Singularly perturbed k-contractive linear systems

Pietro Lorenzetti[†], Mattia Giaccagli[†], Irinel-Constantin Morarescu[†], and Romain Postoyan[†]

***Abstract*—A dynamical system is said to be k-contractive when its trajectories contract k-dimensional volumes. For $k = 1$, this property coincides with the classical notion of contraction. However, for $k > 1$, it allows to characterize a much richer asymptotic behavior. The property of k-contraction has been introduced only recently, thus many analysis tools that are key in relevant applications are currently lacking for k-contractive systems. Motivated by this, we study k-contraction for singularly perturbed systems (SPSs), which naturally arise in many engineering applications. In particular, we focus on singularly perturbed linear time-invariant (LTI) systems. First we show that, for a "sufficiently large" time-scale separation, the k-contraction properties of a SPS can be derived from those of the associated boundary-layer (fast) system, and from the dimension of the reduced order (slow) model. Then, we focus on the case in which the reduced order (slow) model is k-contractive and the boundary-layer (fast) system is 1-contractive. In this setting, we provide a stronger result by showing that the overall system is k-contractive when the time-scale separation is "large".**

***Index Terms*—k-contraction, singular perturbations, linear systems**

I. Introduction

Contraction theory [1], [2] is a great tool for the analysis and design of nonlinear control systems. A system is said to be contractive if the distance between any two trajectories exponentially converges to zero. Contractive systems exhibit many remarkable properties. For instance, under mild assumptions, they possess a unique steady-state bounded solution, which is attractive [2]. Moreover, they are structurally robust to external perturbations [3]. These (and other) properties have been exploited in control problems, such as observer design [4], [5], output regulation [6], [7], and multi-agent synchronization [8], [9], [10], [11]. Thus, given the effectiveness of contraction-based tools, it became natural to search for suitable generalizations of this property, e.g., [12], [13], [14], [15], which could encompass a larger class of dynamical systems.

In this context, we are interested in the notion of k-contraction, introduced in the seminal work [16], and recently used for control purposes in, e.g., [14], [17]. While the trajectories of a contractive system converge to each other, and, thus, contract distances, those of a k-contractive system contract k-dimensional volumes. That is, 1-contraction corresponds to the contraction of lengths (i.e., contraction in the classical sense), 2-contraction corresponds to the contraction of surfaces, 3-contraction to the contraction of volumes, and so on. The interest in this generalization is that k-contractive systems, with $k > 1$, present a much richer asymptotic behavior than contractive ones. For instance, the bounded trajectories of 2-contractive time-invariant systems converge to a (possibly non-unique) equilibrium point, while those of some particular 3-contractive systems to a simple attractor, i.e., a fixed point, a set of fixed points/connecting arcs, or a limit cycle [18].

The notion of k-contraction is promising, and it has been shown to naturally generalize known results for, e.g., Lur'e systems [17], series interconnection [19], and Lyapunov methods [18]. However, being the theory recent, many analysis tools are currently lacking. For instance, analysis tools for singularly perturbed contractive systems have been proposed in [20], [21], where their relevance is shown for applications in biomolecular systems [20], feedback optimization, and game theory [21]. To the best of our knowledge, no generalizations of these tools exist for k-contractive systems. In this work, we address this gap by studying singularly perturbed k-contractive linear time-invariant (LTI) systems, as a first, important step towards a more general theory applicable to nonlinear systems.

Many engineering systems exhibit phenomena happening at different time-scales, which can be captured using the framework of singularly perturbed systems (SPSs) [22]. The idea is then to exploit the time-scale separation by decomposing the original model in two subsystems: a *boundary-layer system*, which describes the fast dynamics, and a *reduced order model*, representing the slow ones. Under suitable assumptions, it is usually shown that if the boundary-layer system and the reduced order model have certain stability properties, then the overall (singularly perturbed) system inherits these properties when the time-scale separation is "sufficiently large" [22].

In this paper, we present analysis tools for singularly perturbed k-contractive LTI systems. We provide two main results. First, we show that, when the boundary-layer system is k-contractive, then the overall system is $(k+n_x)$-contractive where n_x is the dimension of the reduced order model, for a "sufficiently large" time-scale separation. A natural question then is whether having the boundary-layer system k_1-contractive and the reduced order model k_2-contractive ensures that the overall model is $(k_1 + k_2)$-contractive, for a "sufficiently large" time-scale separation. We show that, in general, the answer to this question is no when $k_1 > 1$, via a counter-example. Finally, we focus on the special, relevant case where the boundary-layer system is 1-contractive and the reduced order model k-contractive. Under these assumptions, we show that the overall system is k-contractive when the time-scale separation is "sufficiently large".

Notation. We denote by $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$, respectively, the sets of natural (with $0 \in \mathbb{N}$), real, and complex numbers. Given $a \in \mathbb{C}$, we denote by $\Re(a)$ its real part and by $\Im(a)$ its

[†] Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France. The e-mail addresses are, respectively, pietro.lorenzetti@univ-lorraine.fr, mattia.giaccagli@univ-lorraine.fr, constantin.morarescu@univ-lorraine.fr, and romain.postoyan@univ-lorraine.fr. Research supported by the ANR project COMMITS, number ANR-23-CE25-0005, and the project DECIDE, funded under the PNRR I8 scheme by the Romanian Ministry of Research.

