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Robust semi-global output regulation of uncertain two-time-scale systems with input saturation

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Summary

This paper investigates the robust semi-global output regulation of uncertain linear systems with input saturation. In this setup, one cannot apply off-the-shelf techniques to reject structural uncertainties due to numerical issues caused by the two time scales but also due to technical issues generated by the input saturation. To solve this problem, we combine internal model principle with low gain technique and singular perturbation theory. Explicitly, the regulator design is based on the computation of the solutions of a Sylvester equation which can be ill-conditioned in the two time scales setting. To overcome this, we propose an easily solvable equation and the existence and uniqueness of its solution are guaranteed under some standard assumptions. Accordingly, an internal model based controller is designed such that the semi-global robust output regulation problem can be solved. In order to cope with the input saturation we impose that the ultimate upper bound of the infinity-norm of the exosystem state is limited and satisfies certain constraints. Finally, three examples are provided to illustrate the effectiveness of the results.

KEYWORDS:

Two-time-scale, robust output regulation, input saturation, internal model.

1 | INTRODUCTION

Systems involving processes that evolve on both slow and fast time scales are prevalent in various practical applications including electric power management[1], robotics[2], and biology[3]. To mathematically describe the time scale separation, we can introduce a small positive parameter that multiplies the derivative of the fast variables. Doing so one can either develop standard control techniques but they generally cannot be implemented in practice due to numerical issues. Tailored methodological tools for two-time-scale systems (TTSSs) are therefore needed, see e.g.,[4,5,6,7]. Note that the works on TTSSs mainly focused on stabilization problems, with relatively less attention given to output regulation problem, despite its fundamental significance in control theory since 1970s[8]. In[9,10], the output regulation has been respectively achieved for the nonlinear TTSSs and linear T-S fuzzy TTSSs. In[11], the reinforcement learning-based optimal speed regulation control scheme is proposed for the permanent magnet synchronous motor modeled by linear TTSSs. Note that, the disturbance generated by the exosystem affects only the slow subsystem, and the output regulation error is independent of the fast subsystems in[9,10]. It is still worthwhile to further analyze the impact of fast dynamics on the output regulation and explore robust control technique to achieve external disturbance rejection, as well as the output tracking with fast subsystems. The control design in this specific case requires a special attention because the numerical errors hamper the application the control design techniques developed in[8,9,10,11].

It is also noteworthy that the control designs presented in[9,10] are not robust to structural uncertainty, which is crucial for practical applications, such as the control of benchmark mechanical system[12], electrical circuit[13] or wing system[14]. Over the last decade, the internal model principle has been widely employed to achieve the robust output regulation for uncertain linear systems[15,16], nonlinear systems[17,18,19,20] and time-variant systems[21]. However, with the small positive parameter in the fast dynamics, the off-the-shelf techniques cannot be directly applied to reject structural uncertainty for TTSSs.

Moreover, most works on the robust output regulation problem ignore the input saturation, which often appears in practice due to the limitations of physical devices, such as electronic motor systems[22,23] and ships[24]. If the input saturation is not carefully handled it may either damage the installation or lead to poor control performances. The small gain design technique is an effective method for input saturation problem of asymptotically null-controllable linear system[25,26,27], and has been employed to achieve the output regulation for regular linear systems[28] ands singular linear system[29] with input saturation. Furthermore, for linear systems[30] and singular linear systems[31] with anti-stable eigenvalues and subject to input saturation, the output regulation is achieved within the specific initial region, and the composite nonlinear feedback control technique is further used in[32,33] to improve the transient responses. However the proposed controllers are not robust against structured uncertainty which is an important property in the output regulation setting as pointed out in[16,34,35]. For constant perturbations and references, the use of integral action in the presence of input saturation has been well studied in[36,37,38]. However, the aforementioned results are for single time scale systems and, to the best of our knowledge, no results are available for the more general problem of robust output regulation of TTSSs with input saturation.

In this context, we consider the uncertain linear TTSSs subject to input saturation and studied the linear robust output regulation problems, which has been widely studied since 1970s[15,16,34], and can be applied in various practical applications such as speed regulation of DC motor[39] and permanent magnet synchronous motor[40,11]. To eliminate the need for continuous exosystem monitoring, a post-processing internal model is introduced as in[8,41,42]. We first investigate the semi-global stabilization of the augmented system, which comprises the internal model system and the original uncertain TTSS, without the presence of external disturbance. Note that, different from[43,44], the impact of internal model system should be further considered, which would complicate the stabilization control design in the presence of input saturation nonlinearity. Due to the challenges posed by the two time-scale evolution and the input saturation non-linearity, traditional control techniques for single-time-scale linear systems in[41,16] cannot be directly applied here. Thus, the singular perturbation theory is combined with low gain feedback technique to formulate the stabilizing controller. It is also noted that, unlike[9,10], a more general case is considered, i.e., both the slow and fast dynamics are exposed to external disturbance, and the tracking error relies on both the fast and slow states. In this case, solving the corresponding Sylvester equation as constructed in[41,16] would suffer from numerical issues. Hence, we propose a readily solvable Sylvester equation, ensuring the existence and uniqueness of the solution under some standard assumptions. Accordingly, the stabilizing controller is redesigned to guarantee the asymptotically stability of the origin of the augmented system, regardless some small structural uncertainty. Then, an internal model-based tracking controller is devised to achieve the robust semi-global output regulation. The main contribution of this paper is threefold.

1) The robust semi-global output regulation problem is handled for uncertain linear systems exhibiting two time scales and input saturation, where both the slow and fast subsystems are exposed to external disturbances, and the tracking error relies also on the fast states.

2) An internal model-based regulator is proposed by combining the singular perturbation theory and low gain technique. Consequently, the need for continuous monitoring of the exosystem's states is alleviated, and the output regulation error asymptotically converges to the origin, regardless some small structural uncertainty.

3) To avoid the numerical issues, an easily solvable equation is proposed to provide the solution of the corresponding Sylvester function with numerical issues needed for control design. Besides, the existence and uniqueness of its solution are ensured under some standard assumptions.

The rest of the paper is organized as follows. The problem formulation is stated in Section II. The robust semi-global output regulation of uncertain linear TTSS with input saturation is investigated in Section III. Three illustrative examples are presented in Section IV.

Notation: *For a matrix A, $Vec(A)$ denotes the vectorization of A. For a piecewise continuous bounded function $v : [0, \infty) \to \mathbb{R}^m$, and $T \geq 0$, $\|v(t)\|_{\infty,T} \triangleq \sup_{t\geq T} \|v(t)\|_\infty$. The function $f : [0, \infty)^2 \to \mathbb{R}^{m\times n}$ is said to be $O(\varepsilon)$ if there exist positive constants k and ε^* strictly positive such that $\|f(t, \varepsilon)\| \leq k\varepsilon$, for all $t \in [0, \infty)$ and $\varepsilon \in [0, \varepsilon^*]$.*

2 | PROBLEM STATEMENT

In this work, we address the problem of robust output regulation for the TTSSs described below. Notice that, for convenience, we often neglect the time argument of the variables.

$$\begin{cases} \dot{x} = A_{11}(w)x + A_{12}(w)z + B_1(w)\sigma(u) + F_1(w)v, \\ \varepsilon\dot{z} = A_{21}(w)x + A_{22}(w)z + B_2(w)\sigma(u) + F_2(w)v, \\ y = C_1(w)x + C_2(w)z, \\ e = y + Q(w)v, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^{n_x}$, $z \in \mathbb{R}^{n_z}$, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are the slow states, fast states, control input and output, respectively, $0 < \varepsilon \ll 1$, $e \in \mathbb{R}^q$ is the output regulation error, $w \in \mathbb{R}^{n_w}$ is a vector of unknown constant parameters, $v \in \mathbb{R}^{n_v}$ presents the exosystem's state, encompassing both references input and external disturbances. It is generated by an autonomous exosystem in the form

$$\dot{v} = Sv, \tag{2}$$

where S is a constant matrix with appropriate dimensions. The matrices $A_{ij}(w)$, $B_i(w)$, $C_i(w)$, $F_i(w)$ and $Q(w)$ are continuous matrix functions with appropriate dimensions for $w \in \mathbb{W}$, where $i, j = 1, 2$, and $\mathbb{W}$ is an open neighborhood of the origin. For the sake of convenience, $(A_{ij}(0), B_i(0), F_i(0), C_i(0), Q(0))$ is denoted by $(A_{ij}, B_i, F_i, C_i, Q)$, $i, j = 1, 2$, which are all known constant matrices. $\sigma(\cdot)$ is a vector-valued saturation function with

$$\sigma(u) = (\bar{\sigma}(u_1), \bar{\sigma}(u_2), \ldots, \bar{\sigma}(u_p)), \tag{3}$$

where

$$\bar{\sigma}(u_i) = \begin{cases} u_i, & \text{if } |u_i| \le \Upsilon \\ -\Upsilon, & \text{if } u_i < -\Upsilon \\ \Upsilon, & \text{if } u_i > \Upsilon, \end{cases}$$

where Υ is the saturation level. The goal of this paper is to design a controller

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e, \\ u &= K_1 x + K_2 z + G(x, z, \eta), \end{aligned} \tag{4}$$

where $\eta \in \mathbb{R}^{n_\eta}$, such that the robust semi-global output regulation problem can be handled, as formalized next. It is worth noting that the controller (4) does not necessitate the information about the state of the exosystem and is independent of parameter ε.

