A Frequency Domain Method for Scattering Problems with Moving Boundaries

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Motivation

Radar detection of moving scatterers.

Multiharmonic numerical method for solving wave scattering problems with moving boundaries, where scatterer assumed to move smoothly around equilibrium position. Starting with one dimensional toy model, we extend the applicability to higher dimensions and more general geometries.

Investigate problem in frequency domain, derive how frequency components of the solution must be coupled, compute only ones with significative contributions by solving coupled systems of Helmholtz- type equations.

Provides alternative method to FFT brute force, which combines both accuracy and efficiency.

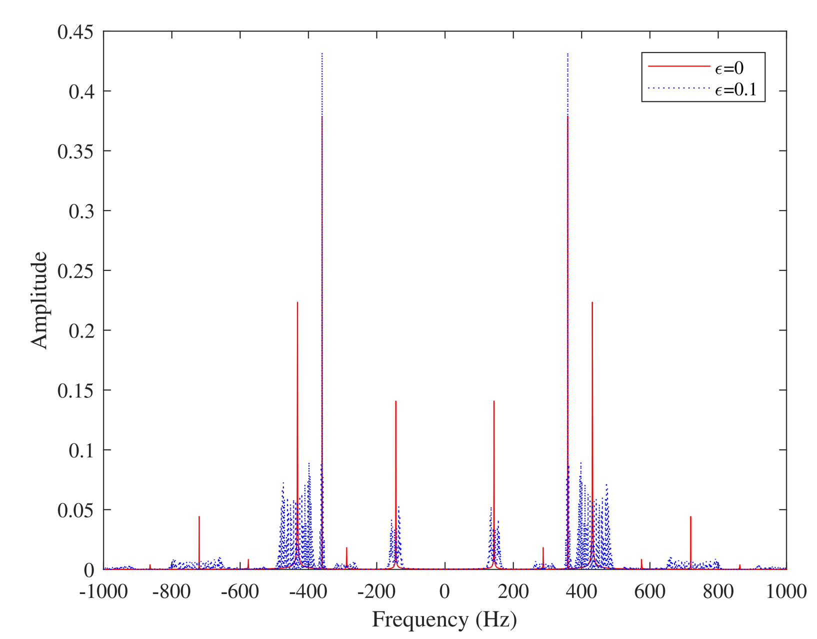

Fig. 1: Computation of the spectrums of the signal without boundary motion (red) and with boundary motion (blue) of amplitude 10cm and frequency $\nu_\ell = 3$ Hz. The emitted wave velocity is $c = 300$ m.s^{-1}, with frequency $\nu_I = 360$ Hz.

Model

Consider bounded spatial domain $\Omega(t) \subset \mathbb{R}^n$ with boundary $\partial\Omega(t) = \Gamma_0 \cup \Gamma(t)$ where $\Gamma(t)$ is an open moving boundary described by time-dependent smooth and bounded field $\boldsymbol{\ell}(t)$. For all $\mathbf{x} \in \Omega(t)$ and $t > 0$, the unknown total wave field $u(\mathbf{x}, t)$ is solution to

$$\partial_{tt}u - c^2\Delta u = 0. \tag{1}$$

Consider single source $\mathbf{x}_s \in \bar\Omega(t)\backslash\Gamma(t)$ such that $u(\mathbf{x}_s, t) = A\sin(2\pi\nu_I t)$ and $u(\mathbf{x}, t) = 0$ on $\partial\Omega(t) \setminus \{\mathbf{x}_s\}$.

Other boundary conditions are investigated in the work.

For well-posedness: $u(\mathbf{x}, 0) = 0$ and $\partial_t u(\mathbf{x}, 0) = 0$.

Reformulation in fixed domain

Detail of the method in case of one dimensional domain $\Omega(t) := \{x : x \in]0, \ell(t)[\}$ with boundary conditions $u(\ell(t), t) = 0$ and $u(0, t) = A\sin(2\pi\nu_I t)$.

