Constant mean curvature surfaces into 3-Homogeneous Manifolds

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Introduction

Surfaces with constant mean curvature (CMC) take an important place in differential geometry since the 18th century. An interesting problem involving the intrinsic curvature K and the mean curvature H is the classification problem of H-CMC surfaces with constant intrinsic curvature K in some ambient 3-manifold.

The aim of the work is classify the CMC surfaces with constant intrinsic curvature in in some homogeneous Riemannian 3-manifolds. We say that a Riemaannian 3-manifold (M, g) is homogeneous if for all $p, q \in M$ there is an isometry f such that $f(p) = q$. Roughly speaking, M looks the same at all points, even though, standing at one point, M can look different in different directions.

A background history

Let Σ be an oriented surface in $\mathbb{R}^3$ and consider a parametrization $X : U \subset \mathbb{R}^2 \to \mathbb{R}^3$ of $\Sigma = X(U)$. For $\Sigma \subset \mathbb{R}^3$, we have two important curvatures:

- Gauss curvature K: depends only the induced metric on Σ (Gauss's Theorema Egregium).
- Mean curvature H: if $N : \Sigma \to \mathbb{R}^3$ is the unit normal vector field of Σ, the mean curvature of Σ at $p \in \Sigma$ is defined by

$$H(p) = \frac{1}{2}\operatorname{tr}(-dN_p) = \frac{k_1(p) + k_2(p)}{2},$$

where $k_1(p)$ and $k_2(p)$ are the principal curvatures of Σ at the point p.

We say that Σ is a constant mean curvature surface if H is constant (H-CMC). If $H = 0$ we say that Σ is a minimal surface.

The study of CMC surfaces was initially motivated by variational problems, such as the minimization of the area with or without a constraint on the volume enclosed by the surface.

The first examples of minimal surfaces in $\mathbb{R}^3$ were given by Lagrange (1761), Meusnier (1776):

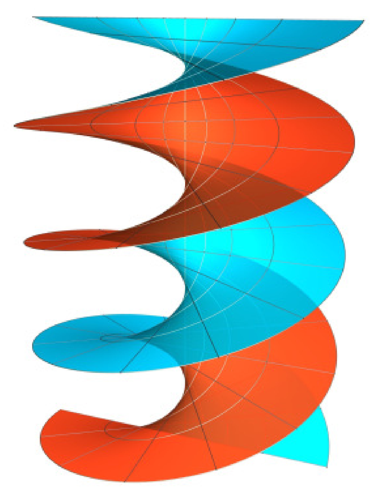


Figure 1: Helicoid

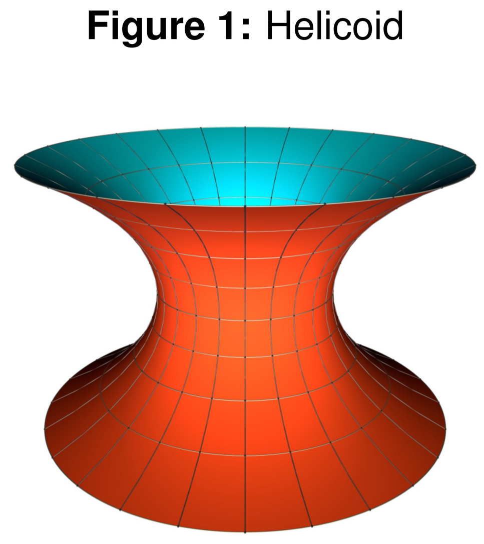


Figure 2: Catenoid

The first examples of CMC surfaces with $H \neq 0$ were the round sphere and the right cylinder. In the 19th century, Delaunay gave a systematic way to obtain non-zero constant mean curvature revolution surfaces in $\mathbb{R}^3$:

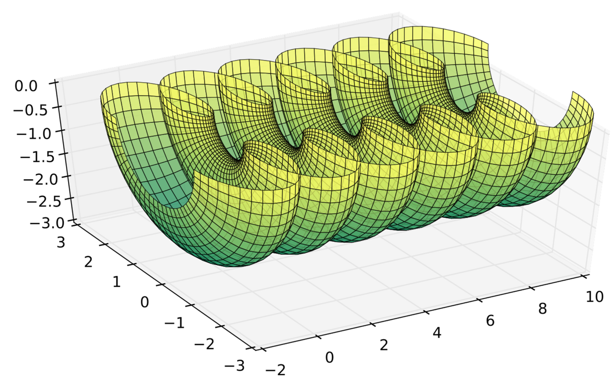


Figure 3: Nodoid

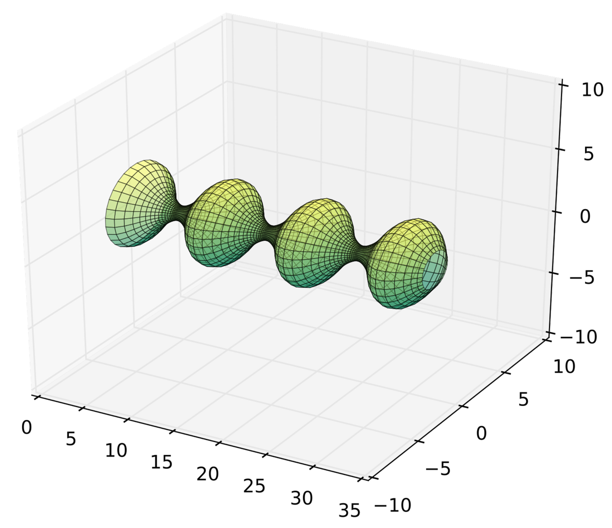


Figure 4: Unduloid

What are the surfaces in $\mathbb{R}^3$ such that K and H are constant?

Levi-Civita (1937): Let Σ be an H-CMC surface in $\mathbb{R}^3$ with K constant. Then

- either $H = 0$, $K = 0$ and Σ is part of a plane;
- or $H \neq 0$, $K = 0$ and Σ is part of a right circular cylinder;
- or $H \neq 0$, $K = H^2$ and Σ is part of a 2-sphere of radius $1/H$.

Classification result in $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$

Let $c \in \mathbb{R}^*$. For $c > 0$, we consider $\mathbb{S}^2_c$ the 2-sphere of radius $1/\sqrt{c}$, in Euclidian space $(\mathbb{R}^3, \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2)$ given by

$$\mathbb{S}^2_c = \Big\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1/c\Big\},$$

endowed with the induced metric by $\mathbb{R}^3$. For $c < 0$, we consider $\mathbb{H}^2_c$ the hyperbolic plane given by

$$\mathbb{H}^2_c = \Big(\{(x, y) \in \mathbb{R}^2 : y > 0\}, -\frac{1}{cy^2}(\mathrm{d}x^2 + \mathrm{d}y^2)\Big).$$

We denote by $\mathbb{M}^2_c$ the 2-sphere $\mathbb{S}^2_c$ when $c > 0$ and the hyperbolic plane $\mathbb{H}^2_c$ when $c < 0$. Consider $\mathbb{M}^2_c \times \mathbb{R} = \{(x, t) : x \in \mathbb{M}^2_c \text{ and } t \in \mathbb{R}\}$ endowed with the product metric.

The classification result for minimal surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ was established in [1, Theorem 4.2]:

Daniel (2015): Let Σ be a minimal surface in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature K. Then

- either Σ is totally geodesic and $K = 0$ or $K = c$;
- or $c < 0$, $K = c$ and Σ is part of an associate surface of the parabolic generalized catenoid.

In this direction, we proved the classification result for CMC-surfaces in $\mathbb{M}^2_c \times \mathbb{R}$ when $H \neq 0$.

Theorem 1. *Let $H \neq 0$ and Σ be an H-CMC surface in $\mathbb{M}^2_c \times \mathbb{R}$ with constant intrinsic curvature K. Then*

- *either $K = 0$ and Σ is part of a vertical cylinder $\gamma \times \mathbb{R}$, where $\gamma \subset \mathbb{M}^2_c$ is a curve of geodesic curvature $2H$,*
- *or $c < 0$, $K = 4H^2 + c < 0$ and Σ is part of either an Abresch-Rosenberg-Leite surface (Figure 5) or an helicoidal surface of Sa Earp and Toubiana (Figure 6).*

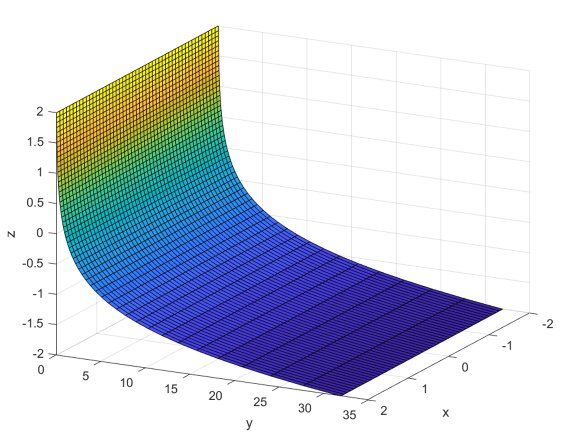


Figure 5: ARL-surface, $H = 1/4$.