imaginary part. The symbol I_n indicates the identity matrix of dimension $n \times n$. Given a matrix $P \in \mathbb{R}^{n\times m}$, we denote by $\mathrm{rank}(P)$ its rank and by $\mathrm{Img}(P)$ its image. If P is square ($m = n$), $\det(P)$ stands for its determinant. Let $\sigma(P)$ denote the spectrum of P and let $\sigma_-(P)$, $\sigma_0(P)$, $\sigma_+(P)$ indicate the number of eigenvalues of P with negative, zero, and positive real part, respectively (counting multiplicity). When P is symmetric, i.e., $P = P^\top$, we define its *inertia* as the triple of integers $\mathcal{I}(P) := (\sigma_-(P), \sigma_0(P), \sigma_+(P))$. We write $P > 0$ if $P = P^\top$ and $x^\top P x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$; equivalently, $\mathcal{I}(P) = (0, 0, n)$. Similarly, we write $P < 0$ if $-P > 0$. Given p matrices $A_1, \ldots, A_p$ with dimension $A_i \in \mathbb{R}^{n_i \times n_i}$, the symbol $\mathrm{blkdiag}(A_1, \ldots, A_p)$ indicates the $\sum_{i=1}^p n_i \times \sum_{i=1}^p n_i$ block diagonal matrix defined by the blocks A_i and 0 everywhere else. The symbol $|\cdot|$ indicates the (matrix) Euclidean norm. Let $\varrho \in \mathbb{N} \setminus \{0\}$ and $Q : \mathbb{R} \to \mathbb{R}^{p\times q}$ be a matrix function of a scalar variable $\varepsilon \in \mathbb{R}$, we write that $Q(\varepsilon) = O(\varepsilon^\varrho)$ if $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^\varrho} |Q(\varepsilon)| < \infty$. Given two Euclidean spaces U and V, we denote by $C^1(U; V)$ the class of continuously differentiable functions mapping U to V.

II. Preliminaries on k-contraction

Consider a system described by

$$\dot{\mathbf{x}} = f(\mathbf{x}), \tag{1}$$

with $\mathbf{x} \in \mathbb{R}^{n_\mathbf{x}}$, $n_\mathbf{x} \in \mathbb{N} \setminus \{0\}$, and $f \in C^1(\mathbb{R}^{n_\mathbf{x}}; \mathbb{R}^{n_\mathbf{x}})$. We denote by $\mathbf{x}(\mathbf{x}_0, t)$ the solution of (1) evaluated at time $t \geq 0$, starting from initial condition $\mathbf{x}_0 \in \mathbb{R}^{n_\mathbf{x}}$. We assume that system (1) is forward complete, i.e., its trajectories exist for all $t \geq 0$ and all initial conditions.

Given $k \in \{1, \ldots, n_\mathbf{x}\}$, let Φ_k be the set of continuously differentiable *immersions* from $[0, 1]^k$ to $\mathbb{R}^{n_\mathbf{x}}$, i.e.,

$$\Phi_k := \left\{ \varphi \in C^1([0,1]^k; \mathbb{R}^{n_\mathbf{x}}) \,|\, \mathrm{rank}\left(\tfrac{\partial\varphi}{\partial s}(s)\right) = k \;\; \forall s \in [0,1]^k \right\}.$$

Let $P = P^\top > 0$. We define the *k-volume* determined by $\varphi \in \Phi_k$ in the metric associated with P as

$$V^k(\varphi(s)) := \int_{[0,1]^k} \sqrt{\det\left(\tfrac{\partial\varphi}{\partial s}^\top(s) P \tfrac{\partial\varphi}{\partial s}(s)\right)}\, \mathrm{d}s\,. \tag{2}$$

Note that, when $k = 1$, $V^1(\varphi(s))$ is the length of the curve defined by φ, see, e.g., [23, Section 2.1.2].

Definition 1 (k-contraction): Let $k \in \{1, \ldots, n_\mathbf{x}\}$. System (1) is *$k$-contractive* on a forward invariant set $\mathcal{S} \subseteq \mathbb{R}^{n_\mathbf{x}}$ if there exist $\lambda, \gamma > 0$ such that

$$V^k(x(\varphi(s), t)) \leq \gamma V^k(\varphi(s)) \exp(-\lambda t)\,, \tag{3}$$

for all $t \geq 0$ and all $\varphi \in \Phi_k$ such that $\mathrm{Img}(\varphi) \subseteq \mathcal{S}$. □

Property (3) means that k-dimensional volumes contract exponentially along the trajectories of system (1), independently of the choice of φ, and, thus, of the k-volume considered.

In this work, we focus on LTI systems, described by

$$\dot{\mathbf{x}} = A\mathbf{x}, \tag{4}$$

with $A \in \mathbb{R}^{n_\mathbf{x} \times n_\mathbf{x}}$. For LTI systems, 1-contraction is equivalent to global exponential stability of the origin, i.e., to A Hurwitz. Similarly, for $k > 1$, the following equivalent conditions on the eigenvalues of A can be derived. We refer the reader to [14, Remark 1] or [18, Lemma 5] for a proof.

Theorem 1: Consider system (4). Let $\sigma(A) = \{\lambda_1, \ldots, \lambda_{n_\mathbf{x}}\}$ be ordered such that $\Re(\lambda_1) \geq \cdots \geq \Re(\lambda_{n_\mathbf{x}})$. The system is k-contractive on $\mathbb{R}^{n_\mathbf{x}}$, with $k \in \{1, \ldots, n_\mathbf{x}\}$, if and only if $\sum_{i=1}^k \Re(\lambda_i) < 0$. □

An equivalent linear matrix inequality (LMI) characterization for k-contraction is provided in [18, Theorem 2], and reported below. The interested reader can refer to [18, Sect. III-A] for further insights. We exploit this result in Section III.

