Synchronization via output feedback for multi-agent singularly perturbed systems with guaranteed cost

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Abstract

This paper focuses on the problem of designing a decentralized output feedback control strategy for synchronization of homogeneous multi-agent systems with global performance guarantees. The agents under investigation are described as linear singularly perturbed dynamics representing a wide class of physical systems characterized by processes evolving on two time-scales. The collaborative decentralized control is achieved using only output information from neighboring agents and considering that the only available graph information consists in its connectivity, that is, there is no centralized information related to the interconnection network structure. As methodology, the synchronization problem is rewritten as a dynamic output feedback robust stabilization of a singularly perturbed uncertain linear system with guaranteed cost. We show that these problems can be solved by the use of convex conditions expressed by LMIs and by decoupling the slow and fast dynamics. As an advantage, the fast dynamic matrix can be singular (nonstandard systems) and unstable. The proposed conditions circumvent some drawbacks of the existing works on this topic by providing a dynamic controller that does not depend on the singular parameter or by allowing the design of slow controllers when the fast system is stable. Numerical examples are presented to demonstrate the effectiveness of the proposed protocol and design method.

Key words: Multi-agent systems; Synchronization; Output feedback; Guaranteed cost control; Uncertain singularly perturbed systems.

1 Introduction

Decentralized coordination control of multi-agent systems has been an important engineering problem in the last decades due to its capacity to deliver solutions in many emerging fields such as unmanned aerial vehicles, distributed optimization, and formation of mobile robots [2]. An important feature of this class of large scale systems is the fact that local information is used to coordinate a group of autonomous systems to cooperatively accomplish a task or reach an agreement [18] without requiring important amounts of communication and computation with a central entity.

The problem of designing protocols for consensus of multi-agent systems when the states are not available for communication is a recent field of research. The solution involves either dynamic output [14,16] or observer-based [17] protocols. The work [14] considers SISO heterogeneous agents, but the design and the implementation of the controller are complex for high order plants. For homogeneous agents, [16] proposes a design algorithm for a reduced-order observer-type distributed dynamic output-feedback protocol respecting a Sylvester equation and [17] solves two sequential algebraic Riccati equations using a set of scalars to be chosen. Both results demand precisely known Laplacian matrix. Finally, it is worthy to mention that the design of decentralized protocols allowing singularly perturbed multi-agent systems to achieve synchronization has

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been studied very recently [19,20]. We highlight that [20] considers a global performance guarantee when the states of all agents are available to communicate.

In this work, we consider the problem of designing a distributed output-feedback consensus protocol for homogeneous singularly perturbed linear multi-agent systems as a robust design problem of full-order dynamic output feedback (DOF) controllers with a guaranteed cost. We design decentralized output feedback controllers able to achieve asymptotically synchronization with global performance guarantees. For the decentralized guaranteed cost control design, the methodology relies on the transformation of the synchronization problem in an uncertain system stabilization one. We consider a multi-agent system under a fixed but unknown undirected interaction graph. Our results only require that the interconnection graph is connected. Therefore, the eigenvalues of the Laplacian matrix are uncertain but belong to known bounds. As a novelty, we propose a time-scale decomposition of the closed-loop system avoiding the design of a composite control law composed by the slow and fast components, separately designed. This approach allows handling nonstandard singularly perturbed systems where the fast dynamic matrix can be singular or unstable. We also propose the design of low order controllers, independent of the singular parameter, for the case where the singular parameter is unknown or when the actuators cannot respond to the fast variables resulting in controllers economically implementable. To the best of the author's knowledge, no solution has been proposed before for the problem of consensus of singularly perturbed systems with the use of neighbors' output information.

It is noteworthy to highlight that the synchronization problem is not solved once it is transformed in a stabilization one. First we note that we consider multiple time-scales dynamics and it is well known that, applying directly standard control methods may lead to high dimensionality and ill-posed numerical conditions for stability and control design. Singular perturbation framework [3,15] adopts a time-scale decomposition as an efficient way to overcome these problems. In this approach, the control design can be performed for each subsystem (slow and fast lower-order dynamics) and their combination allows to obtain a composite control for the global system [15]. This approach has been extensively used for state-feedback control [15] and, less frequent, observer-based controllers [6]. For instance, the problem of state feedback quadratic optimal control design for linear singularly perturbed systems is addressed in [11]. On the other hand, the output feedback design, without the use of the separation principle (independent design of observer and controller gains), for singularly perturbed systems still remains a field less explored. In [6], an observer-type strictly proper controller is proposed to the H_2 guaranteed cost problem for uncertain singularly perturbed systems where two Riccati equations have to be solved, one for the slow subsystem and the other for the fast subsystem. Some works proposed conditions to design output controllers for singularly perturbed systems based on the decomposition on slow and fast dynamics [4,13,12]. However, these works require the fast subsystem to be asymptotically stable [4,12], the transfer function of the boundary layer inputoutput system to be zero [12] or are based on high-gain observer-based controllers [4,13]. Other disadvantages of the above approaches are the impossibility of dealing with nonstandard singularly perturbed systems, the lack of convex design conditions and the difficulty to impose dynamic controllers that do not dependent on the singular parameter. Another approach relies on the the use of a descriptor representation of the system and a convenient choice of the Lyapunov matrix to obtain ϵ-independent conditions when ϵ approaches to zero, as adopted in [1]. In this case, the time-scale separation is not adopted, the controller has a multi-scale structure and the dynamic matrix of the controller explicitly depends on ϵ, that, for this reason, must be known.

This work proposes the following original contributions: (i) a design of dynamic output protocols by a convex approach (no need of algorithms with sequential steps) and when the Laplacian matrix is uncertain; (ii) an output feedback stabilization method with decomposition-based approach for nonstandard singularly perturbed systems which is used to solve the decentralized output feedback synchronization problem; (iii) a convex design of low-order singular parameter-independent output controller and with no fast components; (iv) guaranteed cost controllers for singularly perturbed multi-agent systems using the output information of the plants.

The presentation is structured as follows. In Section 2 we provide some preliminaries related to the network and controller structure as well as the problem formulation. The main results on the DOF decentralized synchronization for singularly perturbed systems are reported in Section 3. To illustrate the effectiveness of our results we provide some numerical examples in Section 4. The paper ends with some brief conclusions.

Notation. The notation $\mathbb{R}^n$, $\mathbb{R}_+$ and $\mathbb{R}^{n\times m}$ respectively denote the sets of n-dimensional real vectors, positive scalars, and $n \times m$-dimensional real matrices. For a matrix A, consider: A^T denotes the transpose of A; A^{-1} and A^{-T} denote the inverse of A and A^T, respectively; and $He\{A\} = A + A^T$, if A is square. The block-diagonal matrix is denoted by $diag(\cdot)$. The identity matrix of order n is denoted by I_n and the null $m \times n$ matrix is denoted by $0_{m,n}$ (or simply I and 0 if no confusion arises). The symbol $\star$ denotes symmetric blocks in partitioned matrices, and $\otimes$ denotes the Kronecker product.

2 Preliminaries

2.1 Network structure

We consider a set of n identical singularly perturbed linear systems (called agents) described by the following dynamics:

$$\begin{aligned}\begin{bmatrix}\dot{x}_i(t)\\ \epsilon\dot{z}_i(t)\end{bmatrix} &= \begin{bmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{bmatrix}\begin{bmatrix}x_i(t)\\ z_i(t)\end{bmatrix} + \begin{bmatrix}B_1\\ B_2\end{bmatrix}u_i(t)\\ y_i(t) &= \begin{bmatrix}C_1 & C_2\end{bmatrix}\begin{bmatrix}x_i(t)\\ z_i(t)\end{bmatrix}, \quad \forall i \in \{1,\ldots,n\}\end{aligned}, \tag{1}$$

where $x_i(t) \in \mathbb{R}^{n_x}$ and $z_i(t) \in \mathbb{R}^{n_z}$ are the states, $u_i(t) \in \mathbb{R}^{n_u}$ is the control input, $y_i(t) \in \mathbb{R}^{n_y}$ is the measured output, $i = 1,\ldots,n$, and $\epsilon > 0$ is a small parameter characterizing the time-scale separation between the dynamics of variables x_i and z_i.

In the sequel, our objective is to design an output feedback consensus protocol guaranteeing the synchronization of all agents. Before giving the structure of the decentralized controllers proposed in this paper, we will present the structure of the interaction network under consideration. Precisely we consider that each agent has access to relative measurements for the output of some neighbors. The interaction structure is captured by an undirected graph $\mathcal{G}$ and the associated weighted adjacency matrix $G = [g_{ij}] \in \mathbb{R}^{n\times n}$. The corresponding weighted Laplacian matrix is $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ defined by $\begin{cases} l_{ii} = \sum_{j=1}^{n} g_{ij}, & \forall i = 1,\ldots,n\\ l_{ij} = -g_{ij} & \text{if } i \neq j\end{cases}$.

Let us consider in this paper the following assumption.

Assumption 1 *The undirected graph $\mathcal{G}$ is connected and all $g_{ij} \in \{0\} \cup [g_m,\ g_M]$, where $g_M \geq g_m > 0$ are known bounds. The weight $g_{ij} = 0$ if and only if (i,j) is not an edge in the graph $\mathcal{G}$.*

Remark 2 *There exist an orthonormal matrix $T \in \mathbb{R}^{n\times n}$ and positive scalars $\delta_1 < \delta_2$ such that $TLT^T = D = diag(\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $0 = \lambda_1 < \delta_1 < \lambda_2 \leq \ldots \leq \lambda_n < \delta_2$, where λ_i, $i = 1,\ldots,n$, are the eigenvalues of the Laplacian matrix L. The bounds δ_1 and δ_2 can be obtained from g_m and g_M, respectively, as shown in [8].*

Besides synchronizing the states of the n singularly perturbed systems, we also want to impose a threshold on the overall control effort required to achieve this task. Consequently, we consider the following global cost associated with synchronization of the dynamics in (1):

$$\begin{aligned}J = \int_0^{\infty} & x(t)^T(L\otimes I_{n_x})x(t) + z(t)^T(L\otimes I_{n_z})z(t)\\ & + u(t)^T(I_n \otimes R)u(t)dt\end{aligned} \tag{2}$$

where $x(t) = (x_1(t)^T,\ldots,x_n(t)^T)^T \in \mathbb{R}^{nn_x}$, $z(t) = (z_1(t)^T,\ldots,z_n(t)^T)^T \in \mathbb{R}^{nn_z}$ and $u(t) = (u_1(t)^T,\ldots, u_n(t)^T)^T \in \mathbb{R}^{nn_u}$, are the vectors collecting the states and the control input of all agents, and R is a positive definite matrix that penalizes the control effort required for synchronization.