Problem Statement. Given any arbitrary large compact sets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$ containing the origin, and $\Upsilon > 0$, find dynamic regulator feedback (4), sets $\mathbb{E} \subset \mathbb{R}^{n_\eta}$, $\mathbb{V} \subset \mathbb{R}^{n_v}$, $\mathbb{W} \in \mathbb{R}^{n_w}$ and $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, the closed-loop system (1)-(4) satisfies the followings:

1. *(Stabilization)* when $v = 0$ and $w = 0$, the equilibrium point $(x, z, \eta) = (0, 0, 0)$ is asymptotically stable with a domain of attraction contains the set $\mathbb{X} \times \mathbb{Z}$.

2. *(Robust Output Regulation)* for any $w \in \mathbb{W}$, for any $v(0) \in \mathbb{V}$, for any initial condition $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{E}$, the trajectories of the closed-loop system are bounded for all $t \geq 0$, and satisfies $\lim_{t\to\infty} \|e(t)\| = 0$.

In this paper, our goal is to determine the correspondingly specific region rather than a uniform region, for ε and w that ensures the stabilization or output regulation of the system with priori given compact initial condition sets and saturation level Γ. Note that in view of the constraints of the problem and in particular of asking for a controller (4) independent of the parameter ε and the presence of a saturation, we cannot directly apply the techniques proposed in[28,30]. To solve the above control objective, we make now the following assumptions.

Assumption 1 ([8]). The eigenvalues of matrix S are semi-simple with zero real parts.

Assumption 1 is common and standard for ensuring the neutral stability of system (2), and is widely used in the output regulation literature, see, e.g.[8,41]. Such an assumption guarantees the external signals to be bounded and the well posedness of the problem. In this respect, note that tracking of unbounded signals in the presence of input saturation can never be achieved.

Assumption 2 ([4]). The matrix A_{22} is invertible.

Assumption 3 ([28]). The pairs (A_0, B_0) and (A_{22}, B_2) are asymptotically null controllable with bounded controls (ANCBC), where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$.

Assumption 2 is essential to decouple the slow and fast dynamics, which is standard in the singularly perturbed literature, see, e.g.,[4]. Assumption 3 is an additional but common assumption which is required for semi-global designing small stabilizing control gains for the decoupled slow and fast subsystems. The similar assumption can also be seen in[28]. Under Assumptions 2-3, the eigenvalues of A_{22} are located in the closed left half of the complex plane, excluding the origin. Note that above assumptions on A_{22} are slightly less restrictive than the asymptotic stability condition (Hurwitz condition) of the fast subsystem in[45,46].

Assumption 4 ([8]). $\begin{pmatrix} A_\varepsilon - \lambda I & B_\varepsilon \\ C & 0 \end{pmatrix}$ has independent rows for each λ being an eigenvalues of S, where $A_\varepsilon = E^{-1}A$, $B_\varepsilon = E^{-1}B$, $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$, $C = \begin{pmatrix} C_1 & C_2 \end{pmatrix}$, $E = \mathrm{diag}\{I_{n_x}, \varepsilon I_{n_z}\}$.

The non-resonance condition in Assumption 4 is standard and instrumental for output regulation of TTSSs (1)-(2), which has also been used in[8,41]. Note that Assumptions 1-4 can be satisfied by various practical systems, including the DC motor in[39] and permanent magnet synchronous motor in[40,11].

3 | MAIN RESULT

In this section, the semi-global stabilization problem and robust output regulation problem of TTSSs exhibiting input saturation are investigated.

3.1 | Semi-global stabilization of TTSSs

The semi-global stabilization is studied in this subsection. The goal is to design the controller

$$u = K_1 x + K_2 z + G(x, z, \eta), \tag{5}$$

to handle the semi-global stabilization problem as in Problem Statement above for the following augmented TTSSs

$$\begin{cases} \dot{\eta} = \Phi\eta + \Gamma(C_1 x + C_2 z), \\ \dot{x} = A_{11}x + A_{12}z + B_1\sigma(u), \\ \varepsilon\dot{z} = A_{21}x + A_{22}z + B_2\sigma(u), \end{cases} \tag{6}$$

where the internal model unit of $\eta \in \mathbb{R}^{n_\eta}$ is designed as in[8,41] with $\Gamma = \bar{\Gamma} \otimes I_p$, $\Phi = \bar{\Phi} \otimes I_p$ and the minimal polynomial of $\bar{\Phi}$ equal to that of S and the pair $(\bar{\Phi}, \bar{\Gamma})$ being controllable.

The controller (5) can be designed following a two-step procedure. First, the control gains K_1 and K_2 are designed for the stabilization of TTSSs of (x, z), where

$$\begin{aligned} K_1(\gamma) &:= (1 - K_2(\gamma)A_{22}^{-1}B_2)K_0(\gamma) + K_2(\gamma)A_{22}^{-1}A_{21}, \\ K_2(\gamma) &:= B_2^\top P_2(\gamma), K_0(\gamma) := B_0^\top P_1(\gamma), \end{aligned} \tag{7}$$

and $P_1(\gamma), P_2(\gamma) > 0$ satisfy

$$A_0^\top P_1(\gamma) + P_1(\gamma)A_0 - 2P_1(\gamma)B_0B_0^\top P_1(\gamma) = -\gamma I_{n_x}, \tag{8}$$

$$A_{22}^\top P_2(\gamma) + P_2(\gamma)A_{22} - 2P_2(\gamma)B_2B_2^\top P_2(\gamma) = -\gamma I_{n_z}, \tag{9}$$

where $\gamma > 0$ is the small gain determined in relation to the given saturation level and compact initial conditions. Since $0 < \varepsilon \ll 1$, the eigenvalues of the system matrix would be hard to obtain. The traditional small gain design techniques for the single-time-scale systems with input saturation in[28,30] are no longer directly applicable here to design (K_1, K_2). Thus, the singular perturbed theory is combined with low gain feedback technique to design the control gain K_1 and K_2. Note that equations (8), (9) admits a solution in view of the controllability of the pairs (A_0, B_0) and (A_{22}, B_2) as in Assumption 3.

The second step for the design of the function $G(\cdot)$ is inspired by the so-called forwarding approach[47]. In particular,

$$G(x, z, \eta) = -G_c(\gamma)\xi + \gamma B^\top \overline{M}^\top P_3(\eta - \overline{M}\,\overline{E}\xi), \tag{10}$$

and $\xi = (x, z)$, $G_c(\gamma) = (B_1^\top P_1(\gamma) - B_2^\top(\Lambda_{12}\Lambda_{22}^{-1})^\top P_1(\gamma) + B_2^\top P_2(\gamma)\Lambda_{22}^{-1}\Lambda_{21}, B_2^\top P_2(\gamma))$, $\overline{E} = \text{diag}\{I_{n_x}, 0\}$, $P_3 > 0$ satisfies

$$\Phi^\top P_3 + P_3\Phi \leq 0.$$

Note that P_3 always exists in view of Assmption 1. The matrix $\overline{M}$ satisfies

$$\overline{M}\Lambda = \Phi\overline{M}\,\overline{E} + \Gamma C, \tag{11}$$

where $\Lambda = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}$ with $\Lambda_{ij} = A_{ij} + B_i K_j$, $i, j = 1, 2$. Note that in standard forwarding approach (see e.g.[47]), the design of the feedback relies on the solution of the following sylvester equation

$$M\Lambda_\varepsilon = \Phi M + \Gamma C, \tag{12}$$

in which $\Lambda_\varepsilon = E^{-1}\Lambda$, $E = \text{diag}\{I_{n_x}, \varepsilon I_{n_z}\}$. The proposed modified Sylvester equation (11) allows to find a solution which is independent of the parameter ε. In the next Lemma we establish the existence of the solution $\overline{M}$ to (11) and we show that it provides an ε approximation to the matrix M solution to (12).