First map $\Omega(t)$ with variable size to $\widetilde\Omega =]0, L[$. As a result, initial scattering problem writes: find solution $\tilde u(\tilde x, t) = u(x, t)$ in $\widetilde\Omega \times \mathbb{R}_*^+$ of:

$$\partial_{tt}\tilde u - c^2\left[\left(\frac{\partial\tilde x}{\partial x}\right)^2\partial_{\tilde x\tilde x}\tilde u + \frac{\partial^2\tilde x}{\partial x^2}\partial_{\tilde x}\tilde u\right] + \left(\frac{\partial\tilde x}{\partial t}\right)^2\partial_{\tilde x\tilde x}\tilde u + 2\frac{\partial\tilde x}{\partial t}\partial_{\tilde x t}\tilde u + \frac{\partial^2\tilde x}{\partial t^2}\partial_{\tilde x}\tilde u = 0, \tag{2}$$

with $\tilde u(0, t) = A\sin(2\pi\nu_I t)$, $\tilde u(L, t) = 0$, $\tilde u(\tilde x, 0) = 0$ and

$$\partial_t\tilde u(\tilde x, 0) = -\left.\frac{\partial\tilde x}{\partial t}\right|_{t=0}\partial_{\tilde x}\tilde u(\tilde x, 0) = 0.$$

Frequency domain method

Time Fourier transform of Eq. (2) writes

$$\partial_{\tilde x\tilde x}\widehat{\tilde u} * \mathcal{F}\left(\left(\frac{\partial\tilde x}{\partial t}\right)^2 - c^2\left(\frac{\partial\tilde x}{\partial x}\right)^2\right) + \partial_{\tilde x}\widehat{\tilde u} * \mathcal{F}\left(\frac{\partial^2\tilde x}{\partial t^2} - c^2\frac{\partial^2\tilde x}{\partial x^2}\right) + 4i\pi\xi\partial_{\tilde x}\widehat{\tilde u} * \mathcal{F}\left(\frac{\partial\tilde x}{\partial t}\right) - 4\pi^2\xi^2\widehat{\tilde u} = 0 \tag{3}$$

Assuming periodic and $\mathcal{C}^\infty$ boundary motion, Fourier transform of fixed domain space variable is approximated by

$$\widehat{\tilde x}(x, \xi) \simeq \sum_{n=-N}^{N} c_{\tilde x,n}(x)\delta_{n\nu_\ell}.$$

Furthermore, shown that unknown field $\tilde u$ writes as sum of Dirac combs of the form

$$\widehat{u}_{\nu_I} = \sum_{j\in\mathbb{Z}} \tilde a_j(\tilde x)\delta_{\nu_I + j\nu_\ell}.$$

Hence, injecting the two last in Eq. (3), obtain the following differential-algebraic coupled system of Helmholtz-like equations:

$$\tilde\kappa_j^2\tilde a_j + \sum_{-2N}^{2N}\Pi^1_{\tilde x,n}\partial_{\tilde x\tilde x}\tilde a_{j-n} + \sum_{-N}^{N}\partial_{\tilde x}\tilde a_{j-n}\left[\Pi^2_{\tilde x,n} + 2k_\ell\tilde\kappa_{j-n}\Pi^3_{\tilde x,n}\right] = 0 \tag{4}$$

with $\Pi^1_{\tilde x,n} := \sum_{k=-|n|}^{|n|}\kappa_\ell^2 k(n-k)c_{\tilde x,k}c_{\tilde x,n-k} + \frac{\partial c_{\tilde x,k}}{\partial x}\frac{\partial c_{\tilde x,n-k}}{\partial x}$, $\Pi^2_{\tilde x,n} := \kappa_\ell^2 n^2 c_{\tilde x,n} + \frac{\partial^2 c_{\tilde x,n}}{\partial x^2}$, $\Pi^3_{\tilde x,n} := nc_{\tilde x,n}$ and $\kappa_\ell = 2\pi\nu_\ell/c$. Shown that inner coefficients $\tilde a_j$ are of exponential decay with respect to $|j|$. Allows to define frequency boundary conditions to previous system.

Below, comparison between computation of spectrum *via* our method (frequency domain method, resolution by means of 1 order FEM) and standard space time + FFT brute force resolution, in case of sine motion $\ell(t)$ of amplitude ϵ and frequency ω_ℓ.