2.2 Controller structure

Let us now introduce the structure of the controller used to synchronize the n systems while keeping the control effort under some threshold. We consider that each system has a local controller that accesses local information, i.e., the output of the system and the output of neighboring systems in the graph. Consequently, we end-up with the problem of designing a distributed dynamic output-feedback consensus protocol with the following structure

$$\begin{aligned}\begin{bmatrix}\dot{\eta}_i(t)\\ \epsilon\dot{\nu}_i(t)\end{bmatrix} &= \begin{bmatrix}A_{c011} & A_{c012}\\ A_{c021} & A_{c022}\end{bmatrix}\begin{bmatrix}\eta_i(t)\\ \nu_i(t)\end{bmatrix}\\ &\quad + \begin{bmatrix}A_{c11} & A_{c12}\\ A_{c21} & A_{c22}\end{bmatrix}\begin{bmatrix}\vartheta_i(\eta(t))\\ \vartheta_i(\nu(t))\end{bmatrix} + \begin{bmatrix}B_{c1}\\ B_{c2}\end{bmatrix}\vartheta_i(y(t))\\ u_i(t) &= \begin{bmatrix}C_{c1} & C_{c2}\end{bmatrix}\begin{bmatrix}\vartheta_i(\eta(t))\\ \vartheta_i(\nu(t))\end{bmatrix} + D_c\vartheta_i(y(t))\end{aligned} \tag{3}$$

where $\eta_i(t) \in \mathbb{R}^{n_x}$ and $\nu_i(t) \in \mathbb{R}^{n_z}$, $i = 1,\ldots,n$, are the states of the distributed controller, $\eta(t) = (\eta_1(t)^T,\ldots,\eta_n(t)^T)^T \in \mathbb{R}^{nn_\eta}$, $\nu(t) = (\nu_1(t)^T,\ldots, \nu_n(t)^T)^T \in \mathbb{R}^{nn_\nu}$, $y(t) = (y_1(t)^T,\ldots,y_n(t)^T)^T \in \mathbb{R}^{nn_y}$, and the function $\vartheta(w) : \mathbb{R}^{nn_w} \to \mathbb{R}^{n_w}$ is defined by $\vartheta_i(w) = \sum_{j=1}^{n} g_{ij}(w_i - w_j)$, where $g_{ij} \neq 0$ means that agent i has access to the output y_j and $g_{ij} = 0$ otherwise.

2.3 Closed-loop and problem formulation

First, observe that the function $\vartheta_i(\cdot)$ is a linear map, then

$$\begin{aligned}\vartheta_i(y) &= \sum_{j=1}^{n} g_{ij}((C_1x_i + C_2z_i) - (C_1x_j + C_2z_j))\\ &= \sum_{j=1}^{n} g_{ij}\left(C_1(x_i - x_j) + C_2(z_i - z_j)\right)\\ &= C_1\vartheta_i(x) + C_2\vartheta_i(z)\end{aligned}$$

and the closed-loop formed by (1) and (3) is given by

$$\begin{bmatrix}\dot{\xi}_i(t)\\ \epsilon\dot{\mu}_i(t)\end{bmatrix}=\begin{bmatrix}\hat{A}_{11} & \hat{A}_{12}\\ \hat{A}_{21} & \hat{A}_{22}\end{bmatrix}\begin{bmatrix}\xi_i(t)\\ \mu_i(t)\end{bmatrix}+\begin{bmatrix}\check{A}_{11} & \check{A}_{12}\\ \check{A}_{21} & \check{A}_{22}\end{bmatrix}\begin{bmatrix}\vartheta_i(\xi(t))\\ \vartheta_i(\mu(t))\end{bmatrix}, \tag{4}$$

with $\xi_i(t) = (x_i(t)^T, \eta_i(t)^T)^T \in \mathbb{R}^{2n_x}$, $\mu_i(t) = (z_i(t)^T, \nu_i(t)^T)^T \in \mathbb{R}^{2n_z}$ and

$$\hat{A}_{ij}=\begin{bmatrix}A_{ij} & 0\\ 0 & A_{c0ij}\end{bmatrix},\quad \check{A}_{ij}=\begin{bmatrix}B_iD_cC_j & B_iC_{cj}\\ B_{ci}C_j & A_{cij}\end{bmatrix},$$

$i, j = 1, 2$.

If we collect the states of all agents, one can rewrite (4) as

$$\begin{bmatrix}\dot{\xi}(t)\\ \epsilon\dot{\mu}(t)\end{bmatrix}=\begin{bmatrix}\bar{A}_{11} & \bar{A}_{12}\\ \bar{A}_{21} & \bar{A}_{22}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \mu(t)\end{bmatrix} \tag{5}$$

with $\xi(t) = (\xi_1(t)^T, \dots, \xi_n(t)^T)^T \in \mathbb{R}^{2nn_x}$, $\mu(t) = (\mu_1(t)^T, \dots, \mu_n(t)^T)^T \in \mathbb{R}^{2nn_z}$ and

$$\begin{aligned}\bar{A}_{11} &= I_n\otimes\hat{A}_{11}-(I_n\otimes\check{A}_{11})(L\otimes I_{2n_x}),\\ \bar{A}_{12} &= I_n\otimes\hat{A}_{12}-(I_n\otimes\check{A}_{12})(L\otimes I_{2n_z}),\\ \bar{A}_{21} &= I_n\otimes\hat{A}_{21}-(I_n\otimes\check{A}_{21})(L\otimes I_{2n_x}),\\ \bar{A}_{22} &= I_n\otimes\hat{A}_{22}-(I_n\otimes\check{A}_{22})(L\otimes I_{2n_z}).\end{aligned}$$

We are now ready to state the problem addressed in this paper.

Problem 3 *For the singularly perturbed multi-agent system* (1) *the design the protocol* (3) *that uses local information such that the closed-loop multi-agent system* (5) *achieves synchronization with a global guaranteed cost* (2) *for a sufficiently small parameter ϵ. In other words, there exist positive scalars $\bar{J}$ and ϵ^* such that*

$$\lim_{t\to\infty}||x_i(t)-x_j(t)||=0,\quad \lim_{t\to\infty}||z_i(t)-z_j(t)||=0$$

and $J \le \bar{J}$ for all $\epsilon \in (0, \epsilon^*)$.

2.4 Reformulation of synchronization as robust stabilization

Following [20], we propose the following change of variable

$$\begin{aligned}\tilde{x}(t) &= (T\otimes I_{n_x})x(t), & \tilde{z}(t) &= (T\otimes I_{n_z})z(t),\\ \tilde{\eta}(t) &= (T\otimes I_{n_x})\eta(t), & \tilde{\nu}(t) &= (T\otimes I_{n_z})\nu(t)\end{aligned} \tag{6}$$

where T is defined in Assumption 1. Then, Problem 3 becomes a robust stability analysis problem for the individual set of $n-1$ dynamics given by

$$\begin{bmatrix}\dot{\tilde{\xi}}_i(t)\\ \epsilon\dot{\tilde{\mu}}_i(t)\end{bmatrix}=\underbrace{\begin{bmatrix}\tilde{A}_{11}(\lambda_i) & \tilde{A}_{12}(\lambda_i)\\ \tilde{A}_{21}(\lambda_i) & \tilde{A}_{22}(\lambda_i)\end{bmatrix}}_{\tilde{A}(\lambda_i)}\begin{bmatrix}\tilde{\xi}_i(t)\\ \tilde{\mu}_i(t)\end{bmatrix},\quad i=2,\dots,n, \tag{7}$$

with $\tilde{\xi}_i(t) = (\tilde{x}_i(t)^T, \tilde{\eta}_i(t)^T)^T \in \mathbb{R}^{2n_x}$, $\tilde{\mu}_i(t) = (\tilde{z}_i(t)^T, \tilde{\nu}_i(t)^T)^T \in \mathbb{R}^{2n_z}$, $\tilde{A}_{k\ell}(\lambda_i) = \hat{A}_{k\ell} - \lambda_i\check{A}_{k\ell}$, $k, \ell = 1, 2$, and $\lambda_i \in [\delta_1,\ \delta_2]$, $i = 2, \dots, n$. Observe that, since $\lambda_1 = 0$, system (7) with $i = 1$ is uncontrolled.

Observe also that the global cost J in (2) can be rewritten as a sum of individual costs associated with the systems in (7): $J = \sum_{i=1}^n J_i$,

$$J_i=\int_0^\infty \lambda_i\tilde{x}_i(t)^T\tilde{x}_i(t)+\lambda_i\tilde{z}_i(t)^T\tilde{z}_i(t)+\tilde{u}_i(t)^TR\tilde{u}_i(t)dt \tag{8}$$

where $\tilde{u}_i(t)$ is the i-th component of $\tilde{u}(t) = (T \otimes I_{n_u})u(t)$. Note that $J_1 = 0$ since $\tilde{u}_i(t) = -\lambda_i[D_cC_1\ C_{c1}]\tilde{\xi}_i(t) - \lambda_i[D_cC_2\ C_{c2}]\tilde{\mu}_i(t)$.

It is interesting to note that the change of variable above decouples the dynamics in n independent ones. The first one describes the synchronization manifold and the rest of them have to be stabilized by the protocol (3). Consequently, solving Problem 3 is equivalent with solving the following stabilization problem.

Problem 4 *Design a protocol* (3) *that uses local information and stabilizes systems* (7) *with guaranteed individual costs J_i, $i = 2, \dots, n$.*

Remark 5 *It is noteworthy that although the further developments may give the sensation that we use λ_i in our design, this is not the case. Indeed, Theorem 11 in Section 3 provides the design of the dynamic feedback controller* (3) *using only decentralized information and the values of δ_1, δ_2.*

Let us finish this section with some lemmas which are instrumental for the further developments.

Lemma 6 *Let a symmetric matrix $M_0 \in \mathbb{R}^{n\times n}$ and matrices $M_1 \in \mathbb{R}^{m\times n}$ and $M_2 \in \mathbb{R}^{m\times n}$. The following conditions are equivalents:*

(i) $M_0 + He\{M_1^TM_2\} < 0$,
(ii) $\exists\ P_1 \in \mathbb{R}^{n\times m}$ *and* $P_2 \in \mathbb{R}^{m\times m}$:

$$\begin{bmatrix}M_0+He\{P_1M_1\} & \star\\ M_2-P_1^T+P_2M_1 & -P_2-P_2^T\end{bmatrix}<0.$$

PROOF. The equivalence can be demonstrated by the well-known Projection Lemma [9] and is omitted for the sake of brevity.

Lemma 7 ([7]) *Consider two symmetric matrices with the following structure*

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} \\ \star & \Theta_{22} \end{bmatrix}, \quad \Upsilon = \begin{bmatrix} 0 & \Upsilon_{12} \\ \star & \Upsilon_{22} \end{bmatrix}$$

Υ_{22} *non singular. Then there exists* ϵ^* *such that for all* $\epsilon \in (0, \epsilon^*)$ *the conditions* (9) *and* (10) *are equivalent.*

$$\Theta + \epsilon^{-1}\Upsilon < 0 \tag{9}$$

$$\Upsilon_{12} = 0, \quad \Theta_{11} < 0, \quad \Upsilon_{22} < 0 \tag{10}$$

3 Main results

In this section we present a solution for Problem 3. Note that the controller (3) is ϵ-dependent and has a two time-scale property. Hence, we suppose that the actuator of the agents are able to respond to the fast variables presented in the signal $u_i(t)$. This assumption will be relaxed later when we present ϵ-independent controllers.

