Construction of Polynomial and Morphic Sequences by Cellular Automata

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Abstract

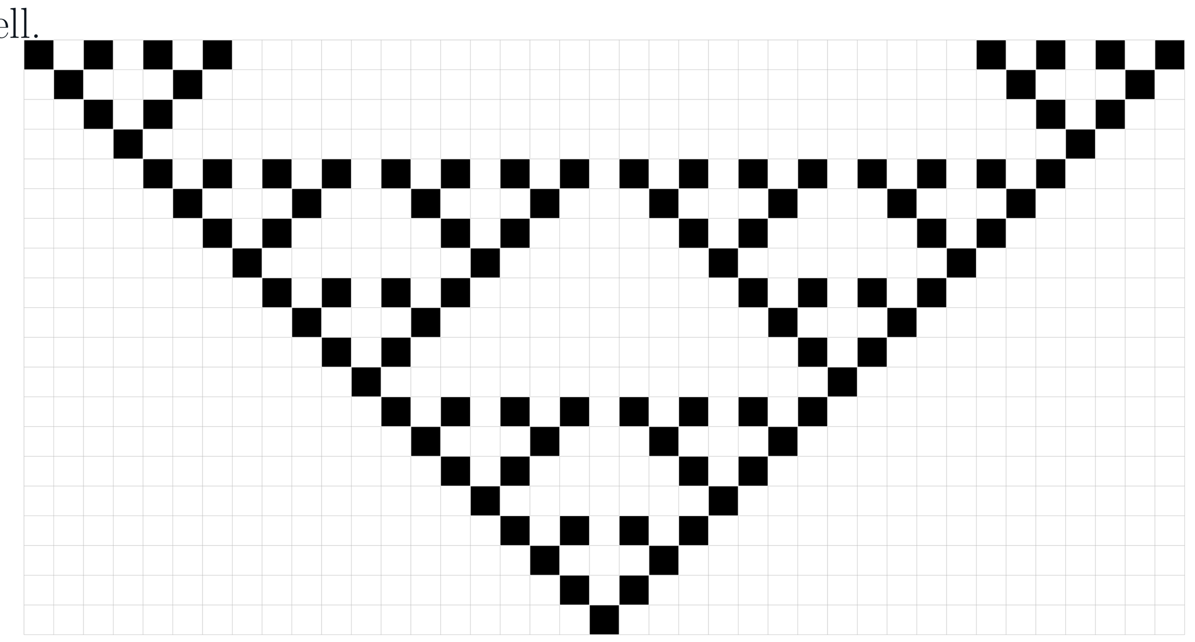
In 2015: Construction for a prime number p of p-automatic sequences by linear cellular automata with memory (Rowland and Yassawi) [2].
In 2018: Other constructions for certain nonautomatic sequences by nonlinear cellular automata, such as the characteristic sequence of the integer polynomials and the Fibonacci word [1].

Cellular automata

A one-dimensional *cellular automaton* (CA) is a dynamical system $(\mathcal{A}^{\mathbb{Z}}, F)$, where $\mathcal{A}$ is a finite set, and where the map $F : \mathcal{A}^{\mathbb{Z}} \to \mathcal{A}^{\mathbb{Z}}$ is defined by a local rule which acts uniformly and synchronously on the configuration space.
Precisely, there exists an integer $r \geq 0$ called the *radius* of the CA, and a *local rule* $f : \mathcal{A}^{2r+1} \to \mathcal{A}$ such that $\forall x \in \mathcal{A}^{\mathbb{Z}}, \forall k \in \mathbb{Z}, F(x)_k = f((x_{k+i})_{-r \leq i \leq r})$.

Example

We consider $\mathcal{A} = \{0, 1\}$ and the local rule $f(a, b, c) = a + c$. We start by the initial condition $x_0 = 1$ and $x_k = 0$ for all $k \neq 0$. We code 0 by a white cell and 1 by a black cell.



