ORIGINAL ARTICLE

Journal Section

Set input-to-state stability for nonlinear time delay systems with disturbances

Pallavi Sinha[1*] | Irinel-Constantin Morarescu[2] | Sukumar Srikant[1]

[1]Systems and Control Engineering, Indian Institute of Technology Bombay, Mumbai, India

[2]Université de Lorraine, CNRS, CRAN, F-54000, Nancy, France and also associated with the Automation Department, Technical University of Cluj-Napoca, 400114 Cluj-Napoca, Romania

Correspondence
Systems and Control Engineering, Indian Institute of Technology Bombay, Powai, Mumbai-400076, India
Email: pallavisinha33@gmail.com

Present address
Systems and Control Engineering, Indian Institute of Technology Bombay, Powai, Mumbai-400076, India

Funding information
This work was supported by CEFIPRA under the grant 6001-1. The work of I.-C. Morărescu was also supported by ANR under the grant NICETWEET ANR-20-CE48-0009 and by project DECIDE funded under the PNRR I8 scheme by the Romanian Ministry of Research..

We propose new results on input-to-state stability (ISS) subject to time delays in the input for compact, invariant sets that contain the origin. First, using nonlinear small-gain theory, we prove a Razumikhin-type theorem that ensures ISS for sets in the context of functional differential equations (FDEs) with delayed disturbances. Next we demonstrate that this theorem can be used to ensure set ISS for nonlinear systems with input delays and disturbances. In comparison to the existing research on robustness of set ISS with respect to time delays at the input, our results are more general, retain the ISS gain, and do not impose constraints on time delayed states. The advantages of the method are illustrated through two case-studies on set-stability for classes of nonlinear oscillators of practical interest.

KEYWORDS
Set-stability; Time delay systems; Nonlinear control.

1 | INTRODUCTION

Time delays are ubiquitous in modern control applications. These delays can be attributed to the presence of communication networks between sensors, plants, and controllers. Delayed states can also be found in a variety of engineering

[*]Corresponding author.

systems that don't involve communication networks (arising from discretization, quantization, and processing limitations), such as advanced defense systems, rolling mills, automotive engines, chemical and biological reactors [1, 2, 3].

It is well established that time delay systems (TDS) are a class of infinite-dimensional FDEs. TDS stability analysis methods can be structured into frequency-domain [4], [5], and time-domain ones. The time-domain methods extend the Lyapunov approach and are based either on Lyapunov-Krasovskii functionals [6] or on the Lyapunov-Razumikhin function [7]. Detailed descriptions of these methods and their applications can be found in [8], [9]. Stability analysis of linear TDS based on linear matrix inequalities (LMI) and Lyapunov-Krasovskii functionals can be found in [10], [11] and references therein. While the Lyapunov-Krasovskii technique is a logical extension of the Lyapunov approach, the choice of a Lyapunov-Krasovskii functional remains a complicated problem. In many nonlinear scenarios, it is easier to determine a Lyapunov-Razumikhin function (see, for instance, [12] and [13]). The rest of this article will, therefore, focus on using the Lyapunov-Razumikhin technique to develop robustness results under time delay.

The concept of ISS was first put forth as a theoretical foundation for the investigation of robustness to external disturbances in [14], and many researchers have since used it in a variety of contexts, from robust control [15] to highly nonlinear small-gain theorems [16]. For FDEs with disturbances, ISS is guaranteed by Razumikhin-type theorems, which were proposed in [17]. The author in [17] also demonstrated that ISS is robust to input time delays.

The ISS property, along with fundamental stability notions, was initially defined with regard to a specific equilibrium state of interest. However, convergence to sets has become critical in modern control applications, and therefore *set ISS* has become prevalent. Several robust stability problems in areas such as incremental stability, leader-follower consensus, consensus tracking, synchronization of oscillators are covered under a more comprehensive framework of ISS with respect to a closed set in [18], compact set in [19, 20, 21, 22]. Seminal articles on set ISS for closed sets include [23], [24] where various notions of set-ISS and Lyapunov characterizations of the same are introduced. Specializations of set ISS notions for compact invariant sets are shown in [25]. The study of the ISS property with respect to a compact set that comprises the origin is a common application of IOS, as stated in [26]. The distance between the states and the set is the output when applying IOS in these instances. However, in order to investigate the robustness of set ISS to time delay, we concentrate on set ISS rather than IOS notions.

The focus of the current article is the robustness of set ISS to input time delays, which is not covered in the aforementioned literature. Time delays naturally lead to FDE, and the various notions of stability of invariant sets for FDEs are introduced in [27]. The authors in [28] provide a Razumikhin-type characterization of set ISS for FDEs in their work. To the best of the author's knowledge, the only rigorous result guaranteeing set ISS for a class of nonlinear systems in the presence of time delay at the input is [28]. However, both the Razumikhin-type theorem and robustness of set ISS to time delay results have conditions (such as the additive nature of the time delay and a constraint on the additive delay term) restricting their general applicability. Our focus primarily is to improve both these results and significantly expand the scope of application of the Razumikhin based robustness results on set ISS.

The article's novel contributions in comparison to the body of literature are listed below.

1. We present a novel variant of the Razumikhin theorem for set ISS of FDEs along with proof.
2. We propose an innovative result on the robustness of set ISS to delays using the new variant of the Razumikhin theorem developed here.
3. It is demonstrated that the robustness results proposed here are more generally applicable and relax some of the conditions required in [28]. Unlike [28] our results are also applicable to non-additive time delays at the input.

The remainder of the paper is set out as follows. The relevant notations, definitions, and theorems are provided in Section 2. The proofs of the set-versions of the Razumikhin-type theorem, ISS, and the major finding of the paper on the robustness of the set ISS to time delays at the input, are all illustrated in Section 3. We also compare the findings of our robustness study to those in the article [28], which is well illustrated using examples in Section 4. The future road map is presented in Section 5 of our study.

2 | PRELIMINARIES

This section aims to introduce some results that help us prove the robustness of set ISS to time delays at the input.

A mapping $\gamma : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to belong to class-$\mathcal{G}$ if it is continuous, zero at zero and non-decreasing; it is of class-$\mathcal{K}$ if it is of class-$\mathcal{G}$ and strictly increasing; it is of class-$\mathcal{K}_\infty$ if it is of class-$\mathcal{K}$ and unbounded. A continuous function $\beta : \mathbb{R}_{\geqslant 0} \times \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\geqslant 0}$ is said to be of class-$\mathcal{KL}$ if $\beta(\cdot, t)$ is of class-$\mathcal{K}$ for each $t \geqslant 0$ and $\beta(s, \cdot)$ is monotonically decreasing to zero for each $s > 0$. Throughout the paper, the following standard notations are used: set $C = C([-\delta, 0], \mathbb{R}^n)$ represent continuous functions mapping the interval $[-\delta, 0]$ into $\mathbb{R}^n$, with the topology of uniform convergence and some $\delta > 0$. At the time t, the vector is $x(t)$, and for the sake of simplicity, when no confusion can arise, we will denote it just by x. The state history function x_t corresponds to the past time-interval $[t - \delta, t]$, where $x_t(\theta) = x(t + \theta)$, where $\theta \in [-\delta, 0]$.

2.1 | Set-stability for systems without time delay

We start by recalling some standard definitions/notations for non delayed nonlinear systems. The readers familiar with these notions can skip this section and go directly to the next one that addresses set-stability for time delay systems. These definitions make the article self-content and will be important in establishing our main findings regarding the set ISS's robustness to systems with time delays in feedback.

Consider nonlinear systems of the following form

$$\dot{x}(t) = f(x(t), u(t)), \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $\forall t \geqslant 0$, and the map $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz on bounded sets [29]. The input u is measurable, locally essentially bounded function of the type $u : \mathbb{R}_{\geqslant 0} \to \mathbb{R}^m$. The space of such functions is denoted by $\mathcal{L}_\infty^m$ with the norm $\|u_{[t_0,\infty)}\| \triangleq \operatorname{ess\,sup}\{u(t) : t \geqslant t_0 \geqslant 0\}$. The space $\mathcal{L}_\infty^m$ consists of functions which are bounded almost everywhere. We use $\|u\| = |u_{[0,\infty)}|$ and let $\|u_{[0,t]}\|$ be the signal norm over the truncated interval $[0, t]$. For each initial $x_0 = x(0) \in \mathbb{R}^n$ and each $u \in \mathcal{L}_\infty^m$, let $x(t, x_0, u)$ denote the solution of (1) at time t. If there is no ambiguity from the context, the solution is simply written as $x(t)$.

For a nonempty closed set $A \in \mathbb{R}^n$ we have, $|y|_A \triangleq \inf\{d(y, z) : z \in A\}$ where $d(y, z) = |y - z|$.

Definition 1 (Finite escape time[25]) *The system* (1) *is finite escape-time detectable through* $|\cdot|_A$ *if a solution's maximal interval of existence is bounded, i.e.,* $x(t)$ *is defined exclusively on* $[0, T)$ *with* T *finite, then* $\lim_{t \to T} |x(t)|_A = \infty$.

Definition 2 (Invariant set w.r.t ODE [25]) *For the associated "zero-input" system*

$$\dot{x}(t) = f(x(t), 0),$$

set A *is said to be a* 0*-invariant set if it holds that for any* $x_0 \in A$, $x(t, x_0, 0) \in A$ *for all* $t \geqslant 0$.

The rest of the document will refer to the 0-invariant set as the invariant set.