3.1 ϵ-dependent DOF controllers

First, we will rewrite the cost (8) as

$$J_i = \int_0^\infty \tilde{y}_{zi}(t)^T \tilde{y}_{zi}(t)dt$$

where

$$\tilde{y}_{zi}(t) = \begin{bmatrix} C_{z1}(\lambda_i) & C_{z2}(\lambda_i) \end{bmatrix} \begin{bmatrix} \tilde{x}_i(t) \\ \tilde{z}_i(t) \end{bmatrix} + D\tilde{u}_i(t) \tag{11}$$

and

$$\left[\, C_{z1}(\lambda_i) \,\middle|\, C_{z2}(\lambda_i) \,\right] = \left[\begin{array}{c|c} \sqrt{\lambda_i} I_{n_x} & 0 \\ 0 & \sqrt{\lambda_i} I_{n_z} \\ 0 & 0 \end{array}\right], \; D = \begin{bmatrix} 0 \\ 0 \\ \sqrt{R} \end{bmatrix}. \tag{12}$$

Considering the controller (3), we can replace the control signal $\tilde{u}_i(t)$ in (11) by $\tilde{u}_i(t) = -\lambda_i[D_c C_1 \;\; C_{c1}]\tilde{\xi}_i(t) - \lambda_i[D_c C_2 \;\; C_{c2}]\tilde{\mu}_i(t)$ yielding

$$\tilde{y}_{zi}(t) = \underbrace{\begin{bmatrix} \tilde{C}_{z1}(\lambda_i) & \tilde{C}_{z2}(\lambda_i) \end{bmatrix}}_{\tilde{C}_z(\lambda_i)} \begin{bmatrix} \tilde{\xi}_i(t) \\ \tilde{\mu}_i(t) \end{bmatrix}, \tag{13}$$

with

$$\tilde{C}_{zj}(\lambda_i) = \begin{bmatrix} C_{zj}(\lambda_i) - \lambda_i D D_c C_j & -\lambda_i D C_{cj} \end{bmatrix}, \quad j = 1, 2.$$

If we consider a scalar $\gamma > 0$ and the Lyapunov function

$$V(\tilde{\xi}_i, \tilde{\mu}_i) = \begin{bmatrix} \tilde{\xi}_i \\ \tilde{\mu}_i \end{bmatrix}^T W^{-1} \begin{bmatrix} \tilde{\xi}_i \\ \tilde{\mu}_i \end{bmatrix}, \tag{14}$$

$W = W^T > 0$, $W \in \mathbb{R}^{2n_x + 2n_z}$, for the closed-loop system (7), one has that the integration of

$$\dot{V}(\tilde{\xi}_i(t), \tilde{\mu}_i(t)) + \gamma^{-1}\tilde{y}_{zi}(t)^T \tilde{y}_{zi}(t) \le 0, \quad i = 2, \dots, n, \tag{15}$$

over that interval $[0, \infty)$ implies

$$J_i \le \gamma V(\tilde{\xi}_i(0), \tilde{\mu}_i(0)), \quad i = 2, \dots, n, \tag{16}$$

that is, the cost J_i is upper limited by the initial condition $(\xi(0), \mu(0))$ weighted by γW^{-1}. For the minimization of the guaranteed cost of J we can exploit the fact the initial condition of the controller can be set arbitrarily to zero, impose some constraint on the trace of W^{-1} or minimizing γ.

LMI conditions to solve Problem 4 may present ill-conditioned numerical issues for small values of ϵ [15]. Therefore, we adopt the time-scale decomposition [3,15] to define two ϵ-independent subsystems associated with the closed-loop system (7). In the literature of singularly perturbed systems the control law is usually decomposed for each lower-order subsystem and separately designed. Then, the control gains are combined to obtain a composite control for the full system (1) [15]. This approach requires A_{22} to be nonsingular and, in the case of designing DOF controllers in the form (3), yields a non-trivial formulation to recovery the gains of the composite controller from its slow and fast components. A way to circumvent such difficulties consists of performing the time-scale decomposition in the closed-loop system (7) instead of (1). As a consequence, we can deal with nonstandard singularly perturbed systems where matrix A_{22} is not required to be nonsingular.

Let us introduce some notation that allows us to completely decouple the slow and fast dynamics that occur in the overall system. Following the lines presented in [15], the slow subsystem is obtained by setting $\epsilon = 0$ in (7) and expressing the slow part of $\mu_i(t)$, denoted by $\mu_{i,s}(t)$, in terms of the slow part of $\xi_i(t)$, denoted by $\xi_{i,s}(t)$, that is,

$$\mu_{i,s}(t) = -G(\lambda_i)\xi_{i,s}(t),$$

where $G(\lambda_i) = \tilde{A}_{22}(\lambda_i)^{-1}\tilde{A}_{21}(\lambda_i)$. Therefore, the reduced-order (slow) system is

$$\dot{\xi}_{i,s}(t) = A_s(\lambda_i)\xi_{i,s}(t), \quad \xi_{i,s}(0) = \xi_i(0), \tag{17}$$

where $A_s(\lambda_i) = \tilde{A}_{11}(\lambda_i) - \tilde{A}_{12}(\lambda_i)G(\lambda_i)$.

The boundary-layer (fast) system is defined by treating $\xi_i(t)$ as a constant variable and removing the slow bias from $\mu_i(t)$, that is, $\mu_{i,f}(t) = \mu_i(t) - \mu_{i,s}(t)$, yielding

$$\epsilon\dot{\mu}_{i,f}(t) = \tilde{A}_{22}(\lambda_i)\mu_{i,f}(t), \quad \mu_{i,f}(0) = \mu_i(0)+G(\lambda_i)\xi_i(0). \tag{18}$$

Then, one has

$$\begin{aligned} y_{zi}(t) &= \tilde{C}_{z1}(\lambda_i)\xi_{i,s}(t) + \tilde{C}_{z2}(\lambda_i)(\mu_{i,f}(t) - G(\lambda_i)\xi_{i,s}(t)) \\ &= C_s(\lambda_i)\xi_{i,s}(t) + \tilde{C}_{z2}(\lambda_i)\mu_{i,f}(t), \end{aligned} \tag{19}$$

where $C_s(\lambda_i) = \tilde{C}_{z1}(\lambda_i) - \tilde{C}_{z2}(\lambda_i)G(\lambda_i)$.

The system (17) is well-defined if $\tilde{A}_{22}(\lambda_i)$ is non singular. This is verified if there exist matrices $(A_{c022}, A_{c22}, B_{c2}, C_{c2}, D_c)$ such that $\tilde{A}_{22}(\lambda_i)$ is Hurwitz, that is, assuring the asymptotic stability of the fast system (18).

Remark 8 *Considering a decentralized control scheme each agent only knows their local weights and, by Assumption 1, lower and uppers bounds of the weights related to the connections of other agents. As a consequence, the eigenvalues cannot be precisely known and stability conditions must be verified for all values of λ_i such that $\delta_1 \le \lambda_i \le \delta_2$, $i = 2, \ldots, n$. Note that we do not need to solve $n-1$ inequalities since all eigenvalues λ_i, $i = 2, \ldots, n$, belong to the same interval. We first present infinite-dimensional conditions that will be useful for the main results, expressed in terms of bounds δ_1 and δ_2.*

Next, we present ϵ-independent conditions for asymptotically stability with guaranteed cost of the close-loop system (7) in terms of its slow and fast decomposition.

Lemma 9 *Suppose there exist symmetric positive definite matrices $W_1 \in \mathbb{R}^{n_x \times n_x}$ and $W_2 \in \mathbb{R}^{n_z \times n_z}$, and a scalar $\gamma \in \mathbb{R}_+$ verifying the following conditions for all $\lambda \in [\delta_1,\ \delta_2]$:*

$$\begin{bmatrix} A_s(\lambda)W_1 + W_1A_s(\lambda)^T & \star \\ C_s(\lambda)W_1 & -\gamma I \end{bmatrix} < 0, \tag{20}$$

$$\tilde{A}_{22}(\lambda)W_2 + W_2\tilde{A}_{22}(\lambda)^T < 0. \tag{21}$$

Then, there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the closed-loop system (7) is asymptotically stable with guaranteed cost given by $\bar{J}_i = \gamma\tilde{\xi}_i(0)^T W_1^{-1}\tilde{\xi}_i(0) + \gamma\tilde{\mu}_{i,f}(0)^T W_2^{-1}\tilde{\mu}_{i,f}(0)$, $i = 2, \ldots, n$.

PROOF. It follows from the proof of Theorem 1 of [7] by considering the system (7) rewritten as

$$\begin{bmatrix} \dot{\tilde{\xi}}_i(t) \\ \dot{\tilde{\mu}}_i(t) \end{bmatrix} = \mathcal{A}(\lambda_i, \epsilon) \begin{bmatrix} \tilde{\xi}_i(t) \\ \tilde{\mu}_i(t) \end{bmatrix}, \tag{22}$$

$$\mathcal{A}(\lambda_i, \epsilon) = E(\epsilon)^{-1}\tilde{A}(\lambda_i),\ E(\epsilon) = \begin{bmatrix} I & 0 \\ 0 & \epsilon I \end{bmatrix},\ i = 2, \ldots, n.$$

Condition (15) with the Lyapunov function given by (14) is equivalent to

$$W\mathcal{A}(\lambda_i, \epsilon)^T + \mathcal{A}(\lambda_i, \epsilon)W + \gamma^{-1}W\tilde{C}_z(\lambda_i)^T\tilde{C}_z(\lambda_i)W < 0. \tag{23}$$

Consider W with the following partition

$$W = \begin{bmatrix} W_1 & -W_1G(\lambda_i)^T \\ \star & W_2 + G(\lambda_i)W_1G(\lambda_i)^T \end{bmatrix}. \tag{24}$$

Then, if we replace (24) in (23), we obtain an expression in the form of (9) with $\Upsilon_{12} = 0$, $\Theta_{11}(\lambda_i) = A_s(\lambda_i)W_1 + W_1A_s(\lambda_i)^T + \gamma^{-1}W_1C_s(\lambda_i)^TC_s(\lambda_i)W_1$ and $\Upsilon_{22}(\lambda_i) = \tilde{A}_{22}(\lambda_i)W_2 + W_2\tilde{A}_{22}(\lambda_i)^T$. From condition (10) of Lemma 7, $\Upsilon_{22}(\lambda_i) < 0$ and $\Theta_{11}(\lambda_i) < 0$ are equivalent to (23) for $i = 2, \ldots, n$. Finally, applying the Schur complement in $\Theta_{11}(\lambda_i) < 0$ and considering $\lambda_i \in [\delta_1,\ \delta_2]$, $i = 2, \ldots, n$, conditions $\Upsilon_{22}(\lambda) < 0$ and $\Theta_{11}(\lambda) < 0$, $\lambda \in [\delta_1,\ \delta_2]$, are equivalent to (20)–(21) assuring (16) and the asymptotically stability of the closed-loop system (7). Observe that W is a candidate Lyapunov matrix in (14) since $W > 0$ is assured by $W_1 > 0$ and, by Schur complement, $W_2 > 0$.

