Network-aware Controller Design with Performance Guarantees for Linear Wireless Systems

A. M. de Oliveira, V. S. Varma, R. Postoyan, I.-C. Morărescu , J. Daafouz, O. L. V. Costa

***Abstract*—We investigate discrete-time closed-loop dynamics consisting of a linear plant, a linear controller and a wireless network that connects the sensors and the actuators to the control unit. The objective, and the main contribution of this work, is the static output feedback control synthesis under given network specifications. Precisely, the network features are formulated in terms of *stochastic allowable transmission interval* (SATI) which is a concept well-suited for the time-triggered control of wireless network control systems (WNCS). Given SATI parameters, we provide sufficient conditions in terms of linear matrix inequalities (LMIs) under which we can design a static output feedback controller that stabilizes the closed-loop WNCS in mean-square sense. Moreover, we guarantee that a quadratic control cost is less than a given bound. Consequently, the results can be used to ensure not only stability but also desired control performances for the WNCS and its SATI characteristics.**

I. Introduction

Wireless networks are increasingly employed in control architectures, leading to the so-called wireless networked control system (WNCS), due to their flexible architectures, reduced costs, ease of implementation and maintenance, to name a few, see, e.g., [1], [2]. The drawback is that they introduce transmission imperfections that could degrade performance and may even lead to instability. Specifically, the random nature of the communication over wireless networks caused by channel fading, shadowing, and collisions need to be carefully handled when designing and implementing the controller, see, e.g., [1], [3], [4] and the references therein.

Transmissions over networks are commonly generated by a clock, we talk about time-triggered control in this case. In the deterministic setting, the *Maximum Allowable Transmission Interval* (MATI) plays a key role for the analysis of the WNCS, see, e.g., [5], [6], [7], [8]. The MATI is a maximum time allowed between two successful transmissions. Due to the stochastic nature of the transmissions in WNCS, it is very hard, not to say impossible, to ensure that the MATI is upper bounded by a fixed value N. To overcome this limitation, a stochastic notion of the MATI, called *Stochastic Allowable Transmission Interval* (SATI), was introduced in [9], [10]. There, the transmission policy is characterized not only by N, a given number of steps since the last successful transmission, but also by η, the *cumulative probability of transmitting before* N steps have elapsed, and after N, δ *the maximum probability of successful transmission*, which is related to the physical limitations of the wireless network. In that way, the time between two successful transmissions is allowed to be larger than N. The notion of SATI is motivated by the design of energy-efficient transmission policies for WNCS, see [9], [10]. Indeed, the power used to send data directly impacts the probability of a successful transmission [11]. This is the main difference with the related works of the literature e.g. [12], [13], which typically assume that packet drops occur with i.i.d random variables. Given the SATI parameters, we can then optimize the power used to send data over the network, while ensuring the control objectives in terms of stability and performance [10]. In [9], [10], the control of WNCS was addressed based on emulation, that is, a controller that stabilizes the origin of the plant without the network is provided as a starting point, and then conditions on the SATI parameters, namely N, η and δ, are derived to ensure stability and performance for the closed-loop system. A limitation of the design strategy employed in [9], [10] is the choice of the controller that may lead to strict requirements on the SATI parameters.

In this paper, we focus instead on network-aware design of static output feedback controllers, that is, we aim at designing the controller under given network characteristics. Regarding the network effects, we focus on packet drops, and we ignore delays, quantization, and scheduling. We adopt the zeroing strategy as in [12] in which no control is applied to the plant in the case of a packet dropout. Since it is not possible to directly design the controller through the conditions of [10] due to non-linearities that arise by considering the controller as a decision variable, we present new linear matrix inequality (LMI) design conditions for the synthesis of static output feedback controllers. Standard techniques cannot be applied "off the shelf" for this purpose because we deal with static output feedback control which does not have an exact (convex) solution. To address this challenge, we exploit properties of the problem structure in order to apply a discrete-time version of the technique recently developed in [14]. The controllers computed by the proposed technique ensure mean square stability of the origin of the closed-loop system and guarantee that a given control cost is less than a desired bound. As a consequence, given the SATI parameters, we can optimize the communication energy used to transmit the packets as explained in [10].

Related literature concerning network-aware design for NCS

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can be found in e.g. [12], [13], [15], [16], and the references therein. In [15], the network-aware design problem consists in finding static state feedback controllers by applying dynamic programming to minimize the sum of the control and communication costs, the channel gains taken as i.i.d. random variables. That is a different approach with respect to our paper as (i) we deal with static output feedback control; (ii) we address a time-triggered control problem as opposed to event-triggered control as in [17] and the references therein; (iii) we rely on LMI conditions, which can easily be checked off-line. In [16], the network-aware design for dynamic output feedback controllers is addressed for linear deterministic systems under MATI constraints, while we tackle stochastic WNCS through the SATI concept.