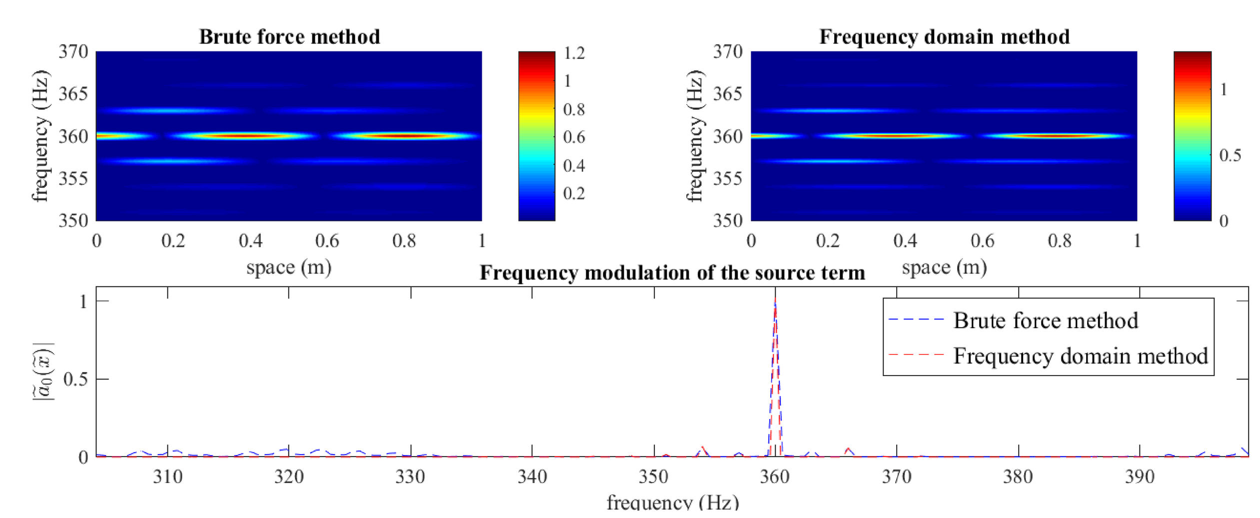

Fig. 2: $c = 300$ m.s^{-1}, $\nu_I = 360$ Hz, $\nu_\ell = 3$ Hz, $\epsilon = 0.1$ m, $\tilde x = 0.45$ m.