From (24), one has

$$W^{-1} = \begin{bmatrix} W_1^{-1} + G(\lambda_i)^TW_2^{-1}G(\lambda_i) & G(\lambda_i)^TW_2^{-1} \\ \star & W_2^{-1} \end{bmatrix}$$

then, considering (14) and (16), one has

$$\begin{aligned} J_i &\le \gamma \begin{bmatrix} \tilde{\xi}_i(0) \\ \tilde{\mu}_i(0) \end{bmatrix}^T W^{-1} \begin{bmatrix} \tilde{\xi}_i(0) \\ \mu(0) \end{bmatrix} \\ &= \gamma\tilde{\xi}_i(0)^TW_1^{-1}\tilde{\xi}_i(0) \\ &\quad + \gamma(\mu(0) + G(\lambda_i)\tilde{\xi}_i(0))^TW_2^{-1}(\mu(0) + G(\lambda_i)\tilde{\xi}_i(0)) \\ &= \gamma\tilde{\xi}_i(0)^TW_1^{-1}\tilde{\xi}_i(0) + \gamma\tilde{\mu}_{i,f}(0)^TW_2^{-1}\tilde{\mu}_{i,f}(0). \end{aligned}$$

Observe from Lemma 9 that $V_1(\tilde{\xi}_{i,s}) = \tilde{\xi}_{i,s}^TW_1^{-1}\tilde{\xi}_{i,s}$ and $V_2(\tilde{\mu}_{i,f}) = \tilde{\mu}_{i,f}^TW_2^{-1}\tilde{\mu}_{i,f}$ can be viewed as Lyapunov functions that assure the stability of the slow and fast systems, respectively, for all $\epsilon \in (0, \epsilon^*)$, which agrees with the arguments of time-decomposition as in [15]. However, conditions of Lemma 9 are non-convex for the design of the controller (3).

The design of (3) such that conditions (20) and (21) hold can be done in two steps. First, the gains $(A_{c022}, A_{c22}, B_{c2}, C_{c2}, D_c)$ are designed by condition (21) such that $\tilde{A}_{22}(\lambda_i)$, $i = 2, \ldots, n$, are Hurwitz using standard conditions from the literature of dynamic output feedback control. After that, the obtained gains are used as input parameters in (20) ($\tilde{A}_{22}(\lambda_i)$ becomes an input parameter) and the remaining gains of (3) are obtained by algebraic manipulations in (20). One may note that the design in two independent steps is not convenient since the gains $(A_{c22}, B_{c2}, C_{c2}, D_c)$ may not be suitable to find a solution for (20) or may yield a conservative guaranteed cost.

Therefore, we propose a one-step procedure to design (3) such that (17) and (18) are asymptotic stable with a guaranteed cost. Firstly, we consider the following parametrization for the Lyapunov matrices as adopted in [22]:

$$W_i = \begin{bmatrix} X_i & U_i^T \\ U_i & H_i \end{bmatrix}, \quad W_i^{-1} = \begin{bmatrix} Y_i & V_i \\ V_i^T & Z_i \end{bmatrix}, \quad i = 1, 2, \quad (25)$$

where $X_1 \in \mathbb{R}^{n_x \times n_x}$, $Y_1 \in \mathbb{R}^{n_x \times n_x}$, $H_1 \in \mathbb{R}^{n_x \times n_x}$, $Z_1 \in \mathbb{R}^{n_x \times n_x}$, $X_2 \in \mathbb{R}^{n_z \times n_z}$, $Y_2 \in \mathbb{R}^{n_z \times n_z}$, $H_2 \in \mathbb{R}^{n_z \times n_z}$, $Z_2 \in \mathbb{R}^{n_z \times n_z}$ are symmetric positive definite matrices, $U_1 \in \mathbb{R}^{n_x \times n_x}$, $V_1 \in \mathbb{R}^{n_x \times n_x}$, $U_2 \in \mathbb{R}^{n_z \times n_z}$, $V_2 \in \mathbb{R}^{n_z \times n_z}$ are full row rank. From $W_1^{-1} W_1 = I$ and $W_2^{-2} W_2 = I$, one has $Y_1 X_1 + V_1 U_1 = I$ and $Y_2 X_2 + V_2 U_2 = I$, respectively.

We observe that matrices $\tilde{A}_{11}(\lambda_i)$, $\tilde{A}_{12}(\lambda_i)$, $\tilde{A}_{21}(\lambda_i)$ and $\tilde{A}_{22}(\lambda_i)$ have the same structure, then the product with the Lyapunov matrices can be handled with the congruence transformation and the change of variables proposed by [22]. Define the following nonsingular matrices

$$T_1 = \begin{bmatrix} I & Y_1 \\ 0 & V_1^T \end{bmatrix}, \quad T_2 = \begin{bmatrix} I & Y_2 \\ 0 & V_2^T \end{bmatrix}. \quad (26)$$

Then, pre- and post-multiplying (20) by $diag(T_1^T, I)$ and its transpose, respectively, and introducing the terms $T_2 T_2^{-1} = I$ and $W_2 T_2 T_2^{-1} W_2^{-1} = I$ in appropriate positions, one can observe that the inequality (20) is equivalent to

$$\begin{bmatrix} \text{He}\left\{ T_1^T \tilde{A}_{11}(\lambda_i) W_1 T_1 - \Delta_1 \right\} & \star \\ \tilde{C}_{z1}(\lambda_i) W_1 T_1 - \Delta_2 & -\gamma I \end{bmatrix} < 0, \quad (27)$$

$i = 2, \ldots, n$, where

$$\begin{aligned} \Delta_1 &= T_1^T \tilde{A}_{12}(\lambda_i)(W_2 T_2 T_2^{-1} W_2^{-1}) \tilde{A}_{22}(\lambda_i)^{-1} \\ &\quad \times (T_2^{-T} T_2^T) \tilde{A}_{21}(\lambda_i) W_1 T_1 \\ \Delta_2 &= \tilde{C}_{z2}(\lambda_i)(W_2 T_2 T_2^{-1} W_2^{-1}) \tilde{A}_{22}(\lambda_i)^{-1} \\ &\quad \times (T_2^{-T} T_2^T) \tilde{A}_{21}(\lambda_i) W_1 T_1, \end{aligned}$$

that can be rewritten as

$$\begin{bmatrix} \text{He}\left\{ \Psi_{11}(\lambda_i) - \Psi_{12}(\lambda_i) \Psi_{22}(\lambda_i)^{-1} \Psi_{21}(\lambda_i) \right\} & \star \\ \Lambda_1(\lambda_i) - \Lambda_2(\lambda_i) \Psi_{22}(\lambda_i)^{-1} \Psi_{21}(\lambda_i) & -\gamma I \end{bmatrix} < 0, \quad (28)$$

where $\Psi_{jk}(\lambda_i) = T_j^T \tilde{A}_{jk}(\lambda_i) W_k T_k$ and $\Lambda_j(\lambda_i) = \tilde{C}_{zj}(\lambda_i) W_j T_j$, $j, k = 1, 2$. If we define the following variables

$$L_j = \begin{bmatrix} D_c C_j & C_{cj} \end{bmatrix} \begin{bmatrix} X_j \\ U_j \end{bmatrix}, \quad F_j = \begin{bmatrix} Y_j & V_j \end{bmatrix} \begin{bmatrix} B_j D_c \\ B_{cj} \end{bmatrix}$$

$$Q_{jk} = \begin{bmatrix} Y_j & V_j \end{bmatrix} \begin{bmatrix} B_j D_c C_k & B_j C_{ck} \\ B_{cj} C_k & A_{cjk} \end{bmatrix} \begin{bmatrix} X_k \\ U_k \end{bmatrix},$$

$$S_{jk} = \begin{bmatrix} Y_j & V_j \end{bmatrix} \begin{bmatrix} A_{jk} & 0 \\ 0 & A_{c0jk} \end{bmatrix} \begin{bmatrix} X_k \\ U_k \end{bmatrix}. \quad (29)$$

the terms Ψ_{jk} and Λ_j can be rewritten as

$$\Psi_{jk}(\lambda_i) = \begin{bmatrix} A_{jk} X_k - \lambda_i B_j L_k & A_{jk} - \lambda_i B_j D_c C_k \\ -\lambda_i Q_{jk} + S_{jk} & Y_j A_{jk} - \lambda_i F_j C_k \end{bmatrix},$$

$$\Lambda_j(\lambda_i) = \begin{bmatrix} C_{zj}(\lambda_i) X_j - \lambda_i D L_j & C_{zj}(\lambda_i) - \lambda_i D D_c C_j \end{bmatrix}. \quad (30)$$

Observe in (30) that $\Psi_{jk}(\lambda_i)$ and $\Lambda_j(\lambda_i)$ are affine in the variables X_j, Y_j, L_j, F_j, Q_{jk} and D_c. Exploring this property, Lemma 6 can be applied to decouple the product involving the terms $\Psi_{jk}(\lambda_i)$ and $\Lambda_j(\lambda_i)$ in (28).

Remark 10 *Note that the conditions in Lemma 9 depend on λ_i, that are assumed to be unknown. To obtain finite dimension conditions we represent λ_i and $\sqrt{\lambda_i}$ as convex combinations of their extreme values, that is, $\lambda_i \in Co\{\delta_1, \delta_2\}$ and $\sqrt{\lambda_i} \in Co\{\sqrt{\delta_1}, \sqrt{\delta_2}\}$, where Co denotes the convex hull.*

Using the structure of the Lyapunov matrix (25) and the change of variables (29), we propose convex conditions for Lemma 9 and, therefore, to solve Problem 3.