Compared to the preliminary version of this work [18], we not only study the stabilization of the closed-loop system, but also the problem of quadratic performance through SATI. Moreover, we address the static output feedback control and not only state feedback controllers, which is much more challenging. Furthermore, we also compare the performance of the controller obtained with our conditions to the ones calculated through [12] in the zeroing case and we show that we retrieve the controller of [12] as a special case.

The rest of the paper is organized as follows. In Section II, we describe the system, the SATI modeling, the control cost, as well as some tools for achieving our results. The main results are presented in Section III. In Section IV, we provide an illustrative example. We conclude the work with final remarks in Section V. The proofs can be found in the appendix.

Notation. For a square matrix $M \in \mathbb{R}^{n\times n}$, the notation $M = M^T > (\geq)\ 0$ indicates that all eigenvalues of M are positive (non-negative). For symmetric matrices, $\bullet$ represents a symmetric block. I_n is the identity matrix of size $n\times n$, $0_{n\times m}$, the zero matrix of dimension $n\times m$, and for simplicity, we set $0_n \triangleq 0_{n\times n}$. For a square matrix M, $\mathrm{Her}(M) \triangleq M + M^T$ and $\mathrm{tr}(M)$ denotes the trace of M. The expected value operator is represented by $\mathbf{E}(\cdot)$ and the probability of an event A is given by $\mathrm{Prob}(A)$.

II. Problem Statement

A. Preliminaries

We consider the following discrete-time plant

$$\mathcal{P} : \begin{cases} x(k+1) &= Ax(k) + B\hat{u}(k) \\ y(k) &= Cx(k) \\ z(k) &= C_z x(k) + D_z \hat{u}(k) \\ x(0) &= x_0, \end{cases} \tag{1}$$

where $k \in \mathbb{Z}_{\geq 0}$ is the time, $x(k) \in \mathbb{R}^n$ is the state, $\hat{u}(k) \in \mathbb{R}^m$ is the networked version of the control signal $u(k) \in \mathbb{R}^m$, $y(k) \in \mathbb{R}^q$ is the measured output, $z(k) \in \mathbb{R}^r$ is the controlled output, and $x_0 \in \mathbb{R}^n$ is the initial condition. We make the next assumption on C.

Assumption 1: C is full row rank and $q \leq n$. □

In the system setup shown in Figure 1, we consider that there is a wireless network connecting the plant and the controller. The measured output signal y is transmitted through the network which introduces packet dropouts between the sensor and the plant. In this sense, the controller has access only to a networked version of the measured output denoted by $\hat{y}$. Similarly the controller output u is transmitted through the same network that also introduces packet drops so that the signal available to the plant is the networked version of u, represented by $\hat{u}$. The packet dropout occurs if the information does not arrive or if it arrives too late or in reversed order to the plant.

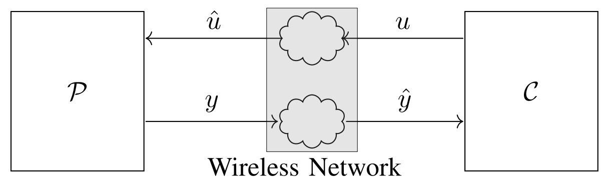

Fig. 1. Schematic of the closed-loop WNCS.

A *zeroing strategy* is considered so that, if a packet dropout occurs, then the information is lost to the controller and set to zero [12]. For modeling this behavior, we define two variables, $\sigma_y(k)$ and $\sigma_u(k)$, that represents the information loss of y and u, respectively. For the transmissions between the sensors and the controller in Figure 1, we set

$$\sigma_y(k) = \begin{cases} 1, & \text{successful transmission of } y \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

For the transmissions between the controller and the actuators, we set

$$\sigma_u(k) = \begin{cases} 1, & \text{successful transmission of } u \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Then the networked versions of y and u are given by $\hat{y}(k) \triangleq \sigma_y(k)y(k)$ and $\hat{u}(k) \triangleq \sigma_u(k)u(k)$. We aim at designing controller $\mathcal{C}$ defined as follows

$$\mathcal{C} \ :\ u(k) = L\hat{y}(k) \tag{4}$$

that is, a static output feedback controller. By setting $\sigma(k) \triangleq \sigma_y(k)\sigma_u(k)$, the dynamics of the closed-loop system $\mathcal{G}_c$ are given by

$$\begin{aligned} x(k+1) &= Ax(k) + BL\sigma_u(k)\hat{y}(k) \\ &= (A + BLC\sigma(k))x(k) \\ &= \begin{cases} \mathcal{A}_1 x(k), & \text{successful transmission,} \\ \mathcal{A}_0 x(k), & \text{otherwise} \end{cases} \end{aligned} \tag{5}$$

where

$$\mathcal{A}_1 \triangleq A + BLC, \quad \mathcal{A}_0 \triangleq A. \tag{6}$$

Similarly, the controlled output is given by

$$\begin{aligned} z(k) &= C_z x(k) + D_z L\sigma_u(k)\hat{y}(k) \\ &= (C_z + D_z LC\sigma(k))x(k) \\ &= \begin{cases} \mathcal{C}_1 x(k), & \text{successful transmission,} \\ \mathcal{C}_0 x(k), & \text{otherwise} \end{cases} \end{aligned} \tag{7}$$

where

$$\mathcal{C}_1 \triangleq C_z + D_z LC, \quad \mathcal{C}_0 \triangleq C_z. \tag{8}$$

