

State and multiplicative sensor fault estimation for nonlinear systems

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Motivation and proposition

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Design a joint state and (multiplicative sensor) fault observer for nonlinear systems





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Design a joint state and (multiplicative sensor) fault observer for nonlinear systems

Proposition

- Rewrite the nonlinear system into a T-S model with unmeasurable premise variables
- 2. Describe the time-varying sensor fault using the sector nonlinearity approach
- 3. Establish the convergence conditions of the state and fault estimation errors



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Outline

- Problem statement
- Observer design
- Illustrative example
- 4. Conclusions and perspectives



T-S approach for modeling

The Takagi-Sugeno structure

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{n} \mu_{i}(\xi(t))(A_{i}x(t) + B_{i}u(t)) \\ y(t) = \sum_{i=1}^{n} \mu_{i}(\xi(t))(C_{i}x(t) + D_{i}u(t)) \end{cases}$$

 $x(t) \in \mathbb{R}^{n_x}$ is the system state variable, $u(t) \in \mathbb{R}^{n_u}$ is the control input and $y(t) \in \mathbb{R}^m$ is the system output. $\xi(t) \in \mathbb{R}^q$ is the decision variable vector.



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• Nonlinear interpolation between linear submodels with adequate weighting functions $\mu_i(\xi(t))$ satisfying the convex sum property

$$\begin{cases} \sum_{i=1}^{n} \mu_i(\xi(t)) = 1 \\ 0 \le \mu_i(\xi(t)) \le 1, \quad i = 1, \dots, n, \quad \forall t \end{cases}$$



How to systematically obtain a T-S model fom a given NL system?

 Sector nonlinearity transformation: a systematic procedure which guarantees an exact model construction for nonlinear systems with bounded nonlinearities.





How to systematically obtain a T-S model fom a given NL system?

- Sector nonlinearity transformation: a systematic procedure which guarantees an exact model construction for nonlinear systems with bounded nonlinearities.
- The nonlinear systems is rewritten as a quasi-LPV model. The T-S form is obtained by using the convex polytopic transformation.
 Each vertex defines a linear submodel and the nonlinearities are rejected into the weighting functions.

$$\mathsf{NL} \left\{ \begin{array}{l} \dot{x}(t) = f_x(x(t), u(t)) \\ y(t) = f_y(x(t), u(t)) \end{array} \right. \Rightarrow \\ \mathsf{Quasi\text{-LPV}} \left\{ \begin{array}{l} \dot{x}(t) = A(x(t), u(t))x(t) + B(x(t), u(t))u(t) \\ y(t) = C(x(t), u(t))x(t) + D(x(t), u(t))u(t) \end{array} \right. \Rightarrow \\ \mathsf{T\text{-S model}} \left\{ \begin{array}{l} \dot{x}(t) = \sum_{i=1}^n \mu_i(\xi(t))(A_ix(t) + B_iu(t)) \\ y(t) = \sum_{i=1}^n \mu_i(\xi(t))(C_ix(t) + D_iu(t)) \end{array} \right. \right.$$



Takagi-Sugeno system with multiplicative time-varying sensor faults

$$\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))(A_ix(t) + B_iu(t)), \quad y(t) = C(t)x(t) = (I_m + F(t))Cx(t) \quad (1)$$

 $F(t) = \sum_{j=1}^{m} f_j(t)F_j$ with F_j matrices of dimension $\mathbb{R}^{m \times m}$ and where the element of coordinate (j,j) is equal to 1 and 0 elsewhere.



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Polytopic decomposition of the sensor faults $f_i(t)$

$$f_{j}(t) = \widetilde{\mu}_{j}^{1}(f_{j}(t))f_{j}^{1} + \widetilde{\mu}_{j}^{2}(f_{j}(t))f_{j}^{2}, \quad f_{j}(t) \in [f_{j}^{2}, f_{j}^{1}]$$

$$\begin{cases}
\widetilde{\mu}_{j}^{1}(f_{j}(t)) &= \frac{f_{j}(t) - f_{j}^{2}}{f_{j}^{1} - f_{j}^{2}} \\
\widetilde{\mu}_{j}^{2}(f_{j}(t)) &= \frac{f_{j}^{1} - f_{j}(t)}{f_{j}^{1} - f_{j}^{2}}
\end{cases}$$

$$\begin{cases}
\widetilde{\mu}_{j}^{1}(f_{j}(t)) + \widetilde{\mu}_{j}^{2}(f_{j}(t)) = 1, \quad \forall t \\
0 \leq \widetilde{\mu}_{j}^{i}(f_{j}(t)) \leq 1, \quad i = 1, 2
\end{cases}$$