Theorem 11 *If there exist symmetric positive definite matrices $X_1 \in \mathbb{R}^{n_x \times n_x}$, $Y_1 \in \mathbb{R}^{n_x \times n_x}$, $X_2 \in \mathbb{R}^{n_z \times n_z}$, $Y_2 \in \mathbb{R}^{n_z \times n_z}$, a scalar $\gamma \in \mathbb{R}_+$ and matrices $F_1 \in \mathbb{R}^{n_x \times n_y}$, $L_1 \in \mathbb{R}^{n_u \times n_x}$, $F_2 \in \mathbb{R}^{n_z \times n_y}$, $L_2 \in \mathbb{R}^{n_u \times n_z}$*

$Q_{11} \in \mathbb{R}^{n_x \times n_x}$, $Q_{12} \in \mathbb{R}^{n_x \times n_z}$, $Q_{21} \in \mathbb{R}^{n_z \times n_x}$, $Q_{22} \in \mathbb{R}^{n_z \times n_z}$, $S_{11} \in \mathbb{R}^{n_x \times n_x}$, $S_{12} \in \mathbb{R}^{n_x \times n_z}$, $S_{21} \in \mathbb{R}^{n_z \times n_x}$, $S_{22} \in \mathbb{R}^{n_z \times n_z}$ *and* $D_c \in \mathbb{R}^{n_u \times n_y}$, *a scalar* $\varsigma > 0$, *and a given matrix* $\mathcal{I} \in \mathbb{R}^{2n_x \times 2n_z}$, *such that*

$$\begin{bmatrix} X_i & I \\ I & Y_i \end{bmatrix} > 0, \quad i = 1, 2, \tag{31}$$

$$\Xi_{\ell,\kappa} < 0, \quad \ell, \kappa = 1, 2, \tag{32}$$

hold with

$$\Xi_{\ell,\kappa} = \begin{bmatrix} He\{\Psi_{11}(\delta_\ell) + \Theta(\delta_\ell)\} & \star & \star \\ \Omega(\delta_\ell) & \varsigma He\{\Psi_{22}(\delta_\ell)\} & \star \\ \tilde{\Lambda}_1(\delta_\ell, \delta_\kappa) + \tilde{\Lambda}_2(\delta_\ell, \delta_\kappa)\mathcal{I}^T & \varsigma\tilde{\Lambda}_2(\delta_\ell, \delta_\kappa) & -\gamma I \end{bmatrix}, \tag{33}$$

and

$$\Theta(\delta_\ell) = \mathcal{I}\Psi_{12}(\delta_\ell)^T, \tag{34}$$

$$\Omega(\delta_\ell) = \Psi_{21}(\delta_\ell) + \Psi_{22}(\delta_\ell)\mathcal{I}^T + \varsigma\Psi_{12}(\delta_\ell)^T, \tag{35}$$

$$\tilde{\Lambda}_i(\delta_\ell, \delta_\kappa) = \begin{bmatrix} C_{zi}(\delta_\kappa)X_i - \delta_\ell D L_i & C_{zi}(\delta_\kappa) - \delta_\ell D D_c C_i \end{bmatrix}, \tag{36}$$

$i = 1, 2$, *with* $\Psi_{ij}(\cdot)$, $i, j = 1, 2$, *given by* (30), *then there exists* $\epsilon^* > 0$ *such that for all* $\epsilon \in (0, \epsilon^*)$ *the controller* (3) *with gains*

$$\begin{bmatrix} A_{cij} & B_{ci} \\ C_{cj} & D_c \end{bmatrix} = \begin{bmatrix} V_i^{-1} & -V_i^{-1}Y_iB_i \\ 0 & I \end{bmatrix} \begin{bmatrix} Q_{ij} & F_i \\ L_j & R \end{bmatrix} \times \begin{bmatrix} U_j^{-1} & 0 \\ -C_jX_jU_j^{-1} & I \end{bmatrix}, \tag{37}$$

$$A_{c0ij} = V_i^{-1}\left(S_{ij} - Y_iA_{ij}X_j\right)U_j^{-1}, \quad i, j = 1, 2, \tag{38}$$

where V_i *and* U_i *are such that* $Y_iX_i + V_iU_i = I$, $i, j = 1, 2$, *asymptotically synchronize with local information the multi-agent system* (1). *Furthermore, the guaranteed cost* $\bar{J}_i$ *such that* $J_i \leq \bar{J}_i$, $i = 2, \ldots, n$, *is given by* $\bar{J}_i = \gamma\tilde{\xi}_i(0)^T W_1^{-1}\tilde{\xi}_i(0) + \gamma\tilde{\mu}_{i,f}(0)^T W_2^{-1}\tilde{\mu}_{i,f}(0)$, *where* $\tilde{\mu}_{i,f}(0) = \tilde{\mu}_i(0) + G\tilde{\xi}_i(0)$.

PROOF. First, observe that $\Lambda_j(\lambda_i) \in \text{Co}\{\tilde{\Lambda}_j(\delta_1, \delta_1), \tilde{\Lambda}_j(\delta_1, \delta_2), \tilde{\Lambda}_j(\delta_2, \delta_1), \tilde{\Lambda}_j(\delta_2, \delta_2)\}$ and $\Psi_{jk}(\lambda_i) \in \text{Co}\{\Psi_{jk}(\delta_1), \Psi_{jk}(\delta_2)\}$, $j, k = 1, 2$, then, from the definition of $\Xi_{i,\kappa}$ in (33), one has $\Xi(\lambda_i) \in \text{Co}\{\Xi_{1,1}, \Xi_{1,2}, \Xi_{2,1}, \Xi_{2,2}\}$, where

$$\Xi(\lambda_i) = \begin{bmatrix} \text{He}\{\Psi_{11}(\lambda_i) + \mathcal{I}\Psi_{12}(\lambda_i)^T\} & \star & \star \\ \Psi_{21}(\lambda_i) + \Psi_{22}(\lambda_i)\mathcal{I}^T + \varsigma\Psi_{12}(\lambda_i)^T & \varsigma\text{He}\{\Psi_{22}(\lambda_i)\} & \star \\ \Lambda_1(\lambda_i) + \Lambda_2(\lambda_i)\mathcal{I}^T & \varsigma\Lambda_2(\lambda_i) & -\gamma I \end{bmatrix}$$

Therefore, (32) implies $\Xi(\lambda_i) < 0$ for all $\lambda_i \in [\delta_1,\ \delta_2]$.

Note that inequality (31) is equivalent to $T_i^T W_i T_i > 0$, with W_1 and W_2 as in (25) and T_1 and T_2 as in (26). Then $W_i > 0$, $i = 1, 2$, since T_i is nonsingular.

Observe that the inequality (28) can be written as condition (i) of Lemma 6 with

$$M_0 = \begin{bmatrix} \text{He}\{\Psi_{11}(\lambda_i)\} & \star \\ \Lambda_1(\lambda_i) & -\gamma I \end{bmatrix}, \tag{39}$$

$$M_1^T = -\begin{bmatrix} \Psi_{12}(\lambda_i) \\ \Lambda_2(\lambda_i) \end{bmatrix} \Psi_{22}(\lambda_i)^{-1}, \quad M_2 = \begin{bmatrix} \Psi_{21}(\lambda_i) & 0 \end{bmatrix}, \tag{40}$$

where the inverse of $\Psi_{22}(\lambda_i)$ is well defined due to the block (2,2) of $\Xi(\lambda_i)$. If we define $P_1^T = -\Psi_{22}(\lambda_i)[\mathcal{I}^T \ \ 0]$ and $P_2 = -\varsigma\Psi_{22}(\lambda_i)^T$, then condition (ii) of Lemma 6 is equivalent to $\Xi(\lambda_i) < 0$. Therefore, by Lemma 6, if (33) holds, then (28) is satisfied. By considering the change of variables (25) and (29), rewriting (28) as (27), pre- and post-multiplying (27) by $diag(T_1^{-T}, I)$ and its transpose, respectively, one obtains (20). Finally, pre- and post-multiplying $\Xi(\lambda_i) < 0$ by $[0\ T_2^{-T}\ 0]$, one obtains (21) for any $\varsigma > 0$. By Lemma 9, we conclude the proof.

Remark 12 *Condition* (33) *becomes an LMI for fixed values of* ς *and* $\mathcal{I}$. *Matrix* $\mathcal{I}$ *is a given matrix used to adjust the dimension in* (33) *for the case* $n_x \neq n_z$. *We have observe good numerical results for the choice* $\mathcal{I} = 0$ *or* $\mathcal{I} = \tau\mathbb{I}$, *where* $\mathbb{I} \in \mathbb{R}^{2n_x \times 2n_z}$ *is a matrix with ones on the main diagonal and zeros elsewhere, and* $\tau \in \mathbb{R}$ *is a scalar that represents an extra degree of freedom.*

Remark 13 *The inequality* (28) *can be also written as condition (i) of Lemma 6 with*

$$M_1 = \Psi_{22}(\lambda_i)^{-1}\begin{bmatrix} \Psi_{21}(\lambda_i) & 0 \end{bmatrix}, \quad M_2^T = -\begin{bmatrix} \Psi_{12}(\lambda_i) \\ \Lambda_2(\lambda_i) \end{bmatrix}$$

and M_0 *as in* (39). *Therefore, the choice* $P_1^T = -[\mathcal{I} \ \ 0]\Psi_{22}(\lambda_i)^T$ *and* $P_2 = -\varsigma\Psi_{22}(\lambda_i)^T$ *yields* $\Xi_{\ell,\kappa}$ *in*

(33) *with*

$$\Theta(\delta_\ell) = \mathcal{I}\Psi_{21}(\delta_\ell), \\ \Omega(\delta_\ell) = \Psi_{12}(\delta_\ell)^T + \Psi_{22}(\delta_\ell)^T\mathcal{I}^T + \varsigma\Psi_{21}(\delta_\ell). \quad (41)$$

The conditions of Theorem 11 with (34) *and* (41) *are not equivalent yielding different controllers and costs.*

3.2 ϵ-independent DOF controllers

In this section, we design an output controller that does not depend on ϵ. We may be motivated by the case where ϵ is uncertain but supposed to be upper bounded by a known value or by the design of controllers that do not contain fast variables due to processing constraints or limitations in the bandwidth of the actuator response.

We propose the following ϵ-independent DOF controller [2]:

$$\dot{\eta}_i(t) = A_{c011}\eta_i(t) + A_{c11}\vartheta_i(\eta(t)) + B_{c1}\vartheta_i(y(t)) \\ u_i(t) = C_{c1}\vartheta_i(\eta(t)) + D_c\vartheta_i(y(t)) \quad (42)$$

yielding the individual set of $n-1$ closed-loop dynamics given by (7) with matrices

$$\tilde{A}_{11}(\lambda_i) = \begin{bmatrix} A_{11} - \lambda_i B_1 D_c C_1 & -\lambda_i B_1 C_{c1} \\ -\lambda_i B_{c1} C_1 & -\lambda_i A_{c11} + A_{c011} \end{bmatrix}, \\ \tilde{A}_{12}(\lambda_i) = \begin{bmatrix} A_{12} - \lambda_i B_1 D_c C_2 \\ -\lambda_i B_{c1} C_2 \end{bmatrix}, \\ \tilde{A}_{21}(\lambda_i) = \begin{bmatrix} A_{21} - \lambda_i B_2 D_c C_1 & -\lambda_i B_2 C_{c1} \end{bmatrix}, \\ \tilde{A}_{22}(\lambda_i) = A_{22} - \lambda_i B_2 D_c C_2, \quad (43)$$

and (13) expressed with

$$\tilde{C}_{z1}(\lambda_i) = \begin{bmatrix} C_{z1}(\lambda_i) - \lambda_i D D_c C_1 & -\lambda_i D C_c \end{bmatrix}, \\ \tilde{C}_{z2}(\lambda_i) = C_{z2}(\lambda_i) - \lambda_i D D_c C_2.$$

The following result is presented to design (42).