We now present in more detail the way we model the network.

Remark 1: The design of dynamic output feedback controllers is technically challenging in the context of non-switching controllers as done in the paper. This extension is left for future work. □

B. Stochastic Allowable Transmission Intervals (SATI)

First, we adopt the following assumption regarding the wireless network.

Assumption 2: The packet transmission status is known via an adequate *acknowledgment scheme* (ACK). □

Assumption 2 is often reasonable in practice and commonly employed in digital communication protocols, see, for instance, [19].

For implementing the network, we introduce the clock which counts the number of steps since the last successful communication $\sigma(k) = 1$,

$$\tau(k+1) = \begin{cases} 1, & \text{successful communication}, \\ \tau(k)+1, & \text{failed communication}, \end{cases} \tag{9}$$

for $k \in \mathbb{Z}_{\geq 0}$. Usually in the WNCS literature, packet dropouts are modeled through i.i.d Bernoulli random variables as in e.g. [12], [13], [15]. In this work we do not make this assumption, instead we assume that the dropout probabilities depend on the clock $\tau(k)$. In other words, the dropout probability depends on the time elapsed since the last successful transmission instant, which is known in view of Assumption 2. Again, this is justified as we can often tune the power with which packets are sent according to $\tau(k)$, thereby impacting the dropout probabilities e.g., [11]. In particular, we characterize the sequence of successful transmission instants using the concept of SATI. The SATI is characterized by three parameters: (i) $N \in \mathbb{Z}_{>0}$, which is a given bound on the number of steps since the last successful transmission; (ii) η, the cumulative probability that a successful transmission occurs as long as $\tau(k) \leq N$, given by

$$\eta \triangleq 1 - \prod_{i=1}^{N} e(i); \tag{10}$$

where $\bar{e} \triangleq (e(1), \ldots, e(N)) \in [0, 1-\delta]^N$ are the dropout probabilities; (iii) the maximum probability of successful transmission at any given time. The idea is the following. When $\tau(k) \leq N$, transmissions may be attempted with less resources, which is represented by the cumulative probability η. If $\tau(k)$ becomes bigger than N, then we can no longer wait and we need to use the maximum resources we have to communicate to the plant, which leads to the maximum probability of transmission δ.

C. WNCS model

We model (5) and (7) as a Markov jump linear system (MJLS) whose Markov chain $\theta(k)$ of $N+1$ states represents the clock values (9). For $\tau(k) \in \{1, 2, \ldots, N\}$, $\theta(k) = \tau(k)$ and for $\tau(k) > N$, $\theta(k) = N+1$, since transmission is always attempted with the same maximum probability δ. Thus, the transition probability matrix is given by

$$\mathbb{P}(\bar{e}) \triangleq \begin{bmatrix} 1-e(1) & e(1) & \ldots & 0 \\ \vdots & \ldots & \ddots & \vdots \\ 1-e(N) & 0 & \ldots & e(N) \\ \delta & 0 & \ldots & 1-\delta \end{bmatrix} \tag{11}$$

where the dropout probabilities $(e(1), \ldots, e(N))$ must respect (10). Then the system (5) and (7) can be rewritten as the following MJLS

$$\mathcal{G}_c : \begin{cases} x(k+1) &= \mathcal{A}_{\sigma(k)} x(k), \\ z(k) &= \mathcal{C}_{\sigma(k)} x(k), \\ x(0) &= x_0,\ \theta(0) = \theta_0, \end{cases} \tag{12}$$

where $\sigma(k) = 1$ if $\theta(k) = 1$ and $\sigma(k) = 0$ if $\theta(k) > 1$, as well as $\mathcal{A}_0$ and $\mathcal{A}_1 \in \mathbb{R}^{n \times n}$ are given in (6), and $\mathcal{C}_0$ and $\mathcal{C}_1 \in \mathbb{R}^{r \times n}$, in (8). We consider that the initial time is a first successful transmission interval, that is, $\theta_0 = 1$, unless stated otherwise.

D. Goals

We define $\mathbb{L}(N, \eta, \delta)$ as the set of mean square stabilizing[1] (MSS) controllers for a fixed (N, η, δ). We state the main goal next.