Lemma 1. Suppose Λ_{22} and $\Lambda_0 := \Lambda_{11} - \Lambda_{12}\Lambda_{22}^{-1}\Lambda_{21}$ are both Hurwitz, and Assumption 1 holds. There exists a $\bar{\varepsilon} > 0$ such that the equation (11) has a unique solution $\overline{M}$ for any $\varepsilon \in (0, \bar{\varepsilon}]$. Besides, the solution M of (12) satisfies

$$ME^{-1} = \overline{M} + O(\varepsilon).$$

Proof. From Corollary 3.1 in[4], there exists $\bar{\varepsilon}_1 > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, Λ_ε is Hurwitz. Then, $\Lambda = E\Lambda_\varepsilon$ and $\Lambda^{-1} = \Lambda_\varepsilon^{-1}E^{-1}$. In this way, solving the equation (11) is equivalent with solving

$$\overline{M} = \Phi\overline{M}\,\overline{E}\Lambda^{-1} + \Gamma C\Lambda^{-1}, \tag{13}$$

which in turn, is equivalent with solving

$$(I_{n_\eta \times (n_x + n_z)} - \Phi \otimes \overline{E}\Lambda^{-1})\text{Vec}(\overline{M}) = \text{Vec}(\Gamma C\Lambda^{-1}). \tag{14}$$

From (26) in Appendix, we have $\Lambda_\varepsilon^{-1} = T_c A_D^{-1} T_c^{-1}$, where $A_D^{-1} = \text{diag}\{\Lambda_s^{-1}, \varepsilon\Lambda_f^{-1}\}$. Thus,

$$\overline{E}\Lambda^{-1} = \overline{E}T_c A_D^{-1}T_c^{-1}E^{-1} = \begin{pmatrix} \Lambda_s^{-1} - \varepsilon\Lambda_s^{-1}HL + \varepsilon^2 H\Lambda_f^{-1}L & \varepsilon^2 H\Lambda_f^{-1} - \Lambda_s^{-1}H \\ 0 & 0 \end{pmatrix}.$$

Obviously, there exists $0 < \bar{\varepsilon}_3 \leq \bar{\varepsilon}_1$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_3]$, $\Lambda_s^{-1}(I_{n_x} - \varepsilon HL) + \varepsilon^2 H\Lambda_f^{-1}L$ is always Hurwitz. Thus, matrix $I_{n_\eta \times (n_x+n_z)} - \Phi \otimes \overline{E}\Lambda^{-1}$ has no zero eigenvalues, which means that equation (11) has an unique solution $\overline{M}$.

Next, note that from (12) and (11), we have

$$(M - \overline{M}E)\Lambda_\varepsilon = \Phi(M - \overline{M}E) + \Phi\overline{M}(E - \overline{E}). \tag{15}$$

Since Λ_ε is Hurwitz, it is obtained that,

$$(ME^{-1} - \overline{M}) = \Phi(ME^{-1} - \overline{M})\Lambda^{-1} + \varepsilon\Phi\overline{M}(I_{n_x+n_z} - \overline{E})\Lambda^{-1}.$$

Thus, we have $(I_{n_\eta \times (n_x+n_z)} - \Phi \otimes \Lambda^{-1})\text{Vec}(ME^{-1} - \overline{M}) = \varepsilon\text{Vec}(\Phi\overline{M}(I_{n_x+n_z} - \overline{E})\Lambda^{-1})$. Since the spectrum of Λ_ε does not intersect with Φ, $I_{n_\eta \times (n_x+n_z)} - \Phi \otimes \Lambda^{-1}$ is invertible. Thus

$$\|\text{Vec}(ME^{-1} - \overline{M})\| = \varepsilon\|(I_{n_\eta \times (n_x+n_z)} - \Phi \otimes \Lambda^{-1})^{-1}\text{Vec}(\Phi\overline{M}(I_{n_x+n_z} - \overline{E})\Lambda^{-1})\| = O(\varepsilon),$$

which also means that $ME^{-1} = \overline{M} + O(\varepsilon)$. □

Then, the following theorem is obtained, and its proof is provided in the appendix.

Theorem 1. Assuming Assumptions 1-4 hold. For the saturation level $\Upsilon > 0$, and any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, and $\mathbb{Z} \subset \mathbb{R}^{n_z}$, all containing the origin, there exist a compact set $\mathbb{E} \subset \mathbb{R}^{n_\eta}$ containing the origin, $\bar{\varepsilon} > 0$, $\gamma^* \in (0, 1]$, and $\mathbb{W} \subset \mathbb{R}^{n_w}$, so that for any $w \in \mathbb{W}$, $\varepsilon \in (0, \bar{\varepsilon}]$ and $\gamma \in (0, \gamma^*]$, the controller (5), (7) and (10) solves the semi-global stabilization problem as defined in Problem Statement.

Remark 1. Note that, the controller designed in Lemma 2 would be hard to apply, since the solutions M of (12) would be hard to obtain when ε is unknown or when faced with numerical issues arising from a very small ε. This is also one of the challenges in this context. Besides, due to the robustness of the controller and the continuity of its solution to the controller parameters, with small enough ε, the controller designed in Theorem 1 can be used to replace the controller design in Lemma 2 for the stabilization of systems (6).

Following the proof of Theorem 1, for the given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$ and $\mathbb{E} \subset \mathbb{R}^{n_\eta}$, the control design procedure can be proposed as follows to choose specific γ and corresponding $\bar{\varepsilon}$ for the stabilization of TTSS (6).

Step 1: The bisection method is utilized to choose the γ. First, choose $\gamma^* \in (0,1]$. Then, $K_1(\gamma)$, $K_2(\gamma)$ and $G(x,z,\eta)$ can be obtained. Following the proof of Lemma 2, the region $\mathcal{B}(r) := \{\{(x,z,\eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(x,z,\eta)\| \le r\}$ could be found with $(x(t),z(t),\eta(t)) \in \mathcal{B}(r), \forall t \ge 0$ with the given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$ and any given compact subset $\mathbb{E} \subset \mathbb{R}^{n_\eta}$ containing the origin. If $\|u\|_\infty \le \Upsilon, \forall (x,z,\eta \in \mathcal{B}(r)$, then the parameter γ^* is feasible. Otherwise, use $\frac{\gamma^*}{2}$ to repeat the above process until $\|u\|_\infty \le \Upsilon, \forall (x,z,\eta \in \mathcal{B}(r)$ is satisfied.

Step 2: With the specific γ determined in Step 1, one of the upper bound of ε is obtained by determining the upper bound of ε for which $\bar{J}$ defined in (35) is Hurwitz. This involves solving the optimization problem in[48] as follows,

$$\max_{Z_1=Z_1^\top, Z_{2,l}=Z_{2,l}^\top, Z_{3,l}=Z_{3,l}^\top, Z_{4,l}=Z_{4,l}^\top, l=1,\dots,N-1} \bar{\varepsilon},$$
$$s.t.\ \Omega_1 > 0, Z_1 > 0, \Omega_1 + \sum_{l=1}^{k-1} \bar{\varepsilon}^i \Omega_{l+1} > 0 \text{ and } \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{l=1}^{k-1} \bar{\varepsilon}^l \begin{pmatrix} Z_{2,l} & Z_{3,l}^\top \\ Z_{3,l}^\top & Z_{4,l} \end{pmatrix} > 0, k = 2, \dots, N,$$

where $\Omega_l = -(E_J \bar{J})^\top \mathcal{Z}_l - \mathcal{Z}_l^\top E_J \bar{J}$, $l = 1, \dots, N$, $E_J = \operatorname{diag}\{I_{n_\eta + n_x}, \varepsilon I_{n_z}\}$, and $\mathcal{Z}_1 = \begin{pmatrix} Z_1 & 0 \\ Z_{3,1} & Z_{4,1} \end{pmatrix}$, $\mathcal{Z}_l = \begin{pmatrix} Z_{2,l-1} & Z_{3,l-1}^\top \\ Z_{3,l} & Z_{4,l} \end{pmatrix}$, $l = 2, \dots, N-1$, $\mathcal{Z}_N = \begin{pmatrix} Z_{2,N-1} & Z_{3,N-1}^\top \\ 0 & 0 \end{pmatrix}$.