Automatic sequence

A deterministic finite automaton with output (DFAO) is a 6-tuple $(\mathcal{Q}, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ where $\mathcal{Q}$ is a finite set of *states*, $\Sigma_k = \{0, 1, \ldots, k-1\}$, $s_0 \in \mathcal{Q}$ is the initial state, $\mathcal{A}$ is a finite alphabet, $\omega : \mathcal{Q} \longrightarrow \mathcal{A}$ is the output function, and $\delta : \mathcal{Q} \times \Sigma_k \longrightarrow \mathcal{Q}$ is the transition function.
A sequence $(u_n)_{n \geqslant 0}$ of elements in $\mathcal{A}$ is k-automatic if there is a DFAO $(\mathcal{Q}, \Sigma_k, \delta, s_0, \mathcal{A}, \omega)$ such that $u_n = \omega(\delta(s_0, (n)_k))$ for all $n \geqslant 0$.

Example (Thue-Morse sequence)

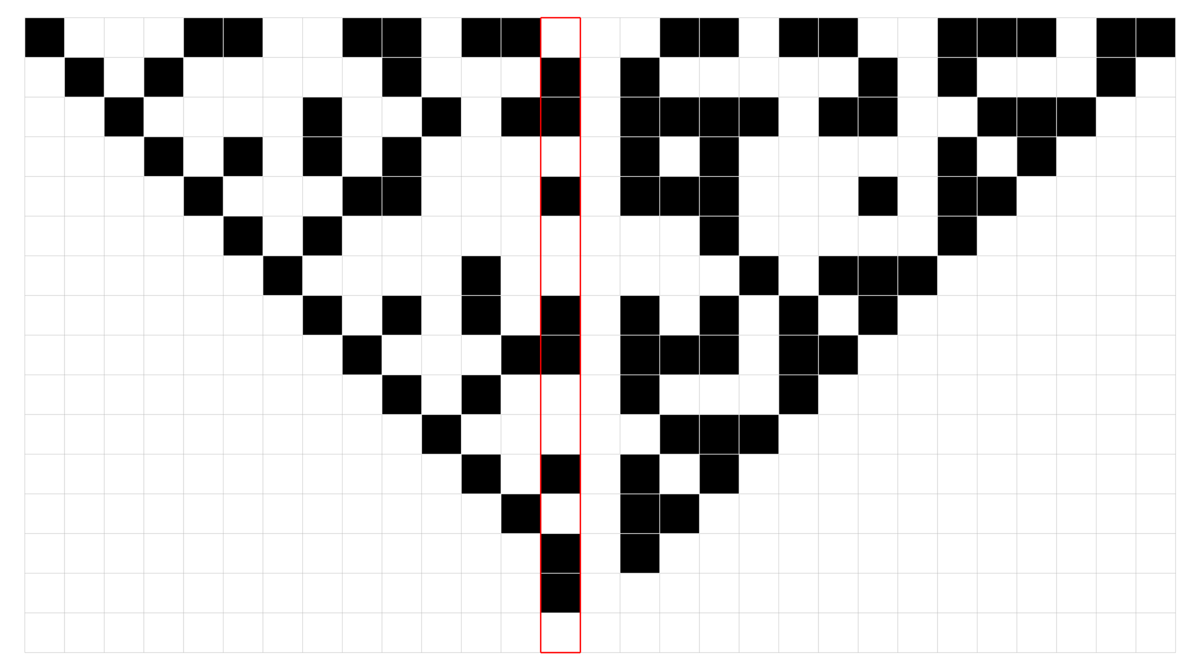
The Thue-Morse sequence is defined by $t_n = \begin{cases} 0, & \text{if the number of 1's in } (n)_2 \text{ is even;} \\ 1, & \text{otherwise.} \end{cases}$

$$(t_n)_{n \geqslant 0} = 0, 1, 1, 0, 1, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 0, \ldots$$

Theorem (Rowland and Yassawi, 2015)

Let p be a prime number and q be a power of p. A sequence of elements in $\mathbb{F}_q$ is p-automatic if and only if it is a column of a spacetime diagram of a linear cellular automaton with memory over $\mathbb{F}_q$ whose initial conditions are eventually periodic in both directions.