Theorem 2: Given $k \in \{1, \ldots, n_\mathbf{x}\}$, system (4) is k-contractive on $\mathbb{R}^{n_\mathbf{x}}$ if and only if there exist:

- a positive integer $\ell \in \{1, \ldots, k\}$;
- ℓ real numbers $\mu_i \in \mathbb{R}$, with $i \in \{0, \ldots, \ell - 1\}$;
- ℓ positive integers $d_i \in \mathbb{N}$, $i \in \{1, \ldots, \ell - 1\}$, satisfying

$$0 = d_0 < d_1 < \cdots < d_{\ell-1} = k - 1;$$

- ℓ symmetric matrices $P_i = P_i^\top \in \mathbb{R}^{n_\mathbf{x} \times n_\mathbf{x}}$, $i \in \{0, \ldots, \ell - 1\}$, with inertia $\mathcal{I}(P_i) = (d_i, 0, n_\mathbf{x} - d_i)$;

such that

$$P_i A + A^\top P_i < 2\mu_i P_i \qquad \forall\, i \in \{0, \ldots, \ell - 1\}\,, \tag{5a}$$

$$\text{with} \qquad \sum_{i=0}^{\ell-1} h_i \mu_i \leq 0, \tag{5b}$$

where $h_0 \geq 1$ and $h_i = d_{i+1} - d_i$ for all $i = \{0, \ldots, \ell - 1\}$, with $d_\ell \in \mathbb{N}$ satisfying $d_{\ell-1} + 1 \leq d_\ell \leq k$. □

Remark 1: Condition $\sum_{i=1}^k \Re(\lambda_i) < 0$ implies that there exists $k_- \in \{1, \ldots, k\}$ such that $\Re(\lambda_{k-}) < 0$. Besides, due to the ordering of $\sigma(A)$ in Theorem 1, $\Re(\lambda_j) < 0$ for all $j \in \{k_-, \ldots, n_\mathbf{x}\}$. Therefore, if (4) is k-contractive for some $k \in \{1, \ldots, n_\mathbf{x}\}$, then it is also $(k + \eta)$-contractive for any $\eta \in \{1, \ldots, n_\mathbf{x} - k\}$. □

III. Main results

A. Class of systems

Consider a singularly perturbed LTI system given by

$$\dot{x} = A_{11}x + A_{12}z\,, \tag{6a}$$

$$\varepsilon\dot{z} = A_{21}x + A_{22}z\,, \tag{6b}$$

where $(x, z) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ is the state, with $n_x, n_z \in \mathbb{N} \setminus \{0\}$, $A_{11}, A_{12}, A_{21}, A_{22}$ are real matrices of suitable dimensions, and $\varepsilon > 0$ is the small parameter inducing the time-scale separation between the dynamics of x and the dynamics of z, i.e., (6) exhibits dynamics evolving on two time-scales. The fast dynamics are described by the *boundary-layer system*

$$\varepsilon\dot{x}_f = A_{22}x_f, \tag{7}$$

where $x_f \in \mathbb{R}^{n_z}$ is the "fast" variable. Assuming that A_{22} is an invertible matrix, the slow dynamics are described by the *reduced order model*, given by

$$\dot{x}_s = A_s x_s\,, \tag{8a}$$

with

$$A_s := A_{11} - A_{12}A_{22}^{-1}A_{21}, \tag{8b}$$

where $x_s \in \mathbb{R}^{n_x}$ is the "slow" variable.

We are interested in inferring the k-contraction properties of the SPS (6) from the contractive properties of the boundary-layer system (7) and of the reduced order model (8), when $\varepsilon > 0$ is "sufficiently small".

B. Boundary-layer system k-contractive

We start by assuming that only the boundary-layer system is k-contractive, which yields the following result for (6).

Theorem 3: Consider system (6), and suppose that the following holds.

(A1) The matrix A_{22} is non-singular.
(A2) The boundary-layer system (7) is k-contractive on $\mathbb{R}^{n_z}$ for some $k \in \{1, \ldots, n_z\}$.

Then, there exists $\varepsilon^\star > 0$ such that (6) is $(n_x + k)$-contractive on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ for all $\varepsilon \in (0, \varepsilon^\star)$. □

Proof. For any $\varepsilon > 0$, we denote the state matrix associated to (6) by

$$\widetilde{A}(\varepsilon) := \begin{pmatrix} A_{11} & A_{12} \\ \frac{1}{\varepsilon} A_{21} & \frac{1}{\varepsilon} A_{22} \end{pmatrix}, \tag{9}$$

and its eigenvalues by

$$\{\lambda_1(\varepsilon), \ldots, \lambda_{n_x+n_z}(\varepsilon)\} = \sigma(\widetilde{A}(\varepsilon)).$$

We further denote by $\{\Omega_1, \ldots, \Omega_{n_x}\} = \sigma(A_s)$, ordered as

$$\Re(\Omega_1) \geq \cdots \geq \Re(\Omega_{n_x}), \tag{10}$$

and by $\{\Lambda_1, \ldots, \Lambda_{n_z}\} = \sigma(A_{22})$, ordered as

$$\Re(\Lambda_1) \geq \cdots \geq \Re(\Lambda_{n_z}). \tag{11}$$

From (A2) and Theorem 1, it follows that

$$\gamma := \sum_{j=1}^{k} \Re(\Lambda_j) < 0\,. \tag{12}$$

Since A_{22} is non-singular by (A1), it follows from [22, Theorem 3.1 (Section 2.3)] that, as $\varepsilon \to 0$,

$$\{\lambda_1(\varepsilon), \ldots, \lambda_{n_x}(\varepsilon)\} \to \{\Omega_1, \ldots, \Omega_{n_x}\},$$

while $\{\lambda_{n_x+1}(\varepsilon), \ldots, \lambda_{n_x+n_z}(\varepsilon)\}$ tend to infinity, with rate $\frac{1}{\varepsilon}$, along asymptotes defined by $\Lambda_1, \ldots, \Lambda_{n_z}$, i.e.,

$$\lim_{\varepsilon \to 0} |\lambda_{n_x+i}(\varepsilon) - \tfrac{1}{\varepsilon}\Lambda_i| \to 0 \qquad \forall\, i \in \{1, \ldots, n_z\}.$$