Theorem 14 *If there exist symmetric positive definite matrices $X_1 \in \mathbb{R}^{n_x \times n_x}$, $Y_1 \in \mathbb{R}^{n_x \times n_x}$, $W_2 \in \mathbb{R}^{n_z \times n_z}$, a scalar $\gamma \in \mathbb{R}_+$ and matrices $F_1 \in \mathbb{R}^{n_x \times n_y}$, $L_1 \in \mathbb{R}^{n_u \times n_x}$, $Q_{11} \in \mathbb{R}^{n_x \times n_x}$, $S_{11} \in \mathbb{R}^{n_x \times n_x}$, and $D_c \in \mathbb{R}^{n_u \times n_y}$, a scalar $\varsigma > 0$, and a given matrix $\mathcal{I} \in \mathbb{R}^{2n_x \times n_z}$, such that* (31) *and*

$$\Xi_{\ell,\kappa} < 0, \quad \Upsilon_\ell < 0, \quad \ell, \kappa = 1, 2, \quad (44)$$

[2] We maintained the same subscription pattern for the controller's gains so that we benefit from the previous variable definitions.

hold with

$$\Xi_{\ell,\kappa} = \begin{bmatrix} He\left\{\Psi_{11}(\delta_\ell) + \mathcal{I}\varphi_{12}(\delta_\ell)^T\right\} & \star & \star \\ \varphi_{21}(\delta_\ell) + \tilde{A}_{22}(\delta_\ell)\mathcal{I}^T + \varsigma\varphi_{12}(\delta_\ell)^T & \varsigma He\left\{\tilde{A}_{22}(\delta_\ell)\right\} & \star \\ \tilde{\Lambda}_1(\delta_\ell, \delta_\kappa) + \tilde{\Lambda}_2(\delta_\ell, \delta_\kappa)\mathcal{I}^T & \varsigma\tilde{\Lambda}_2(\delta_\ell, \delta_\kappa) & -\gamma I \end{bmatrix}, \quad (45)$$

$$\varphi_{12}(\delta_\ell) = \begin{bmatrix} A_{12} - \delta_\ell B_1 D_c C_2 \\ Y_1 A_{12} - \delta_\ell F_1 C_2 \end{bmatrix}, \\ \varphi_{21}(\delta_\ell) = \begin{bmatrix} A_{21}X_1 - \delta_\ell B_2 L_1 & A_{21} - \delta_\ell B_2 D_c C_1 \end{bmatrix}, \\ \tilde{\Lambda}_2(\delta_\ell, \delta_\kappa) = C_{z2}(\delta_\kappa) - \delta_\ell D D_c C_2, \\ \Upsilon_\ell = \begin{bmatrix} He\{A_{22}W_2 - \delta_\ell B_2 D_c C_2\} & \star \\ C_2 W_2 - C_2 - \delta_\ell D_c^T B_2^T & -2I \end{bmatrix}, \quad (46)$$

where $\Psi_{11}(\cdot)$ and $\tilde{\Lambda}_1(\cdot)$ given by (30) *and* (36)*, respectively. Then there exists $\epsilon^* > 0$ such that for all $\epsilon \in (0, \epsilon^*)$ the controller* (42) *asymptotically synchronize with local information the multi-agent system* (1)*. The gains of* (42) *are given by* (37) *and* (38) *with $i = j = 1$ where V_1 and U_1 are such that $Y_1X_1 + V_1U_1 = I$. Furthermore, the guaranteed cost $\bar{J}_i$ such that $J_i \leq \bar{J}_i$, $i = 2, \ldots, n$, is given by $\bar{J}_i = \gamma\tilde{\xi}_i(0)^T W_1^{-1}\tilde{\xi}_i(0) + \gamma\tilde{\mu}_{i,f}(0)^T W_2^{-1}\tilde{\mu}_{i,f}(0)$.*

PROOF. First, observe that the inequality (20) with matrices (43) is equivalent to

$$\begin{bmatrix} \text{He}\left\{T_1^T\tilde{A}_{11}(\lambda_i)W_1T_1 - \Delta_1\right\} & \star \\ \tilde{C}_{z1}(\lambda_i)W_1T_1 - \Delta_2 & -\gamma I \end{bmatrix} < 0$$

with

$$\Delta_1 = T_1^T\tilde{A}_{12}(\lambda_i)\tilde{A}_{22}(\lambda_i)^{-1}\tilde{A}_{21}(\lambda_i)W_1T_1 \\ \Delta_2 = \tilde{C}_{z2}(\lambda_i)\tilde{A}_{22}(\lambda_i)^{-1}\tilde{A}_{21}(\lambda_i)W_1T_1,$$

or, alternatively,

$$\begin{bmatrix} \text{He}\left\{\Psi_{11}(\lambda_i) - \varphi_{12}(\lambda_i)\tilde{A}_{22}(\lambda_i)^{-1}\varphi_{21}(\lambda_i)\right\} & \star \\ \Lambda_1(\lambda_i) - (C_{z2}(\lambda_i) - \lambda_i D D_c C_2)\tilde{A}_{22}(\lambda_i)^{-1}\varphi_{21}(\lambda_i) & -\gamma I \end{bmatrix} < 0, \quad (47)$$

for $\lambda_i \in [\delta_1, \ \delta_2]$, where $\varphi_{12}(\lambda_i) = T_1^T\tilde{A}_{12}(\lambda_i)$ and $\varphi_{21}(\lambda_i) = \tilde{A}_{21}(\lambda_i)W_1T_1$. Following similar steps of the proof of Theorem 11, if we apply Lemma 6 with

$$M_0 = \begin{bmatrix} \mathrm{He}\{\Psi_{11}(\lambda_i)\} & \star \\ \Lambda_1(\lambda_i) & -\gamma I \end{bmatrix},$$

$$M_1^T = -\begin{bmatrix} \varphi_{12}(\lambda_i) \\ C_{z2}(\lambda_i) - \lambda_i D D_c C_2 \end{bmatrix} \tilde{A}_{22}(\lambda_i)^{-1},$$

$$M_2 = \begin{bmatrix} \varphi_{21}(\lambda_i) & 0 \end{bmatrix}, \quad P_1^T = -\tilde{A}_{22}(\lambda_i)[\mathcal{I}^T \;\; 0],$$

$P_2 = -\varsigma \tilde{A}_{22}(\lambda_i)^T$, then $\Xi_{\ell,\kappa} < 0$, $\ell, \kappa = 1, 2$, implies (47), and consequently (20), for all $\lambda_i \in [\delta_1,\ \delta_2]$. Finally, pre- and post-multiplying $\Upsilon_\ell < 0$ by $[I \;\; B_2 D_c]$ and its transpose, one gets (21).

Remark 15 *Observe that, unlike Theorem 11, Theorem 14 needs an extra LMI to satisfy* (21)*. Condition* (21) *with $\tilde{A}_{22}$ given in* (43) *is interpreted as the static output feedback control problem with gain D_c to be designed. We could use standard conditions from the literature (see [21] for a survey) to design D_c, however most of them does not present the static gain as an explicit variable, that is, D_c is recovered from other decision variables. In this case, condition* (21) *need to be solved as a previous step and the gain D_c used as an input parameter to solve $\Xi_{\ell,\kappa} < 0$ yielding more a conservative result. Condition $\Upsilon_\ell < 0$ circumvent this problem allowing the design of D_c and the other gains of the controller* (42) *concurrently.*

Remark 16 *Observe that the control signal of controller* (42) *is composed by slow and fast variables, $u_i = C_c\vartheta_i(\eta) + D_c C_1 \vartheta_i(x) + D_c C_2 \vartheta_i(z)$. If we consider the boundary-layer (fast) system open-loop stable, that is, A_{22} Hurwitz, one can design strictly proper controllers by imposing $D_c = 0$ in Theorem 14. In this case, the control signal $u_i(t)$ does not contain the fast variable $z_i(t)$ avoiding the necessity fast actuators that can be expensive or even impossible to use.*

Remark 17 *The techniques proposed in this paper can also be adapted to the problem of designing DOF controllers for singularly perturbed systems. The dynamic is described by* (1) *with $n = 1$ (one agent) and the objective is to design a DOF controller that minimizes the following quadratic cost function:*

$$J = \int_0^\infty x(t)^T Q_x x(t) + z(t)^T Q_z z(t) + u(t)^T R u(t) dt,$$

where $Q_x \geq 0$, $Q_z \geq 0$ and $R > 0$ are symmetric matrices that weights the effort of the control action and convergence of the trajectories.

The ϵ-dependent and ϵ-independent DOF controllers are given by

$$\begin{aligned} \begin{bmatrix} \dot{\eta}(t) \\ \epsilon\dot{\nu}(t) \end{bmatrix} &= \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix} \begin{bmatrix} \eta(t) \\ \nu(t) \end{bmatrix} + \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} y(t) \\ u(t) &= \begin{bmatrix} C_{c1} & C_{c2} \end{bmatrix} \begin{bmatrix} \eta(t) \\ \nu(t) \end{bmatrix} + D_c y(t) \end{aligned} \tag{48}$$

and

$$\begin{aligned} \dot{\eta}_c(t) &= A_{c11}\eta(t) + B_{c1} y(t) \\ u(t) &= C_{c1}\eta(t) + D_c y(t), \end{aligned} \tag{49}$$

respectively.

In Theorems 11 and 14 just impose $S_{ij} = 0$, $ij = 1, 2$, $\delta_\ell = -1$, $\ell = 1, 2$ and replace $C_{zi}(\delta_\kappa)$, $i = 1, 2$, by

$$\left[C_{z1} \middle| C_{z2} \right] = \left[\begin{array}{c|c} \sqrt{Q_x} & 0 \\ 0 & \sqrt{Q_z} \\ 0 & 0 \end{array} \right]$$

in matrices $\Xi_{\ell,\kappa}$ and Υ_ℓ. The controller gains are recovery by

$$\begin{bmatrix} A_{cij} & B_{ci} \\ C_{cj} & D_c \end{bmatrix} = \begin{bmatrix} V_i^{-1} & -V_i^{-1} Y_i B_i \\ 0 & I \end{bmatrix} \times \begin{bmatrix} Q_{ij} - Y_i A_{ij} X_j & F_i \\ L_j & D_c \end{bmatrix} \begin{bmatrix} U_j^{-1} & 0 \\ -C_j X_j U_j^{-1} & I \end{bmatrix}.$$

Despite the existence of many works for this problem, we propose a more suitable solution with the following advantages: output stabilization of nonstandard singularly perturbed systems (A_{22} singular), the design of ϵ-independent DOF controllers and the minimization of a guaranteed cost by LMIs.

