The Topology of Plane Curves

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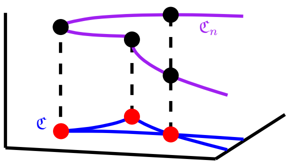
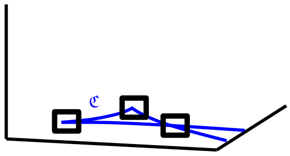
Summary

Input: curve $\mathfrak{C}_n$ in $\mathbb{R}^n$ given by $f_1(x) = \cdots = f_{n-1}(x) = 0$, $x \in \mathbb{R}^n$ and f_i are infinitely differentiable

Goal: Drawing with certified topology the projection $\mathfrak{C}$ of $\mathfrak{C}_n$ in $\mathbb{R}^2$

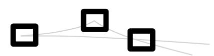
Contribution: A certified numerical algorithm to isolate the singular points* of $\mathfrak{C}$ (main step for drawing that curve)





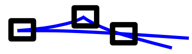
*Singular point = point where the curve is not smooth

Usual way to compute the topology of 2D curves



1) Isolate the singular points

2) Draw the curve around each singular point



3) Connect the singular points properly

Our approach and its difficulties

Our approach: Isolating the singular points using certified numerical methods: Interval Newton

Requirements:

1. Square: the number of variables is equal to the number of equations
2. Regular: the Jacobian matrix of this system is invertible

Difficulties: If the curve is given by the equation $f(x, y) = 0$, then the usual system that represents the singular points is

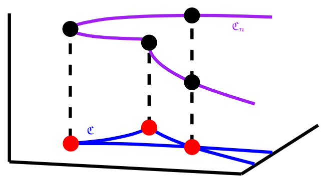
$$\begin{cases} f(x,y) = 0 \\ \frac{\partial f}{\partial x}(x,y) = 0 \\ \frac{\partial f}{\partial y}(x,y) = 0 \end{cases}$$

⚠ Interval Newton requirements are not satisfied

We design a system in $\mathbb{R}^{2n-1}$: (1) encoding the singularity of 2D curve
(2) satisfying interval Newton requirements

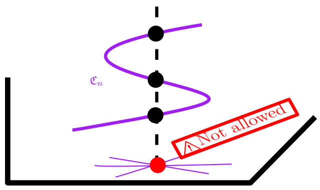
Our Assumptions

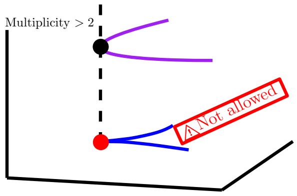
We consider plane curves that are projections of smooth curves in $\mathbb{R}^n$ such that:



1) Singular points are only self-intersections and ordinary cusps

2) They lift to finite sets in $\mathfrak{C}_n$





3) No more than 2 points in $\mathfrak{C}_n$ (counted with multiplicity) have the same projection

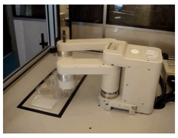
Main results

1. Almost all curves in $\mathbb{R}^n$ satisfy these assumptions
2. Under these assumptions, there exists a square regular system in $\mathbb{R}^{2n-1}$ that characterizes the singularities of the 2D curve
3. Semi-algorithm for checking our assumptions: terminates iff the assumptions are satisfied

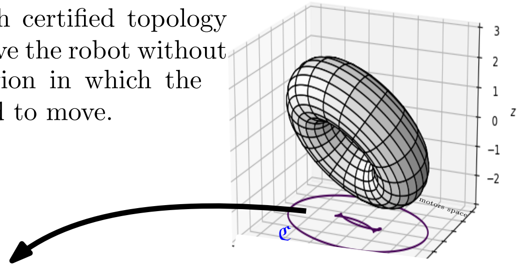
Future work

- Proving that a generic silhouette curve satisfies our assumptions
- Implementing the algorithms
- Computing the topology of generic singular surfaces in $\mathbb{R}^n$

Applications



2-degrees-of-freedom robots: Drawing $\mathfrak{C}$ with certified topology allows us to move the robot without leaving the region in which the robot is allowed to move.



projection of its silhouette in motors space ($\mathbb{R}^2$)

geometric modeling of the robot's motion with 2 degrees of freedom

References

George Krait, Sylvain Lazard, Guillaume Moroz, Marc Pouget. Numerical Algorithm for the Topology of Singular Plane Curves. EuroCG'19 - 35th European Workshop on Computational Geometry, Utrecht, Netherlands.