Let $(\eta_-, \eta_0, \eta_+) = (\sigma_-(A_{22}), \sigma_0(A_{22}), \sigma_+(A_{22}))$ and, similarly, $(\rho_-, \rho_0, \rho_+) = (\sigma_-(A_s), \sigma_0(A_s), \sigma_+(A_s))$. It follows from the orderings (10) and (11) that there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_1)$, we have

$$\begin{aligned} &\tfrac{1}{\varepsilon}\Re(\Lambda_1) \geq \cdots \geq \tfrac{1}{\varepsilon}\Re(\Lambda_{\eta_+}) > \Re(\Omega_1) \geq \cdots \geq \Re(\Omega_{\rho_+}) \\ &> \tfrac{1}{\varepsilon}\Re(\Lambda_{\eta_++1}) = \cdots = \tfrac{1}{\varepsilon}\Re(\Lambda_{\eta_++\eta_0}) = 0 \\ &= \Re(\Omega_{\rho_++1}) = \cdots = \Re(\Omega_{\rho_++\rho_0}) \\ &> \Re(\Omega_{\rho_++\rho_0+1}) \geq \cdots \geq \Re(\Omega_{n_x}) \\ &> \tfrac{1}{\varepsilon}\Re(\Lambda_{\eta_++\eta_0+1}) \geq \cdots \geq \tfrac{1}{\varepsilon}\Re(\Lambda_{n_z}). \end{aligned} \tag{13}$$

Clearly, it follows from (12) that $k > \eta_+ + \eta_0$. We denote by $\eta_k := k - (\eta_+ + \eta_0)$. In light of (13), when summing the $n_x + k$ largest eigenvalues of $\widetilde{A}(\varepsilon)$ for $\varepsilon \to 0$, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} \sum_{i=1}^{n_x+k} \Re(\lambda_i(\varepsilon)) \\ &= \lim_{\varepsilon \to 0} \Bigg(\tfrac{1}{\varepsilon} \sum_{i=1}^{\eta_+} \Re(\Lambda_i) + \sum_{j=1}^{\rho_+} \Re(\Omega_j) + \tfrac{1}{\varepsilon} \sum_{i=1}^{\eta_0} \Re(\Lambda_{\eta_++i}) \\ &+ \sum_{j=1}^{\rho_0} \Re(\Omega_{\rho_++j}) + \sum_{j=1}^{\rho_-} \Re(\Omega_{\rho_++\rho_0+j}) + \tfrac{1}{\varepsilon} \sum_{i=1}^{\eta_k} \Re(\Lambda_{\eta_++\eta_0+i}) \Bigg) \\ &= \sum_{j=1}^{n_x} \Re(\Omega_j) + \lim_{\varepsilon \to 0} \tfrac{1}{\varepsilon} \sum_{i=1}^{k} \Re(\Lambda_i) = \sum_{j=1}^{n_x} \Re(\Omega_j) + \lim_{\varepsilon \to 0} \tfrac{\gamma}{\varepsilon}. \end{aligned}$$

As $\gamma < 0$ by (12), there exists $\varepsilon^\star \in (0, \varepsilon_1)$ such that

$$\sum_{i=1}^{n_x+k} \Re(\lambda_i(\varepsilon)) < 0 \qquad \forall\, \varepsilon \in (0, \varepsilon^\star).$$

The proof concludes using the converse part of Theorem 1. ■

C. Boundary-layer system k_1-contractive, reduced order model k_2-contractive

Given the result of Theorem 3, it is natural to ask the following question: If the boundary-layer system (7) is k_1-contractive and the reduced order model (8) is k_2-contractive, does there always exist $\varepsilon^\star > 0$ such that the SPS (6) is $(k_1 + k_2)$-contractive? In general, the answer to this question is no as shown by the example below.

Example. Consider (6) with $A_{11} = \mathrm{blkdiag}(-4.5, -4.5, 4)$, $A_{12} = A_{21} = I_3$, and $A_{22} = \mathrm{blkdiag}(-2, -2, 1)$, which leads to $A_s = \mathrm{blkdiag}(-4, -4, 3)$ in (8). By Theorem 1, both the boundary-layer system and the reduced order model are 2-contractive. Recall the notation $\widetilde{A}(\varepsilon)$ from (9). For $\varepsilon \to 0$, by [22, Theorem 3.1 (Section 2.3)], it follows that

$$\sigma(\widetilde{A}(\varepsilon)) \to \{-\tfrac{2}{\varepsilon}, -\tfrac{2}{\varepsilon}, \tfrac{1}{\varepsilon}\} \cup \{-4, -4, 3\}.$$

Clearly, there exists $\varepsilon_1 > 0$ such that

$$\tfrac{1}{\varepsilon} - 4 - 4 + 3 > 0 \qquad \forall\, \varepsilon \in (0, \varepsilon_1).$$

Therefore, it follows from Theorem 1 that the overall system cannot be 4-contractive for any $\varepsilon > 0$ "sufficiently small". Instead, the best we achieve is for the overall system to be 5-contractive for any $\varepsilon > 0$ "sufficiently small", which coincides with the statement of Theorem 3. □

D. Boundary-layer system 1-contractive, reduced order model k-contractive

Despite the negative answer to the question of Section III.C, there is a relevant, special case for which stronger guarantees can be concluded: when the boundary-layer system is 1-contractive and the reduced order model is k-contractive.