3.2 | Robust Semi-global Output Regulation of TTSSs

The robust semi-global output regulation problem of TTSSs is studied in this subsection.

Based on the stabilizing controller in Theorem 1, the output regulation controller is designed as

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e, \\ u &= K(\gamma)\xi - G_c(\gamma)\xi + \gamma B^\top \overline{M}^\top P_3(\eta - \overline{M}\,\overline{E}\xi), \end{aligned} \tag{16}$$

where $K(\gamma) = (K_1(\gamma), K_2(\gamma))$, and the matrices $G_c(\gamma)$, $K_1(\gamma)$, $K_2(\gamma)$, $\overline{M}$ and P_3 have same definition as in Theorem 1. Then, the following theorem is obtained, and its proof is provided in the appendix.

Theorem 2. Assuming Assumptions 1-4 hold. For the saturation level $\Upsilon > 0$, and any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, and $\mathbb{Z} \subset \mathbb{R}^{n_z}$, all containing the origin, there exist compact sets $\mathbb{E} \subset \mathbb{R}^{n_\eta}$, $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin, $\bar{\varepsilon} > 0$, $\gamma^* \in (0,1]$ and $\mathbb{W} \subset \mathbb{R}^{n_w}$, so that for any $w \in \mathbb{W}$, $\varepsilon \in (0, \bar{\varepsilon}]$ and $\gamma \in (0, \gamma^*]$, the controller (16) solves the robust semi-global output regulation problem as defined in Problem Statement.

Remark 2. Using the internal model design, the output regulation problem can be reformulated as a stabilization problem through the appropriate coordinate transformation. As a result, the regulator (16) takes on a similar structure to the stabilizing controller outlined in Theorem 1. Benefiting from such control design, the proposed controller exhibits structural robustness and achieve the external disturbance rejection in the fast subsystems, which wider applicability contrast with the one in[9,10]. It is important to highlight that, due to the influence of the exosystems, the feasible range of γ in Theorem 2 would deviate from that in Theorem 1.

Following the proof of Theorem 2, for the given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, $\mathbb{E} \subset \mathbb{R}^{n_\eta}$, and compact set $\mathbb{V}$ determined in (42), the control design procedure can be proposed as follows to choose specific γ and corresponding $\bar{\varepsilon}$ and $\mathbb{W}$ for the output regulation of TTSS composed of (1), (2) and (16).

Step 1: The bisection method is utilized to choose the γ. First, choose $\gamma^* \in (0,1]$. Then, $K_1(\gamma)$, $K_2(\gamma)$ and $G(x,z,\eta)$ can be obtained. Similarly, the region $\overline{\mathcal{B}}(r) := \{\{(\bar{\xi},\bar{\eta}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(\bar{\xi},\bar{\eta})\| \le r\}$ can be obtained with $(\bar{\xi}(t),\bar{\eta}(t)) \in \overline{\mathcal{B}}(r), \forall t \ge 0$ with the given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, any given compact subset $\mathbb{E} \subset \mathbb{R}^{n_\eta}$ containing the origin and $\mathbb{V}$ determined in (42). If $\|u\|_\infty \le \Upsilon, \forall (\bar{\xi}(t),\bar{\eta}(t)) \in \overline{\mathcal{B}}(r)$ and $v \in \mathbb{V}$, then the parameter γ^* is feasible. Otherwise, use $\frac{\gamma^*}{2}$ to repeat the above process until $\|u\|_\infty \le \Upsilon, \forall (x,z,\eta) \in \mathcal{B}(r)$ is satisfied.

Step 2: With the specific γ determined in Step 1, one of the upper bound of ε and set $\mathbb{W}$ are derived by handling the optimization problem as follows,

$$\max_{Z_1=Z_1^\top, Z_{2,l}=Z_{2,l}^\top, Z_{3,l}=Z_{3,l}^\top, Z_{4,l}=Z_{4,l}^\top, l=1,\dots,N-1} \bar{\varepsilon} \textit{ and } \mathbb{W},$$

$$s.t.\ \Omega_1 > 0, Z_1 > 0, \Omega_1(w) + \sum_{l=1}^{k-1} \bar{\varepsilon}^l \Omega_{l+1}(w) > 0, \forall w \in \mathbb{W}, \text{and} \begin{pmatrix} Z_1 & 0 \\ 0 & 0 \end{pmatrix} + \sum_{l=1}^{k-1} \bar{\varepsilon}^l \begin{pmatrix} Z_{2,l} & Z_{3,l}^\top \\ Z_{3,l}^\top & Z_{4,l} \end{pmatrix} > 0, k = 2, \dots, N,$$

where $\Omega_l(w) = -(E_J \tilde{J}(w))^\top \mathcal{Z}_l - \mathcal{Z}_l^\top E_J \tilde{J}(w)$, $l = 1, \dots, N$, $\tilde{J}(w)$ is defined in (37), and E_J, $\mathcal{Z}_l$ have the same definition as in Subsection 3.1.

Remark 3. Unlike the approaches in[9,10,11], the proposed control design procedure allows for the further specification of the applicable upper bound $\bar{\varepsilon}$ for the two-time-scale factor ε.

3.3 | The application to single time scale linear systems

The above result is also suitable for single time scale linear systems. Consider system

$$\begin{cases} \dot{x} = A_0(w)x + B_0(w)\sigma(u) + F_0(w)v, \\ e = C(w)x + Q(w)v, \end{cases} \tag{17}$$

in which $(A_0(w), B_0(w), F_0(w), C_0(w), Q_0(w))$ are all continuous matrix functions of $w \in \mathbb{W}$ with appropriate dimensions and v is generated by exosystem (2). For convenience, $(A_0(0), B_0(0), F_0(0), C_0(0), Q_0(0))$ is denoted by $(A_0, B_0, F_0, C_0, Q_0)$. Suppose the matrix $\begin{pmatrix} A_0 - \lambda I & B_0 \\ C_0 & 0 \end{pmatrix}$ has independent rows for each λ being an eigenvalues of S.

Then, the next corollary is proposed. The proof is similar with the one of Theorem 2, thus it is omitted here.

Corollary 1. Assuming Assumptions 1, 3 hold. For the saturation level $\Upsilon > 0$, and any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, and $\mathbb{Z} \subset \mathbb{R}^{n_z}$, all containing the origin, there exist compact sets $\mathbb{E} \subset \mathbb{R}^{n_\eta}$, $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin, $\gamma^* \in (0, 1]$ and $\mathbb{W} \subset \mathbb{R}^{n_w}$, such that for any $\gamma \in (0, \gamma^*]$, and $w \in \mathbb{W}$, the regulator

$$\begin{aligned} \dot{\eta} &= \Phi\eta + \Gamma e, \\ u &= K_0(\gamma)x - B_0^\top P_1(\gamma)x + \gamma B_0^\top M_0{}^\top P_3(\eta - M_0 x), \end{aligned} \tag{18}$$

with $K_0(\gamma)$, $P_1(\gamma)$, P_3, have the same definition in Theorem 1, and M_0 being the solution of

$$M_0(A_0 + B_0 K_0) = \Phi M_0 + \Gamma C, \tag{19}$$

solves the robust semi-global output regulation problem.

Note that we here recover a solution mixing the forwarding approach in[47] and the small-gain feedback in[28]. The control design procedure for choosing the specific γ and corresponding $\mathbb{W}$ is similar to the one in Subsection 3.2.

4 | SIMULATION EXAMPLE

To illustrate the effectiveness of the obtained results, we present three examples including a illustrative case, a comparative case, and a permanent magnet synchronous motor.