Problem statement

The time-varying matrice F(t) is expressed as:

$$\begin{cases}
F(t) &= \sum_{j=1}^{m} \sum_{k=1}^{2} \widetilde{\mu}_{j}^{k} (f_{j}(t)) f_{j}^{k} F_{j} \\
&= \sum_{j=1}^{2^{m}} \widetilde{\mu}_{j} (f(t)) \overline{F}_{j}
\end{cases}$$

with

$$\begin{cases} \widetilde{\mu}_{j}(f(t)) = \prod_{k=1}^{m} \widetilde{\mu}_{k}^{\sigma_{j}^{k}}(f_{k}(t)) \\ \overline{F}_{j} = \sum_{k=1}^{m} f_{k}^{\sigma_{j}^{k}} F_{j} \end{cases}$$

where the $\widetilde{\mu_i}(f(t))$ satisfy the convex sum property.



Equivalent representation of the system

$$\begin{cases} \dot{x}(t) &= g(x(t), u(t)) \\ y(t) &= h(x(t), u(t), f(t)) \\ &\equiv \\ \begin{cases} \dot{x}(t) &= \sum_{i=1}^{r} \mu_{i}(x(t))(A_{i}x(t) + B_{i}u(t)) \\ y(t) &= (I_{m} + F(t))Cx(t) \\ &\equiv \\ \begin{cases} \dot{x}(t) &= \sum_{i=1}^{r} \mu_{i}(x(t))(A_{i}x(t) + B_{i}u(t)) \\ y(t) &= \sum_{i=1}^{r} \widetilde{\mu}_{i}(f(t))\widetilde{C}_{j}x(t) \end{cases}$$



Joint state and time-varying faults observer

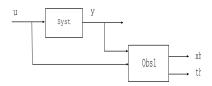


Figure: Joint state and time-varying fault observer

$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i\hat{x}(t) + B_i u(t) \\ + L_i(y(t) - \hat{y}(t))) \\ \dot{\hat{f}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(-\alpha_i \hat{f}(t) \\ + K_i(y(t) - \hat{y}(t))) \\ \hat{y}(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))\widetilde{C}_j\hat{x}(t) \end{cases}$$

Unknown gain matrices $L_i \in \mathbb{R}^{n_X \times m}$, $K_i \in \mathbb{R}^{m \times m}$ and $\alpha_i \in \mathbb{R}^{m \times m}$ must be computed to minimize the \mathcal{L}_2 gain from f(t) to the state and fault estimation errors:

- $e_x(t) = x(t) \hat{x}(t)$ the state estimation error
- $e_f(t) = f(t) \hat{f}(t)$ the time-varying fault estimation error



Difficulty

The estimation problem is not trivial since the weighting functions of the system depend on f(t) and x(t), while those of the observer depend on their estimate $\hat{f}(t)$ and $\hat{x}(t)$.

$$\operatorname{system} \begin{cases} \dot{x}(t) &= \sum_{i=1}^{r} \mu_{i}(x(t))(A_{i}x(t) + B_{i}u(t)) \\ y(t) &= \sum_{j=1}^{2^{m}} \widetilde{\mu}_{j}(f(t))\widetilde{C}_{j}x(t) \end{cases}$$

$$\operatorname{observer} \begin{cases} \dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_{i}(\hat{x}(t))(A_{i}\hat{x}(t) + B_{i}u(t) + L_{i}(y(t) - \hat{y}(t))) \\ \dot{\hat{f}}(t) &= \sum_{i=1}^{r} \mu_{i}(\hat{x}(t))(-\alpha_{i}\hat{f}(t) + K_{i}(y(t) - \hat{y}(t))) \\ \hat{y}(t) &= \sum_{j=1}^{2^{m}} \widetilde{\mu}_{j}(\hat{f}(t))\widetilde{C}_{j}\hat{x}(t) \end{cases}$$

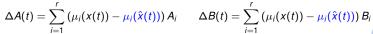


Solution: rewritting of the state equation

Based on the convex sum property of the weighting functions, rewrite the system equation as an uncertain-like system:

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \left[\mu_{i}(\hat{x}(t))(A_{i}x(t) + B_{i}u(t)) + (\mu_{i}(x(t)) - \mu_{i}(\hat{x}(t)))(A_{i}x(t) + B_{i}u(t)) \right] \\ y(t) = \sum_{j=1}^{2^{m}} \left[\widetilde{\mu}_{j}(\hat{f}(t))\widetilde{C}_{j}x(t) + \underbrace{(\widetilde{\mu}_{j}(f(t)) - \widetilde{\mu}_{j}(\hat{f}(t)))\widetilde{C}_{j}}_{\Delta C(t)} x(t) \right] \\ = \end{cases}$$

$$\begin{cases} \dot{x}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t)) \\ y(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))(\widetilde{C}_j + \Delta C(t))x(t) \end{cases}$$