Theorem 4: Consider system (6) and suppose that the following holds.

(A3) The boundary-layer system (7) is 1-contractive on $\mathbb{R}^{n_z}$.

(A4) The reduced order model (8) is k-contractive on $\mathbb{R}^{n_x}$, for some $k \in \{1, \dots, n_x\}$.

Then there exists $\varepsilon^\star > 0$ such that system (6) is k-contractive on $\mathbb{R}^{n_x} \times \mathbb{R}^{n_z}$ for all $\varepsilon \in (0, \varepsilon^\star)$. □

Proof. We start by introducing a change of coordinates[1] for system (6), which simplifies the stability analysis. Let

$$\begin{pmatrix} x_s \\ x_f \end{pmatrix} := T(\varepsilon) \begin{pmatrix} x \\ z \end{pmatrix}, \tag{14}$$

where $T(\varepsilon) := \begin{pmatrix} I_{n_x} - \varepsilon H(\varepsilon) L(\varepsilon) & -\varepsilon H(\varepsilon) \\ L(\varepsilon) & I_{n_z} \end{pmatrix}$. The matrix A_{22} is non-singular, being Hurwitz by Assumption (A3). Therefore, by applying Lemma 1 from the Appendix, we transform system (6) into the system

$$\begin{pmatrix} \dot{x}_s \\ \dot{x}_f \end{pmatrix} = \begin{pmatrix} A_{11} - A_{12} L(\varepsilon) & 0 \\ 0 & \frac{1}{\varepsilon} A_{22} + L(\varepsilon) A_{12} \end{pmatrix} \begin{pmatrix} x_s \\ x_f \end{pmatrix}, \tag{15}$$

for all $\varepsilon \in (0, \varepsilon_1)$, with ε_1 as in Lemma 1. We define

$$\begin{aligned} F(\varepsilon) &:= -\varepsilon A_{12} A_{22}^{-2} A_{21} A_s - A_{12} M(\varepsilon), \\ G(\varepsilon) &:= L(\varepsilon) A_{12}, \end{aligned} \tag{16}$$

where $M(\varepsilon) := L(\varepsilon) - A_{22}^{-1} A_{21} + \varepsilon A_{22}^{-2} A_{21} A_s$. From (34) one has

$$M(\varepsilon) = 2\varepsilon A_{22}^{-2} A_{21} A_s + O(\varepsilon^2), \tag{17}$$

$$G(\varepsilon) = \left(A_{22}^{-1} A_{21} + \varepsilon A_{22}^{-2} A_{21} A_s + O(\varepsilon^2)\right) A_{12}. \tag{18}$$

As a consequence, it follows that $M(\varepsilon) \in O(\varepsilon)$. With the above notation, we can rewrite system (15) as

$$\begin{pmatrix} \dot{x}_s \\ \dot{x}_f \end{pmatrix} = A(\varepsilon) \begin{pmatrix} x_s \\ x_f \end{pmatrix}, \tag{19a}$$

with

$$A(\varepsilon) := \begin{pmatrix} A_s + F(\varepsilon) & 0 \\ 0 & \frac{1}{\varepsilon} A_{22} + G(\varepsilon) \end{pmatrix}, \tag{19b}$$

where, by (16) and by (18),

$$\lim_{\varepsilon \to 0} |F(\varepsilon)| = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} |G(\varepsilon)| = \left|A_{22}^{-1} A_{21} A_{12}\right|. \tag{20}$$

By (A3), A_{22} is Hurwitz. Thus, there exist a symmetric matrix $P_f \in \mathbb{R}^{n_z \times n_z}$, with $P_f > 0$, and $\mu_f < 0$ such that

$$P_f A_{22} + A_{22}^\top P_f < 2\mu_f P_f. \tag{21}$$

On the other hand, by (A4), it follows from the converse part of Theorem 2 that there exist: a positive integer $\ell \in \{1, \dots, k\}$; $\mu_i \in \mathbb{R}$ with $i \in \{1, \dots, \ell-1\}$; $d_i \in \mathbb{R}$ with $i \in \{1, \dots, \ell-1\}$ satisfying $0 = d_1 < \cdots < d_{\ell-1} = k-1$; and symmetric matrices $P_i^s \in \mathbb{R}^{n_x \times n_x}$ with inertia $\mathcal{I}(P_i^s) = (d_i, 0, n_x - d_i)$, $i \in \{1, \dots, \ell-1\}$; such that

$$\begin{aligned} &P_i^s A_s + A_s^\top P_i^s < 2\mu_i P_i^s \qquad \forall\, i \in \{0, \dots, \ell-1\}, \\ &\text{with} \quad \sum_{i=0}^{\ell-1} h_i \mu_i \le 0, \end{aligned} \tag{22}$$

where $h_0 \ge 1$ and $h_i = d_{i+1} - d_i$ and $d_{\ell-1} + 1 \le d_\ell \le k$.

[1]See the Appendix for the definition of the change of coordinates.

