WELL-POSEDNESS OF THE DIRICHLET PROBLEM FOR TWO TANGENT SPHERES

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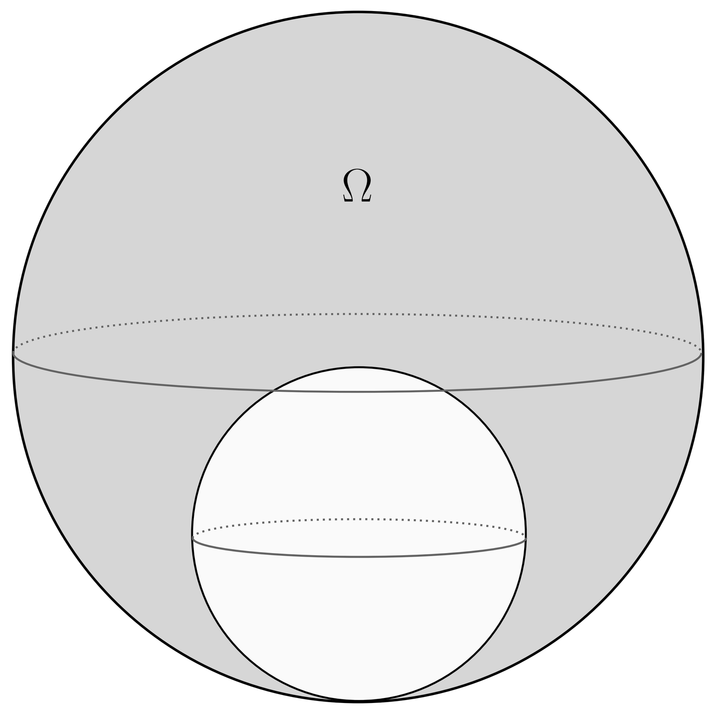
Setting of the problem

Consider the domain Ω given by the inclusion of a three-dimensional ball inside annother:



The contact point is a *singularity*.

The *Dirichlet problem* is stated as follows: find the solution u to

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{D}$$

for a given data $f \in C^\infty(\Omega)$. Here $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$.

Definition

Let $\mathcal{H}, \mathcal{H}'$ be normed vector spaces containing $C^\infty(\Omega)$. The problem (D) is *well-posed* from $\mathcal{H}$ to $\mathcal{H}'$ if, for any $f \in \mathcal{H}'$, there is a unique $u \in \mathcal{H}$ solving (D), together with a stability estimate

$$\|u\| \leq C\|f\|.$$

The problem (D) is *essentially well-posed* if it is well-posed up to some finite dimensional vector spaces.

Question: can we find function spaces $\mathcal{H}, \mathcal{H}'$ such that the problem (D) is well-posed from $\mathcal{H}$ to $\mathcal{H}'$?

Context

Classical results

On a bounded, *smooth* domain $\Omega \subset \mathbb{R}^n$, the Dirichlet problem is known to be well-posed between Sobolev spaces. For $m \in \mathbb{N}$, define

$$H^m(\Omega) = \{v : \Omega \to \mathbb{C} \mid \partial^\alpha v \in L^2(\Omega), \text{for all } |\alpha| \leq m\}.$$

Then (D) is essentially well-posed from $H^2(\Omega)$ to $L^2(\Omega)$.

Loss of regularity and convergence of numerical schemes

An important feature of the presence of singularities is the loss of regularity for the solutions. If $f \in L^2(\Omega)$ and u is a solution to (D), then we don't necessarily have $u \in H^2(\Omega)$.
A consequence is that the usual numerical methods to approximate u (e.g. finite element methods) have a sub-optimal convergence.

Two dimensions

In two dimensions, i.e. for two tangent disks in $\mathbb{R}^2$, the problem is essentially well-posed between *weighted* Sobolev spaces. For $m \in \mathbb{N}$ and $a, b \in \mathbb{R}$, define

$$\mathcal{K}^m_{a,b}(\Omega) = \{v : \Omega \to \mathbb{C} \mid r^{2(b-|\alpha|)}\partial^\alpha(e^{a/r}v) \in L^2(\Omega), \text{for all } |\alpha| \leq m\},$$

with r the distance to the singular point.