Rewritting of the state equation

$$\text{System} \left\{ \begin{array}{l} \dot{x}(t) = \displaystyle \sum_{i=1}^{r} \mu_i(\hat{x}(t))((A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t)) \\ y(t) = \displaystyle \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))(\widetilde{C}_j + \Delta C(t))x(t) \end{array} \right.$$

Observer
$$\begin{cases} \dot{\hat{x}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))) \\ \dot{\hat{f}}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(-\alpha_i\hat{f}(t) + K_i(y(t) - \hat{y}(t))) \\ \hat{y}(t) = \sum_{j=1}^{2^m} \widetilde{\mu}_j(\hat{f}(t))\widetilde{C}_j\hat{x}(t) \end{cases}$$



Estimation errors dynamics

$$\dot{\mathbf{e}}_{x}(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^{m}} \mu_{i}(\hat{x}(t))\widetilde{\mu}_{j}(\hat{f}(t))((A_{i} - L_{i}\widetilde{C}_{j})\mathbf{e}_{x}(t) + (\Delta A(t) - L_{i}\Delta C(t))x(t) + \Delta B(t)u(t))$$

$$\dot{\mathbf{e}}_{f}(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^{m}} \mu_{i}(\hat{x}(t))\widetilde{\mu}_{j}(\hat{f}(t))(-K_{i}\widetilde{C}_{j}\mathbf{e}_{x}(t) - \alpha_{i}\mathbf{e}_{f}(t) + K_{i}\Delta C(t)x(t) + \alpha_{i}f(t))$$



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Let us consider the augmented vectors

$$e_{a}(t) = \begin{pmatrix} e_{x}^{T}(t) & e_{f}^{T}(t) \end{pmatrix}^{T} \text{ and } \omega(t) = \begin{pmatrix} x^{T}(t) & f^{T}(t) & \dot{f}^{T}(t) & u^{T}(t) \end{pmatrix}^{T}$$

$$\dot{e}_{a}(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^{m}} \mu_{i}(\hat{x}(t)) \widetilde{\mu}_{j}(\hat{f}(t)) \left(\Phi_{ij} e_{a}(t) + \Psi_{i}(t) \omega(t) \right)$$
(2)



Augmented system dynamic

$$\dot{e}_{a}(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^{m}} \mu_{i}(\hat{x}(t)) \widetilde{\mu}_{j}(\hat{f}(t)) \left(\Phi_{ij} e_{a}(t) + \Psi_{i}(t) \omega(t) \right)$$

$$\Phi_{ij} = \begin{pmatrix} A_{i} - L_{i} \widetilde{C}_{j} & 0 \\ -K_{i} \widetilde{C}_{j} & -\alpha_{i} \end{pmatrix}$$

$$\Psi_{i}(t) = \begin{pmatrix} \Delta A(t) - L_{i} \Delta C(t) & 0 & 0 & \Delta B(t) \\ -K_{i} \Delta C(t) & \alpha_{i} & l & 0 \end{pmatrix}$$

The objective is to guarantee the stability of the augmented system and the boundedness of the transfer from the input $\omega(t)$ to $e_a(t)$ (to attenuate the effect of $\omega(t)$ on the estimation)



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 $\Delta A(t) = \mathcal{A}\Sigma_A(t)E_A$, $\Delta B(t) = \mathcal{B}\Sigma_B(t)E_B$ and $\Delta C(t) = \mathcal{C}\Sigma_C(t)E_C$ are time-varying matrices such that $\Sigma_A^T(t)\Sigma_A(t) < I$, $\Sigma_B^T(t)\Sigma_B(t) < I$ and $\Sigma_C^T(t)\Sigma_C(t) < I$



Procedure

- 1. Consider a quadratic Lyapunov function $V(e_a(t)) = e_a^T(t)Pe_a(t), P = P^T > 0$
- 2. Consider the \mathcal{L}_2 criterion

$$\dot{V}(e_a(t)) + e_a^T(t)e_a(t) - \omega^T(t)\Gamma_2\omega(t) < 0$$

$$\Gamma_2 = \operatorname{diag}(\Gamma_2^k), \ \Gamma_2^k < \beta I, \text{ for } k = 0, 1, 2, 3$$
(3)

- guarantee the stability of $e_a(t)$ and a bounded transfer from $\omega(t)$ to $e_a(t)$.
- \circ Γ_2 allows to attenuate the transfer of some $\omega(t)$ components to $e_a(t)$ components