Let $P_i := \mathrm{blkdiag}(P_i^s, P_f)$ for all $i \in \{1, \dots, \ell-1\}$. For any $\varepsilon > 0$, we define

$$\mathcal{J}_i(\varepsilon) := P_i A(\varepsilon) + A(\varepsilon)^\top P_i - 2\mu_i P_i = \begin{pmatrix} \overline{J}_i(\varepsilon) & 0 \\ 0 & \underline{J}_i(\varepsilon) \end{pmatrix}, \tag{23}$$

with $i \in \{1, \dots, \ell-1\}$, where $A(\varepsilon)$ is as in (19b) and

$$\begin{aligned} \overline{J}_i(\varepsilon) &:= P_i^s (A_s + F(\varepsilon)) + (A_s + F(\varepsilon))^\top P_i^s - 2\mu_i P_i^s, \\ \underline{J}_i(\varepsilon) &:= P_f (\tfrac{1}{\varepsilon} A_{22} + G(\varepsilon)) + (\tfrac{1}{\varepsilon} A_{22} + G(\varepsilon))^\top P_f - 2\mu_i P_f. \end{aligned}$$

We need to establish that both $\overline{J}_i(\varepsilon)$ and $\underline{J}_i(\varepsilon)$ are negative definite for $\varepsilon \in (0, \varepsilon^\star)$, for some $\varepsilon^\star > 0$ sufficiently small. This is the purpose of the next two claims.

Claim 1. There exists $\varepsilon_3 > 0$ sufficiently small such that, for all $\varepsilon \in (0, \varepsilon_3)$, we have $\overline{J}_i(\varepsilon) < 0$ for all $i \in \{1, \dots, \ell-1\}$.

Proof of Claim 1. The LMIs (22) imply that there exist $\delta_i > 0$ sufficiently small, for all $i \in \{1, \dots, \ell-1\}$, such that

$$P_i^s A_s + A_s^\top P_i^s - 2\mu_i P_i^s < -\delta_i I_{n_x} \quad \forall\, i \in \{0, \dots, \ell-1\}. \tag{24}$$

Therefore, by definition of $\overline{J}_i(\varepsilon)$, for any $\varepsilon > 0$ we have

$$\overline{J}_i(\varepsilon) < -\delta_i I_{n_x} + P_i^s F(\varepsilon) + F(\varepsilon)^\top P_i^s \quad \forall\, i \in \{0, \dots, \ell-1\}.$$

Equivalently, for all $\varepsilon > 0$ and all $x \in \mathbb{R}^{n_x} \setminus \{0\}$, we have

$$x^\top \overline{J}_i(\varepsilon) x < -\delta_i x^\top x + 2x^\top P_i^s F(\varepsilon) x \quad \forall\, i \in \{0, \dots, \ell-1\}.$$

Let $x \in \mathbb{R}^{n_x} \setminus \{0\}$ and $\varepsilon > 0$. Using Young's inequality[2],

$$x^\top \overline{J}_i(\varepsilon) x < -\delta_i x^\top x + \varepsilon x^\top (P_i^s)^2 x + \tfrac{1}{\varepsilon} x^\top F(\varepsilon)^\top F(\varepsilon) x$$

for all $i \in \{0, \dots, \ell-1\}$. From (16) and (17) there exist $\alpha > 0$ (independent of ε) and $\varepsilon_2 \in (0, \varepsilon_1)$ such that

$$\left|F(\varepsilon)^\top F(\varepsilon)\right| \le \alpha \varepsilon^2 \qquad \forall\, \varepsilon \in (0, \varepsilon_2). \tag{25}$$

Using (25), for any $\varepsilon \in (0, \varepsilon_2)$ we have

$$\begin{aligned} x^\top \overline{J}_i(\varepsilon) x &< -\delta_i x^\top x + \varepsilon x^\top (P_i^s)^2 x + \alpha \varepsilon x^\top x \\ &\le x^\top (-\delta_i + \varepsilon(\overline{\lambda}_i + \alpha)) x \quad \forall\, i \in \{0, \dots, \ell-1\}, \end{aligned}$$

where $\overline{\lambda}_i := \max(\sigma(P_i^s))^2 > 0$. Let $\varepsilon_3 \in (0, \varepsilon_2)$ such that

$$\varepsilon_3 < \frac{\delta_i}{\overline{\lambda}_i + \alpha} \qquad \forall\, i \in \{0, \dots, \ell-1\}. \tag{26}$$

It follows that $\overline{J}_i(\varepsilon) < 0$, for any $i \in \{0, \dots, \ell-1\}$ and any $\varepsilon \in (0, \varepsilon_3)$. This concludes the proof of Claim 1. ■

Claim 2. There exists $\varepsilon_4 > 0$ sufficiently small such that, for all $\varepsilon \in (0, \varepsilon_4)$, we have $\underline{J}_i(\varepsilon) < 0$ for all $i \in \{1, \dots, \ell-1\}$.

Proof of Claim 2. Using (21), for all $i \in \{0, \dots, \ell-1\}$ and $\varepsilon > 0$, we have that

$$\underline{J}_i(\varepsilon) < 2(\tfrac{\mu_f}{\varepsilon} - \mu_i) P_f + P_f G(\varepsilon) + G(\varepsilon)^\top P_f.$$

Again, for all $z \in \mathbb{R}^{n_z} \setminus \{0\}$, the above is equivalent to

$$z^\top \underline{J}_i(\varepsilon) z < 2(\tfrac{\mu_f}{\varepsilon} - \mu_i) z^\top P_f z + 2z^\top P_f G(\varepsilon) z.$$

[2]Young's inequality : $2a^\top b \le c a^\top a + \frac{b^\top b}{c}$ for any $a, b \in \mathbb{R}^n$ and $c > 0$. We choose $a = x^\top P_i^s$, $b = F(\varepsilon) x$, and $c = \varepsilon$.