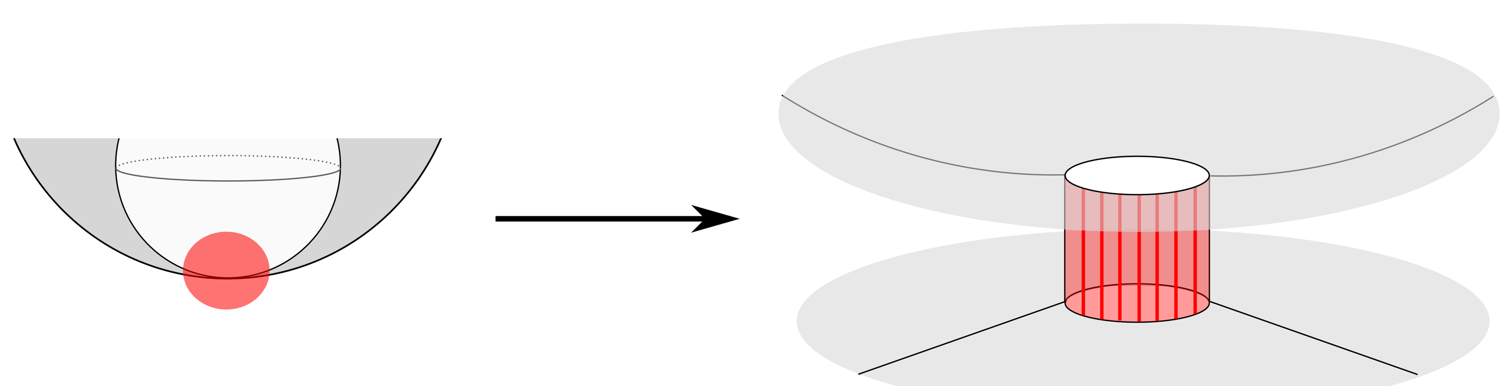
Theorem (Schulze-Sternin-Shatalov 98, Kozlov-Maz′ya-Rossmann 97): Let ρ_s be the radius of the small disk and ρ_b the radius of the big one. Set

$$A := \frac{1}{2}\left(\frac{1}{\rho_s} - \frac{1}{\rho_b}\right),$$

Then for any $|a| < A$, the problem (D) is essentially well-posed from $\mathcal{K}^2_{a,3}(\Omega)$ to $\mathcal{K}^0_{a,1}(\Omega)$.

Blowing up the singularity

A typical approach is to perform a change of variable near the singularity to obtain a blown-up space $\Sigma(\Omega)$:



The singularity is replaced by a cylinder. **Vertical lines** are highlighted.

Limit problems

The blown-up space $\Sigma(\Omega)$ is used to construct an algebra of operators with a prescribed behavior near the singularity.
A careful study of the representations of this algebra shows that the well-posedness of the Dirichlet problem (D) is controlled by a family of *limit problems* on the **vertical lines** on the blown-up space.

Theorem (C. 2019): For any $a, b \in \mathbb{R}$, the Dirichlet problem (D) is essentially well-posed from $\mathcal{K}^2_{a,b+2}(\Omega)$ to $\mathcal{K}^2_{a,b}(\Omega)$ if, and only if, all the limit problems

$$\begin{cases} v'' - (i\xi + a)^2 + \eta^2 = g \\ v(0) = v(1) = 0. \end{cases} \tag{$L_{\xi,\eta}$}$$

on $[0,1]$, indexed by $(\xi,\eta) \in \mathbb{R}^2$, are well-posed from $H^2(0,1)$ to $L^2(0,1)$.

Conclusions

Results

The above theorem may be used to obtain a well-posedness result for the 3D problem:

Theorem (C. 2019): There is an $A > 0$ such that, for all $|a| < A$ and $b \in \mathbb{R}$, the problem (D) is essentially well-posed from $\mathcal{K}^2_{a,b+2}(\Omega)$ to $\mathcal{K}^0_{a,b}(\Omega)$. Moreover, we have

$$A = \frac{1}{2}\left(\frac{1}{\rho_s} - \frac{1}{\rho_b}\right),$$

with ρ_s the radius of the small ball and ρ_b the radius of the big one.

The result is quite general. It applies to:

1. **More general geometries:** for instance cusps given by $r \mapsto r^\gamma$, for any $\gamma > 1$,
2. **Other boundary conditions** such as mixed Dirichlet/Neumann.

Further work

Problem: the pure Neumann problem, given by

$$\begin{cases} \Delta u = f & \text{on } \Omega \\ \partial_\nu u = 0 & \text{on } \partial\Omega, \end{cases} \tag{N}$$

is equally interesting, but much harder: in that case, the limit problems ($L_{\xi,\eta}$) are *never* all simultaneously well-posed, no matter the value of the weight a.
A current goal is to use operator algebraic methods to refine the limit problems to subspaces of our weighted spaces. This should yield a well-posedness result on a sum of weighted spaces, with different weights on each component.