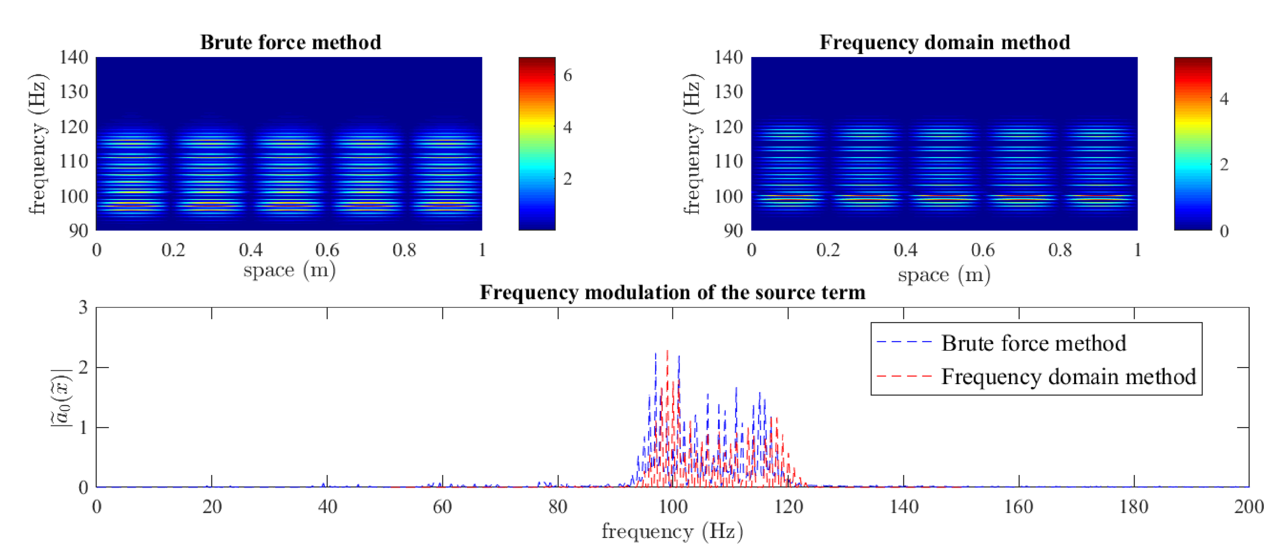

Fig. 3: $c = 43$ m.s^{-1}, $\nu_I = 101$ Hz, $\nu_\ell = 1$ Hz, $\epsilon = 0.1$ m, $\tilde x = 0.18$ m.

Impact of the boundary motion on the frequency modulation

Description of domain deformation provided by knowledge of Fourier coefficients of $\tilde x$.

Numerically obtained via FFT algorithm, need to be computed only once for given boundary motion.

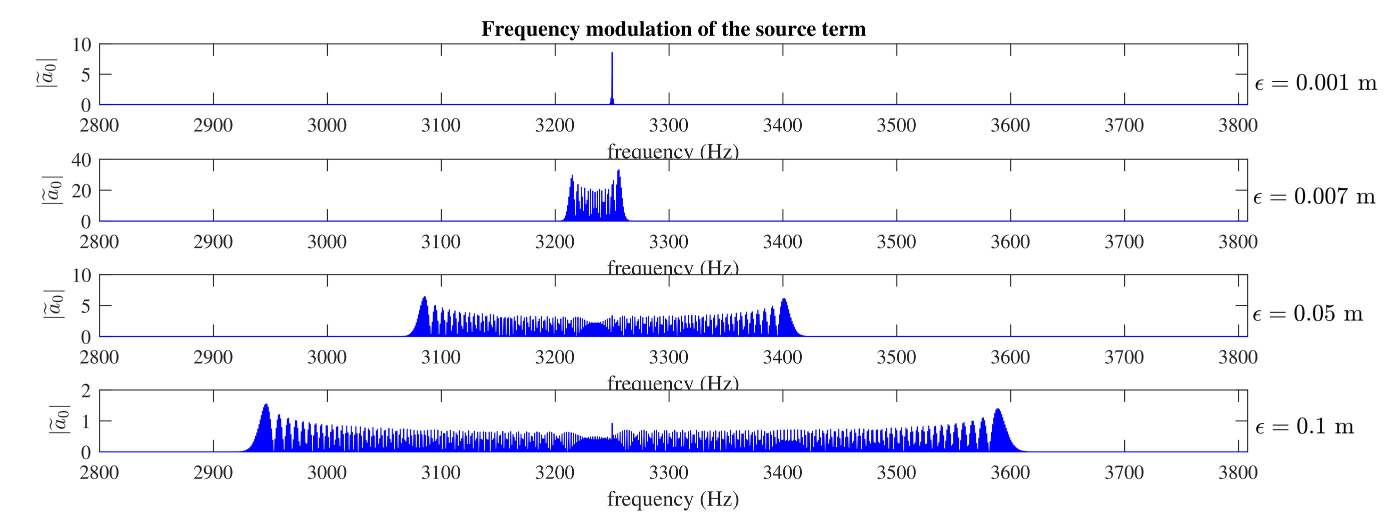

Fig. 4: $c = 1080$ m.s^{-1}, $\nu_I = 3250$ Hz, $\nu_\ell = 1$ Hz, $\tilde x = 0.55$ m.

Conclusion

As a frequency domain method, two main advantages:

- allows to define the optimal frequency grid over which the coupled system has to be solved, frequency step corresponds to fundamental frequency of modulating signal (*i.e.* boundary motion).
- provides simple criterion to truncate the system in order to keep minimal number of frequencies that are not negligible. Hence, optimize size of coupled system to solve without loss of accuracy.

As a consequence, the computationnal costs only depends on the parameter η and the spatial discretization.

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