Definition 3 (Set ISS using ISS-Lyapunov function w.r.t. sets [30]) *A smooth ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$ for* (1) *with respect to the closed set A is a function that satisfies*

1. *there exists $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for any $x \in \mathbb{R}^n$,*

$$\alpha_1(|x|_A) \leqslant V(x) \leqslant \alpha_2(|x|_A).$$

2. *there exist $\alpha_3 \in \mathcal{K}$ and $\chi \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$|x|_A \geqslant \chi(|u|) \implies \frac{\partial V(x)}{\partial x} f(x,u) \leqslant -\alpha_3(|x|_A). \tag{2}$$

The ISS gain in this scenario is given by $\alpha_1^{-1} \circ \alpha_2 \circ \chi$. For compact set A, an equivalent representation of (2) is:

3. *there exists $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ such that for all $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$,*

$$\frac{\partial V(x)}{\partial x} f(x,u) \leqslant -\alpha_3(|x|_A) + \alpha_4(|u|).$$

Assume the closed set A is invariant for (1)*. The system* (1) *is ISS with respect to A if it admits a smooth ISS-Lyapunov function with respect to A (see [31]) and is finite escape-time detectable through $|\cdot|_A$.*

In order to take into account a system in which we have control inputs and disturbances, we can formulate the following ISS property:

$$\dot{x}(t) = \phi(x(t), u_1(t), u_2(t)) \tag{3}$$

where $x(t) \in \mathbb{R}^n$ denotes the vector of state variables, $u_1(t) \in \mathbb{R}^p$ is the controlled input , $u_2(t) \in \mathbb{R}^m$ denote the vector of uncontrolled disturbances, and ϕ is Lipschitz on bounded sets on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$.

Definition 4 (Set ISS for two input ODE) *System (3) is set ISS if there exists $\gamma_{u_1}, \gamma_{u_2} \in \mathcal{K}$, $\beta' \in \mathcal{KL}$ such that for each $x(0) \in \mathbb{R}^n$ and for each pair of measurable, essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (3) with $x(0) = x_0$ exists for all $t \geq 0$ and the following inequality holds:*

$$|x(t)|_A \leq \beta'(|x_0|)_A, t) + \gamma_{u_1}(||u_1||) + \gamma_{u_2}(||u_2||). \tag{4}$$

Lemma 1 (Set ISS for two input ODE using ISS-Lyapunov function w.r.t sets) *The system in* (3) *is said to be set ISS with gain $(\gamma_{u_1}, \gamma_{u_2})$ with two functions $\gamma_{u_1} \in \mathcal{K}$ and $\gamma_{u_2} \in \mathcal{K}$, if there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, three functions $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and two functions $\tilde{\gamma}_{u_1}, \tilde{\gamma}_{u_2} \in \mathcal{K}$ with $\gamma_{u_1}(e) = \alpha_1^{-1}(\alpha_2(\tilde{\gamma}_{u_1}(e)))$, $\gamma_{u_2}(e) = \alpha_1^{-1}(\alpha_2(\tilde{\gamma}_{u_2}(e)))$ such that for all $x(t) \in \mathbb{R}^n$, all $u_1 \in \mathbb{R}^p$, all $u_2 \in \mathbb{R}^m$ and all $e \geq 0$:*

1. $\alpha_1(|x|_A) \leqslant V(x) \leqslant \alpha_2(|x|_A)$
2. $|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), \tilde{\gamma}_{u_2}(|u_2|)\} \implies \frac{\partial V}{\partial x}\phi(x, u_1, u_2) \leqslant -\alpha_3(|x|_A)$,

Note that the proposition above is an extension of [32, Definition 1] to sets.

Proof From the hypothesis of the Lemma 1 in [17] or Proposition 2.1, for system (3 with ISS-Lyapunov function satisfying the two conditions of Lemma 1, there exists a $\mathcal{KL}$ function $\beta' = \max\{\alpha_1^{-1}(\beta(|x(0)|_A, 0))\}$ with $\beta \in \mathcal{KL}$ such that for each pair of measurable, essentially bounded inputs $u_1(\cdot), u_2(\cdot)$, the solution of (3) with $x(0) = x_0$ exists for all $t \geq 0$ and satisfies

$$|x(t)|_A \leq \beta'(|x_0|)_A, t) + \alpha_1^{-1}(\alpha_2(\max\{\tilde{\gamma}_{u_1}(||u_1||), \tilde{\gamma}_{u_2}(||u_2||)\})).$$

From the fact that for any pair of nonnegative real numbers (c_1, c_2)

$$\begin{aligned}\alpha_1^{-1}(\alpha_2(\max\{c_1, c_2\})) &\leq \max\{\alpha_1^{-1}(\alpha_2(c_1)), \alpha_1^{-1}(\alpha_2(c_2))\}\\ &\leq \alpha_1^{-1}(\alpha_2(c_1)) + \alpha_1^{-1}(\alpha_2(c_2)).\end{aligned}$$

We obtain

$$|x(t)|_A \leq \beta'(|x_0|)_A, t) + \gamma_{u_1}(||u_1||) + \gamma_{u_2}(||u_2||)$$

with $\gamma_{u_1} = \alpha_1^{-1}(\alpha_2(\tilde{\gamma}_{u_1})), \gamma_{u_2} = \alpha_1^{-1}(\alpha_2(\tilde{\gamma}_{u_2}))$.

2.2 | Set-stability for systems with time delay

The notions of set-stability for systems with time delays and disturbances are introduced in this subsection. Consider the autonomous TDS with no perturbations,

$$\dot{x}(t) = f(x_t), \quad x_0 = \phi \in C, \tag{5}$$

where $f : C \to \mathbb{R}^n$ is Lipschitz on bounded sets and $\phi : [-\delta, 0] \mapsto \mathbb{R}^n$ is the initial condition. The vector $x(t)$ is the solution at time t while x_t is the state history in TDS (5).

Definition 5 (Invariant set w.r.t FDE) *A set $A \subseteq C$ is said to be an invariant set (with respect to (5)) if for any ϕ in A, there is a solution $x(\cdot)$ of (5) that is defined on $[0, \infty)$ such that $x_t \in A$, $\forall t \geq 0$.*

We further introduce FDEs with delayed disturbances and some notations related to the same as given in [17]. We analyze FDEs of the form,

$$\dot{x}(t) = f(x_t, w_t), \quad x_0 = \xi \in C, \tag{6}$$

where $x : C \to \mathbb{R}^n$ and the initial data is continuous. w takes values in $\mathbb{R}^m$ and is measurable and locally essentially bounded [33]. The signal w is an exogenous input. We assume that there exists $T_f > 0$ and an unique maximal solution $x(\cdot)$ defined on $[t_0 - \delta, t_0 + T_f)$ for each initial data, input, and starting time $t_0 > 0$. We are allowed to use $T_f = +\infty$ in further calculations. The norms are defined as follows

$$|x_t| \triangleq \max_{-\delta \leqslant s \leqslant 0} |x(t+s)|, \quad \|x\|_{t_0} \triangleq \sup_{t \geq t_0} |x_t|.$$

The definition of norms with regard to other variables is similar as well. Given continuous functions $x : [-\delta, \infty) \to \mathbb{R}^n$

and $V : [-\delta, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, we are using the shorthand $V(t, x(t)) \triangleq V(t)$ and define $V_t(\tau) \triangleq V(t-\tau)$ for $t \geqslant 0$ and $\tau \in [0, \delta]$. Additionally, the upper right hand derivative of V along the solution of (6), $x(t)$ is defined as $D^+V(t) = \lim\sup_{h \to 0^+} \dfrac{V(t+h) - V(t)}{h}$.

To conclude this section, we present an instrumental lemma and introduce a final relevant definition.

Proposition 2.1 (Lemma 1 in [17]) *Let $\tilde{\xi} \geqslant 0$ and $\alpha \in \mathcal{K}$. If $V(t) \geqslant \tilde{\xi}$ implies $D^+V(t) \leqslant -\alpha(V(t))$, then there exists $\beta \in \mathcal{KL}$ (independent of $\tilde{\xi}$) with $\beta(s, 0) \geqslant s$, such that $V(t) \leqslant \max\{\beta(V(t_0), t - t_0), \tilde{\xi}\}$.*

A compact, invariant set $A \subset \mathbb{R}^n$ that contains the origin serves as the cornerstone for the entire discussion in the article. The norms w.r.t set A are defined as follows

$$|x_t|_A \triangleq \inf_{k \in A} \max_{-\delta \leqslant s \leqslant 0} |k - x(t+s)|,$$

$$||x_t||_A \triangleq \max_{-\delta \leqslant s \leqslant 0} |x(t+s)|_A = \max_{-\delta \leqslant s \leqslant 0} \inf_{k \in A} |k - x(t+s)|,$$

$$\|x\|_{t_0, A} = \sup_{t \geqslant t_0} \{|x_t|_A\}.$$

We also have $\|x_t\|_A \leqslant |x_t|_A$ (under the assumption that the system has no finite escape time through $|\cdot|_A$) [34]. Invariant set w.r.t (6) is similar to that in Definition 5 with $w_t = 0$ in (6).

The following definition is the set-modified version of [17, Definition 1].

Definition 6 *Let $\gamma \in \mathcal{G}, \mu \in \mathbb{R}_{\geqslant 0}$, and $\Delta_x, \Delta_w \in \mathbb{R}_{\geqslant 0} \cup \infty$. A set A (compact, invariant, and contains origin) with respect to (6) is said to be uniformly ISS with gain γ [and offset μ and restriction (Δ_x, Δ_w)] if $|x_0|_A < \Delta_x$ and $\|w\|_{t_0} < \Delta_w$ imply $T_f = \infty$ and that the following properties hold uniformly in $t_0 > 0$*

1. *for each $\epsilon > 0$ there exists $\underline{\delta} > 0$ such that $|x_0|_A \leqslant \underline{\delta}$ implies $\|x\|_{t_0, A} \leqslant \max\{\epsilon, \gamma(\|w\|_{t_0}), \mu\}$ and,*
2. *for each $\epsilon > 0, \eta_x \in (0, \Delta_x), \eta_w \in (0, \Delta_w)$ there exists $T > 0$ such that $|x_0|_A \leqslant \eta_x$ and $\|w\|_{t_0} \leqslant \eta_w$ imply $\|x\|_{(t_0+T), A} \leqslant \max\{\epsilon, \gamma(\|w\|_{t_0}), \mu\}$.*

This forms the standard definition of uniform asymptotic stability for the set A of the FDEs for $\mu = 0$ and $w(t) \equiv 0$. It can be proved with the help of [27, Definition 2.6]. Note that Definition 6 imposes bounds on the perturbations and in this sense indeed refers to local properties. On the other hand the input to state stability is ensured for the same set of initial conditions as for the system without delays.

3 | MAIN RESULTS

3.1 | Set ISS versions of the Razumikhin-type theorem

In this subsection, we present the set-ISS theorems based on Lyapunov-Razumikhin functions.

The following result is a slight reformulation of the global set ISS version of the Razumikhin-type theorem in [28].

