State and multiplicative sensor fault estimation for nonlinear systems

Systol 2013, Nice
October 7, 2013

S. Bezzaoucha, B. Marx, D. Maquin, J. Ragot

Research Center for Automatic Control, Nancy, France
Motivation and proposition

Motivation

Design a joint state and (multiplicative sensor) fault observer for nonlinear systems
Motivation and proposition

**Motivation**

Design a joint state and (multiplicative sensor) fault observer for nonlinear systems

**Proposition**

1. Rewrite the nonlinear system into a T-S model with unmeasurable premise variables
2. Describe the time-varying sensor fault using the sector nonlinearity approach
3. Establish the convergence conditions of the state and fault estimation errors
Motivation and proposition

Motivation

Design a joint state and (multiplicative sensor) fault observer for nonlinear systems

Proposition

1. Rewrite the nonlinear system into a T-S model with unmeasurable premise variables
2. Describe the time-varying sensor fault using the sector nonlinearity approach
3. Establish the convergence conditions of the state and fault estimation errors

Outline

1. Problem statement
2. Observer design
3. Illustrative example
4. Conclusions and perspectives
1. Problem statement

T-S approach for modeling

- The Takagi-Sugeno structure

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{n} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\
y(t) &= \sum_{i=1}^{n} \mu_i(\xi(t))(C_i x(t) + D_i u(t))
\end{align*}
\]

\(x(t) \in \mathbb{R}^{n_x}\) is the system state variable, \(u(t) \in \mathbb{R}^{n_u}\) is the control input and \(y(t) \in \mathbb{R}^{m}\) is the system output. \(\xi(t) \in \mathbb{R}^{q}\) is the decision variable vector.
1. Problem statement

T-S approach for modeling

- The Takagi-Sugeno structure

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{n} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\
y(t) &= \sum_{i=1}^{n} \mu_i(\xi(t))(C_i x(t) + D_i u(t))
\end{align*}
\]

\(x(t) \in \mathbb{R}^{nx}\) is the system state variable, \(u(t) \in \mathbb{R}^{nu}\) is the control input and \(y(t) \in \mathbb{R}^{m}\) is the system output. \(\xi(t) \in \mathbb{R}^{q}\) is the decision variable vector.

- Nonlinear interpolation between linear submodels with adequate weighting functions \(\mu_i(\xi(t))\) satisfying the convex sum property

\[
\begin{align*}
\sum_{i=1}^{n} \mu_i(\xi(t)) &= 1 \\
0 \leq \mu_i(\xi(t)) &\leq 1, \quad i = 1, \ldots, n, \quad \forall t
\end{align*}
\]
1. Problem statement

How to systematically obtain a T-S model from a given NL system?

- **Sector nonlinearity transformation**: a systematic procedure which guarantees an exact model construction for nonlinear systems with bounded nonlinearities.
1. Problem statement

How to systematically obtain a T-S model from a given NL system?

- **Sector nonlinearity transformation**: a systematic procedure which guarantees an exact model construction for nonlinear systems with bounded nonlinearities.

- The nonlinear systems is rewritten as a quasi-LPV model. The T-S form is obtained by using the convex polytopic transformation. Each vertex defines a linear submodel and the nonlinearities are rejected into the weighting functions.

\[
\begin{align*}
\text{NL} & \quad \begin{cases}
\dot{x}(t) = f_x(x(t), u(t)) \\
y(t) = f_y(x(t), u(t))
\end{cases} \\
\Rightarrow \\
\text{Quasi-LPV} & \quad \begin{cases}
\dot{x}(t) = A(x(t), u(t))x(t) + B(x(t), u(t))u(t) \\
y(t) = C(x(t), u(t))x(t) + D(x(t), u(t))u(t)
\end{cases} \\
\Rightarrow \\
\text{T-S model} & \quad \begin{cases}
\dot{x}(t) = \sum_{i=1}^{n} \mu_i(\xi(t))(A_i x(t) + B_i u(t)) \\
y(t) = \sum_{i=1}^{n} \mu_i(\xi(t))(C_i x(t) + D_i u(t))
\end{cases}
\end{align*}
\]
1. Problem statement

Takagi-Sugeno system with multiplicative time-varying sensor faults

\[
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)), \quad y(t) = C(t)x(t) = (I_m + F(t))Cx(t) \quad (1)
\]

\[
F(t) = \sum_{j=1}^{m} f_j(t)F_j \quad \text{with } F_j \text{ matrices of dimension } \mathbb{R}^{m \times m} \text{ and where the element of coordinate } (j,j) \text{ is equal to 1 and 0 elsewhere.}
\]
1. Problem statement

