Dynamics and stability of a cold magnetic plasma

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Motivation

Nuclear fusion is a way to produce energy from the nuclei of atoms. It requires that two hydrogen atoms meet: deuterium + tritium → helium.
This requires a very high temperature ⇝ Extremely hot gas, ionized:
Plasma: "soup" of various species of charged particles (ions, electrons...)

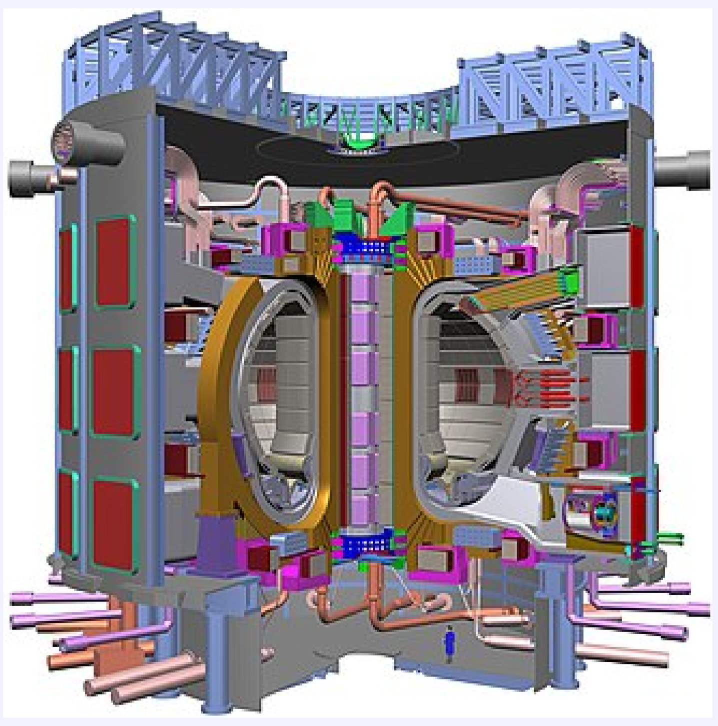

Figure: Tokamak

The tokamak is a toric chamber intended for the study of plasmas and the nuclear fusion. To produce energy, one has to

- fill the chamber with plasma,
- confine the plasma by an external magnetic field (limited volume, away from the material wall),
- heat the plasma: percussion of atoms

To reach the needed temperature ($\simeq 10^8$ K) required for nuclear fusion, one of plasma heating processes is by electromagnetic waves. We send them by antennas, which cover part of the walls of the chamber. The electromagnetic waves will be absorbed by the plasma and transfer their energy to the particles. Also, waves can be used for diagnostics.

Figure: Antenna

Physics problem: plasma-wave coupling.

Mathematical Model

Let $\Omega \subset \mathbb{R}^3$ be a bounded open set, which represents the plasma volume in the tokamak. We consider the following model for $(t,x) \in \mathbb{R}^+ \times \Omega$:

$$\frac{\partial J_1}{\partial t} = \varepsilon_0\, \omega_{p1}^2\, E + \Omega_{c1}\, J_1 \times b - \nu_1\, J_1\,; \quad (1)$$

$$\frac{\partial J_2}{\partial t} = \varepsilon_0\, \omega_{p2}^2\, E + \Omega_{c2}\, J_2 \times b - \nu_2\, J_2\,; \quad (2)$$

$$\frac{\partial E}{\partial t} = c^2 \,\mathbf{curl}\, B - \frac{1}{\varepsilon_0}\sum_s J_s\,; \quad (3)$$

$$\frac{\partial B}{\partial t} = -\,\mathbf{curl}\, E\,; \quad (4)$$

with: $b(x)$: direction of the external magnetic field ;
$\omega_{ps},\ \Omega_{cs},\ \nu_s$: depending on x ;
Initial conditions: $J_{s,0},\ E_0,\ B_0.$ (5)

Indices 1 and 2: particle species ion and electron.

Boundary condition: $\partial\Omega = \Gamma = \overline{\Gamma}_A \cup \overline{\Gamma}_P :=$ antenna + the rest

$$E \times n = 0 \quad \text{on } \Gamma_P, \quad \text{Perfectly conducting} \quad (6)$$

$$E \times n + c\, B_\top = g(t,x) \quad \text{on } \Gamma_A. \quad \text{Silver–Müller} \quad (7)$$

We suppose that $\Gamma_P \neq \emptyset$. Then, according to Γ_A, there are two cases:
Case 1: $\Gamma_A = \emptyset$, the condition (6) is imposed on the entire boundary.
Case 2: $\Gamma_A \neq \emptyset$.

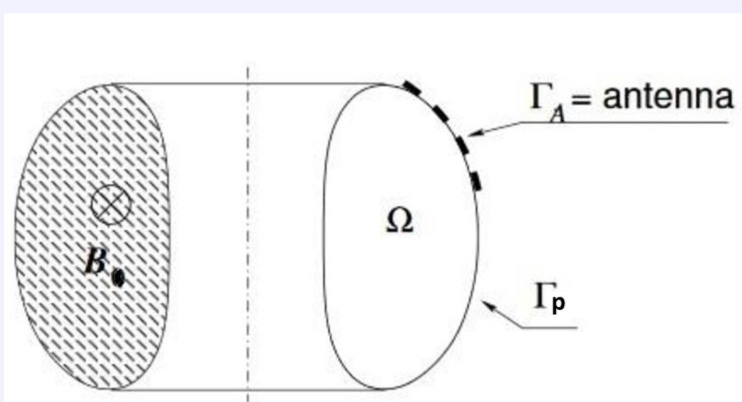

Figure: A cross-section of the domain Ω

Objective

Prove in both cases:
1. Well-posedness of the model (existence of solution)
2. Strong stability of the model (convergence of solution)
3. Exponential decay of the energy

