ -1.94 & -1 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & 2.04 \\ -0.5 & -1.5 \end{pmatrix}$$

and the activation functions (figure 1) are as follow

$$\mu_1(x_1(t)) = \frac{\omega_1(x_1(t))}{\omega_1(x_1(t)) + \omega_2(x_1(t)) + \omega_3(x_1(t))}$$

$$\mu_2(x_1(t)) = \frac{\omega_2(x_1(t))}{\omega_1(x_1(t)) + \omega_2(x_1(t)) + \omega_3(x_1(t))}$$

$$\mu_3(x_1(t)) = \frac{\omega_3(x_1(t))}{\omega_1(x_1(t)) + \omega_2(x_1(t)) + \omega_3(x_1(t))}$$

where

$$\omega_1(x_1(t)) = \exp\left(-\frac{1}{2}\left(\frac{x_1(t)+5}{2}\right)^2\right)$$

$$\omega_2(x_1(t)) = \exp\left(-\frac{1}{2}\left(\frac{x_1(t)}{2}\right)^2\right)$$

$$\omega_3(x_1(t)) = \exp\left(-\frac{1}{2}\left(\frac{x_1(t)-5}{2}\right)^2\right)$$

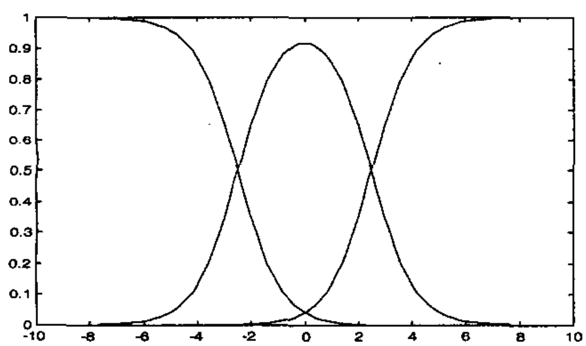

Figure 1. Activation functions of (16).

Although the three LTI local models are stable and simulation indicate that the T-S model (16) is stable (figure 2), the resolution of the three LMIs obtained from (5) shows that there is no quadratic Lyapunov function ensuring stability of the above T-S model. This verification can be made by solving the dual problem stated in lemma1:

$$X_1 > 0, X_2 > 0, X_3 > 0$$

$$X_1 A_1^T + A_1 X_1 + X_2 A_2^T + A_2 X_2 + X_2 A_2^T + A_2 X_2 > 0 \quad (17)$$

which is feasible and gives

$$X_1 = \begin{pmatrix} 156.4691 & 49.2586 \\ 49.2586 & 20.6692 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 74.3123 & -89.4586 \\ -89.4586 & 112.5173 \end{pmatrix},$$

$$X_3 = \begin{pmatrix} 42.8931 & 21.6010 \\ 21.6010 & 15.1008 \end{pmatrix}$$

The system (16) is then not quadratically stable. Moreover the activation functions used in the T-S model (16) have global support, consequently the stability conditions given in [13] is reduced to the search of a common global quadratic Lyapunov function and then fail to prove the stability of the T-S model (16). However, with the following choice of parameters:

$$\tau_{112} = \tau_{113} = 0, \tau_{212} = \tau_{213} = 1, \tau_{312} = \tau_{313} = 0$$

$$\tau_{121} = \tau_{123} = 0, \tau_{221} = \tau_{223} = 1, \tau_{321} = \tau_{323} = 0$$

$$\tau_{131} = \tau_{132} = 1, \tau_{231} = \tau_{232} = 0, \tau_{331} = \tau_{332} = 1 \quad (18)$$

our stability conditions derived in theorem 1 lead to nine LMIs in P_1, P_2 and P_3. By solving those LMIs, we obtain :

$$P_1 = \begin{pmatrix} 71.5815 & 49.0675 \\ 49.0675 & 78.6195 \end{pmatrix}, P_2 = \begin{pmatrix} 71.5815 & 49.0675 \\ 49.0675 & 78.6195 \end{pmatrix}$$

$$P_3 = \begin{pmatrix} 70.8724 & 14.7524 \\ 14.7524 & 69.1407 \end{pmatrix} \quad (19)$$

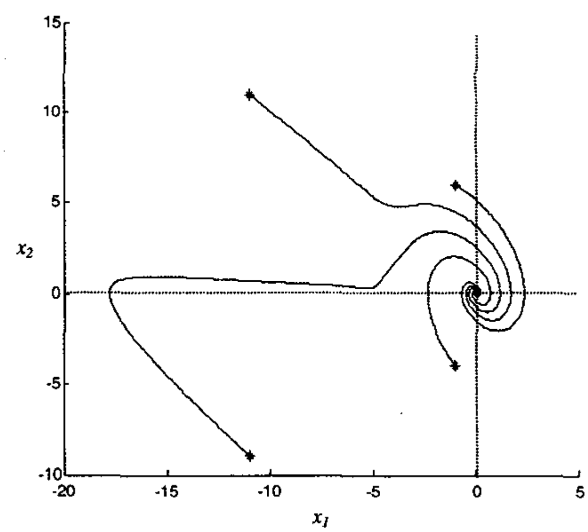

Figure 2. Example of simulation of the T-S system (16)

It is important to note that the LMIs set in P_i is obtained from (9) by fixing the real parameters τ_{ijk}. To pick those one, if any, an iteration method modifying the τ_{ijk} parameters is used.