3.3 Guaranteed cost evaluation

To evaluate the guaranteed cost $\bar{J}_i$, $i = 2, \ldots, n$, it is necessary to compute W_1^{-1} and W_2^{-1} from the variables X_j and Y_j, $j = 1, 2$, obtained in Theorems 11 and 14. From the relation $W_j^{-1} W_j = I$ and (25), one has $Z_j = -U_j^{-T} X_j V_j$, where U_j and V_j are square and nonsingular matrices satisfying $Y_j X_j + V_j U_j = I$. The choice $U_j = -X_j$, with no loss of generality, yields $Z_j = V_j$, that is,

$$W_j^{-1} = \begin{bmatrix} Y_j & V_j \\ V_j & V_j \end{bmatrix}, \quad V_j = Y_j - X_j^{-1}. \tag{50}$$

Note that Theorems 11 and 14 provide guaranteed costs for the individual plants. The global guaranteed cost $\bar{J}$ can be obtained by the summation of all guaranteed costs, that is, $\bar{J} = \sum_{i=2}^{n} \bar{J}_i$.

Remark 18 *The guaranteed costs given by Theorems 11 and 14 does not depend on the singular perturbation parameter ϵ. However the obtained $\bar{J}_i$ depends on the initial conditions of the agents. As a matter of simplicity, we can consider the dynamic controller with zero initial conditions, that is, $\tilde{\xi}_i(0) = (\tilde{x}_i(0), 0)$ and $\tilde{\mu}_i(0) = (\tilde{z}_i(0), 0)$, yielding*

$$\bar{J} = \gamma \tilde{x}_-(0)^T (I_{n-1} \otimes Y_1) \tilde{x}_-(0) + \gamma \tilde{\mu}_{-f}(0)^T (I_{n-1} \otimes W_2^{-1}) \tilde{\mu}_{-f}(0),$$

with $\tilde{x}_-(t) = (\tilde{x}_2(t)^T, \dots, \tilde{x}_n(t)^T)^T \in \mathbb{R}^{(n-1)n_x}$ and $\tilde{\mu}_{-f}(t) = (\tilde{\mu}_{2,f}(t)^T, \dots, \tilde{\mu}_{n,f}(t)^T)^T \in \mathbb{R}^{(n-1)n_z}$.

The dependence on the initial conditions can be completely removed if we consider $(\tilde{\xi}_i(0), \tilde{\mu}_i(0)) \in \Sigma_i$, where

$$\Sigma_i = \{(\tilde{\xi}_i, \tilde{\mu}_i) \in \mathbb{R}^{2n_x+2n_z} : V(\tilde{\xi}_i, \tilde{\mu}_i) \leq 1\}.$$

In this case, one has $J_i \leq \gamma V(\tilde{\xi}_i(0), \tilde{\mu}_i(0)) \leq \gamma$, $i = 2, \dots, n$, and the global guaranteed cost is $\bar{J} = (n-1)\gamma$.

Remark 19 *The minimization of the guaranteed cost is obtained by solving the following optimization problem:*

$$\min \quad \gamma$$

subjected to relations of Theorem 11 (Theorem 14).

Remark 20 *It is possible to determine ϵ^* by taking the controllers (3) or (42) obtained in Theorems 11 and 14, respectively, and solving $\epsilon^* = \sup_{\epsilon>0}$ subject to (23). To relax the structure on W in (24) and to avoid numerical problems due to the ill conditioning, the following Lyapunov function is considered*

$$\mathcal{V}(\tilde{\xi}_i, \tilde{\mu}_i, \epsilon) = \begin{bmatrix} \tilde{\xi}_i \\ \tilde{\mu}_i \end{bmatrix}^T P(\epsilon) E(\epsilon) \begin{bmatrix} \tilde{\xi}_i \\ \tilde{\mu}_i \end{bmatrix}, \quad P(\epsilon) = \begin{bmatrix} P_1 & P_{12} \\ \epsilon P_{12}' & P_2 \end{bmatrix}$$

for the closed-loop system described by (22), yielding the following optimization problem:

$$\epsilon^* = \sup_{\epsilon>0, P_1>0, P_2>0, P_{12}, \gamma>0} \epsilon$$

such that

$$P(\epsilon)E(\epsilon) > 0 \tag{51}$$

$$\Pi(\delta_\ell, \delta_\kappa, \epsilon) = \begin{bmatrix} P(\epsilon)\tilde{A}(\delta_\ell) + \tilde{A}(\delta_\ell)^T P(\epsilon)^T & \star \\ \bar{C}_z(\delta_\ell, \delta_\kappa) & -\gamma I \end{bmatrix} < 0, \tag{52}$$

$\ell, \kappa = 1, 2$, where $\tilde{A}(\cdot)$ is given by (7) and

$$\bar{C}_z(\delta_\ell, \delta_\kappa) = \begin{bmatrix} C_{z1}(\delta_\kappa) - \delta_\ell D D_c C_1 & -\delta_\ell D C_{c1} \\ C_{z2}(\delta_\kappa) - \delta_\ell D D_c C_2 & -\delta_\ell D C_{c2} \end{bmatrix}.$$

Observe that $P(\epsilon)E(\epsilon) > 0$ and $\Pi(\delta_\ell, \delta_\kappa, \epsilon) < 0$ can be written as $P_1 + \mathcal{O}(\epsilon) < 0$ and $\Pi(\delta_\ell, \delta_\kappa, 0) + \mathcal{O}(\epsilon) < 0$, respectively, where $\mathcal{O}(\epsilon) \to 0$ when $\epsilon \to 0$. Therefore, there is no numerical problems to solve (51)–(52) for small values of ϵ. Furthermore, the guaranteed cost can be recalculated by $\bar{J} = \sum_{i=2}^{n} \bar{J}_i$, with $\bar{J}_i = \gamma \mathcal{V}(\tilde{\xi}_i(0), \tilde{\mu}_i(0), \epsilon^*)$.

4 Numerical simulations

In the numerical examples we adopt $\mathcal{I} = \tau I$ with $\tau \in \mathbb{R}$. Therefore, a search must be performed in the scalar τ in Theorems 11– 14. The same for the scalar ς. In the numerical examples, the choice $(\varsigma, \tau) = (1, 0)$ has been adopted whenever the conditions are feasible otherwise a search has been done in the following sets $\varsigma \in \mathcal{U} = \{1,\ 10^{-1},\ 10^{-2},\ 10^{-3},\ 10^{-6},\ 10^{-9}\}$ and $\tau \in \{0\} \cup \mathcal{U} \cup -\mathcal{U}$.

The first and second example illustrate the main results where in the first one the agents converge to a static manifold and in the second one to an oscillating trajectory. The third example shows the design of a DOF controller for singularly perturbed systems, as presented in Remark 17.

Example 21 *Consider the synchronization of three agents as in (1) where [20]:*

$$A_{11} = \begin{bmatrix} 2.5 & -6 \\ -2 & 2 \end{bmatrix}, \ A_{12} = \begin{bmatrix} 2 & 3 \\ 0 & -2 \end{bmatrix}, \ A_{21} = \begin{bmatrix} 0.5 & 2 \\ -1 & 1 \end{bmatrix},$$
$$A_{22} = \begin{bmatrix} -2 & 1 \\ 0 & -1 \end{bmatrix}, \ B_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ C_2 = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The communication network is described by an undirected graph $\mathcal{G}$, connected, and with weights belonging to the interval $[1,\ 2] \cup \{0\}$ yielding $\delta_1 = 0.0278$ and $\delta_2 = 6$. The nominal Laplacian matrix is given by:

$$L = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}.$$

In the design, we consider $R = 1$ and matrices $C_{z1}(\lambda_i)$, $C_{z2}(\lambda_i)$, and D are given by (12) where $\lambda_i \in [\delta_1,\ \delta_2]$.

Theorem 11 provided solution for several values of ς and τ such that (7) is stable for all $\epsilon \in (0, \epsilon^*)$ and $\lambda_i \in [\delta_1,\ \delta_2]$. For the time simulations presented next, we adopted zero initial condition for the controller (3) and the following initial conditions for the three agents: $[2.5,\ 2.0,\ -0.5,\ -1.5]$, $[1.5,\ 1.0,\ 4.0,\ -2.0]$ and $[0.5,\ -1.0,\ 3.0,\ 1.0]$. We present in Figures 1–3 the time-simulation adopting $\epsilon = 0.01$ that satisfies (51)–(52). The controller (3) is obtained by Theorem 11 and the minimization problem as in Remark 19 with $\mathcal{I} = \tau I$, $\tau = \varsigma = 0.1$. The gains are given by:

$$\left[\begin{array}{c|c} A_{c11} & A_{c12} \\ \hline A_{c21} & A_{c22} \end{array}\right] = \left[\begin{array}{cc|cc} -292.57 & -8556.06 & -109.18 & -125.39 \\ 482.77 & 3453.70 & 180.19 & 206.90 \\ \hline 172.64 & -401.76 & 64.44 & 73.99 \\ 174.74 & -410.90 & 65.22 & 74.89 \end{array}\right],$$

$$\left[\begin{array}{c} B_{c1} \\ \hline B_{c2} \end{array}\right] = \left[\begin{array}{c} -9244.45 \\ 4589.69 \\ \hline 4.48 \\ 0.28 \end{array}\right],\ D_c = -0.11,$$

$$\left[\begin{array}{c|c} C_{c1} & C_{c2} \end{array}\right] = \left[\begin{array}{cc|cc} -169.42 & 398.54 & -63.24 & -72.60 \end{array}\right],$$

$$\left[\begin{array}{c|c} A_{c011} & A_{c012} \\ \hline A_{c021} & A_{c022} \end{array}\right] = \left[\begin{array}{cc|cc} 108.25 & -198.85 & 18.31 & 71.13 \\ -54.47 & 97.69 & -8.09 & -35.81 \\ \hline 0.69 & 1.56 & -2.02 & 0.99 \\ -0.69 & 0.27 & -0.04 & -1.02 \end{array}\right]. \tag{53}$$

Figures 1 and 2 show the synchronization of the slow, fast and controller (slow and fast) state variables with the change of variable (6). As expected, the transformed variables go to zero in the consensus. Figure 3 illustrates the consensus of the slow and fast states of the closed-loop system. We see the trajectories of agents 2 and 3 converging to the synchronization manifold given by the trajectories of agent 1. Finally, it is worthy to mention that, differently from [20], only the slow state x_2 is measured.

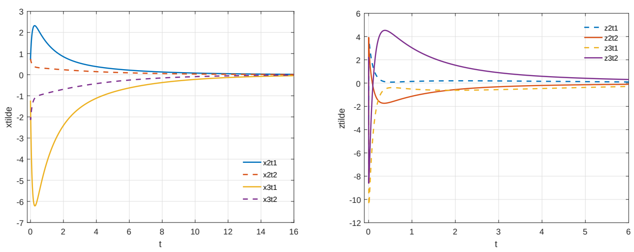

Fig. 1. Trajectories of $\tilde{x}(t)$ (left) and $\tilde{z}(t)$ (right) for Example 21.