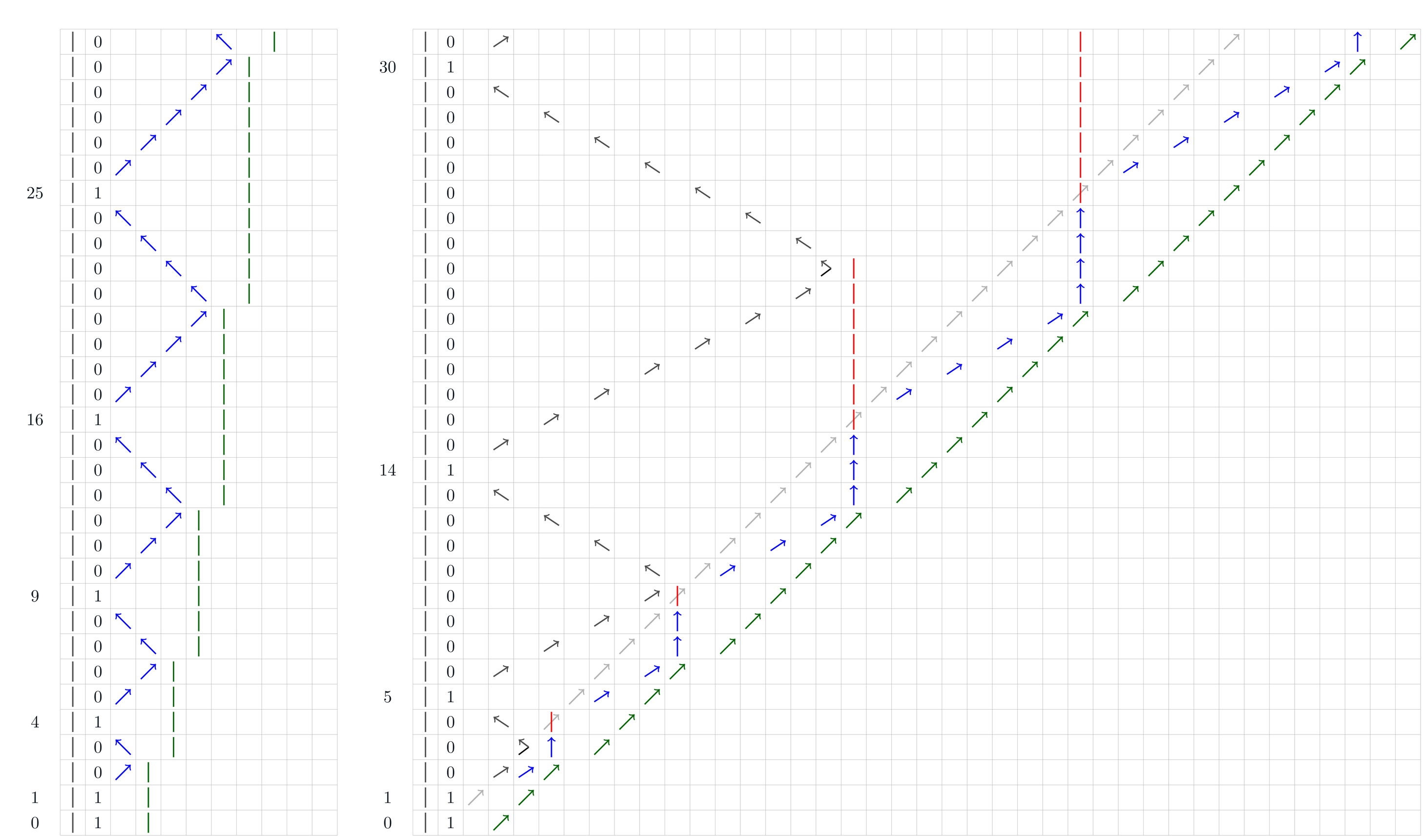
Example (construction of the Thue–Morse sequence)



Construction of polynomial sequences

Let $P(X) \in \mathbb{Q}[X]$ denote a polynomial of degree $d \geq 1$ with $P(\mathbb{N}) \subset \mathbb{N}$. Then the sequence $u = \mathbf{1}_{P(\mathbb{N})}$ can be obtained by a CA.