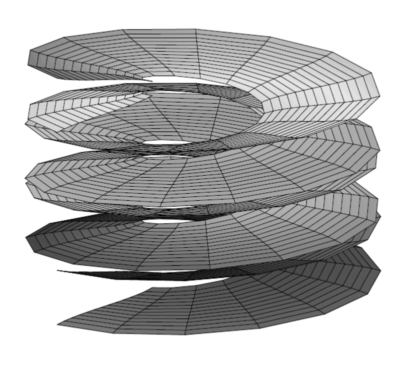


Figure 6: Helicoidal surface, $H = 1/4$. (Sa Earp and Toubiana [2, Figure 12])

Classification result in $\mathbb{E}(\kappa, \tau)$ with $\tau \neq 0$

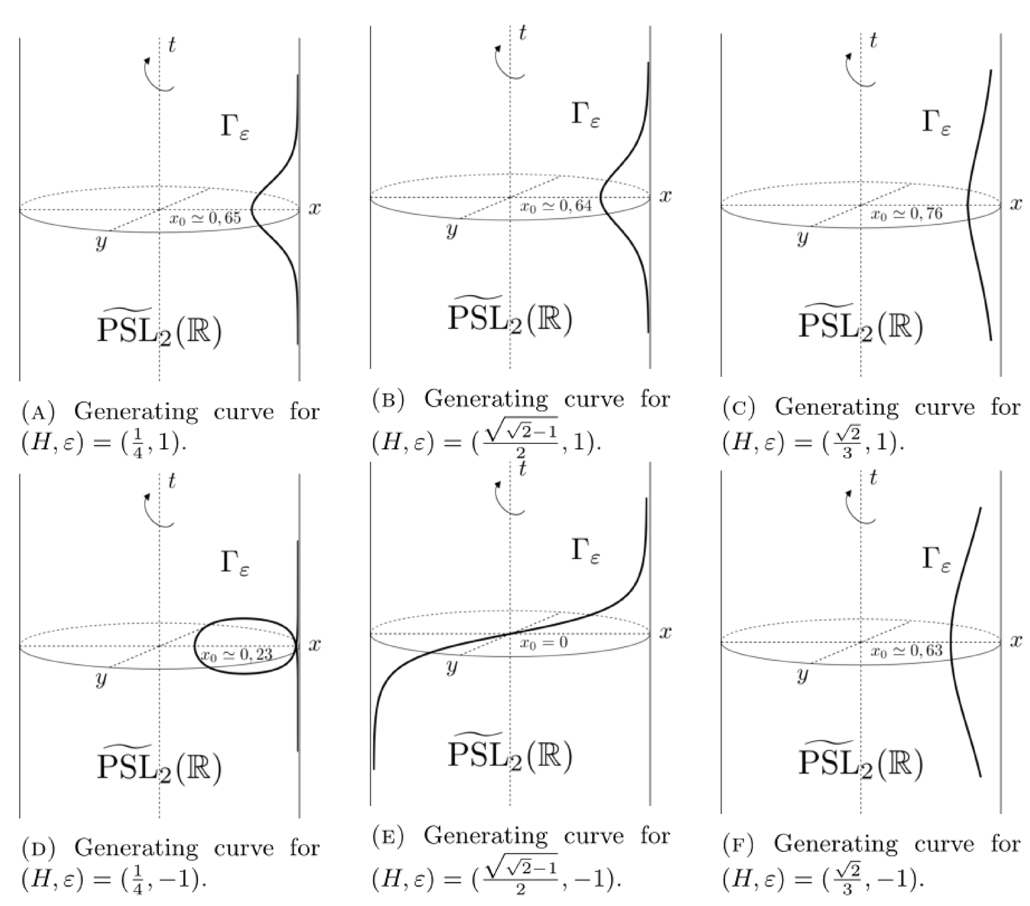
The manifold $\mathbb{E}(\kappa, \tau)$ is a 3-homogeneous space with a 4-dimensional isometry group; it is a Riemannian fibration of bundle curvature τ over $\mathbb{M}^2_\kappa$. These spaces are classified as follows:

- When $\tau = 0$, $\mathbb{E}(\kappa, 0)$ is the product space $\mathbb{M}^2_\kappa \times \mathbb{R}$,
- When $\tau \neq 0$ and $\kappa > 0$, $\mathbb{E}(\kappa, \tau)$ is a Berger sphere,
- When $\tau \neq 0$ and $\kappa = 0$, $\mathbb{E}(0, \tau)$ is the Heisenberg group with a left invariant metric.
- When $\tau \neq 0$ and $\kappa < 0$, $\mathbb{E}(\kappa, \tau)$ is the universal cover of $\mathrm{PSL}_2(\mathbb{R})$ with a left invariant metric, and we denote by $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$.