Theorem 3.1 [28] *If there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, a continuous function $V : [-\delta, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}, \gamma_v, \gamma_w \in \mathcal{G}$ and $\alpha_3 \in \mathcal{K}$ such that*

1. *$\alpha_1(|x(t)|_A) \leqslant V(t) \leqslant \alpha_2(|x(t)|_A)$,*

2. $V(t) \geqslant \max\{\gamma_v(|V_t|), \gamma_w(|w_t|)\} \implies D^+V(t) \leqslant -\alpha_3(|x(t)|_A)$,
3. $\gamma_v(e) < e$ *for* $e > 0$,

then the set A *for* (6) *is uniformly globally ISS with gain* $\alpha_1^{-1} \circ \gamma_w$.

The next version follows from a small-gain argument working with $\|x_t\|_A$ rather than $|V_t|_A$. More significantly, it addresses the case where the small-gain condition does not hold on all of $(0, \infty)$. This situation will arise in our upcoming results on the robustness of the input-to-state stability of sets due to time delays at the input.

Theorem 3.2 *Suppose there exists* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, *a continuous function* $V : [-\delta, \infty) \times \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}, \gamma_v, \gamma_w \in \mathcal{G}$ *and* $\alpha_3 \in \mathcal{K}$, *and non negative real numbers* μ, Δ *with* $\mu < \Delta$ *such that*

1. $\alpha_1(|x(t)|_A) \leqslant V(t) \leqslant \alpha_2(|x(t)|_A)$,
2. $|x(t)|_A \geqslant \max\{\gamma_x(|x_t|_A), \gamma_w(|w_t|)\} \implies D^+V(t) \leqslant -\alpha_3(|x(t)|_A)$,
3. $\alpha_1^{-1} \circ \alpha_2 \circ \gamma_x(e) < e$ *for* $\mu < e < \Delta$.

Let $\beta \in \mathcal{KL}$ *be as in conclusion of Proposition 2.1 when* $\alpha = \alpha_3 \circ \alpha_2^{-1}$. *Then the set* A *for* (6) *is uniformly ISS with gain* $\tilde{\gamma}_w \triangleq \alpha_1^{-1} \circ \alpha_2 \circ \gamma_w$ *(offset* μ *and restriction* (Δ_x, Δ_w)*) such that* $\max\{\alpha_1^{-1}(\beta(\alpha_2(s_1), 0)), \tilde{\gamma}_w(s_2)\} < \Delta$ *when* $s_1 < \Delta_x$, $s_2 < \Delta_w$.

Proof In this case, the pertinent inequalities are

$$|x_t|_A \leqslant \max\{|x_0|_A.\phi(t - t_0), |x|_{t_0,A}\}, \tag{7}$$

$$|x(t)|_A \leqslant \max\{\tilde{\beta}(|x(t_0)|_A, t - t_0), \tilde{\gamma}_x(||x\|_{t_0,A}), \tilde{\gamma}_w(\|w\|_{t_0})\}, \tag{8}$$

where, $\phi(s) = 0.5(1 - sgn(s - \delta))$, $|x|_{t_0,A} = \sup_{t \geqslant t_0}|x(t)|_A$, $\tilde{\beta}(e, t) = \alpha_1^{-1}(\beta(\alpha_2(e), t)$, $\tilde{\gamma}_x = \alpha_1^{-1} \circ \alpha_2 \circ \gamma_x$, and $\tilde{\gamma}_w = \alpha_1^{-1} \circ \alpha_2 \circ \gamma_w$. Inequality (8) follows by modifying Point 2) of the theorem, i.e., if $V(t) \geqslant \alpha_2(\xi)$ implies $|x|_A \geqslant \xi$. Thus, in accordance with the theorem's Point 2), we will obtain $V(t) \geqslant \max\{\alpha_2 \circ \gamma_x(|x_t|_A), \alpha_2 \circ \gamma_w(|w_t|)\} \implies D^+V(t) \leqslant -\alpha_3 \circ \alpha_2^{-1}(|V(t)|)$. We use Proposition 2.1 (which introduces a $\beta \in \mathcal{KL}$ function) and Point 1) of the theorem to obtain

$$\begin{aligned} &V(t) \leqslant \max\{\beta(V(t_0), t - t_0), \alpha_2 \circ \gamma_x(|x_t|_A), \alpha_2 \circ \gamma_w(|w_t|)\} \\ \implies &|x(t)|_A \leqslant \max\{\tilde{\beta}(|x_0|_A, 0), \tilde{\gamma}_x(||x\|_{t_0,A}), \tilde{\gamma}_w(\|w\|_{t_0})\}. \end{aligned}$$

We here make use of the fact that $|x(t_0)|_A \leqslant \|x_0\|_A \leqslant |x_0|_A$. Using $\max\{\alpha_1^{-1}(\beta(\alpha_2(s_1), 0)), \tilde{\gamma}_w(s_2)\} < \Delta$ when $s_1 < \Delta_x$, $s_2 < \Delta_w$, guarantees $|x_0|_A < \Delta$. We have $\alpha_1^{-1}(\beta(\alpha_2(s_1), 0)) < \Delta$ when $s_1 < \Delta_x$. For $s_1 = |x_0|_A$, we have $\alpha_1^{-1}(\beta(\alpha_2(|x_0|_A), 0)) < \Delta$ and,

$$\begin{aligned} &\alpha_2(|x_0|_A) \leqslant \beta(\alpha_2(|x_0|_A), 0), \\ &\alpha_1(|x_0|_A) \leqslant \alpha_2(|x_0|_A) \leqslant \beta(\alpha_2(|x_0|_A), 0). \end{aligned}$$

The existence of the aforementioned truncation guarantees $|x_t|_A < \Delta$, and $\|x\|_{t_0,A} < \Delta$. Taking sup norm on both sides of (7) and (8) and substituting the bound of $\|x\|_{t_0,A}$ in (8), we obtain

$$\|x\|_{t_0,A} \leqslant \max\{\tilde{\beta}((|x(t_0)|_A),0),\mu,\tilde{\gamma}_w(\|w\|_{t_0})\}. \tag{9}$$

Following Point 3) of the theorem, we can deduce that for $a,b,c \in [0,\Delta)$, $a \leqslant \max\{b,\tilde{\gamma}_x(a),c\}$ imply $a \leqslant \max\{b,\mu,c\}$. Moreover, we are aware that $\tilde{\gamma}_w(s_2) < \Delta$. Thus, the first condition of the Definition 6 holds if $|x_0|_A \leqslant \underline{\delta}$, for every $\epsilon > 0$, $\exists$ a $\underline{\delta} > 0$ such that $\tilde{\beta}(\underline{\delta},0) \leqslant (\epsilon)$. From (9) we have, $\|x\|_{t_0,A} \leqslant \max\{\tilde{\beta}((|x_0|_A),0),\mu,\tilde{\gamma}_w(\|w\|_{t_0})\}$ and, $\|x\|_{t_0,A} \leqslant \max\{\epsilon,\tilde{\gamma}_w(\|w\|_{t_0}),\mu\}$.

In order to demonstrate uniform convergence, we follow a similar procedure to that used in Theorem 3.1. For given strictly positive real values ϵ,η_x,η_w, let $\kappa = \max\{\tilde{\beta}(\eta_x,0),\tilde{\gamma}_w(\eta_w),\mu\}$, where $|x_0|_A \leqslant \eta_x$ and $\|w\|_{t_0} \leqslant \eta_w$ to obtain $\|x\|_{t_0,A} \leqslant \kappa$. Let $\rho_2 > 0$ be such that $\tilde{\beta}(\kappa,\rho_2) \leqslant (\epsilon)$. Also let $\rho_1 > \mu$. Using (7) and (8), we can then generate

$$\|x\|_{(t_0+\rho_1+\rho_2),A} \leqslant \|x\|_{(t_0+\rho_2),A} \leqslant \max\{\epsilon,\tilde{\gamma}_x(\||x\|_{t_0,A}),\tilde{\gamma}_w(\|w\|_{t_0})\}.$$

Since $\tilde{\gamma}_x(e) < e$ for all $\mu < e < \Delta$, there exists $n(\kappa,\epsilon)$ such that $\tilde{\gamma}_x^n(\kappa) \leqslant \max\{\epsilon,\tilde{\gamma}_w(\|w\|_{t_0}),\mu\}$. We may therefore deduce that

$$\|x\|_{(t_0+n(\rho_1+\rho_2)),A} \leqslant \max\{\epsilon,\tilde{\gamma}_w(\|w\|_{t_0}),\mu\}.$$

Thereby, we demonstrate that the second condition of the Definition 6 also holds here and set A for (6) is uniformly ISS with gain $\tilde{\gamma}_w$ [offset μ and restriction (Δ_x,Δ_w)]. □

Remark 3.1 *In contrast to Theorem 3.1 from [28], which employs $|V_t|_A$, Theorem 3.2 is a novel variation of Razumikhin-type theorems for set ISS of FDEs with delayed disturbances. It is based on a small-gain argument using $\|x_t\|_A$. The uniform stability and convergence property, as stated in Definition 6, provides a basis for the proof of Theorem 3.2. The offset and restriction conditions in set ISS are also addressed in Theorem 3.2, as opposed to Theorem 3.1. Our main finding concerning the robustness of set ISS to a feedback delay is based on Theorem 3.1, making it applicable to systems of a broader class.*

3.2 | Set input-to-state stability

Next, we state a lemma pertinent to this subsection.

Consider a system

$$\dot{x} = \phi(x,u_1,u_2), \tag{10}$$

where $x \in \mathbb{R}^n$ denotes the vector of state variables, $u_1 \in \mathbb{R}^p, u_2 \in \mathbb{R}^m$ denote the vector of input variables (disturbances), and ϕ is Lipschitz on bounded sets on $\mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^m$.