5 Extension to discrete T-S models

A discrete T-S model is based on the interpolation between several LTI local discrete models as follows:

$$x(k+1)=\sum_{i=1}^{n}\mu_i(z(k))(A_i x(k)+B_i u(k)) \qquad (20)$$

where n is the number of submodels, $x(k)\in \mathbb{R}^p$ is the state vector, $u(k)\in \mathbb{R}^m$ is the input vector, $A_i\in \mathbb{R}^{p.p}$, $B_i\in \mathbb{R}^{p.m}$ and $z(k)\in \mathbb{R}^q$ is the decision variable vector.

To prove the stability of the unforced T-S model of (20), sufficient conditions are derived using a quadratic Lyapunov function $V(x(k))=x^T(k)Px(k),\ P>0$. So, if there exists a symmetric matrix $P>0$ such that [2]:

$$A_i^T P A_i - P < 0 \qquad \forall i\in\{1,\ldots,n\} \qquad (21)$$

then the unforced T-S model of (20) is globally asymptotically stable.

To reduce the conservativness of the quadratic method, a necessary and sufficient conditions for the computation of the polyquadratic Lyapunov function of the form $V(x(k))=x^T(k)\sum_{i=1}^{n}\mu_i(z(t))P_i x(k),\ P_i>0$ are given in [6]. These stability conditions can be directly applied to discrete T-S systems.

For less of conservatism, the following part extends results derived in section 4.1 to the discrete T-S models

5.1 Stability analysis

The stability conditions of the unforced discrete T-S model of (20) is presented in the following theorem.

Theorem 2 : Suppose that there exists symmetric matrices $P_i, \forall\, i\in\{1,..,n\}$ and scalars $\tau_{ijk}\geq 0$ such that

$$\begin{pmatrix} A_i^T P_j A_i - P_j + \sum\limits_{k=1,k\neq j}^{n}\tau_{ijk}\left(P_j-P_k\right) & 0 \\ 0 & -P_i \end{pmatrix} < 0 \qquad (22)$$

$\forall\, i,j\in\{1,..,n\}$. Then the unforced discrete T-S model of (20) is globally asymptotically stable. ■

Proof: The proof is obtained as in theorem 1, by using the nonquadratic Lyapunov function (8) and the S-procedure lemma.

The result obtained in (22) is less conservative than those of (21). We can prove easily that the quadratic conditions are included in the derived conditions by substituting $P_i, \forall\, i\in\{1,..,n\}$ by P.

5.2 Numerical example

Consider the following discrete T-S model with $n=2$

$$x(k+1)=\sum_{i=1}^{2}\mu_i(z(k))A_i x(k) \qquad (23)$$

where

$$A_1=\begin{pmatrix}0.749 & -1\\ 0.4 & 0.8\end{pmatrix},\ A_2=\begin{pmatrix}0.932 & 0.4\\ 0.1 & 0.4\end{pmatrix}$$

Quadratic conditions (21) and polyquadratic conditions given in [6] fail to prove the stability of the discrete T-S model (23). However the resolution of constraints (22) with the following choice of parameters

$\tau_{112}=0, \tau_{212}=1, \tau_{121}=2, \tau_{221}=0$

give a set of four LMIs which are feasible in P_1 and P_2:

$$P_1=\begin{pmatrix}11.7060 & 0.7443\\ 0.7443 & 29.2653\end{pmatrix},\ P_2=\begin{pmatrix}11.0630 & 5.1254\\ 5.1254 & 14.8184\end{pmatrix}$$

Consequently we conclude that the T-S model (23) is globally asymptotically stable. Example of simulation of this system with the following activation functions is given in figure 3 with two different starting points.

$$\mu_1(x_1(k))=\frac{\left(1-\tanh(x_1(k))\right)}{2},\ \mu_2(x_1(k))=1-\mu_1(x_1(k)) \qquad (24)$$

6 Conclusion

In this paper, the stability analysis of nonlinear model described by T-S model is considered. Using the S-procedure and nonquadratic Lyapunov function candidate, sufficient conditions for the global asymptotic stability are derived. Despite the fact that the obtained conditions are not directly convex, it is proved that the derived stability conditions allow to improve the results obtained by the quadratic method. Two examples which are not quadratically stable are given to illustrate the advantage of the proposed results. The proposed stability conditions will be extended to the stabilization of T-S models.

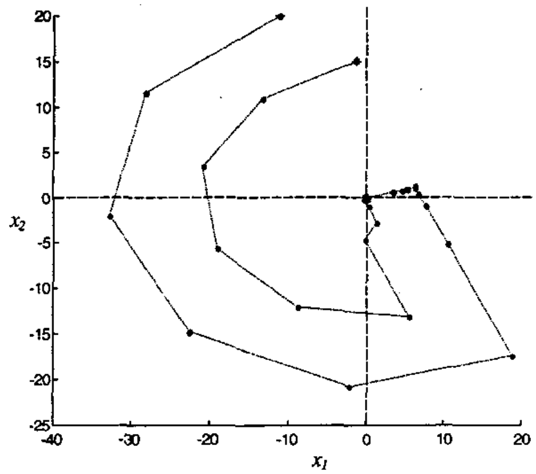

Figure 3. Example of simulation of the T-S model (23) with activation functions (24).

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