Lie Groups Representations and Dirac Operators

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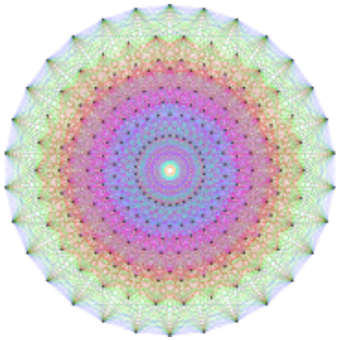
Lie groups

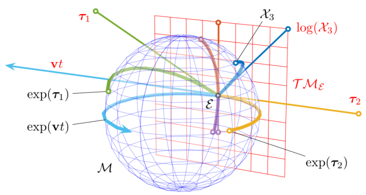
A Lie group G is a differential manifold equipped with a multiplicative group structure such that the group multiplication and the inversion are both smooth for the manifold structure. In other words, a Lie a group is a smooth group of symmetries.



Examples

- The Euclidean vector space $\mathbb{R}^n$ endowed with the usual structure of additive group is an abelian Lie group.
- The multiplicative group $GL(n,\mathbb{R})$ of invertible $n \times n$ real matrices, $n \geq 2$, is a non commutative Lie group.

To understand the structure of a Lie group G, we let it act on some vector space V. This action is called representation of G on V.
If W is a G-stable subspace of V , we say that W is a subrepresentation of G. A representation of G on a vector space V is said to be irreducible if the only subrepresentations W of V are the trivial ones.



Questions

Given a Lie group G

- Find a method to detect/parametrize all the irreducible representations.
- Given a non-irreducible representation of G, can we decompose it in irreducible subrepresentations?
- Find realizations for families of representations.

Dirac Operators

Let G be a Lie group and H a closed subgroup of G. Let E be a representation of H. If $T_{eH}G/H$ splits into two maximal dual isotropic subspaces V and V^* we define the space of spinors $S = \wedge V^*$ endowed with a Clifford multiplication. Let E be a finite dimensional representation of H such that $S \otimes E$ is a representation of H. We set

$C^\infty(G/H, \mathcal{S}\otimes\mathcal{E}) = \{f : G \to S\otimes E$ sm. s.t. $f(gh^{-1}) = h \cdot f(g), g \in G, h \in H\}$

$C^\infty(G/H, \mathcal{S} \otimes \mathcal{E})$ admits a canonical action of G by left translations and it can be seen as a representation of G.
We define the Dirac operator
$D : C^\infty(G/H, \mathcal{S}\otimes\mathcal{E}) \to C^\infty(G/H, \mathcal{S}\otimes\mathcal{E})$
with the property D^2 =laplacian. This operator commutes with the action of G and thus its kernel is a representation of G.

$$\left(\beta mc^2 + \sum_{k=1}^{3} \alpha_k p_k c\right) \psi(\mathbf{x},t) = i\hbar \frac{\partial \psi(\mathbf{x},t)}{\partial t}$$

Figure 1: P. Dirac with his famous equation

In representation theory we have two important families of representations: **principal** and **discrete** series representations.

Important Results

- **Harish-Chandra** Classif. of discrete series representations (60's)
- **Parthasarathy** Realization of some discrete series representations using Dirac op. (72')
- **Atiyah-Schmid** Realization of all discrete series representations using Dirac op. (77')
- **Mehdi-Zierau** Some principal series representations live in kernels of Dirac op. (06',14')

My results

- We found an operator whose kernel, under certain conditions, contains the kernel of D.
- Examples of many different cases
- Relation between this operator and Dirac operators

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