Example 22 *Example 21 is adapted to illustrate oscillating trajectories for the consensus manifold considering the same graph and number of agents,*

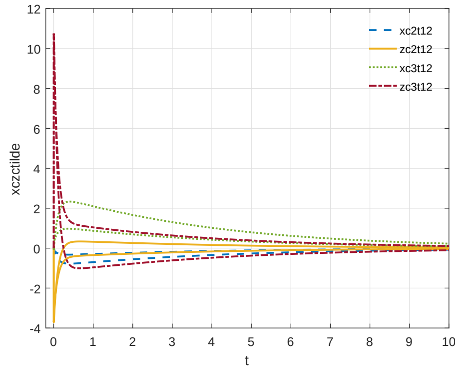

Fig. 2. Trajectories of the states of the controller $(\tilde{\eta}_i(t), \tilde{\nu}_i(t))$, $i = 1, 2$, for Example 21.

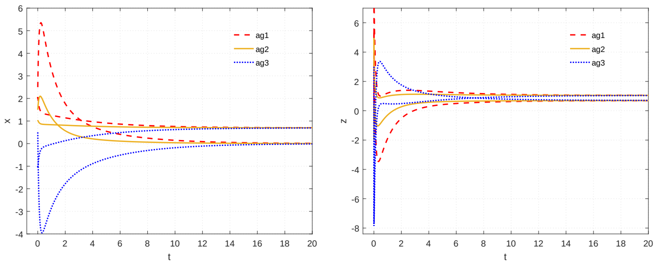

Fig. 3. State trajectories of the system for Example 21.

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ \hline -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right], \quad C_2 = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

and, B_1, B_2 and C_1 the same of Example 21. Note that A is the rotation matrix (with rotation angle of $\pi/2$) in $\mathbb{R}^4$ to get a pair of pure imaginary eigenvalues, and C_2 is modified to make system (1) observable.

The controller (3) is obtained by Theorem 11 with $\varsigma = 0.1$ and $\mathcal{I} = 0.1I$. The synchronization of the slow and fast state variables of the closed-loop system can be observed from the trajectories of the transformed variables $\tilde{x}$ and $\tilde{z}$ presented in Figure 4 with the same initial states of Example 21. Figure 5 and 6 depict slow and fast state trajectories of the closed-loop system, respectively. One observes two manifolds, the first state of each agent converges to an oscillating trajectory and the second one to a fixed value. Clearly, the figures demonstrated that consensus is reached.

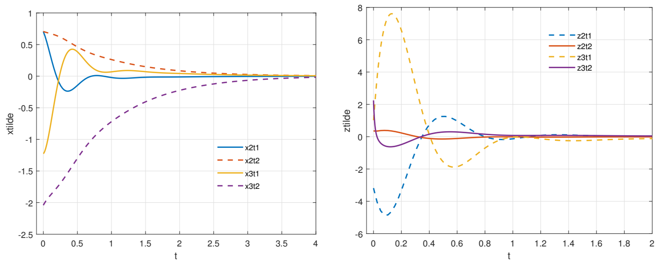

Fig. 4. Trajectories of $\tilde{x}(t)$ and $\tilde{z}(t)$ for Example 22.

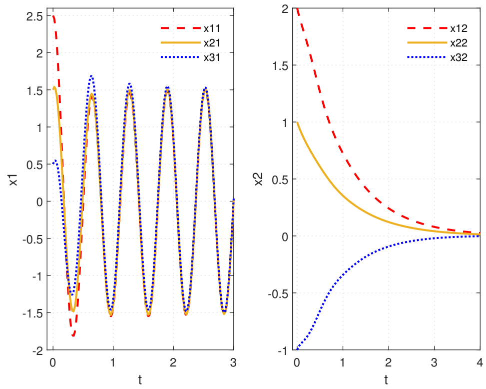

Fig. 5. Trajectories of $x_{i,1}$ (left) and $x_{i,2}$ (right) for agent $i \in \{1, 2, 3\}$ for Example 22.

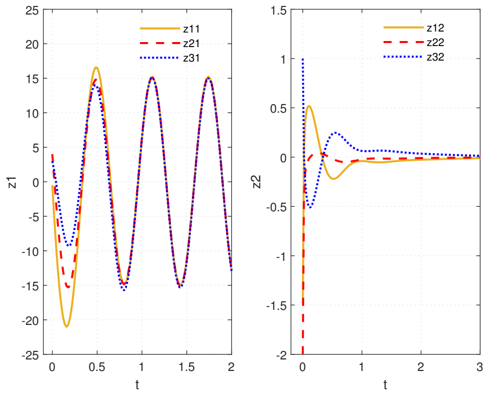

Fig. 6. Trajectories of $z_{i,1}$ (left) and $z_{i,2}$ (right) for agent $i \in \{1, 2, 3\}$ for Example 22.

Example 23 *Let system* (1) *with* $n = 1$ *given by the nominal singularly perturbed in [15,6,10] with the following matrices:*

$$A_{11} = \begin{bmatrix} -0.195 & -0.676 \\ 1.478 & 0 \end{bmatrix},\ A_{12} = \begin{bmatrix} -0.917 & 0.109 \\ 0 & 0 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -0.052 & 0 \\ 0.014 & 0 \end{bmatrix},\ A_{22} = \begin{bmatrix} -0.368 & 0.438 \\ -2.103 & -0.215 \end{bmatrix},$$
$$B_1 = \begin{bmatrix} -0.023 \\ -16.945 \end{bmatrix},\ B_2 = \begin{bmatrix} -0.048 \\ -3.811 \end{bmatrix},\ D = \begin{bmatrix} 0 \\ 10 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ C_2 = \begin{bmatrix} 0.921 & -0.161 \\ 0 & 1 \end{bmatrix},\ C_{z1} = \begin{bmatrix} 0.1 & 0 \\ 0 & 0 \end{bmatrix}$$

and $C_{z2} = 0$*. Theorem 11 with* $\varsigma = 1$, $\mathcal{I} = 0$ *provides the following gains for the controller* (48)

$$\left[\begin{array}{c|c} A_{c11} & A_{c12} \\ \hline A_{c21} & A_{c22} \end{array}\right] = \left[\begin{array}{cc|cc} -0.35 & -0.49 & -1.0 & -1.3 \\ 1.3 & -1.9 & -1.4 & -5.7 \\ \hline -0.16 & 0.023 & -0.76 & 0.19 \\ 0.17 & 2.4 & 0.1 & -0.53 \end{array}\right],$$
$$\left[\begin{array}{c} B_{c1} \\ \hline B_{c2} \end{array}\right] = \left[\begin{array}{cc} 0.16 & -1.3 \\ -1.4 & -5.6 \\ \hline 0.016 & -0.14 \\ 2.1 & 0.038 \end{array}\right],\ D_c = \begin{bmatrix} -0.59 & 0.31 \end{bmatrix} 10^{-9},$$
$$\left[\begin{array}{c|c} C_{c1} & C_{c2} \end{array}\right] = \left[\begin{array}{cc|cc} -2.68 & -21.44 & -3.94 & 1.13 \end{array}\right] 10^{-3}.$$

The singular perturbation parameter is $\epsilon = 0.0336$ *[15] and we verify that the open and closed-loop systems are stable for arbitrarily large values of* ϵ*. We would like to highlight the following scenarios that cannot be handled by the classical approaches [15,5,6,10].*

We consider now the case of designing ϵ*-independent DOF controllers. The following gains for the controller* (49) *are obtained by Theorem 14 with* $\varsigma = 10^{-3}$*, and* $\mathcal{I} = -0.01I$*:*

$$A_{c11} = \begin{bmatrix} 3.13 & 14.34 \\ -1.01 & -4.04 \end{bmatrix},\ B_{c1} = \begin{bmatrix} -0.81 & -18.63 \\ -1.44 & -5.71 \end{bmatrix},$$
$$C_{c1} = \begin{bmatrix} -0.58 & -1.69 \end{bmatrix},\ D_c = \begin{bmatrix} -0.063 & 1.063 \end{bmatrix}.$$

We can also impose $D_c = 0$ *to obtain a control signal* $u(t)$ *independent of the fast variable as a matter of actuator rate constraints, as point out in Remark 16, yielding the following gains for the controller* (49) *obtained by Theorem 14 with* $\varsigma = 10^{-9}$ *and* $\mathcal{I} = 0$*:*

$$A_{c11} = \begin{bmatrix} -0.39 & -2.10 \\ 1.33 & -1.39 \end{bmatrix},\ B_{c1} = \begin{bmatrix} -1.36 & -3.96 \\ -1.30 & -1.67 \end{bmatrix},$$
$$C_{c1} = \begin{bmatrix} -2.00 & -0.32 \end{bmatrix} \cdot 10^{-2}.$$

We would like to stress the advantage of the proposed conditions over the existing results in the literature by imposing matrix A_{22} *singular and with an unstable eigenvalue. The arbitrary choice is made (eigenvalues* 0 *and* 0.07*):* $A_{22} = \begin{bmatrix} -0.368 & 0.438 \\ -0.368 & 0.438 \end{bmatrix}$*. The following gains for the controller* (48) *are obtained with Theorem 11 with* $\varsigma = 1$ *and*

$\mathcal{I} = 0$:

$$\left[\begin{array}{c|c} A_{c11} & A_{c12} \\ \hline A_{c21} & A_{c22} \end{array}\right] = \left[\begin{array}{cc|cc} -0.24 & -0.56 & -61.21 & -386.77 \\ 1.26 & -3.13 & -780.86 & -4751.98 \\ \hline -0.11 & -1.43 & -48.77 & -283.93 \\ 0.07 & 3.36 & -102.80 & -646.94 \end{array}\right],$$

$$\left[\begin{array}{c} B_{c1} \\ \hline B_{c2} \end{array}\right] = \left[\begin{array}{cc} 0.11 & -19.01 \\ -1.62 & -81.69 \\ \hline -1.35 & -12.00 \\ 3.51 & 1.41 \end{array}\right], \; D_c = \left[2.90 \;\; 2.12\right] \cdot 10^{-8},$$

$$\left[\begin{array}{c|c} C_{c1} & C_{c2} \end{array}\right] = \left[\begin{array}{cc|cc} -0.010 & -0.086 & -31.10 & -183.62 \end{array}\right].$$

5 Conclusion

In this work we presented results on the design of decentralized dynamic output feedback protocols for the synchronization of singularly perturbed systems. The designs proposed do not require the fast dynamic matrix to be nonsingular, the knowledge on the singular perturbation parameter and fast actuators to stabilize the fast dynamics. On top of that we are able to guaranty that the overall synchronization cost is upper bounded by a value that can be a priori computed. The results are also extended for implementation oriented output feedback stabilization methods. Numerical simulation emphasize the efectiveness of our results.

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