Lemma 3.1 *If the system* (10) *with $u_2 \equiv 0$ is ISS w.r.t a set A (compact and containing the origin) with gain γ_{u_1}, then there exists a $m \times m$ matrix $B(x,u_1)$ of smooth functions, invertible for all $x \in \mathbb{R}^n$, $u_1 \in \mathbb{R}^p$ and satisfies $B(x,u_1) \equiv I_{m\times m}$ in a* *neighborhood of the origin of set A, and a function* $\gamma_{u_2} \in \mathcal{K}_\infty$*, such that the system*

$$\dot{x} = \phi(x, u_1, B(x, u_1)u_2), \tag{11}$$

is ISS w.r.t set A with gain $(\gamma_{u_1}, \gamma_{u_2})$*. More specifically,* $\exists\, \alpha_i \in \mathcal{K}$ *for* $i = 1, 2, 3$*, such that* $\alpha_1(|x|_A) \leqslant V(x) \leqslant \alpha_2(|x|_A)$ *and the following hold true*

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), \tilde{\gamma}_{u_2}(|u_2|)\} \implies \frac{\partial V}{\partial x}\phi(x, u_1, B(x, u_1)u_2) \leqslant -0.5\alpha_3(|x|_A),$$

where, $\tilde{\gamma}_{u_1}(e) = \alpha_2^{-1}(\alpha_1(\gamma_{u_1}(e)))$ *and* $\tilde{\gamma}_{u_2}(e) = \alpha_2^{-1}(\alpha_1(\gamma_{u_2}(e)))$*.*

The proof of the aforementioned lemma is provided in Appendix A and is an extension of the corresponding result on equilibrium points in [32].

3.3 | Robustness of set ISS to time delays at input

The key result of this article is now ready to be stated. We first give the results of the robustness analysis as provided in [28], but without the use of any practical constants. Additionally, we will compare the outcomes for the robustness of set ISS to input time delays.

Consider the following system

$$\dot{x}(t) = f(x(t), w(t)), \tag{12}$$

where $\tilde{\delta} > 0$, $x \in \mathbb{R}^n$, and $w \in \mathbb{R}^m$.

Theorem 3.3 [28] *If set A for* (12) *is ISS, i.e., there exists an ISS-Lyapunov function* $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *which satisfies the following*

$$\alpha_1(|x|_A) \leqslant V(x) \leqslant \alpha_2(|x|_A),$$
$$\text{and } \frac{\partial V}{\partial x} f(x, w) \leqslant -\alpha_3(|x|_A) + \gamma(|w|),$$

and the ISS gain is $\tilde{\gamma}_1 \triangleq \alpha_1^{-1} \circ \alpha_2 \circ \gamma_1$*, where* $\gamma_1 = \alpha_3^{-1}(\frac{1}{1-\epsilon}\gamma)$ *and* $\epsilon \in (0, 1)$ *in the absence of delay. Then there exists some* $\tilde{\delta} > 0$ *such that the set A for* $\dot{x}(t) = f(x(t), w + g(x_t))$ *is ISS and gain* $\tilde{\gamma}_2 \triangleq \alpha_1^{-1} \circ \alpha_4^{-1} \circ 8\gamma$*, where* $\alpha_4(e) = \alpha_3 \circ 0.5\alpha_2^{-1}$ *under the assumption that g is a continuous function and satisfies,*

$$|g(x_t)| \leqslant h(|V_t|), \text{ for some } h \in \mathcal{K}_\infty \tag{C1}$$

We next utilize Theorem 3.2 to demonstrate the relaxed version of the robustness of set ISS before comparing the two results.

Consider the system

$$\dot{x}(t) = f(x(t), u(t - \tilde{\delta}), w(t)), \tag{13}$$

where $\tilde{\delta} > 0$, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $w \in \mathbb{R}^m$. If there exists $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, smooth functions $k : \mathbb{R}^n \to \mathbb{R}^m$, $V : \mathbb{R}^n \to$

$\mathbb{R}_{\geqslant 0}$ and $\gamma \in \mathcal{G}$ such that $\alpha_1(|x(t)|_A) \leqslant V(t) \leqslant \alpha_2(|x(t)|_A)$, $|x|_A \geqslant \gamma(|w|) \implies \frac{\partial V}{\partial x} f(x, k(x), w) \leqslant -\alpha_3(|x(t)|_A)$, we say that set A for (13) is ISS with gain $\tilde{\gamma} \triangleq \alpha_1^{-1} \circ \alpha_2 \circ \gamma$ in the absence of delay [35].

Theorem 3.4 *If set A for (13) is ISS with gain $\tilde{\gamma}$ in the absence of delay, then there exists $\delta^* > 0$ such that for all $\tilde{\delta} \in (0, \delta^*)$, the set A is uniformly ISS with gain $\tilde{\gamma}$, nonzero offset and nonzero restriction. In addition, the offset becomes arbitrarily small, and the restriction becomes arbitrarily large as $\tilde{\delta}$ approaches zero.*

Proof Lemma 3.1 states that if the system (13) is ISS w.r.t. a set A in the absence of delay, then there exists a smooth invertible $m \times m$ matrix G and $\gamma_\theta \in \mathcal{K}_\infty$ such that the system

$$\dot{x} = f(x, k(x) + G(x, w)\theta, w), \tag{14}$$

satisfies

$$|x(t)|_A \geqslant \max\{\gamma_\theta(|\theta(t)|), \gamma(|w(t)|)\} \implies \dot{V} \leqslant -0.5\alpha_3(|x(t)|_A). \tag{15}$$

The original system (13) can be expressed as (14) with $\theta(t) = G(x(t), w(t))^{-1}[k(x(t - \tilde{\delta})) - k(x(t))]$. The function k and f are locally Lipschitz and $\xi(t) = k(x(t - \tilde{\delta})) - k(x(t))$. Since k is locally Lipschitz, we have,

$$|\xi(t)| \leqslant L|x(t - \tilde{\delta}) - x(t)| \leqslant L\left|\int_{t-\tilde{\delta}}^{t} \frac{dx}{d\bar{s}} d\bar{s}\right| \leqslant L\left|\int_{t-\tilde{\delta}}^{t} f(x(\bar{s}), k(x(\bar{s} - \tilde{\delta})), w(\bar{s}))d\bar{s}\right|,$$

for some Lipschitz constant $L > 0$. We arrive at the last inequality using (13). When we apply the integral mean value theorem to the equation above, we get,

$$|\xi(t)| \leqslant L\tilde{\delta}|f(x(c), k(x(c - \tilde{\delta})), w(c))|,$$

where, $c \in (t - \tilde{\delta}, t)$. Therefore, a $\gamma_1^* \in \mathcal{K}$ will exist such that $|\xi(t)| \leqslant \tilde{\delta}\max\{\gamma_1^*(|x_t|_A), \gamma_1^*(|w_t|_A)\}$. One of such constructions of $\gamma_1^* \in \mathcal{K}$ can be as follows,

$$\gamma_1^*(r_1) = \max_{|\kappa|_A \leqslant r_1, |\lambda|_A \leqslant r_1, |\psi|_A \leqslant r_1} |f_1(\kappa, \lambda, \psi)|,$$

where $f_1(x(t), x(t - \tilde{\delta}), w(t)) = f(x(t), k(x(t - \tilde{\delta}), w(t))$ and we assume that $f_1(\kappa, \lambda, \psi) = 0$ when $\kappa, \lambda, \psi \in A$ (this ensures $\gamma_1^*(0) = 0$). We will have,

$$\begin{aligned}
|f_1(x(c), x(c - \tilde{\delta}), w(c))| &\leqslant \max_{|\kappa|_A \leqslant |x(c)|_A, |\lambda|_A \leqslant |x(c-\tilde{\delta})|_A, |\psi|_A \leqslant |w(c)|_A} |f_1(\kappa, \lambda, \psi)| \\
&\leqslant \max_{|\kappa|_A \leqslant |x_t|_A, |\lambda|_A \leqslant |x_t|_A, |\psi|_A \leqslant |w_t|_A} |f_1(\kappa, \lambda, \psi)| \\
&\leqslant \max\left\{\max_{|\kappa|_A \leqslant |x_t|_A, |\lambda|_A \leqslant |x_t|_A, |\psi|_A \leqslant |x_t|_A} |f_1(\kappa, \lambda, \psi)|, \max_{|\kappa|_A \leqslant |w_t|_A, |\lambda|_A \leqslant |w_t|_A, |\psi|_A \leqslant |w_t|_A} |f_1(\kappa, \lambda, \psi)|\right\} \\
&\leqslant \max\{\gamma_1^*(|x_t|_A), \gamma_1^*(|w_t|_A)\},
\end{aligned}$$

where, $|x_t|_A = \inf_{k \in A} \max_{-2\tilde{\delta} \leqslant s \leqslant 0} (|k - x(t + s)|)$, and $|w_t|_A = \inf_{k \in A} \max_{-2\tilde{\delta} \leqslant s \leqslant 0} (|k - w(t + s)|)$. The above calculation shows

$$|\xi(t)| \leqslant L\tilde{\delta}|f_1(x(c), x(c - \tilde{\delta}), w(c))| \leqslant L\tilde{\delta}\max\{\gamma_1^*(|x_t|_A), \gamma_1^*(|w_t|_A)\}.$$

Additionally, some $\gamma_1, \gamma_2 \in \mathcal{K}$ will exist such that

$$|\theta(t)| \leqslant \tilde{\delta} \max\{\gamma_1(|x_t|_A), \gamma_2(|w_t|_A)\}. \tag{16}$$

A is a compact set which contains origin so, $|w_t|_A \leqslant |w_t|$, thus (16) can be written as

$$|\theta(t)| \leqslant \tilde{\delta} \max\{\gamma_1(|x_t|_A), \gamma_2(|w_t|)\}.$$

Defining $\hat{\gamma} \in \mathcal{G}$ by $\hat{\gamma} \triangleq \max\{\gamma_\theta(\tilde{\delta}.\gamma_2(e)), \gamma(e)\}$, it follows from (15) that

$$|x(t)|_A \geqslant \max\{\gamma_\theta(\tilde{\delta}.\gamma_1(|x_t|_A)), \hat{\gamma}(|w_t|)\} \implies \dot{V} \leqslant -0.5\alpha_3(|x(t)|_A).$$

The subsequent analysis follows from [17]. Since $\gamma_\theta(0) = 0$, for each pair of strictly positive real numbers μ, and Δ, there exists $\delta^* > 0$ such that $\tilde{\delta} \in (0, \delta^*)$ implies

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(\tilde{\delta}.\gamma_1(e)) < e, \quad \forall e \in (\mu, \Delta).$$