4.1 | Example 1: Illustrative example

We consider TTSS (1) and exosystem (2) with $x \in \mathbb{R}^3$, $z \in \mathbb{R}^2$, $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and

$$A(w) = \begin{pmatrix} -2 & -3+10w_1 & -1+2w_2 & 1+w_3 & 0 \\ -5 & -6+w_3 & 3 & -2 & -3 \\ 2 & 4-w2 & -2 & w_1 & 2 \\ 1+w3 & 1+w_1 & 1+2w_2 & w_3 & 1 \\ -1 & 1-w_2 & 1 & -2-w_2 & w_1 \end{pmatrix}, B(w) = \begin{pmatrix} -2 \\ w_1+7 \\ w_3-2 \\ w_2-1 \\ 2+w_1 \end{pmatrix}, F(w) = \begin{pmatrix} w_3-0.2 & 0.3+w_2 & \\ w_1 & 0.1 & \\ 0.2 & -0.1 & \\ 0.1(w_2-1) & 0-w_1 & 0.1 \end{pmatrix}^\top,$$

$$C(w) = \begin{pmatrix} 1+w_3 & w_2 & -1 & 1 & 1+w_1 \end{pmatrix}, Q(w) = \begin{pmatrix} -0.5-w_1 & 0 \end{pmatrix}.$$

Thus, Assumptions 1-4 are satisfied. The simulation is presented with $\varepsilon \in (0, \bar{\varepsilon}]$, $w \in \mathbb{W}$, $x(0) \in \mathbb{X} := \{x | x_1, x_2, x_3 \in (-3, 3)\}$, $z(0) \in \mathbb{Z} := \{z | z_1, z_2 \in (-3, 3)\}$, $v(0) = \mathbb{V} := \{v | v_1, v_2 \in (-1, 1)\}$ and $\Upsilon = 1$. By applying the control design procedure under Theorem 2, we can choose $\gamma = 0.004$, $\Phi = S$, $\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$, $P_3 = 3I_2$, and then choose $\bar{\varepsilon} = 0.01$, $\mathbb{W} = \{w | w_1, w_2, w_3 \in (-0.01, 0.01)\}$. Let $\eta = (0, 0)$. Then, controller (16) is designed. Simulation results are shown in Fig. 1, demonstrating that the error e asymptotically converges to origin without input being saturated for any $\varepsilon \in (0, 0.01]$ and $w \in \mathbb{W}$, which confirms the effectiveness of Theorem 2. We note that the proposed control design procedure is applicable for any arbitrary large initial compact sets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$ containing the origin, i.e, the specific region for ε and w can always be determined to ensures the output regulation of the system with priori given compact initial condition sets.

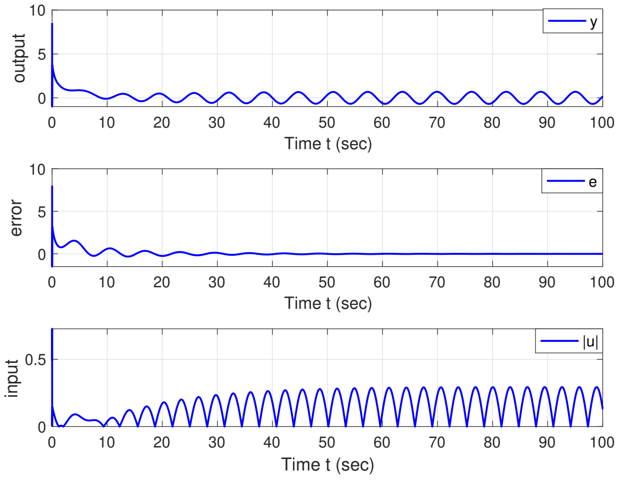

Figure 1 Evolution of y, e and $\|u\|_\infty$.

4.2 | Example 2: Comparative example

For the purpose of comparison, we also consider a case similar to the one described in[9,10]. In this case, we simulate systems (1)-(2) with

$$F(w) = \begin{pmatrix} w_3 - 0.2 & w_1 & 0.2 & 0 & 0 \\ 0.3 + w_2 & 0.1 & -0.1 & 0 & 0 \end{pmatrix}^\top, C(w) = \begin{pmatrix} 1 + w_3 & w_2 & -1 & 0 & 0 \end{pmatrix},$$

while keeping the other parameters and initial conditions as defined above. Then, Assumptions 1-4 are satisfied. Let $\gamma = 0.004$, $\Phi = S$, $\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$, $P_3 = 3I_2$ and $\eta = (0, 0)$. Then, controller (16) is designed. Simulation results are shown in Fig. 2, demonstrating that the regulation error e asymptotically converges to origin without input being saturated.

The simulation is also run for the controller designed in the following form by directed solving the output regulator equation as in[9,10],

$$u = K_1 x + K_2 z + G v, \tag{20}$$

where K_1, K_2 have same definition as in above, $G = \Gamma_c - K_1 \Pi_c$, and Γ_c, Π_c are the solution of the following output regulator equation,

$$\begin{aligned} A_{11}\Pi_c + B_1\Gamma_c + F_1 &= \Pi_c S, \\ C_1\Pi_c + Q &= 0. \end{aligned} \tag{21}$$

The obtained evolution of the regulation error and input are given in Fig. 3. Obviously, under the proposed controller (16), the system (1) has better performance. It is noted that the input is not saturated since K_1 and K_2 are designed by applying the design technique proposed in this paper. However, as shown in Fig. 3, the error e can not converges to origin, since the controller (20) is not robust to the structure uncertainty.

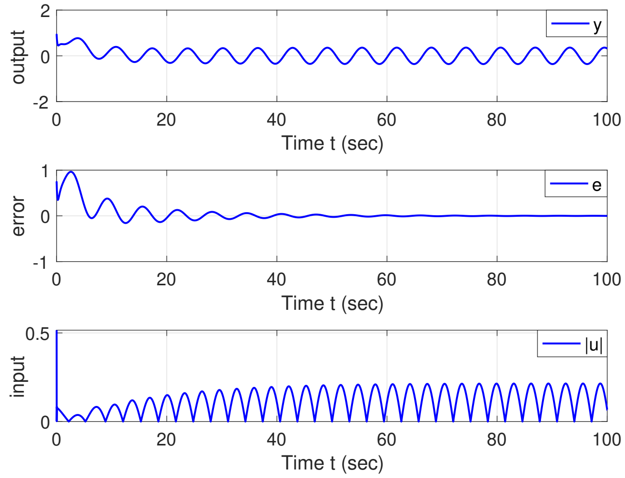

Figure 2 Evolution of y, e and $\|u\|_\infty$ under controller (16).

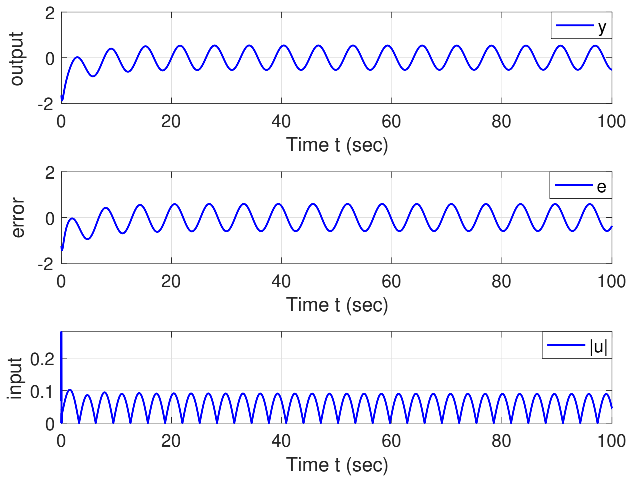

Figure 3 Evolution of y, e and $\|u\|_\infty$ under controller (20).

4.3 | Example 3: Permanent Magnet Synchronous Motor

Consider the permanent magnet synchronous motor (PMSM) as in[40] subject to input saturation,

$$\begin{aligned} J_m\dot{\omega} &= n_p\varphi_f i_q - B\omega - T_L, \\ \bar{L}\dot{i}_q &= -R_s i_q - n_p\varphi_f\omega + \sigma(u), \end{aligned} \tag{22}$$

where ω and i_q are the mechanical rotor speed and stator currents respectively, u is the stator voltages, T_L is the load torque, and J_m, $\bar{L}$, n_p, φ_f, B, R_s are respectively the rotor inertia, stator inductance, number of pole pairs and stator resistance. The goal is to achieve the speed servo of PMSM, i.e., dive the rotating speed tracking the given reference signal ω_{ref}. In this case, the regulation error is defined as $e = \omega_{ref} - \omega$.