Condition to solve

$$\sum_{i=1}^{r} \sum_{j=1}^{2^{m}} \mu_{i}(\hat{x}(t)) \mu_{j}(\hat{f}(t)) \begin{pmatrix} e_{a}(t) \\ \omega(t) \end{pmatrix}^{T} \begin{pmatrix} \Phi_{ij}^{T} P + P \Phi_{ij} + I_{2n_{x}} & P \Psi_{i}(t) \\ \Psi_{i}^{T}(t) P & -\Gamma \end{pmatrix} \begin{pmatrix} e_{a}(t) \\ \omega(t) \end{pmatrix} < 0$$



2. Observer design: theorem

There exists a joint robust state and multiplicative sensor fault observer for the considered TS model with an \mathcal{L}_2 gain from $\omega(t)$ to $e_a(t)$ bounded by β ($\beta > 0$) if there exists matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, Γ_1 , Γ_2 , Γ_3 , $\Gamma_4 > 0$, $\overline{\alpha}_i$, \overline{K}_i , R_i and scalars β , λ_1 , $\lambda_{1C} > 0$, $\lambda_{2C} > 0$ and $\lambda_B > 0$ solutions of the optimization problem (4) under LMI constraints (5) and (6)

$$\min_{P_1, P_2, R_i, \overline{K}_i, \overline{\alpha}_i, \lambda_1, \lambda_{1C}, \lambda_{2C}, \lambda_B} \beta \tag{4}$$

$$\Gamma_k < \beta I \text{ for } k = 1, 2, 3, 4 \tag{5}$$



Process description

- A reduced form of an activated sludge reactor model with modelling errors is considered.
- The process consists in mixing used waters with a rich mixture of bacteria in order to degrade the organic matter.

Nonlinear system

$$\dot{x}_1(t) = \frac{a(t)x_1(t)x_2(t)}{x_2(t)+b} - x_1(t)u(t)
\dot{x}_2(t) = -\frac{ca(t)x_1(t)x_2(t)}{x_2(t)+b} + (d-x_2(t))u(t)$$
(7)

 $x_1(t)$ and $x_2(t)$ represent the biomass and the substrat concentration respectively.

u(t) is the dwell-time in the treatment plant.

The biomass concentration is measured $(y(t) = x_1(t))$



Modelling errors

- Parameters a, b, c, d have been identified and set to a = 0.5, b = 0.07, c = 0.7 et d = 2.5.
- It is assumed that a bounded multiplicative sensor fault f₁(t) affects the output y(t) such that:

$$y(t) = (1 + f_1(t))x_1(t)$$

with $min(f_1(t)) = f_1^2 = 0.125$ and $max(f_1(t)) = f_1^1 = 0.625$.



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Starting with the nonlinear system, a quasi-LPV state representation is established. The T-S form is obtained by using the sector nonlinearity transformation.



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T-S model of the process

$$\dot{x}(t) = \sum_{i=1}^{4} \mu_i(x(t))(A_ix(t) + Bu(t)); \ y(t) = \sum_{j=1}^{2} \widetilde{\mu}_j(f_1(t))\widetilde{C}_jx(t)$$



• Nominal output $y_n(t)$: Cx(t)

• Faulty system output : y(t) : C(t)x(t)

The output deviation caused by the time-varying parameter (multiplicative sensor fault)

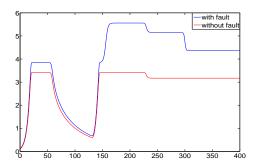


Figure: Output with and without $f_1(t)$



Initial conditions $x_0=(0.1 \ 1.5), \hat{x}_a(0)=(0.09 \ 2.3 \ 0)$ for the joint state and fault observer

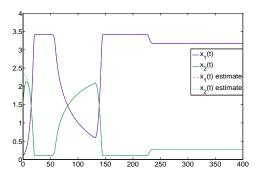


Figure: System states and their estimates



Actual and estimated time-varying parameter

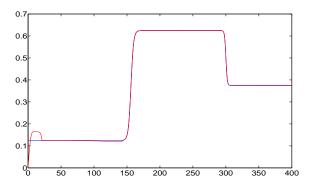


Figure: Time-varying fault $f_1(t)$ (blue) and its estimate (red)



Conclusions and perspectives

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- A new systematic procedure was presented to deal with the state and multiplicative sensor fault estimation for nonlinear systems.
- Based on a T-S representation (by the sector nonlinearity approach).
- The estimation problem and observer synthesis are expressed in terms of LMI optimization.
- No assumption on the time-varying parameter and/or the system



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Perspectives

- Practical application (Benchmark of a Wastewater Treatment Plant)
- Use the results for Fault Tolerant Control (FTC)