Takagi-Sugeno system with multiplicative time-varying sensor faults

\[
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)), \quad y(t) = C(t)x(t) = (l_m + F(t))Cx(t) \quad (1)
\]

\[F(t) = \sum_{j=1}^{m} f_j(t)F_j \quad \text{with } F_j \text{ matrices of dimension } \mathbb{R}^{m \times m} \text{ and where the element of coordinate } (j, j) \text{ is equal to } 1 \text{ and } 0 \text{ elsewhere.}
\]

Polytopic decomposition of the sensor faults \(f_j(t)\)

\[f_j(t) = \tilde{\mu}_j^1(f_j(t))f_j^1 + \tilde{\mu}_j^2(f_j(t))f_j^2, \quad f_j(t) \in [f_j^2, f_j^1]
\]

\[
\begin{align*}
\tilde{\mu}_j^1(f_j(t)) &= \frac{f_j(t) - f_j^2}{f_j^1 - f_j^2} \\
\tilde{\mu}_j^2(f_j(t)) &= \frac{f_j^1 - f_j(t)}{f_j^1 - f_j^2}
\end{align*}
\]

\[
\begin{align*}
\tilde{\mu}_j^1(f_j(t)) + \tilde{\mu}_j^2(f_j(t)) &= 1, \quad \forall t \\
0 &\leq \tilde{\mu}_j^i(f_j(t)) \leq 1, \quad i = 1, 2
\end{align*}
\]
1. Problem statement

Problem statement

The time-varying matrix $F(t)$ is expressed as:

$$
F(t) = \sum_{j=1}^{m} \sum_{k=1}^{2} \tilde{\mu}^k_j(f_j(t)) f^k_j F_j
$$

$$
= \sum_{j=1}^{2^m} \tilde{\mu}_j(f(t)) \bar{F}_j
$$

with

$$
\tilde{\mu}_j(f(t)) = \prod_{k=1}^{m} \tilde{\mu}^k_{j_k}(f_k(t))
$$

$$
\bar{F}_j = \sum_{k=1}^{m} f^k_{j_k} F_j
$$

where the $\tilde{\mu}_j(f(t))$ satisfy the convex sum property.
1. Problem statement

Equivalent representation of the system

\[
\begin{align*}
\dot{x}(t) &= g(x(t), u(t)) \\
y(t) &= h(x(t), u(t), f(t))
\end{align*}
\]

\[
\equiv
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\
y(t) &= (I_m + F(t))C x(t)
\end{align*}
\]

\[
\equiv
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(x(t))(A_i x(t) + B_i u(t)) \\
y(t) &= \sum_{j=1}^{2^m} \tilde{\mu}_j(f(t))\tilde{C}_j x(t)
\end{align*}
\]
2. Observer design

Joint state and time-varying faults observer

![Diagram of Joint state and time-varying fault observer](image)

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))) \\
\dot{\hat{f}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))(-\alpha_i\hat{f}(t) + K_i(y(t) - \hat{y}(t))) \\
\hat{y}(t) &= \sum_{j=1}^{2^m} \tilde{\mu}_j(\hat{f}(t)\tilde{C}_j\hat{x}(t))
\end{align*}
\]

Unknown gain matrices \(L_i \in \mathbb{R}^{n_x \times m}\), \(K_i \in \mathbb{R}^{m \times m}\) and \(\alpha_i \in \mathbb{R}^{m \times m}\) must be computed to minimize the \(L_2\) gain from \(f(t)\) to the state and fault estimation errors:

- \(e_x(t) = x(t) - \hat{x}(t)\) the state estimation error
- \(e_f(t) = f(t) - \hat{f}(t)\) the time-varying fault estimation error
2. Observer design

**Difficulty**

The estimation problem is not trivial since the weighting functions of the system depend on \( f(t) \) and \( x(t) \), while those of the observer depend on their estimate \( \hat{f}(t) \) and \( \hat{x}(t) \).

\[
\begin{align*}
\text{system} & \quad \left\{ \begin{array}{l}
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(x(t))(A_ix(t) + B_iu(t)) \\
y(t) = \sum_{j=1}^{2^m} \tilde{\mu}_j(f(t))\tilde{C}_jx(t)
\end{array} \right.
\\
\text{observer} & \quad \left\{ \begin{array}{l}
\dot{x}(t) = \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i\hat{x}(t) + B_iu(t) + L_i(y(t) - \hat{y}(t))) \\
\dot{\hat{f}}(t) = \sum_{i=1}^{2^m} \mu_i(\hat{x}(t))(-\alpha_i\hat{f}(t) + K_i(y(t) - \hat{y}(t))) \\
\hat{y}(t) = \sum_{j=1} \tilde{\mu}_j(\hat{f}(t))\tilde{C}_j\hat{x}(t)
\end{array} \right.
\end{align*}
\]
2. Observer design