Given an arbitrarily small offset $\mu > 0$ and an arbitrarily large (finite) restriction (Δ_x, Δ_w), it follows from the Theorem 3.2 that there exists $\delta^* > 0$ such that $\tilde{\delta} \in (0, \delta^*)$ indicates the set A is uniformly ISS with gain $\alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma}$. Given the offset δ and the restriction Δ_w it is always feasible to further refine $\tilde{\delta}$ such that

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(\tilde{\delta}.\gamma_2(\Delta_w)) \leqslant \mu,$$

with μ being arbitrarily small. At this point, $\alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma} = \max\{\delta, \alpha_1^{-1} \circ \alpha_2 \circ \gamma\}$. Thus, $\alpha_1^{-1} \circ \alpha_2 \circ \hat{\gamma}$ can be replaced by $\alpha_1^{-1} \circ \alpha_2 \circ \gamma = \tilde{\gamma}$. □

Remark 3.2 *Theorem 3.4 is a local result that holds for some suitably small neighborhood of set A determined by the ISS gain lying inside the appropriate restriction and offset range.*

3.4 | Comparative review

We compare our work with [28] because, to the best of our knowledge, [28] is the only pertinent study that analyzes the robustness of set ISS to input delays. The novel contributions of our work w.r.t [28] are as below

- The robustness of the set ISS of a nonlinear system to input is demonstrated in the existing literature, in particular, [28], where the input is dependent on a state history as an additive effect, which may not always be the case in practical applications. We are putting forth a more general setup to address this discrepancy and demonstrate the set ISS of nonlinear system's robustness to input time delays, where the control input is replaced by its time delayed version. In mathematical terms, [28] prove that if $\dot{x}(t) = f(x(t), w(t))$ is set ISS, then $\dot{x}(t) = f(x(t), w(t) + g(x_t))$ will also be set ISS for some time delay $\delta > 0$. On the other hand, our study establishes that if $\dot{x}(t) = f(x(t), u(t), w(t))$ is set ISS for some $u = k(x(t))$, then the system $\dot{x}(t) = f(x(t), k(x_t), w(t))$ is also set ISS for some time delay $\delta > 0$, with k, g being continuous functions.

- The state history function is yet again constrained in [28] as $|g(x_t)| \leqslant h(|V_t|)$, for some $h \in \mathcal{K}_\infty$, however in our work, the state history functions are not subject to any such limitations. This condition indicated in [28] is applicable to fewer systems to show set ISS robustness to time delay at the input as opposed to our method. We will demonstrate via examples that it will be difficult for this criterion in [28] to hold in some systems.

We essentially compare Section 4.1 of [28] (the main result therein has also been restated in our article as Theorem 3.3, with practical constants as zero) against Theorem 3.4 of our paper. The result in Theorem 3.4 is a relaxation since it allows more general non-additive delays to the input and removes constraint (C1) from Theorem 3.3. This significantly increases the scope of robustness results under time delay for set stable systems. These contrasts are further highlighted with the help of examples in the next section.

Remark 3.3 *The proofs demonstrated in this article go beyond the simple substitution of $|x|$ to $|x|_A$. The novel variant of Razumikhin theorem (Theorem 3.2) for uniform set ISS of FDEs is established by thoroughly demonstrating uniform stability and convergence w.r.t sets, providing a foundation for our paper's major result on robustness w.r.t. input time delay. Lemma 3.1 derived as an extension of the results in [32] for ISS of compact sets containing the origin also feeds into the proof of Theorem 3.4. Another crucial and challenging aspect of the proof is constructing a general function $\gamma_1^*(s)$ to demonstrate set ISS for time delayed systems in Theorem 3.4.*

4 | COMPARISON USING EXAMPLES

This section uses examples to show the robust set ISS analysis of a nonlinear system with delays described in Section 3.3.

The first example uses a set that just contains the origin to show how the set ISS is robust to input time delays using Theorem 3.4. This serves as an additional example of the limitation of Theorem 3.3 where it is challenging to satisfy a restrictive assumption given singleton set circumstances. The second example also illustrates the same with sets rather than the origin.

4.1 | Example 1

Consider a nonlinear oscillator:

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\bar{k}\,(x_1) - \bar{\mu}\,(x_2) + w, \end{cases} \tag{17}$$

where the continuous functions $\bar{k} : \mathbb{R} \to \mathbb{R}$ and $\bar{\mu} : \mathbb{R} \to \mathbb{R}$ satisfy the following conditions $\bar{k}(0) = \bar{\mu}(0) = 0$ and that for any $r \neq 0$ we have

$$r\bar{k}(r) > 0, \quad r\bar{\mu}(r) > 0.$$

w stands for a disturbing force. Under the above assumptions, the origin is the only equilibrium point of the unperturbed $(w = 0)$ system. Let $\bar{\mu}(s) = \bar{\mu}s$ be linear, where $\bar{\mu} > 0$ and $\bar{k}$ be continuous with $\bar{k}(0) = 0$ and $r\bar{k}(r) > 0$ for any $r \in \mathbb{R}$ and in addition $\liminf_{r\to\pm\infty} \bar{k}(r) \neq 0$, and $\eta \in \mathcal{K}_\infty$ be such that $\bar{k}(|e|) \geqslant \eta(|e|)$ for any real $e \neq 0$. Consider the following Lyapunov function and its time derivative:

$$V(x) = \frac{\bar{\mu}^2}{2}x_1^2 + \bar{\mu}x_1x_2 + x_2^2 + 2\int_0^{x_1} \bar{k}(r)dr$$
$$\dot{V}(x,w) = -\bar{\mu}\left(x_1\bar{k}\left(x_1\right) + x_2^2\right) + \left(\bar{\mu}x_1 + 2x_2\right)w \leq -\frac{1}{2}\bar{\mu}\left(|x_1|\,\eta\left(|x_1|\right) + x_2^2\right)$$

if

$$\frac{1}{2}\bar{\mu}\left(|x_1|\,\eta\left(|x_1|\right) + x_2^2\right) \geq \left(\bar{\mu}\,|x_1| + 2\,|x_2|\right)|w|. \tag{18}$$

It is well known from [36] that

$$|x| \geqslant \max\left\{\sqrt{1+\frac{4}{\bar{\mu}^2}}\frac{8|w|}{\bar{\mu}}, \sqrt{1+\frac{\bar{\mu}^2}{4}}\eta^{-1}\left(4|w|\right)\right\} \triangleq \gamma(|w|) \tag{19}$$

implies (18). So, we have V as an ISS-Lyapunov function. By Definition 3, the system is ISS with the gain $\gamma \in \mathcal{K}_\infty$ defined in (19). We can also represent the nonlinear oscillator system (17) as

$$\dot{x} = f(x) + w_1,$$

where $x = (x_1, x_2)^\top \in \mathbb{R}^2$, $w_1 = (0, w)^\top \in \mathbb{R}^2$ and $f(x) = (-x_2, -\bar{k}(x_1) - x_2)^\top \in \mathbb{R}^2$. The subsequent equation with an input delay perturbation is

$$\dot{x} = f(x) + w_1 + g(x_t),$$

with $g(x_t) = \begin{pmatrix} -x_2(t) + x_2(t-\tau) \\ x_2(t) - x_2(t-\tau), \end{pmatrix}$, $\tau > 0$, and $\bar{\mu} = 1$, which results in the following time delayed modification of (17):

$$\begin{cases} \dot{x}_1(t) = x_2(t-\tau) \\ \dot{x}_2(t) = -\bar{k}\left(x_1(t)\right) - \bar{\mu}x_2(t-\tau) + w(t). \end{cases}$$

Applying the mean value theorem, we obtain

$$|g(x_t)| = \sqrt{2}\tau \max_{t-\tau\leqslant s\leqslant t} |-\bar{k}\left(x_1(\phi)\right) - x_2(\phi-\tau) + w(\phi)|,$$

where $\phi \in [s-\tau, s]$ and $|x_t| = \sup_{-2\tau\leqslant s\leqslant 0}|x(t+s)|$. It is difficult to establish the requirement that there exists a $h \in \mathcal{K}_\infty$, such that $|g(x_t)| \leqslant h(|V_t|)$, even in situations where Theorem 3.3 is applicable. This is because we don't have the information on the shared relationship between $w(\phi)$ and $x_1(\phi), x_2(\phi-\tau)$. We propose that the ISS of the time delayed system can be demonstrated using Theorem 3.4) rather than having to go through the tedious process of validating the above requirements of Theorem 3.3. In accordance with (17), we take into consideration the following system

$$\begin{cases} \dot{x}_1(t) = u \\ \dot{x}_2(t) = -\bar{k}\left(x_1(t)\right) - \bar{\mu}u + w(t), \end{cases} \tag{20}$$

with feedback $u = x_2$. Alternatively, we can write the above nonlinear oscillator (20) with $\bar{\mu} = 1$ as,

$$\dot{x} = f(x) + u + w_1,$$

where $x = (x_1, x_2)^\top \in \mathbb{R}_2, u = (u_1, u_2)^\top \in \mathbb{R}^2$, $w_1 = (0, w)^\top \in \mathbb{R}^2$ and $f(x) = (0, -\bar{k}(x_1))^\top \in \mathbb{R}^2$. We choose $u = (u_1, u_2)^\top = (x_2, -x_2)^\top$, such that the nonlinear oscillator (20) is ISS without any time delay at input. If we have a time delay introduced in the feedback, we will then have the following:

$$\dot{x} = f(x) + \begin{pmatrix} x_2(t-\tau) \\ -x_2(t-\tau) \end{pmatrix} + w_1. \tag{21}$$

Given that the aforementioned system (20) is ISS without any time delay, (21) rewritten as,

$$\dot{x} = f(x) + u + G(x, w)\theta + w_1,$$

where $G(x, w) = I_2$ and

$$\theta = (\theta_1, \theta_2)^\top = (x_2(t-\tau) - x_2(t), -(x_2(t-\tau) - x_2(t)))^\top$$

satisfies the following

$$\begin{aligned} |x(t)|_A = |x(t)| &\geqslant \max\{\gamma(|w(t)|), \gamma_\theta(|\theta(t)|)\}, \\ \gamma(|w(t)|) &\triangleq \max\left\{\sqrt{1+4}(8|w|), \sqrt{1+\frac{1}{4}\eta^{-1}}\ (4|w|)\right\} \\ \gamma_\theta(|\theta(t)|) &\triangleq \max\left\{\max\{10(\eta^{-1}(|\theta|))^4, 10(\eta^{-1}(|\theta|))^2\}, 24\sqrt{10}(|\theta|)\right\} \\ \implies D^+V(t) &\leqslant -\frac{1}{4}\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right). \end{aligned} \tag{22}$$