Let $z \in \mathbb{R}^{n_z} \setminus \{0\}$. Using, again, Young's inequality[3] we get

$$z^\top \underline{J}_i(\varepsilon) z < 2(\tfrac{\mu_f}{\varepsilon} - \mu_i) z^\top P_f z + z^\top P_f^2 z + z^\top G(\varepsilon)^\top G(\varepsilon) z$$

for all $i \in \{0, \dots, \ell - 1\}$. We denote by $\overline{\lambda}_f := \max \sigma(P_f)$ and by $\overline{\lambda}_G(\varepsilon) := \max \sigma(G(\varepsilon)^\top G(\varepsilon))$. Since $\mu_f < 0$ and (20) holds, there exists $\varepsilon_4 \in (0, \varepsilon_3)$ such that, for all $\varepsilon \in (0, \varepsilon_4)$,

$$2(\tfrac{\mu_f}{\varepsilon} - \mu_i) + \overline{\lambda}_f + \frac{\overline{\lambda}_G(\varepsilon)}{\overline{\lambda}_f} < 0 \qquad \forall\, i \in \{0, \dots, \ell - 1\}$$

With the above choice, $\underline{J}_i(\varepsilon) < 0$ for any $i \in \{0, \dots, \ell - 1\}$ and any $\varepsilon \in (0, \varepsilon_4)$. This concludes the proof of Claim 2. ■

From Claims 1 and 2, it follows that $\forall \varepsilon \in (0, \varepsilon_4)$, $\mathcal{J}_i(\varepsilon) < 0$, for any $i \in \{0, \dots, \ell - 1\}$. From (23), this is equivalent to

$$P_i A(\varepsilon) + A(\varepsilon)^\top P_i < 2\mu_i P_i \qquad \forall\, \varepsilon \in (0, \varepsilon_4)\,, \tag{27}$$

for any $i \in \{0, \dots, \ell - 1\}$, where, by construction,

$$\mathcal{I}(P_i) = (d_i, 0, n_x + n_z - d_i) \qquad \forall\, i \in \{0, \dots, \ell - 1\}.$$

Therefore, by Theorem 1 system (19a) is k-contractive for any $\varepsilon \in (0, \varepsilon_4)$. Indeed, (27) corresponds to condition (5a), and the second inequality in (22) corresponds to condition (5b).

It remains to show that also (6) in the original coordinates is k-contractive. Since the change of coordinates (14) is linear, it follows from [18, Lemma 1] that system (6) is k-contractive for all $\varepsilon \in (0, \varepsilon_4)$, i.e., we can choose $\varepsilon^\star = \varepsilon_4$. ■

Note that the result of Theorem 4 does not contradict Theorem 3. In fact, as discussed in Remark 1, if a system is k-contractive, then it is also k_0-contractive for any $k_0 > k$.

Remark 2: System (6) inherits the contraction rates μ_i, $i \in \{0, \dots, \ell - 1\}$, of the reduced order (slow) model (8). This is a consequence of two facts: (a) the LMIs in (5a) being *strict*; (b) the contraction rate μ_f of the boundary-layer (fast) system (7) becoming larger and larger (negatively) for $\varepsilon \to 0$. □

Remark 3: It follows from [18, Lemma 1] that the SPS (6) satisfies Theorem 2 with the same μ_i, $i \in \{0, \dots, \ell - 1\}$, of (19a), and with inertia matrices $\widetilde{P}_i := T(\varepsilon)^{-\top} P_i T(\varepsilon)^{-1}$, $i \in \{0, \dots, \ell - 1\}$, where P_i are the inertia matrices defined after (22) and $T(\varepsilon)$ is defined as in (14). The derivation of $\widetilde{P}_i$ follows arguments similar to, e.g., [24, Lemma 2] or [25, Proposition 4.6]. Indeed, since P_i are symmetric and, thus, diagonalizable, $\mathcal{I}(P_i) = \mathcal{I}(\widetilde{P}_i)$, i.e., the inertia is preserved under (well-defined) linear change of coordinates. □

Remark 4: Results in the spirit of Theorem 4 can be found in [26] for p-dominant systems [27]. Although p-dominance and k-contraction are related, they express different system's properties, as discussed thoroughly in [18, Section V.A]. □

IV. Application to Integral Control

Consider the singularly perturbed plant $\mathbf{P}$ described by

$$\mathbf{P} : \begin{cases} \dot{e}_1 = e_2, \\ \varepsilon \dot{e}_2 = e_1 + u, \\ y = e_2, \end{cases} \tag{28}$$

where $y \in \mathbb{R}$ is the measured output, $u \in \mathbb{R}$ is the control input, and $\varepsilon > 0$ is a "small" parameter. The control objective is to design an output feedback controller such that $y(t) \to 0$ as $t \to \infty$ for any initial condition, while guaranteeing boundedness of the closed-loop trajectories. This control problem cannot be solved via a static output feedback design $u = \kappa e_2$ with gain $\kappa \in \mathbb{R}$, since the resulting closed-loop characteristic polynomial would be $p(\lambda) := \lambda^2 - \frac{\kappa}{\varepsilon}\lambda - \frac{1}{\varepsilon}$, which exhibits one root with positive real part for any $\kappa \in \mathbb{R}$. Therefore, we choose instead a proportional-integral controller $\mathbf{C}$ of the form

$$\mathbf{C} : \begin{cases} \dot{\eta} = y, \\ u = -2\eta - y. \end{cases} \tag{29}$$

The resulting closed-loop system $\mathbf{CL}$ is given by

$$\mathbf{CL} : \begin{cases} \dot{e}_1 = e_2, \\ \varepsilon \dot{e}_2 = e_1 - 2\eta - e_2, \\ \dot{\eta} = e_2. \end{cases} \tag{30}$$

System (30) can be written in the form (6), with $x = (e_1, \eta)$, $z = e_2$, $A_{11} = 0$, $A_{12} = (1, 1)^\top$, $A_{21} = (1, -2)$, $A_{22} = -1$. Then, the corresponding boundary-layer (7) is given by

$$\varepsilon \dot{x}_f = -x_f\,, \tag{31}$$

and the corresponding reduced order model (8) is given by

$$\dot{x}_s = A_s x_s\,, \qquad A_s = \begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix}. \tag{32}$$

Clearly, the boundary-layer (31) is 1-contractive. Similarly, since $\sigma(A_s) = \{0, -1\}$, the reduced order model (32) is 2-contractive. Therefore, by Theorem 4, the closed-loop system (30) is 2-contractive for $\varepsilon > 0$ sufficiently small.