Solution: rewritting of the state equation

Based on the convex sum property of the weighting functions, rewrite the system equation as an uncertain-like system:

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \left[ \mu_i(\hat{x}(t))(A_i x(t) + B_i u(t)) + (\mu_i(x(t)) - \mu_i(\hat{x}(t))) (A_i x(t) + B_i u(t)) \right] \\
y(t) &= \sum_{j=1}^{2^m} \left[ \tilde{\mu}_j(\hat{f}(t)) \tilde{C}_j x(t) + \left( \tilde{\mu}_j(f(t)) - \tilde{\mu}_j(\hat{f}(t)) \right) \Delta C(t) x(t) \right]
\end{align*}
\]

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))((A_i + \Delta A(t)) x(t) + (B_i + \Delta B(t)) u(t)) \\
y(t) &= \sum_{j=1}^{2^m} \tilde{\mu}_j(\hat{f}(t))(\tilde{C}_j + \Delta C(t)) x(t)
\end{align*}
\]

\[
\begin{align*}
\Delta A(t) &= \sum_{i=1}^{r} (\mu_i(x(t)) - \mu_i(\hat{x}(t))) A_i \\
\Delta B(t) &= \sum_{i=1}^{r} (\mu_i(x(t)) - \mu_i(\hat{x}(t))) B_i
\end{align*}
\]
2. Observer design

Rewritting of the state equation

\[
\begin{align*}
\dot{x}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i + \Delta A(t))x(t) + (B_i + \Delta B(t))u(t) \\
y(t) &= \sum_{j=1}^{2^m} \tilde{\mu}_j(\hat{f}(t))(\tilde{C}_j + \Delta C(t))x(t)
\end{align*}
\]

\[
\begin{align*}
\dot{\hat{x}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))(A_i \hat{x}(t) + B_i u(t) + L_i(y(t) - \hat{y}(t))) \\
\dot{\hat{f}}(t) &= \sum_{i=1}^{r} \mu_i(\hat{x}(t))(-\alpha_i \hat{f}(t) + K_i(y(t) - \hat{y}(t))) \\
\hat{y}(t) &= \sum_{j=1}^{2^m} \tilde{\mu}_j(\hat{f}(t))\tilde{C}_j \hat{x}(t)
\end{align*}
\]
2. Observer design

Estimation errors dynamics

\[ \dot{e}_x(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\tilde{\mu}_j(\hat{f}(t))((A_i - L_i\tilde{C}_j)e_x(t) + (\Delta A(t) - L_i\Delta C(t))x(t) + \Delta B(t)u(t)) \]

\[ \dot{e}_f(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\tilde{\mu}_j(\hat{f}(t))(-K_i\tilde{C}_j e_x(t) - \alpha_i e_f(t)) \]

\[ f(t) - K_i\Delta C(t)x(t) + \alpha_i f(t) \]
2. Observer design

Estimation errors dynamics

\[
\dot{e}_x(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\mu_j(\hat{f}(t))((A_i - L_i\tilde{C}_j)e_x(t) + (\Delta A(t) - L_i\Delta C(t))x(t) + \Delta B(t)u(t))
\]

\[
\dot{e}_f(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\mu_j(\hat{f}(t))(-K_i\tilde{C}_j e_x(t) - \alpha_i e_f(t) - f(t) - K_i\Delta C(t)x(t) + \alpha_i f(t))
\]

Let us consider the augmented vectors

\[
e_a(t) = \begin{pmatrix} e_x^T(t) & e_f^T(t) \end{pmatrix}^T \quad \text{and} \quad \omega(t) = \begin{pmatrix} x^T(t) & f^T(t) & \dot{f}^T(t) & u^T(t) \end{pmatrix}^T
\]

\[
\dot{e}_a(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\mu_j(\hat{f}(t)) (\Phi_{ij}e_a(t) + \Psi_i(t)\omega(t))
\]
2. Observer design

Augmented system dynamic

\[
\dot{e}_a(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\hat{\mu}_j(\hat{f}(t)) (\Phi_{ij}e_a(t) + \Psi_i(t)\omega(t))
\]

\[
\Phi_{ij} = \begin{pmatrix}
A_i - L_i\tilde{C}_j & 0 \\
-K_i\tilde{C}_j & -\alpha_i
\end{pmatrix}
\]

\[
\Psi_i(t) = \begin{pmatrix}
\Delta A(t) - L_i\Delta C(t) & 0 & 0 & \Delta B(t) \\
-K_i\Delta C(t) & \alpha_i & 1 & 0
\end{pmatrix}
\]