Consider $J_m = 0.021\text{kg}\cdot\text{m}^2$, $n_p = 4$, $\varphi_f = w_1 + 0.081\text{Wb}$, $B = 0.0571\text{N}\cdot\text{m}\cdot\text{s/rad}$, $R_s = w_2 + 1.06\Omega$, and T_L, ω_{ref} are described by the exosystem of the form (2) with $T_L = F_1(w)v$ $\omega_{ref} = Q(w)v$, $F_1 = (w_3 - 0.1, 0.1)$, $Q(w) = (0.5 + w_4, 0)$ and

$S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, in which $w = (w_1, w_2, w_3, w_4)$ is the unknown constant parameter vector presenting the structure uncertainty. Obviously, Assumptions 1-4 are satisfied.

The simulation is presented with $\overline{L} \in (0, 0.06]$, $\mathbb{W} = \{w|w_1, w_2, w_3, w_4 \in (-0.01, 0.01)\}$, $\xi(0) \in \{\xi|\omega, i_q \in (-3, 3)\}$, $v(0) = \{v|v_1, v_2 \in (-1, 1)\}$ and $\Upsilon = 1$. Let $\gamma = 0.001$, $\Phi = S$, $\Gamma = \begin{pmatrix} 1 & 0 \end{pmatrix}^\top$, $P_3 = 10 I_2$ and $\eta = (0, 0)$. Then, controller (16) is designed. Simulation results are shown in Fig. 4, demonstrating that the error e asymptotically converges to origin without input being saturated for any $\overline{L} \in (0, 0.06]$ and $w \in \mathbb{W}$, which confirms the effectiveness of Theorem 2.

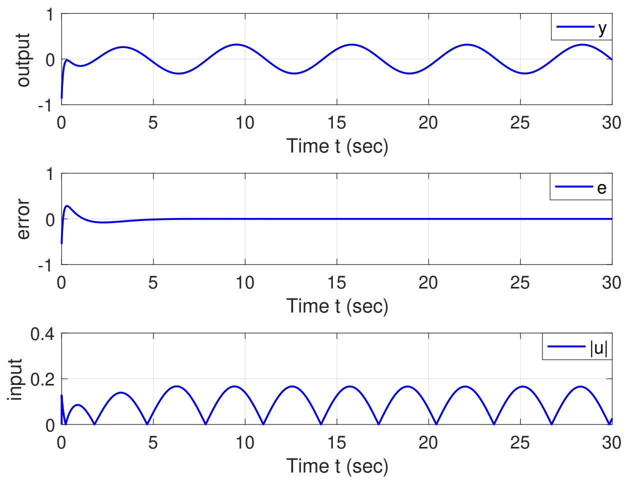

Figure 4 Evolution of y, e and $\|u\|_\infty$ of PMSM.

5 | CONCLUSION

The semi-global robust output regulation was studied for linear TTSSs with input saturation and structural uncertainty. An internal model based state feedback control scheme has been presented with combining the singular perturbed theory and low gain feedback technique. As a result, the regulation error can converge to the origin asymptotically regardless small structural uncertain parameters. In future work, it would be interesting to explore the robust output regulation problem for nonlinear TTSSs with input saturation, and its application to the power systems.

6 | APPENDIX

6.1 | Proof of Theorem 1

The following Lemma is presented firstly for system (6) disregarding the input saturation, for proving Theorem 1.

Lemma 2. Assuming Assumptions 1-4 hold. Consider

$$\begin{cases} \dot{\eta} = \Phi\eta + \Gamma(C_1 x + C_2 z), \\ \dot{x} = A_{11}x + A_{12}z + B_1 u, \\ \varepsilon\dot{z} = A_{21}x + A_{22}z + B_2 u, \\ u = K_1(\gamma)x + K_2(\gamma)z + G_\varepsilon(x, z, \eta), \end{cases} \tag{23}$$

where $K_1(\gamma)$, $K_2(\gamma)$ are defined in (7) and $G_\varepsilon(x, z, \eta) = -B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1}\xi + \gamma B_\varepsilon^\top M^\top P_3(\eta - M\xi)$ with $P_\varepsilon(\gamma) := \mathrm{diag}\{P_1(\gamma), \varepsilon P_2(\gamma)\}$, M and T_c^{-1} as defined (12) and (24), respectively. Then, for any priori given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$, $\mathbb{Z} \subset \mathbb{R}^{n_z}$, and $\mathbb{E} \subset \mathbb{R}^{n_\eta}$ all containing the origin, and for any $\gamma \in (0, 1]$, for all $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{E}$, there exist $\bar{\varepsilon} > 0$

such that, for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\lim_{t\to\infty} \|x(t)\| = 0$, $\lim_{t\to\infty} \|z(t)\| = 0$, $\lim_{t\to\infty} \|\eta(t)\| = 0$. Moreover, there exists $r > 0$, such that for any $t \geq 0$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(x, z, \eta)\| \leq r\}$.

Proof. Firstly, the Chang transformation is introduced as in[4] to conduct the stability analysis, which is presented as follows

$$\begin{pmatrix} x_s \\ z_f \end{pmatrix} := T_c^{-1} \begin{pmatrix} x \\ z \end{pmatrix}, \tag{24}$$

where $T_c^{-1} := \begin{pmatrix} I_{n_x} - \varepsilon HL & -\varepsilon H \\ L & I_{n_z} \end{pmatrix}$, and the matrices L and H are the solution of the following equations

$$\begin{aligned} &\Lambda_{21} - \Lambda_{22} L + \varepsilon L \Lambda_{11} - \varepsilon L \Lambda_{12} L = 0, \\ &\Lambda_{12} - H\Lambda_{22} + \varepsilon \Lambda_{11} H - \varepsilon \Lambda_{12} L H - \varepsilon H L \Lambda_{12} = 0. \end{aligned} \tag{25}$$

As a result, the system can be rewritten as

$$\begin{pmatrix} \dot{x}_s \\ \dot{z}_f \end{pmatrix} = \begin{pmatrix} \Lambda_s & 0 \\ 0 & \frac{\Lambda_f}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_s \\ z_f \end{pmatrix} + T_c^{-1} B_\varepsilon G_\varepsilon(x, z, \eta), \tag{26}$$

where $\Lambda_s = (1 + O(\varepsilon))(A_0 + B_0 K_0)$, $\Lambda_f = (1 + O(\varepsilon))(A_{22} + B_2 K_2)$. From (8) and (9), both $A_0 + B_0 K_0$ and $A_{22} + B_2 K_2$ are Hurwitz. Thus, there is $\bar{\varepsilon}_1 > 0$ so that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, Λ_s, Λ_f are both Hurwitz.

Recall that $P_\varepsilon(\gamma) := \mathrm{diag}\{P_1(\gamma), \varepsilon P_2(\gamma)\}$ and $\xi := (x, z)$. Consider the Lyapunov function candidate

$$\begin{aligned} V :=& \xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \xi + \gamma(\eta - M\xi)^\top P_3(\eta - M\xi) \\ =& x_s^\top P_1 x_s + \varepsilon z_f^\top P_2 z_f + \gamma(\eta - M\xi)^\top P_3(\eta - M\xi). \end{aligned} \tag{27}$$

Thus,

$$\begin{aligned} \dot{V} =& x_s^\top (\Lambda_s^\top P_1(\gamma) + P_1(\gamma)\Lambda_s) x_s + z_f^\top (\Lambda_f^\top P_2(\gamma) + P_2(\gamma)\Lambda_f) z_f + 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} B_\varepsilon G_\varepsilon(x, z, \eta) \\ &+ 2\gamma(\eta - M\xi)^\top P_3(\Phi\eta + \Gamma C\xi - M(\Lambda_\varepsilon \xi + B_\varepsilon G_\varepsilon(x, z, \eta))) \\ \leq& -(1 - O(\varepsilon))(\gamma x_s^\top x_s + \gamma z_f^\top z_f) + 2\xi^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} B_\varepsilon G_\varepsilon(x, z, \eta) + 2\gamma(\eta - M\xi)^\top P_3(\Phi(\eta - M\xi) - M B_\varepsilon G_\varepsilon(x, z, \eta)). \end{aligned}$$

Thus, there exists $0 < \bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$, so that for all $\varepsilon \in (0, \bar{\varepsilon}_2]$, $O(\varepsilon) < \frac{1}{2}$. Thus, for $\varepsilon \in (0, \bar{\varepsilon}_2]$,

$$\dot{V} \leq -\frac{\gamma}{2} x_s^\top x_s - \frac{\gamma}{2} z_f^\top z_f - 2G_\varepsilon(x, z, \eta)^\top G_\varepsilon(x, z, \eta). \tag{28}$$

By applying La Salle's arguments, it is proven that the state of the closed-loop system (23) would converge to the set $\{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : G_\varepsilon(x, z, \eta) = 0, x = 0, z = 0\} = \{B_\varepsilon^\top M^\top P_3 \eta = 0\} \times \{0\} \times \{0\}$. Then, $\lim_{t\to\infty} \|x(t)\| = 0$, $\lim_{t\to\infty} \|z(t)\| = 0$. Once ξ converges to origin, the dynamics of η simplifies to

$$\dot{\eta} = \Phi\eta.$$

According to Proposition 2 in[47], the pair $(B_\varepsilon^\top M^\top P_3, \Phi)$ is observable under Assumption 4. Thus, it has $\lim_{t\to\infty} \|\eta(t)\| = 0$.