As an application of Theorem 1, we classify constant mean curvature surfaces in $\mathbb{E}(\kappa, \tau)$, for $\kappa - 4\tau^2 \neq 0$, with constant intrinsic curvature.

Theorem 2. *Let κ and τ be real numbers such that $\tau \neq 0$ and $\kappa - 4\tau^2 \neq 0$, and Σ be an H-CMC surface in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature K. Then*

- *either $K = 0$ and Σ is part of a vertical cylinder over a curve $\gamma \subset \mathbb{M}^2_\kappa$ with geodesic curvature $2H$;*
- *or $\kappa < 0$, $K = \kappa$ and Σ is part of Peñafiel minimal surface invariant by parabolic isometries;*
- *or $\kappa < 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of a generalized Abresch-Rosenberg-Leite surface;*
- *or $\kappa < 0$, $K = 4H^2 + \kappa < 0$ and Σ is part of one of twin helicoidal surfaces (Figure 7 and 8).*



(A) Generating curve for $(H, \varepsilon) = (\frac{1}{4}, 1)$. (B) Generating curve for $(H, \varepsilon) = (\frac{\sqrt{\sqrt{2}-1}}{2}, 1)$. (C) Generating curve for $(H, \varepsilon) = (\frac{\sqrt{2}}{3}, 1)$. (D) Generating curve for $(H, \varepsilon) = (\frac{1}{4}, -1)$. (E) Generating curve for $(H, \varepsilon) = (\frac{\sqrt{\sqrt{2}-1}}{2}, -1)$. (F) Generating curve for $(H, \varepsilon) = (\frac{\sqrt{2}}{3}, -1)$.

Figure 7: Complete generating curve of a complete screw motion surface in $\widetilde{\mathrm{PSL}}_2(\mathbb{R})$, with $\tau = 1/2$.

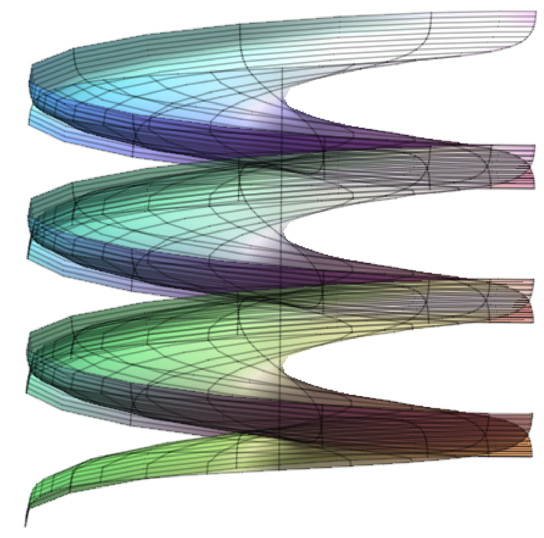


Figure 8: Complete screw motion surface with $(H, \varepsilon, \tau) = (\frac{\sqrt{\sqrt{2}-1}}{2}, -1, 1/2)$.

We remark that Theorems 1 and 2 together with [1, Theorem 4.2] give a complete classification of CMC surfaces in $\mathbb{E}(\kappa, \tau)$ with constant intrinsic curvature.

References

[1] Benoît Daniel, *Minimal isometric immersions into $\mathbb{S}^2 \times \mathbb{R}$ and $\mathbb{H}^2 \times \mathbb{R}$*, Indiana Univ. Math. J. **64** (2015), no. 5, 1425–1445. MR 3418447

[2] Ricardo Sa Earp and Eric Toubiana, *Screw motion surfaces in $\mathbb{H}^2 \times \mathbb{R}$ and $\mathbb{S}^2 \times \mathbb{R}$*, Illinois J. Math. **49** (2005), no. 4, 1323–1362. MR 2210365