The objective is to guarantee the stability of the augmented system and the boundedness of the transfer from the input \(\omega(t)\) to \(e_a(t)\) (to attenuate the effect of \(\omega(t)\) on the estimation).
2. Observer design

Augmented system dynamic

\[
\dot{e}_a(t) = \sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t))\tilde{\mu}_j(\hat{f}(t)) (\Phi_{ij} e_a(t) + \Psi_i(t) \omega(t))
\]

\[
\Phi_{ij} = \begin{pmatrix}
A_i - L_i \tilde{C}_j & 0 \\
-K_i \tilde{C}_j & -\alpha_i
\end{pmatrix}
\]

\[
\Psi_i(t) = \begin{pmatrix}
\Delta A(t) - L_i \Delta C(t) & 0 & 0 & \Delta B(t) \\
-K_i \Delta C(t) & \alpha_i & I & 0
\end{pmatrix}
\]

The objective is to guarantee the stability of the augmented system and the boundedness of the transfer from the input \(\omega(t)\) to \(e_a(t)\) (to attenuate the effect of \(\omega(t)\) on the estimation)

\[
\Delta A(t) = A \Sigma A(t) E_A, \Delta B(t) = B \Sigma B(t) E_B \text{ and } \Delta C(t) = C \Sigma C(t) E_C
\]

are time-varying matrices such that \(\Sigma_A^T(t) \Sigma_A(t) \leq I\), \(\Sigma_B^T(t) \Sigma_B(t) \leq I\) and \(\Sigma_C^T(t) \Sigma_C(t) \leq I\)
2. Observer design

Procedure

1. Consider a quadratic Lyapunov function \( V(e_a(t)) = e_a^T(t)P e_a(t), \ P = P^T > 0 \)
2. Consider the \( \mathcal{L}_2 \) criterion

\[
\dot{V}(e_a(t)) + e_a^T(t) e_a(t) - \omega^T(t) \Gamma_2 \omega(t) < 0
\]

\[
\Gamma_2 = \text{diag}(\Gamma^k_2), \quad \Gamma^k_2 < \beta I, \quad \text{for} \ k = 0, 1, 2, 3
\]

- guarantee the stability of \( e_a(t) \) and a bounded transfer from \( \omega(t) \) to \( e_a(t) \).
- \( \Gamma_2 \) allows to attenuate the transfer of some \( \omega(t) \) components to \( e_a(t) \) components.

Condition to solve

\[
\sum_{i=1}^{r} \sum_{j=1}^{2^m} \mu_i(\hat{x}(t)) \mu_j(\hat{f}(t)) \begin{pmatrix} e_a(t) \\ \omega(t) \end{pmatrix}^T \begin{pmatrix} \Phi_{ij}^T P + P \Phi_{ij} + I_{2n_x} & P \psi_i(t) \\ \psi_i^T(t) P & -\Gamma \end{pmatrix} \begin{pmatrix} e_a(t) \\ \omega(t) \end{pmatrix} < 0
\]
2. Observer design: theorem

There exists a joint robust state and multiplicative sensor fault observer for the considered TS model with an $\mathcal{L}_2$ gain from $\omega(t)$ to $e_a(t)$ bounded by $\beta$ ($\beta > 0$) if there exists matrices $P_1 = P_1^T > 0$, $P_2 = P_2^T > 0$, $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 > 0$, $\alpha_i, K_i, R_i$ and scalars $\beta, \lambda_1, \lambda_1C > 0, \lambda_2C > 0$ and $\lambda_B > 0$ solutions of the optimization problem (4) under LMI constraints (5) and (6).

$$\min \beta$$

$$\Gamma_k < \beta I \text{ for } k = 1, 2, 3, 4$$

$$Q_{ij}^{11} - \bar{C}_j^T \bar{K}_i^T = 0, 0, 0, 0, 0, P_1A, P_1B, R_iC, 0$$

$$Q_{i}^{22} = -\alpha_i - \alpha_i^T + I_m$$

$$Q_{i}^{33} = -\Gamma_1 + \lambda_1 E_A^T E_A + \lambda_1 C E_C^T E_C + \lambda_2 C E_C^T E_C$$

$$Q_{ij}^{66} = -\lambda_1 I, 0, 0, 0$$

$$Q_{i}^{66} = -\lambda_B I, 0, 0$$

$$Q_{ij}^{11} = P_1A_i + A_i^T P_1 - R_i \bar{C}_j - \bar{C}_j^T R_i^T + I_n x$$

$$Q_{i}^{22} = -\alpha_i - \alpha_i^T + I_m$$

$$Q_{ij}^{33} = -\Gamma_1 \lambda_1 C E_C^T E_C + \lambda_2 C E_C^T E_C$$

$$Q_{i}^{66} = -\lambda_4 + \lambda_B E_B^T E_B$$
3. Illustrative example

Process description

- A reduced form of an activated sludge reactor model with modelling errors is considered.
- The process consists in mixing used waters with a rich mixture of bacteria in order to degrade the organic matter.