We will prove the claim made above using the same Lyapunov function as we did for nonlinear oscillator (17):

$$\begin{aligned} V(x) &= \frac{1}{2}x_1^2 + x_1 x_2 + x_2^2 + 2\int_0^{x_1} \bar{k}(r)dr \\ D^+V(t) &= -\left(x_1 \bar{k}\,(x_1) + x_2^2\right) + (x_1 + 2x_2)\,w + (2\bar{k}(x_1) + x_2 + x_1)\theta_1 + (x_1 + 2x_2)\theta_2 \\ &\leqslant -\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right) + (|x_1| + 2\,|x_2|)\,|w| + (2|\bar{k}(x_1)| + 3|x_2| + 2|x_1|)\,|\theta| \\ &\leqslant -\frac{1}{4}\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right) \end{aligned}$$

if

$$\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right) \geq \max\{2\,(|x_1| + 2\,|x_2|)\,|w|, 4(2|\bar{k}(x_1)| + 3|x_2| + 2|x_1|)\,|\theta|\}. \tag{23}$$

We will now prove that (22) implies (23): We have already shown that $|x| \geqslant \gamma(|w(t)|)$ implies $\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right) \geqslant \{2\,(|x_1| + 2\,|x_2|)\,|w|\}$ in the case where there is no time delay in the oscillator. Let us now prove that

$$|x| \geqslant \gamma_\theta(|\theta(t)|) \tag{24}$$

implies

$$\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right) \geqslant 4(2|\bar{k}(x_1)| + 3|x_2| + 2|x_1|)\,|\theta|\}. \tag{25}$$

We consider two cases for this: $2|\bar{k}(x_1)| + 2|x_1| \geqslant 3|x_2|$ and $2|\bar{k}(x_1)| + 2|x_1| < 3|x_2|$. We consider $\bar{k}(x_1) = x_1^3$ here onwards. If $2|\bar{k}(x_1)| + 2|x_1| \geqslant 3|x_2|$, then $|\theta| \leqslant \frac{\eta(|x_1|)}{8(2|x_1|^2+2)} \leqslant \eta(|x_1|)$, because otherwise we have a contradiction

$$\begin{aligned} |x| &= \sqrt{x_1^2 + x_2^2} = \sqrt{x_1^2 + \frac{4}{9}(x_1^3 + x_1)^2} \\ &\leqslant \max\{10|x_1|^4, 10|x_1|^2\} \leqslant \max\{10(\eta^{-1}(|\theta|))^4, 10(\eta^{-1}(|\theta|))^2\}. \end{aligned}$$

Next we have, $|\theta| \leqslant \frac{\eta(|x_1|)}{8(2|x_1|^2+2)}$ and it follows that $4(2|x_1|^3 + 3|x_2| + 2|x_1|)|\theta|\} \leqslant 8(2|x_1|^2 + 2)|x_1| \frac{\eta(|x_1|)}{8(2|x_1|^2+2)} \leqslant |x_1|\eta(|x_1|) \leqslant \left(|x_1|\,\eta\,(|x_1|) + x_2^2\right)$. In the opposite case, $2|x_1|^3 + 2|x_1| < 3|x_2|$, then $|\theta| \leqslant \frac{|x_2|}{24}$, because otherwise we have a contradiction,

$$|x| = \sqrt{x_1^2 + x_2^2} \leq \sqrt{(2x_1^3 + 2x_1)^2 + x_2^2} \leqslant \sqrt{9x_2^2 + x_2^2} \leqslant \sqrt{10}|x_2| \leqslant 24\sqrt{10}|\theta|.$$

Hence, we have $|\theta| \leqslant \frac{|x_2|}{24}$ and it follows that $4(2|x_1|^3 + 3|x_2| + 2|x_1|)|\theta|\} \leqslant 8(3|x_2|)|\theta| \leqslant |x_2|^2 \leqslant (|x_1|\,\eta\,(|x_1|) + x_2^2)$ as desired, which proves (24) implies (25). With $\theta(t) = (x_2(t-\tau) - x_2(t), -(x_2(t-\tau) - x_2(t)))^\top$ and $x_2(t)$ being locally Lipschitz, we can further obtain the following using integral mean value theorem,

$$|\theta(t)| \leqslant \tau L|(-x_1^3(c) - x_2(c-\tau) + w(c))| \leqslant \tau \max\{\gamma_1(|x_t|), \gamma_2(|w_t|)\},$$

where $c \in (t-\tau, t)$ and L is the Lipschitz constant corresponding to $\bar{k} : [a, b] \to \mathbb{R}$ and $a, b \in \mathbb{R}$. Define

$$\gamma_1(r_1) = \max_{|\lambda_1| \leqslant r_1, |\lambda_2| \leqslant r_1} 10L|\lambda_1^3 + \lambda_2|, \gamma_2(r_2) = 10L|r_2|.$$

We now define $\hat{\gamma} \in \mathcal{G}$ by $\hat{\gamma} \triangleq \max\{\gamma_\theta(\tau.\gamma_2(e)), \gamma(e)\}$, and following (22), we obtain

$$|x(t)|_A \geqslant \max\{\gamma_\theta(\tau.\gamma_1(|x_t|)), \hat{\gamma}(|w_t|)\} \implies D^+V(t) \leqslant -\frac{1}{4}\left(|x_1|\,\eta\,(|x_1|) + x_2^2\right).$$

We can now choose $\tau^* > 0$ such that for $\tau \in (0, \tau^*)$,

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(\tau.\gamma_1(e)) < e, \quad \forall e \in (\mu, \Delta),$$

holds, for each pair of strictly positive real numbers μ and Δ.

For $e \in (\mu, \Delta)$, where $\mu = 1$, we have $\gamma_1(e) = 10L|e^3 + e|$. The Lipschitz constant L corresponds to $\bar{k} : [a, b] \to \mathbb{R}$

and $a, b \in \mathbb{R}$. For given $\bar{k}$, $\eta(x_1) = x_1^3$ implies that $\gamma_\theta(|\theta(t)| = 24\sqrt{10}(|\theta|)$ for $e \in (1, \Delta)$. Functions $\alpha_1(x) = \|x\|^2$ and $\alpha_2(x) = 2(1 + \|x\|^2)\|x\|^2$ are the lower and upper bounds respectively for the selected Lyapunov function $V(x) = \frac{\bar{\mu}^2}{2}x_1^2 + \bar{\mu}x_1x_2 + x_2^2 + 2\int_0^{x_1}\bar{k}(r)dr$, with $\bar{\mu} = 1$. Next, we satisfy

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(\tau.\gamma_1(e)) < e, \quad \forall e \in (1, \Delta),$$

to obtain

$$755.88 \times \tau L|e^3 + e| < \sqrt{\frac{-1 + \sqrt{1 + 2|e|^2}}{2}}, \quad e \in (1, \Delta)$$
$$\tau < \frac{\sqrt{-1 + \sqrt{1 + 2|\Delta|^2}}}{1073.95 \times L|\Delta^3 + \Delta|}.$$

The upper bound of τ can vary based on the constants μ, Δ, and the Lipschitz constant L.

The analysis that follows assumes that τ can be further refined so that we achieve the same ISS gain as is for (17), in accordance with [17] and Theorem 3.4. The next example shows how Theorems 3.3 and 3.4 differ by considering a system with an equilibrium set as a compact, invariant set containing the origin.

4.2 | Example 2

Here, we focus on Stuart-Landau oscillator systems, which are used to model the behavior of complex systems in various contexts. For instance, they can be used to characterize electronic oscillators, semiconductor lasers, chemical reaction-diffusion systems, and neuro-physiological phenomena. Studies on the Stuart-Landau oscillator's stability can be found in [37]. A forced generalized Stuart-Landau oscillator is described by

$$\dot{z} = -\nu|z|^2 z + \zeta z + u, \tag{26}$$

where $z \in \mathbb{C}$ represents the oscillator's state, $\nu, \zeta \in \mathbb{C}$ are parameters defined as $\nu = \nu_\mathrm{R} + \mathrm{i}\nu_I$ and $\zeta = \zeta_\mathrm{R} + \mathrm{i}\zeta_I$, respectively, and $u \in \mathbb{C}$ is an input to the oscillator. Here, we consider $\zeta_R, \nu_R > 0$. The compact set of equilibrium points for (26) is given by

$$A \triangleq \left\{z \in \mathbb{C} : |z| = \sqrt{\frac{\zeta_\mathrm{R}}{\nu_\mathrm{R}}}\right\} \bigcup \{z = 0\}.$$

We define the norm $|\cdot|_A$ as follows.

$$|z|_A = \begin{cases} |z| & \text{if} \quad |z| \leqslant 0.7\sqrt{\alpha}, \\ \sqrt{||z|^2 - \alpha|} & \text{if} \quad |z| > 0.7\sqrt{\alpha} \\ \alpha \triangleq \zeta_\mathrm{R}/\nu_\mathrm{R}. \end{cases}$$

Let us consider the following candidate ISS-Lyapunov function

$$V(z) = \frac{1}{4\nu_\mathrm{R}}\left[|z|^2 - \alpha\right]^2,$$

where $\alpha = \zeta_R/\nu_R$ and $|z|^2 = \bar{z}z$ and $\bar{z}$ signify the conjugate of z.