The eigenvalues of the state matrix $\widetilde{A}(\varepsilon)$ associated to (30), defined as in (9), are illustrated in Figure 1 for $\varepsilon \in (0, 1]$. It can be seen (and trivially computed) that $\widetilde{A}(\varepsilon)$ has one eigenvalue at zero and two eigenvalues with real part negative for all $\varepsilon > 0$. Thus, the closed-loop system (30) is stable, in the sense of boundedness, for any $\varepsilon \in (0, 1)$. The equilibrium points of (30) are all $(e_1^\star, e_2^\star, \eta^\star) \in \mathbb{R}^3$ such that $e_2^\star = 0$ and $e_1^\star = 2\eta^\star$, which correspond to $y = 0$. Since the closed-loop system (30) is 2-contractive and has bounded trajectories, it follows that every trajectory converges to an equilibrium point for any $\varepsilon \in (0, 1)$, see [28], and the control objective is fulfilled. As a final comment, note that the double integrator in (30) causes the solution (Π, Ψ) of the Sylvester equations $0 = A\Pi + B\Psi$ and $0 = C\Pi$ (with A, B, C defined by the right-hand-side of (28), (29)) to be non-unique even if the system has the same number of inputs and outputs. Hence, the proposed control design does not fit into "classical" output stabilization frameworks, see, e.g., [29, Lemma 4.1].

V. Conclusions

We have presented k-contraction results for singularly perturbed LTI systems. Future works will aim at extending these results to singularly perturbed nonlinear systems. This extension is currently hindered by the lack of tractable conditions for k-contraction of nonlinear systems allowing for a non-constant matrix P. Indeed, we believe that a similar proof strategy as the one used in the proof of Theorem 4 could be employed for nonlinear systems. For this, a nonlinear version of the change of coordinates (14) can be found in [30,

[3] With $a = z^\top P_f$, $b = G(\varepsilon) z$, and $c = 1$.

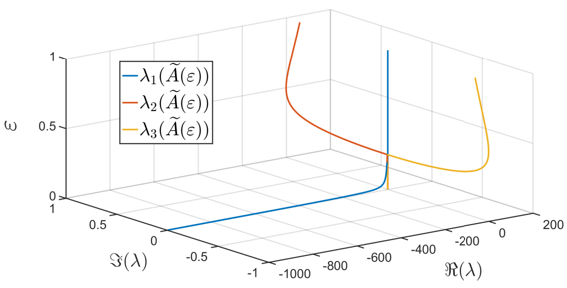

Fig. 1: The eigenvalues of the state matrix $\widetilde{A}(\varepsilon)$ associated to (30), defined as in (9), for $\varepsilon \in (0, 1]$ with a sampling of 10^{-3}. In the legend, we denote $\lambda_i(\widetilde{A}(\varepsilon)) \in \sigma(\widetilde{A}(\varepsilon))$, $i \in \{1, 2, 3\}$.

Chapter 11]. However, this would require a nonlinear version of Theorem 2 allowing for a non-constant matrix P, which is not the case in [18, Theorem 4] (i.e., the nonlinear version of Theorem 2 that is currently available). As suggested by the application in Section V, we believe that the combination of singular perturbation tools with k-contraction properties could provide novel methodologies for stabilization and set-point tracking problems for systems that do not fit into classical output regulation theory framework, such as systems with multiple equilibrium points as those studied in [31], [32], [33].

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Appendix

A. Technical lemma for singularly perturbed linear systems

We report below a technical result, adapted from [22, Sections 2.2 and 2.4], which is used in the proof of Theorem 4.

Lemma 1: Consider system (6) and assume that A_{22} is non-singular. Then for any $\varepsilon \in [0, \varepsilon_1]$, where

$$\varepsilon_1 := \frac{1}{\left|A_{22}^{-1}\right|\left(|A_s|+|A_{12}|\left|A_{22}^{-1}A_{21}\right|+2(|A_s||A_{12}|\left|A_{22}^{-1}\right||A_{21})^{1/2}\right)} > 0\,, \tag{33}$$

there exist $L(\varepsilon) \in \mathbb{R}^{m\times n}$ and $H(\varepsilon) \in \mathbb{R}^{n\times m}$ such that

$$A_{21} - A_{22}L(\varepsilon) + \varepsilon L(\varepsilon)A_{11} - \varepsilon L(\varepsilon)A_{12}L(\varepsilon) = 0\,,$$
$$\varepsilon(A_{11} - A_{12}L(\varepsilon))H(\varepsilon) - H(\varepsilon)(A_{22} + \varepsilon L(\varepsilon)A_{12}) + A_{12} = 0\,.$$

Moreover, with A_s as in (8), $L(\varepsilon)$ and $H(\varepsilon)$ satisfy

$$\begin{aligned} L(\varepsilon) &= A_{22}^{-1}A_{21} + \varepsilon A_{22}^{-2}A_{21}A_s + O(\varepsilon^2), \\ H(\varepsilon) &= A_{12}A_{22}^{-1} + O(\varepsilon). \end{aligned} \tag{34}$$

□