Nonlinear system

\[
\dot{x}_1(t) = \frac{a(t)x_1(t)x_2(t)}{x_2(t)+b} - x_1(t)u(t) \\
\dot{x}_2(t) = -\frac{ca(t)x_1(t)x_2(t)}{x_2(t)+b} + (d - x_2(t))u(t)
\]

(7)

\(x_1(t)\) and \(x_2(t)\) represent the biomass and the substrat concentration respectively.

\(u(t)\) is the dwell-time in the treatment plant.

The biomass concentration is measured \((y(t) = x_1(t))\)
3. Illustrative example

Modelling errors

- Parameters $a$, $b$, $c$, $d$ have been identified and set to $a = 0.5$, $b = 0.07$, $c = 0.7$ et $d = 2.5$.
- It is assumed that a bounded multiplicative sensor fault $f_1(t)$ affects the output $y(t)$ such that:

$$y(t) = (1 + f_1(t))x_1(t)$$

with $\min(f_1(t)) = f_1^2 = 0.125$ and $\max(f_1(t)) = f_1^1 = 0.625$. 
3. Illustrative example

Modelling errors

- Parameters $a$, $b$, $c$, $d$ have been identified and set to $a = 0.5$, $b = 0.07$, $c = 0.7$ et $d = 2.5$.
- It is assumed that a bounded multiplicative sensor fault $f_1(t)$ affects the output $y(t)$ such that:

$$y(t) = (1 + f_1(t))x_1(t)$$

with $\min(f_1(t)) = f_1^2 = 0.125$ and $\max(f_1(t)) = f_1^1 = 0.625$.

T-S representation

Starting with the nonlinear system, a quasi-LPV state representation is established. The T-S form is obtained by using the sector nonlinearity transformation.
3. Illustrative example

Modelling errors

- Parameters $a, b, c, d$ have been identified and set to $a = 0.5$, $b = 0.07$, $c = 0.7$ et $d = 2.5$.
- It is assumed that a bounded multiplicative sensor fault $f_1(t)$ affects the output $y(t)$ such that:

$$y(t) = (1 + f_1(t))x_1(t)$$

with $\min(f_1(t)) = f_1^2 = 0.125$ and $\max(f_1(t)) = f_1^1 = 0.625$.

T-S representation

Starting with the nonlinear system, a quasi-LPV state representation is established. The T-S form is obtained by using the sector nonlinearity transformation.

T-S model of the process

$$\dot{x}(t) = \sum_{i=1}^{4} \mu_i(x(t))(A_i x(t) + Bu(t)); \ y(t) = \sum_{j=1}^{2} \tilde{\mu}_j(f_1(t)) \tilde{C}_j x(t)$$
6. Illustrative example

- Nominal output $y_n(t) : Cx(t)$

- Faulty system output : $y(t) : C(t)x(t)$

The output deviation caused by the time-varying parameter (multiplicative sensor fault)

Figure: Output with and without $f_1(t)$
6. Illustrative example

Initial conditions $x_0 = (0.1 \ 1.5)$, $\hat{x}_a(0) = (0.09 \ 2.3 \ 0)$ for the joint state and fault observer

**Figure:** System states and their estimates
Actual and estimated time-varying parameter

Figure: Time-varying fault $f_1(t)$ (blue) and its estimate (red)
Conclusions and perspectives

Conclusions

• A new systematic procedure was presented to deal with the state and multiplicative sensor fault estimation for nonlinear systems.
• Based on a T-S representation (by the sector nonlinearity approach).
• The estimation problem and observer synthesis are expressed in terms of LMI optimization.
• No assumption on the time-varying parameter and/or the system.
Conclusions and perspectives

Conclusions

• A new systematic procedure was presented to deal with the state and multiplicative sensor fault estimation for nonlinear systems.
• Based on a T-S representation (by the sector nonlinearity approach).
• The estimation problem and observer synthesis are expressed in terms of LMI optimization.
• No assumption on the time-varying parameter and/or the system

Perspectives

• Practical application (Benchmark of a Wastewater Treatment Plant)
• Use the results for Fault Tolerant Control (FTC)