Well-posedness of the model

We define the energy space $\mathbf{X} = [\mathbf{L}^2(\Omega)]^4$ endowed with the norm

$$\|(u_1,u_2,u_3,u_4)\|_{\mathbf{X}}^2 = \frac{1}{2\varepsilon_0}\sum_{s=1}^{2}\left\|\frac{u_s}{\omega_{ps}}\right\|^2 + \frac{\varepsilon_0}{2}\|u_3\|^2 + \frac{\varepsilon_0 c^2}{2}\|u_4\|^2.$$

The system (1)–(5) can be rewritten as an evolution equation :

$$\partial_t U + \mathbb{A} U = 0, \qquad U(0) = U_0 \quad (8)$$

where $\mathbb{A}$ is unbounded operator in $\mathbf{X}$. The domain of $\mathbb{A}$ is defined according to each case [3]:

Case 1: we consider $\mathbb{A}_1 := \mathbb{A}$ and

$$D(\mathbb{A}_1) = \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}_0(\mathbf{curl};\Omega) \times \mathbf{H}(\mathbf{curl};\Omega)$$

Case 2: we study first the case $g = 0$. We define $\mathbb{A}_2 := \mathbb{A}$ and

$$D(\mathbb{A}_2) = \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathcal{H}, \qquad \text{where}$$

$$\mathcal{H} = \{(v,w) \in \mathbf{H}_{0,\Gamma_P}(\mathbf{curl};\Omega) \times \mathbf{H}(\mathbf{curl};\Omega) : v \times n + c w_\top = 0 \text{ on } \Gamma_A\}.$$

⇝ The case $g \neq 0$ is a consequence of the case $g = 0$.

Theorem 1

The operator $-\mathbb{A}_\ell$ generates a C_0-semigroup of contractions $(T_\ell(t))_{t\geq 0}$ on the energy space $\mathbf{X}$ for $\ell = 1,2$. Therefore for all $U_0 \in \mathbf{X}$, the problem (8) has a unique solution $U_\ell \in C([0,\infty);\mathbf{X})$ given by $U_\ell(t) = T_\ell(t)U_0$, for all $t \geq 0$.

Decay of the energy

We define the energy $\mathcal{E} := \frac{1}{2}\|(J_1,J_2,E,B)\|_{\mathbf{X}}^2$ and its derivative

$$\frac{d}{dt}\mathcal{E} = -\frac{1}{2\varepsilon_0}\sum_s\left\|\frac{\sqrt{\nu_s}\,J_s}{\omega_{ps}}\right\|^2 - \int_{\Gamma_A}\left(c\,|B_\top|^2 - g\cdot B_\top\right)d\Gamma$$

$$\leq 0 \quad \text{if} \quad g = 0.$$

Stability results

The following results are related to the spectral analysis of the operator $\mathbb{A}_\ell$ on the imaginary axis [1] [2] .

- Strong stability

We introduce the Hilbert space $\widetilde{\mathbf{X}}_\ell = \mathbf{X} \cap \ker(\mathbb{A}_\ell)^\perp$. The space $\ker(\mathbb{A}_\ell)$ is the set of stationary solutions, then the model (1)-(4) is invariant in $\widetilde{\mathbf{X}}_\ell$.

Theorem 2

The problem (8) is strongly stable on the energy space $\widetilde{\mathbf{X}}_\ell$, for $\ell = 1,2$, in the sense that

$$\lim_{t\to+\infty}\|T_\ell(t)\widetilde{U}_0\|_{\widetilde{\mathbf{X}}_\ell} = 0, \quad \forall \widetilde{U}_0 \in \widetilde{\mathbf{X}}_\ell.$$

- Exponential decay of the energy

Let $\mathbf{K} := \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{L}^2(\Omega) \times \mathbf{H}_{0,\Gamma_P}(\mathrm{div}0;\Omega)$.

Theorem 3

The problem (8) is exponentially stable on $\widetilde{\mathbf{X}}_\ell$, for $\ell = 1,2$, i.e., there exist two constants $C,\omega > 0$ such that

$$\|T_\ell(t)\widetilde{U}_0\|_{\widetilde{\mathbf{X}}_\ell} \leq Ce^{-\omega t}\|\widetilde{U}_0\|_{\widetilde{\mathbf{X}}_\ell}, \quad \forall \widetilde{U}_0 \in \widetilde{\mathbf{X}}_\ell, \quad \forall t \geq 0.$$

Furthermore, under assumptions on B_0, there exist a constant $M > 0$ such that the solution satisfies

$$\|T_\ell(t)U_0 - (0,0,0,B_0)^\top\|_{\mathbf{K}} \leq Me^{-\omega t}\|U_0\|_{\mathbf{K}}, \ \forall U_0 \in \mathbf{K}, \ \forall t \geq 0.$$

Physical interpretation of the results

The energy of the wave is absorbed by the plasma and transformed into heat to heat the plasma. The decay of the energy is expected since there is absorption: collisions between particles (ν_s: fluid friction).

References

[1] W. Arendt & C. J. K Batty. Tauberian theorems and stability of one-parameter semigroups. *Trans. Amer. Math. Soc.*, 305(2) (1988) 837-852.

[2] J. Prüss. On the spectrum of C_0-semigroups. *Trans. Amer. Math. Soc.*, 284(2) (1984), 847-857.

[3] S. Labrunie & I. Zaafrani. Dynamique d'un plasma magnétique froid. 2017. Prépublication HAL