Let $|z| \leqslant 0.7\sqrt{\alpha}$ then, $V(z) = |z|^4 - 2\alpha|z|^2 + \alpha^2$ and, since $-|z|^2 \geqslant -0.49\alpha$, we obtain $V(z) = |z|^4 + 0.02\alpha^2 \geqslant |z|^4$. Now, since $|z|_A = |z|$, which implies that $V(z) \geqslant |z|_A^4$. Next, we can do the same when $|z| > 0.7\sqrt{\alpha}$ and $|z|_A = \sqrt{|(|z|^2 - \alpha)\ |}$. As obtained in [37], the time derivative of V along the trajectories of (26) are

$$\dot{V}(z) \leqslant -\frac{1}{2}\left[|z|^2 - \alpha\right]^2 |z|^2 + \frac{1}{2\nu_R^2}|u|^2. \tag{27}$$

The right-hand side of the abovementioned inequality is then bounded in terms of $|z|_A$. To bound the term $\left[|z|^2 - \alpha\right]^2 |z|^2$ in the two cases ($|z|_A \leqslant 0.7\sqrt{\alpha}$ and $|z|_A > 0.7\sqrt{\alpha}$) independently. Denoting $c_3(|z|_A) \triangleq \min\left\{0.51\alpha^2|z|_A^2, 0.49\alpha\big|z|_A^4\right\}$ and combining the two cases together, we obtain

$$\dot{V}(z) \leqslant -c_3(|z|_A) + \frac{1}{2\nu_R^2}|u|^2,$$

which can also be rewritten as

$$|z|_A \geqslant c_3^{-1}\left(\frac{4}{\nu_R^2}|u|^2\right) \triangleq \gamma(|u|) \implies \dot{V} \leqslant -\frac{7}{8}c_3(|z|_A).$$

The Lyapunov function $V(z)$ satisfies the conditions of Definition 3. Thus, the system (26) is set ISS w.r.t A with gain $\gamma \in \mathcal{K}_\infty$.

We now add input delay perturbation, which is $g(z_t) = \nu|z(t)|^2 z(t) - \nu|z(t-\tau)|^2 z(t-\tau) - \zeta z(t) + \zeta z(t-\tau)$. As a result, the time delay modification of (26) is as follows:

$$\dot{z}(t) = -\nu|z(t-\tau)|^2 z(t-\tau) + \zeta z(t-\tau) + u(t).$$

We now employ the mean value theorem to obtain

$$|g(z_t)| \leq \tau|\zeta(-\nu|z(\phi-\tau)|^2 z(\phi-\tau) + \zeta z(\phi-\tau) + u(\phi))| + \tau|3|z(\phi)|^2\nu(-\nu|z(\phi-\tau)|^2 z(\phi-\tau) + \zeta z(\phi-\tau) + u(\phi))|.$$

It is difficult to establish the requirement that there exists a $k \in \mathcal{K}_\infty$, such that $|g(z_t)| \leqslant k(|V_t|)$, even in situations where Theorem 3.3 is applicable. This is because we are unaware of any shared relationships between $u(\phi)$ and $z(\phi), z(\phi-\tau)$, where $\phi \in [t-\tau, t]$.

We propose that the set ISS of the time delayed system can be demonstrated using Theorem 3.4 rather than validating the above requirements of Theorem 3.3. In accordance with (26), we take into consideration the following system

$$\dot{z} = w + u, \tag{28}$$

where $u \in \mathbb{C}$. We take into account the feedback $w = k(z) = -\nu|z|^2 z + \zeta z$ with the parameters $\nu = \nu_R + i\nu_I$ and $\zeta = \zeta_R + i\zeta_I$, and $\zeta_R, \nu_R > 0$. This system (28) is set ISS with respect to A. Now, we'll examine whether the system (28) is set ISS with a time delay in feedback. To do the same, we first demonstrate that the subsequent system

$$\dot{z} = -\nu|z|^2 z + \zeta z + u + \theta, \tag{29}$$

with

$$\theta = \nu|z(t)|^2 z(t) - \nu|z(t-\tau)|^2 z(t-\tau) - \zeta z(t) + \zeta z(t-\tau),$$

satisfies $D^+V(t) \leqslant -\frac{1}{2}\left[|z|^2 - \alpha\right]^2 |z|^2 + \frac{1}{\nu_R^2}|u|^2 + \frac{1}{\nu_R^2}|\theta|^2$, or

$$|z|_A \geqslant \max\left\{c_3^{-1}\left(\frac{2}{\nu_R^2}|\theta|^2\right), c_3^{-1}\left(\frac{4}{\nu_R^2}|u|^2\right)\right\} \implies D^+V(t) \leqslant -\frac{1}{4}c_3(|z|_A). \tag{30}$$

This is inferred from (27) when the time derivative of the ISS-Lyapunov function is taken along the trajectories (29). It is important to note that we were unable to determine $\gamma_\theta(|\theta|)$, which yields $D^+V(t) \leqslant -\frac{7}{16}c_3(|z|_A)$. However, this won't impact our analysis as we have some definite negative function in $|z|_A$. $\theta(t) = k(z(t-\tau)) - k(z(t))$ and k is a locally Lipschitz function. Thus, we obtain

$$|\theta(t)| \leqslant L\left|\int_{t-\tau}^{t}(-\nu|z(s-\tau)|^2 z(s-\tau) + \zeta z(s-\tau) + u(s))ds\right|,$$

for a particular Lipschitz constant when the function k is specified in a local domain. The results of applying the integral mean value theorem to the aforementioned equation are as follows:

$$|\theta(t)| \leqslant L\tau| - \nu|z(c-\tau)|^2 z(c-\tau) + \zeta z(c-\tau) + u(c)|,$$

where $c \in (t-\tau, t)$, and

$$L\tau| - \nu|z(c-\tau)|^2 z(c-\tau) + \zeta z(c-\tau) + u(c)| \leqslant \tau \max\{\gamma_1(|z_t|_A), \gamma_2(|u_t|)\}.$$

Define $\gamma_1(r_1) = \max_{|\lambda|_A \leqslant r_1} 10L| - \nu|\lambda|^2\lambda + \zeta\lambda|$, where we assume for $\lambda \in A$, then $10L| - \nu|\lambda|^2\lambda + \zeta\lambda| = 0$, and $\gamma_2(r_2) = 10L|r_2|$. We now define $\hat{\gamma} \in \mathcal{G}$ by $\hat{\gamma}(e) \triangleq \max\{\gamma_\theta(\tau.\gamma_2(e)), \gamma(e)\}$, where $\gamma_\theta(e) = c_3^{-1}(\frac{2}{\nu_R^2}|e|^2)$. As a result of (30),

$$|z(t)|_A \geqslant \max\{\gamma_\theta(\tau.\gamma_1(|z_t|_A)), \hat{\gamma}(|u_t|)\} \implies \dot{V} \leqslant -\frac{c_3(|z|_A)}{4}.$$

is implied. We now perform the identical procedures as in Subsection 4.1 to prove the findings from the Theorem 3.4 and to establish the set ISS of (29).

To compute the upper bound on the time delay at the input, we have $\gamma_1(e) = 10L| - \nu|e^2 + \alpha|\sqrt{e^2+\alpha} + \zeta\sqrt{e^2+\alpha}|$, $\gamma_\theta(e) = c_3^{-1}(\frac{2}{\nu_R^2}|e|^2)$. Here we take into account that $e \in (2, \Delta)$ and $\alpha = \frac{\zeta_R}{\nu_R} = 2$ to choose $c_3(e) = 0.51 \times 4|e|^2$ from $c_3(e) \triangleq \min\left\{0.51\alpha^2|e|^2, 0.49\alpha\left|e\right|^4\right\}$, using which we obtain $\gamma_\theta(e) = \frac{1}{\nu_R}0.99|e|$ for $e \in (2, \Delta)$. Functions $\alpha_1(e) = \frac{1}{4\nu_R}|e|^4$ and $\alpha_2(e) = \frac{1}{\nu_R}|e|^4$ represent the upper and lower bounds respectively for the chosen Lyapunov function

$V(z) = \frac{1}{4\nu_R}\left[|z|^2 - \alpha\right]^2$. We further satisfy

$$\alpha_1^{-1} \circ \alpha_2 \circ \gamma_\theta(\tau.\gamma_1(e)) < e, \quad \forall e \in (2, \Delta),$$

to obtain

$$\frac{1}{\nu_R} \times \frac{1}{\nu_R^4} \times 9.9^4 \times L^4 \times \tau^4 | - \nu|e^2 + 2|\sqrt{e^2+2} + \zeta\sqrt{e^2+2}|^4 < \frac{1}{4\nu_R}|e|^4, \quad e \in (2, \Delta)$$
$$\tau < \frac{|\Delta|\nu_R}{13.99L| - \nu|\Delta^2 + 2|\sqrt{\Delta^2+2} + \zeta\sqrt{\Delta^2+2}|}.$$

The upper bound of τ can vary based on the constants μ, Δ, and the Lipschitz constant L. The analysis that follows assumes that τ can be further refined so that we achieve the same ISS gain as is for (26),

5 | CONCLUSIONS

A Razumikhin-type set-stability theorem has been interpreted in terms of the ISS nonlinear small-gain theorem and extended to cover the case of persistent disturbances. Set input-to-state stability (and set-global asymptotic stability as a special case) is shown to be robust, in an appropriate sense, to time delays at the input for nonlinear control systems also described by ordinary differential equations. We also compare our result on the robustness of set ISS to time delay with a related result using illustrative examples.

6 | ACKNOWLEDGMENT

The authors thank Emilia Fridman (Tel Aviv University, Israel) for her valuable input.

references

[1] Hale JK, Lunel SMV. Introduction to functional Differential Equations, vol. 99. Springer; 1993.

[2] Fridman E. Introduction to Time-Delay Systems. Springer International Publishing; 2014.

[3] Kolmanovskii V, Myshkis A. Introduction to the Theory and Applications of Functional Differential Equations. Springer Netherlands; 1999.

[4] Olgac N, Sipahi R. An exact method for the stability analysis of time-delayed linear time-invariant (LTI) systems. IEEE Transactions on Automatic Control 2002;47(5):793–797.

[5] Morărescu CI, Niculescu SI, Gu K. Stability Crossing Curves of Shifted Gamma-Distributed Delay Systems. SIAM Journal on Applied Dynamical Systems 2007;6(2):475–493.

[6] Krasovskii NN. Stability of motion. Stanford University Press 1963;.

[7] Razumikhin BS. On the stability of systems with a delay. Prikl Mat Mekh 1956;20(4):500 – 512.

[8] Michiels W, Niculescu SI. Stability and Stabilization of Time-Delay Systems. An eigenvalue based approach. Advances in Design and Control, Society for Industrial and Applied Mathematics; 2007.