Denote

$$J := \begin{pmatrix} \Phi & \Gamma C \\ \gamma B_\varepsilon B_\varepsilon^\top M^\top P_3 & \tilde{\Lambda}_\varepsilon \end{pmatrix},$$

where $\tilde{\Lambda}_\varepsilon = \Lambda_\varepsilon - B_\varepsilon B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon T_c^{-1} - \gamma B_\varepsilon B_\varepsilon^\top M^\top P_3 M$.

Then, it has

$$\begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = J \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{29}$$

Thus, for $\varepsilon \in (0, \bar{\varepsilon}_2]$, J is Hurwitz.

Meanwhile, from (28), there is a class $\mathcal{KL}$ function β_s, so that $\forall t \geq 0$,

$$\|(x_s(t), \sqrt{\varepsilon} z_f(t), \eta(t) - M\xi(t))\| \leq \beta_s(\|(x_s(0), z_f(0), \eta(0))\|, t).$$

Since $(x(0), z(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{E}$, and $\mathbb{X}$, $\mathbb{Z}$, $\mathbb{E}$ are compact subsets, there exists $r_1 > 0$ such that

$$\|(x_s(t), \eta(t) - M\xi(t))\| \leq r_1, \forall t \geq 0. \tag{30}$$

One notices that an ultimate upper bound of $\|z_f(t)\|$ independent of ε cannot be obtained from (28). Thus, the next Lyapunov function candidate is further introduced,

$$V_f := \varepsilon z_f^\top P_2 z_f. \tag{31}$$

With a similar proof in above, it has, for $\varepsilon \in (0, \bar{\varepsilon}_2]$,

$$\begin{aligned}\dot{V}_f \leq & -\frac{\gamma}{2} z_f^\top z_f - 2G_\varepsilon^\top(x,z,\eta)G_\varepsilon(x,z,\eta) + 2\gamma(\eta - M\xi)^\top P_3 M B_\varepsilon G_\varepsilon(x,z,\eta) - 2x_s^\top P_1(B_1 - HB_2 - \varepsilon HLB_1)G_\varepsilon(x,z,\eta)\\ \leq & -\frac{\gamma}{2} z_f^\top z_f + \|\gamma B_\varepsilon^\top M^\top P_3(\eta - M\xi)\|^2 + \|(B_1 - HB_2 - \varepsilon HLB_1)^\top P_1 x_s\|^2\end{aligned}$$

From Lemma 1, we have $B_\varepsilon^\top M^\top = B^\top \overline{M}^\top + O(\varepsilon)$. Then, from (30), it can be obtained that,

$$\dot{V}_f \leq -\frac{\gamma}{2} z_f^\top z_f + r_2. \tag{32}$$

where $r_2 > (\|\gamma B^\top \overline{M}^\top P_3\|^2 + \|(B_1 - HB_2)^\top P_1\|^2 + O(\varepsilon))r_1^2$. Let $\underline{p}$ and $\bar{p}$ denote the minimum and maximum eigenvalue of P, respectively. From (32), it has, for $V_f \geq \frac{4\varepsilon\bar{p}r_2}{\gamma}$,

$$\dot{V}_f \leq -\frac{\gamma}{4} z_f^\top z_f.$$

Obviously, once the state z_f reaches the boundary of $\mathcal{B}_{z_f} := \{z_f | \|z_f(t)\| \leq \sqrt{\frac{2\bar{p}r_2}{\underline{p}\gamma}}\} \supseteq \{z_f | V_f \leq \frac{4\varepsilon r_2}{\gamma}\}$, $\dot{V}_f \leq 0$. Thus, if $z_f(\tau) \in \mathcal{B}_{z_f}$, it follows that for any $t \geq \tau$, $z_f(t) \in \mathcal{B}_{z_f}$. Moreover, if $\|z_f(0)\| \geq \sqrt{\frac{2\bar{p}r_2}{\underline{p}\gamma}}$, and $\|z_f(\tau)\| \geq \sqrt{\frac{2\bar{p}r_2}{\underline{p}\gamma}}$, then for any $t \in [0, \tau)$, it has $\dot{V}_f(t) \leq -\frac{\gamma}{4} z_f^\top z_f$. This implies that there exists a class $\mathcal{KL}$ function β_f, such that for any $t \in [0, \tau)$, $\|z_f(t)\| \leq \beta_f(\|(z_f(0))\|, t)$. In conclusion, for any $t \geq 0$, it follows $\|z_f(t)\| \leq \beta_f(\|(x(0), z(0))\|, t) + \sqrt{\frac{2\bar{p}r_2}{\underline{p}\gamma}}$. There are $\varepsilon_2 > 0$ and $r > 0$ such that for any $t \geq 0$ and $\varepsilon \in (0, \bar{\varepsilon}]$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{\{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(x, z, \eta)\| \leq r\}$. □

From the definition of T_c^{-1} in (24), it has

$$\begin{aligned}& B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1}\\ =& \begin{pmatrix} (B_1^\top - B_2^\top H^\top + O(\varepsilon))P_1(\gamma) & (B_2^\top + O(\varepsilon))P_2(\gamma) \end{pmatrix} T_c^{-1}\\ =& \begin{pmatrix} B_1^\top P_1(\gamma) - B_2^\top (H^\top P_1(\gamma) + P_2(\gamma)L) & B_2^\top P_2(\gamma) \end{pmatrix} + O(\varepsilon).\end{aligned}$$

Meanwhile, from (25), it can be obtained that

$$L = \Lambda_{22}^{-1}\Lambda_{21} + O(\varepsilon),\ H = \Lambda_{12}\Lambda_{22}^{-1} + O(\varepsilon).$$

Thus

$$B_\varepsilon^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} = G_c + O(\varepsilon). \tag{33}$$

From Lemmas 1 and 2, there exists $\bar{\varepsilon}_1 > 0$, such that for any $\varepsilon \in (0, \bar{\varepsilon}_1]$, the matrix J is Hurwitz, and

$$M = \overline{M}E + O(\varepsilon) = \overline{M}\,\overline{E} + O(\varepsilon). \tag{34}$$

Define

$$\overline{J} := \begin{pmatrix} \Phi & \Gamma C \\ \gamma B_\varepsilon B^\top \overline{M}^\top P_3 & \overline{\Lambda}_\varepsilon \end{pmatrix}, \tag{35}$$

where $\overline{\Lambda}_\varepsilon = \Lambda_\varepsilon - B_\varepsilon(G_c + \gamma B^\top \overline{M}^\top P_3 \overline{M}\,\overline{E})$. From (33) and (34),

$$\overline{J} = J + O(\varepsilon).$$

Thus, with a similar proof of theorem 1, there is a small enough $0 < \bar{\varepsilon}_2 \leq \bar{\varepsilon}_1$, so that for any $\varepsilon \in (0, \bar{\varepsilon}]$, J and $\overline{J}$ are both Hurwitz.