Example



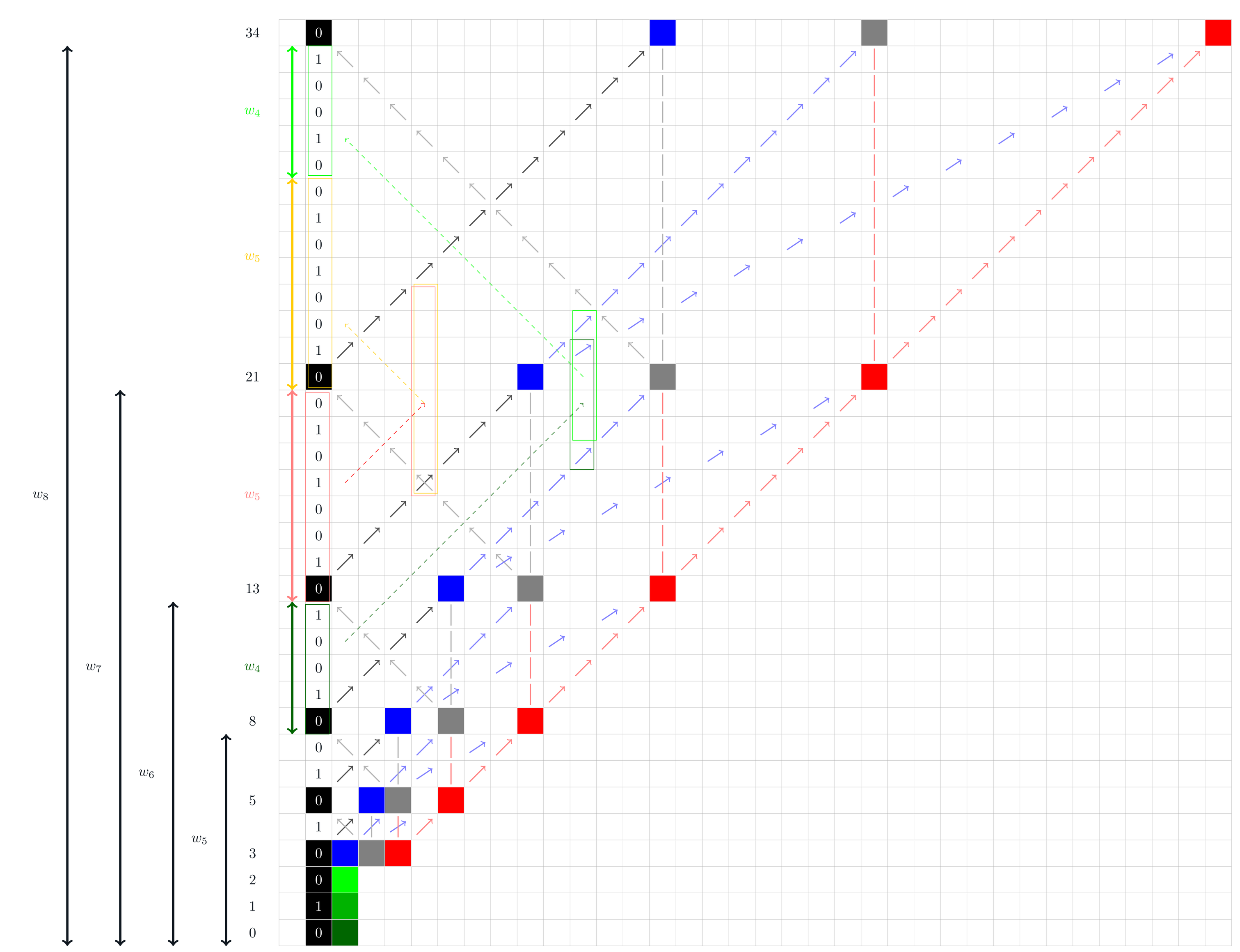
Fibonacci word

Fibonacci word: morphic sequence which is the unique fixed point of the substitution σ defined by $0 \mapsto 01$ and $1 \mapsto 0$.
Let $w_1 = 0$ and for $n \geq 1, w_{n+1} = \sigma(w_n)$. Then

$$\begin{aligned} w_1 &= 0 \\ w_2 &= 01 \\ &\vdots \\ w_7 &= 010010100100101001010 \end{aligned}$$

and the Fibonacci word is the limit of these words.

Construction of the Fibonacci sequence and the Fibonacci word



Open Questions

1. More generally, can any morphic sequence be obtained by a CA?
2. There is a close connection between one-dimensional linear CA and p-automatic sequences. Is there an analogous statement for higher dimensional linear CA?

References

[1] I. Marcovici, T. Stoll, P.-A. Tahay, Construction of Some Nonautomatic Sequences by Cellular Automata, *Cellular automata and discrete complex systems*, Lecture Notes in Comput. Sci., 10875 (2018), 113-126.
[2] E. Rowland, R. Yassawi, A characterization of p-automatic sequences as columns of linear cellular automata, Adv. Appl. Math. (2015), 68-89.