[9] Niculescu SI. Delay Effects on Stability. A Robust Control Approach, vol. 269 of Lecture Notes in Control and Information Sciences. Springer London; 2001.

[10] Fridman E, Shaked U. An improved stabilization method for linear time-delay systems. IEEE Transactions on Automatic Control 2002;47(11):1931–1937.

[11] Fridman E, Seuret A, Richard JP. Robust sampled-data stabilization of linear systems: an input delay approach. Automatica 2004;40(8):1441–1446.

[12] Efimov D, Polyakov A, Perruquetti W, Richard JP. Weighted Homogeneity for Time-Delay Systems: Finite-Time and Independent of Delay Stability. IEEE Transactions on Automatic Control 2016;61(1):210–215.

[13] Efimov D, Aleksandrov A. On estimation of rates of convergence in Lyapunov–Razumikhin approach. Automatica 2020;116:108928.

[14] Sontag ED. Smooth stabilization implies coprime factorization. IEEE Transactions on Automatic Control 1989;34(4):435–443.

[15] Tsinias J. Sufficient Lyapunov-like conditions for stabilization. Mathematics of Control, Signals, and Systems 1989 12;2(4):343–357.

[16] Jiang ZP, Teel AR, Praly L. Small-gain theorem for ISS systems and applications. MCSS Mathematics of control, signals and systems 1994;.

[17] Teel AR. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. IEEE Transactions on Automatic Control 1998;43(7):960–964.

[18] Noroozi N, Geiselhart R, Grüne L, Rüffer BS, Wirth FR. Nonconservative Discrete-Time ISS Small-Gain Conditions for Closed Sets. IEEE Transactions on Automatic Control 2018;63(5):1231–1242.

[19] Peng Z, Wang D, Li T, Wu Z. Leaderless and leader-follower cooperative control of multiple marine surface vehicles with unknown dynamics. Nonlinear Dynamics 2013;74:95–106.

[20] Zhao D, Ni W, Zhu Q. A framework of neural networks based consensus control for multiple robotic manipulators. Neurocomputing 2014;140:8–18.

[21] Liang H, Zhang Y, Huang T, Ma H. Prescribed Performance Cooperative Control for Multiagent Systems With Input Quantization. IEEE Transactions on Cybernetics 2020;50(5):1810–1819.

[22] Panteley E, Loría A. Synchronization and Dynamic Consensus of Heterogeneous Networked Systems. IEEE Transactions on Automatic Control 2017;62(8):3758–3773.

[23] Lin Y, Sontag ED, Wang Y. A Smooth Converse Lyapunov Theorem for Robust Stability. SIAM Journal on Control and Optimization 1996;34(1):124–160.

[24] Lin Y, Sontag ED, Wang Y. Input to state stabilizability for parametrized families of systems. International Journal of Robust and Nonlinear Control 1995;5(3):187–205.

[25] Sontag ED, Wang Y. On characterizations of the input-to-state stability property. Systems & Control Letters 1995;24(5):351–359.

[26] Karafyllis I, Pepe P, Jiang ZP. Input-to-Output Stability for Systems Described by Retarded Functional Differential Equations. European Journal of Control 2008;14(6):539–555.

[27] Bernfeld SR, Corduneanu C, Ignatyev AO. On the stability of invariant sets of functional differential equations. Nonlinear Analysis: Theory, Methods & Applications 2003;55(6):641–656.

[28] Efimov D, Schiffer J, Ortega R. Robustness of delayed multistable systems with application to droop-controlled inverter-based microgrids. International Journal of Control 2016;89(5):909–918.

[29] Chaillet A, Karafyllis I, Pepe P, Wang Y. The ISS framework for time-delay systems: a survey. Mathematics of Control, Signals, and Systems 2023;p. 1–70.

[30] Lin Y, Sontag E, Wang Y. Various results concerning set input-to-state stability. In: Proceedings of 1995 34th IEEE Conference on Decision and Control, vol. 2; 1995. p. 1330–1335 vol.2.

[31] Sontag E, Wang Y. Lyapunov Characterizations of Input to Output Stability. SIAM Journal on Control and Optimization 2000;39(1):226–249.

[32] Christofides PD, Teel AR. Singular perturbations and input-to-state stability. IEEE Transactions on Automatic Control 1996;41(11):1645–1650.

[33] Haidar I, Pepe P. Lyapunov–Krasovskii Characterization of the Input-to-State Stability for Switching Retarded Systems. SIAM Journal on Control and Optimization 2021;59(4):2997–3016.

[34] Rudin W. Principles of mathematical analysis. 3rd ed. McGraw-Hill; 1976.

[35] Gasparri A, Williams RK, Leccese A, Ulivi G. Set Input-to-State Stability for spatially interacting multi-agent systems. In: 53rd IEEE Conference on Decision and Control; 2014. p. 5381–5386.

[36] Dashkovskiy S. Practical examples of ISS systems. IFAC-PapersOnLine 2019;52(16):1–6. 11th IFAC Symposium on Nonlinear Control Systems NOLCOS 2019.

[37] Panteley E, Loria A, Ati AE. On the Stability and Robustness of Stuart-Landau Oscillators. IFAC-PapersOnLine 2015;48(11):645–650.

[38] Chaillet A, Panteley E. Stability of sets for nonlinear systems in cascade. In: Int. Symp. on Math. Theory on Networks and Systems (MTNS), Kyoto, Japan; 2006. .

Appendix A.

To prove Lemma 3.1, we follow the same steps as in the proof of [32, Lemma 1].

With the hypothesis of the lemma and Definition 1, that there exists a smooth function $V : \mathbb{R}^n \to \mathbb{R}_{\geqslant 0}$, $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$, and $\tilde{\gamma}_{u_1} \in \mathcal{K}$ such that $\alpha_1(|x|_A) \leqslant V(x) \leqslant \alpha_2(|x|_A)$ holds and

$$|x|_A \geqslant \tilde{\gamma}_{u_1}(|u_1|) \implies \dot{V} = \frac{\partial V}{\partial x}\phi(x, u_1, 0) \leqslant -\alpha_3(|x|_A),$$

where $\gamma_{u_1}(e) = \alpha_1^{-1} \circ \alpha_2 \circ \tilde{\gamma}_{u_1}(e) \in \mathcal{K}$. We define $\tilde{u}_2 = B(x, u_1)u_2$ and calculate the time derivative of the function V along the trajectories of (11),

$$\dot{V} = \frac{\partial V}{\partial x}\phi(x, u_1, B(x, u_1)u_2) = \frac{\partial V}{\partial x}\phi(x, u_1, 0) + \frac{\partial V}{\partial x}(\phi(x, u_1, \tilde{u}_2) - \phi(x, u_1, 0)). \tag{31}$$

As ϕ is locally Lipschitz, there exist a $\psi \in \mathcal{K}_\infty$ and a positive real number $L \geqslant 1$ such that using (31), we get

$$|x|_A \geqslant \tilde{\gamma}_{u_1}(|u_1|) \implies \dot{V} \leqslant -\alpha_3(|x|_A) + |\tilde{u}_2|(L + \psi(\max\{|x|, |u_1|, |\tilde{u}_2|\})).$$

Let us now define $\tilde{\gamma}_{u_2}(e) \triangleq \max\{e, \alpha_3^{-1}(2(1+L)(e))\} \in \mathcal{K}_\infty$ and consider $b(e)$ to be a smooth function which satisfies $b(e) = 1$ in neighborhood of origin of set A (i.e., $\exists$ a $\delta_1 > 0$ such that $b(e) = 1, \forall e \in [0, \delta_1])$ and specifically is chosen so that the following inequality

$$0 < b(e) \leqslant \min\left\{\frac{1}{0.5 + \psi(e)}, 1\right\} \tag{32}$$

holds for all $e \in \mathbb{R}_{\geqslant 0}$. For $X = [x^\top, u_1^\top]^\top$, we assert that the matrix $B(x, u_1) \triangleq b(|X|)I_{m\times m}$ and the function $\gamma_{u_2}(e) = \alpha_1^{-1} \circ \alpha_2 \circ \tilde{\gamma}_{u_2}(e)$ satisfies the requirement of the lemma, i.e.,

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), \tilde{\gamma}_{u_2}(|u_2|)\} \implies \frac{\partial V}{\partial x}\phi(x, u_1, B(x, u_1)u_2) \leqslant -0.5\alpha_3(|x|_A). \tag{33}$$

Using the fact from (32), that $b(|X|) \leqslant 1$, we have

$$|x|_A \geqslant \tilde{\gamma}_{u_1}(|u_1|) \implies \dot{V} \leqslant -\alpha_3(|x|_A) + b(|X|)|u_2|(L + \psi(\max\{|x|, |u_1|, |u_2|\})).$$

Now,

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|\} \implies \dot{V} \leqslant -\alpha_3(|x|_A) + b(|X|)|u_2|(L + \psi(\max\{|x|, |u_1|\})).$$

We use the fact from [38] that $|x|_A \leqslant |x|$ when a compact set contains the origin, which implies $|u_2| \leqslant |x|_A \leqslant |x|$ for the above inequality. So,

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|\} \implies \dot{V} \leqslant -\alpha_3(|x|_A) + b(|X|)|u_2|(L + \psi(|X|)).$$

From (32) we have, $b(e) \leqslant 1$ and $b(e)\psi(e) \leqslant 1$ for $e \geqslant 0$ and so,

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|\} \implies \dot{V} \leqslant -0.5\alpha_3(|x|_A) - 0.5\alpha_3(|x|_A) + (L+1)|u_2|,$$

which can be rewritten as

$$|x|_A \geqslant \max\{\tilde{\gamma}_{u_1}(|u_1|), |u_2|, \alpha_3^{-1}(2(L+1)(|u_2|))\} \implies \dot{V} \leqslant -0.5\alpha_3(|x|_A),$$

where $\tilde{\gamma}_{u_2}(e) \triangleq \max\{s, \alpha_3^{-1}(2(1+L)(e))\}$ and $\gamma_{u_2}(e) = \alpha_1^{-1} \circ \alpha_2 \circ \tilde{\gamma}_{u_2}(e)$. Hence, (33) is satisfied.