Then, from (34) and the continuity of the solution to the controller parameters, for any $\gamma \in (0, 1]$, similarly, there exist $0 < \bar{\varepsilon} < \varepsilon_2$ and $r > 0$ such that $\forall t \geq 0$ and $\varepsilon \in (0, \bar{\varepsilon}]$, $(x(t), z(t), \eta(t)) \in \mathcal{B}(r) := \{\{(x, z, \eta) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(x, z, \eta)\| \leq r\}$. with a similar proof in[27], it can be obtained that there exists $\gamma^* \in (0, 1]$, so that for any $\gamma \in (0, \gamma^*]$ and for any $(x(0), z(0), \eta(0)) \in$

$\mathbb{X} \times \mathbb{Z} \times \mathbb{E}$, $\|u(t)\|_{\infty,0} \leq \Upsilon$. Choose $\gamma \in (0, \gamma^*]$. Then, $\sigma(u(t)) = u(t)$, $\forall t \geq 0$ and TTSS (6) with controller (5) can always be rewritten as follows

$$\begin{pmatrix} \dot{\eta} \\ \dot{\xi} \end{pmatrix} = \bar{J} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \tag{36}$$

Since $\overline{J}$ is Hurwitz, the origin of the closed-loop system (5)-(6) would be asymptotically stable. The semi-global stabilization of system (5)-(6) is achieved.

6.2 | Proof of Theorem 2

Let

$$\tilde{J}(w) := \begin{pmatrix} \Phi & \Gamma C(w) \\ \gamma B_\varepsilon(w) B^\top(0) \overline{M}^\top P_3 & \tilde{\Lambda}_\varepsilon \end{pmatrix}, \tag{37}$$

where $C(w) = \begin{pmatrix} C_1(w) & C_2(w) \end{pmatrix}$ and $\tilde{\Lambda}_\varepsilon = A_\varepsilon(w) + B_\varepsilon(w)K - B_\varepsilon(w)(G_c + \gamma B^\top(0)\overline{M}^\top P_3 \overline{M}\,\overline{E})$. Based on Theorem 1, there exists small enough $\bar{\varepsilon}_1 > 0$, such that for all $\varepsilon \in (0, \bar{\varepsilon}_1]$, the matrix $\overline{J} = \tilde{J}(0)$ is Hurwitz. Thus, there exists $\mathbb{W}_1$, so that for any $w \in \mathbb{W}_1$, the matrix $\tilde{J}(w)$ is always Hurwitz. In this case, there are uniquely defined matrices Π and Σ satisfying

$$\begin{pmatrix} \Sigma \\ \Pi \end{pmatrix} S = \tilde{J}(w) \begin{pmatrix} \Sigma \\ \Pi \end{pmatrix} + \begin{pmatrix} \Gamma Q(w) \\ F_\varepsilon \end{pmatrix}, \tag{38}$$

where $F_\varepsilon = (F_1^\top, \frac{F_2^\top}{\varepsilon})^\top$. Since S and Φ have the same eigenvalues, the above first equation would lead to $C(w)\Pi + Q(w) = 0$[8, Theorem 1.7, pages 24-26].

Let

$$\bar{\xi} := \xi - \Pi v, \bar{\eta} := \eta - \Sigma v. \tag{39}$$

When $\|u(t)\|_{\infty,0} \leq \Upsilon$, the closed-loop system is rewritten as

$$\begin{pmatrix} \dot{\bar{\eta}} \\ \dot{\bar{\xi}} \end{pmatrix} = \tilde{J}(w) \begin{pmatrix} \bar{\eta} \\ \bar{\xi} \end{pmatrix}, \; e = C\bar{\xi}. \tag{40}$$

As Assumption 4 holds, there exists $\mathbb{W}_2 \subset \mathbb{W}_1$, so that for any $w \in \mathbb{W}_2$, the solution of the closed-loop system exists and can be defined as

$$\boldsymbol{\xi} := \Pi_c v, \; \boldsymbol{u} := \Gamma_c v,$$

where $\Pi_c := (\Pi_x, \Pi_z)$ and Γ_c satisfy

$$\begin{aligned} \Pi_x S &= A_{11}(w)\Pi_x + A_{12}(w)\Pi_z + B_1(w)\Gamma_c + F_1(w), \\ \varepsilon \Pi_z S &= A_{21}(w)\Pi_x + A_{22}(w)\Pi_z + B_2(w)\Gamma_c + F_2(w), \\ 0 &= C_1(w)\Pi_x + C_2(w)\Pi_z + Q(w). \end{aligned} \tag{41}$$

It can be easily obtained that the robust output regulation of system (1) can be ensured under controller (16), when there is no input saturation. Thus, we also have $\boldsymbol{\xi} = \Pi v$ and $\boldsymbol{u} = K_1\Pi_x v + K_2\Pi_z v - G_c\Pi v + \gamma B_\varepsilon^\top E\overline{M}^\top P_3(\Sigma v - E\overline{M}\Pi v)$. Thus, $\Pi = \Pi_c$ and $\lim_{\gamma \to 0} K_1\Pi_x + K_2\Pi_z - G_c\Pi + \gamma B_\varepsilon^\top E\overline{M}^\top P_3(\Sigma - E\overline{M}\Pi) = \lim_{\gamma \to 0} \gamma B_\varepsilon^\top E\overline{M}^\top P_3\Sigma = \Gamma_c$.

Then, with a similar proof of Lemma 2, for any given compact sets $\mathbb{X}$, $\mathbb{Z}$, $\mathbb{V}$, $\mathbb{E}$ all containing the origin, there always exist compact sets $\mathbb{X}_r \subset \mathbb{R}^{n_x}$, $\mathbb{Z}_r \subset \mathbb{R}^{n_z}$, $\mathbb{E}_r \subset \mathbb{R}^{n_\eta}$ containing the origin such that for all $(x(0), z(0), v(0), \eta(0)) \in \mathbb{X} \times \mathbb{Z} \times \mathbb{V} \times \mathbb{E}$, $(\bar{\xi}(0), \bar{\eta}(0)) \in \mathbb{X}_r \times \mathbb{Z}_r \times \mathbb{E}_r$. Let

$$\overline{V} := \bar{\xi}^\top (T_c^{-1})^\top P_\varepsilon(\gamma) T_c^{-1} \bar{\xi} + \gamma(\bar{\eta} - M\bar{\xi})^\top P_3(\bar{\eta} - M\bar{\xi}).$$

According to Theorem 1, there exist $0 < \bar{\varepsilon}_3 \leq \varepsilon_2$ and $r > 0$ so that $\forall \varepsilon \in (0, \bar{\varepsilon}_3]$, and $\forall t \geq 0$, $(\bar{\xi}(t), \bar{\eta}(t)) \in \overline{\mathcal{B}}(r) := \{\{(\bar{\xi}, \bar{\eta}) \in \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_\eta} : \|(\bar{\xi}, \bar{\eta})\| \leq r\}$. From (16), we have $u = (K - G_c)(\bar{\xi} + \Pi v) + \gamma B_\varepsilon^\top E\overline{M}^\top P_3(\bar{\eta} - E\overline{M}(\bar{\xi} + \Pi v)) + \gamma B_\varepsilon^\top E\overline{M}^\top P_3\Sigma v$, where $K = \begin{pmatrix} K_1 & K_2 \end{pmatrix}$. Then, $\lim_{\gamma \to 0} u = \gamma B_\varepsilon^\top E\overline{M}^\top P_3\Sigma v = \Gamma_c(w)v = \Gamma_c(0)v + O(\varepsilon)v + O(\|w\|)v$. Denote

$$\mathbb{V} = \{v \in \mathbb{R}^{n_v} : \|v\| \leq \bar{v}\}, \tag{42}$$

with $\bar{v} > 0$ and $\sup_{|v|\leq\bar{v},t\geq t_0} |\Gamma_c(0)e^{S(t-t_0)}v|_\infty < \Upsilon$. Then,

$$\lim_{\gamma\to 0,\varepsilon\to 0,\|w\|\to 0} \|u\| < \Upsilon.$$

In this way, there are $0 < \bar{\varepsilon} \leq \varepsilon_3$ and $\mathbb{W} \subset \mathbb{W}_2$, so that there is a small enough $\gamma^* \in (0,1]$, so that for any $\varepsilon \in (0,\bar{\varepsilon}]$, $w \in \mathbb{W}$ and $\gamma \in (0,\gamma^*]$, $\|u(t)\|_{\infty,0} \leq \Upsilon$. Choose $\gamma \in (0,\gamma^*]$, then TTSS (1), (16) can always be rewritten as (40). Since the matrix $\tilde{J}(w)$ is Hurwitz, the origin of the system is asymptotically stable, which means that $\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|C(w)\bar{\xi}\| = 0$.

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Conflict of interest

The authors declare no potential conflict of interests.

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