Problem 1: Given $\delta \in (0, 1]$, $S^T = S > 0$, find, if possible, triplets (N, η, L) with $L \in \mathbb{L}(N, \eta, \delta)$ such that

$$\mathcal{J}(x_0, \theta_0 = 1, L) \triangleq \sum_{k=0}^{\infty} \mathbf{E}(\|z(k)\|^2) \leq x_0^T S x_0 \tag{13}$$

for any $x_0 \in \mathbb{R}^n$. □

Problem 1 requires (12) to be mean square stable and that the quadratic cost in (13) less than a given upper bound which we can adjust through the freely selected matrix S. Of course, if S is "too small" loosely speaking, no solutions can be found. Guidelines on how to select S are provided in Section III. The constant δ is fixed as it typically depends on the maximum power available to transmit, which is often given. Note that $\theta_0 = 1$ in (13) simply means that the initial time is a successful transmission time as already mentioned after (15).

Remark 2: Once (N, η, δ) and the controller $\mathcal{C}$ are calculated, we can resort to the numerical approach in [10] to optimize the power used to transmit packets. □

Remark 3: An alternative setup to the one employed in this paper would be to use a time-varying Kalman filter that would compensate the missing samples, therefore leading to the design of full-order controllers. Since we are dealing with MJLS, the time-varying Kalman filter for this class of systems would be sample path dependent, see [20], [21], [22], meaning that the filter gains may not converge and therefore must be computed on-line at each time step. In this case, mean square stability of the closed-loop system, which is required in Problem 1, is difficult to ensure, see [22], [23]. Besides the aim of this work is to design static output feedback controllers depending on the aforementioned parameters N and η that could easily be implemented with

[1]See [20, Definition 3.8] for the definition of mean square stability.

low computational resources. Note also that in the case that the network is between the controller and the actuator, smart actuators capable of implementing the time-varying controller are needed, which are not necessary in our setting. $\square$

III. Main Results

We first provide conditions for designing the controller in (4) for given $N \in \mathbb{Z}_{>0}$, $\eta \in [0,1]$, and $\delta \in (0,1]$ such that the closed-loop system is MSS and (13) holds. The network-aware design result is then derived in Theorem 1 and summarized in Algorithm 1. Afterwards we specialize our result to state feedback control and compare the controller performance with the classic discrete-time LQ control.

A. Controller Design

We propose and exploit a discrete-time version of the parametrization presented in [14] that makes use of slack variables to design the controller gain L in (4). In view of Assumption 1, there exists a non-singular $T \in R^{n\times n}$ such that

$$U \triangleq CT = C\begin{bmatrix} C_{\parallel} & C_{\perp}\end{bmatrix} = \begin{bmatrix} I_q & 0_{q\times(n-q)}\end{bmatrix} \tag{14}$$

where $C_{\parallel} \in \mathbb{R}^{n\times q}$ and $C_{\perp} \in \mathbb{R}^{n\times(n-q)}$. A straightforward choice for $C_{\parallel}$ in T shown in (14) is $C_{\parallel} = C^T(CC^T)^{-1}$ and, for $C_{\perp}$, a basis of the null space of C. Note that CC^T is non singular due to Assumption 1.

For the slack variable $G_1 \in \mathbb{R}^{n\times n}$, we set the following partition

$$G_1 \triangleq \begin{bmatrix} G_{11} & 0_{q\times n-q} \\ G_{12} & G_{22}\end{bmatrix} \tag{15}$$

where $G_{11} \in \mathbb{R}^{q\times q}$ and $G_{22} \in \mathbb{R}^{n-q\times n-q}$. We introduce the following notation $\eta_d \triangleq \sqrt{1-\eta}$, $\eta_n \triangleq \sqrt{\eta}$, and

$$M_{11} \triangleq \begin{bmatrix} \mathrm{Her}(TG_1) - Q_1 & \bullet & \bullet & \bullet \\ \eta_d A^{N-1}(ATG_1 + BYU) & Q_{N+1} & \bullet & \bullet \\ \eta_n(ATG_1 + BYU) & 0 & X & \bullet \\ C_zTG_1 + D_zYU & 0 & 0 & I_r \end{bmatrix},$$

as well as

$$M_{21} \triangleq \begin{bmatrix} C_z(ATG_1 + BYU) & 0_{r\times 2n+r} \\ C_zA(ATG_1 + BYU) & 0_{r\times 2n+r} \\ \vdots & \vdots \\ C_zA^{N-2}(ATG_1 + BYU) & 0_{r\times 2n+r} \end{bmatrix}$$

for $N > 1$. We define the following matrices

$$\bar{\Phi}_1(N) \triangleq \begin{bmatrix} M_{11} & \bullet \\ M_{21} & I_{r(N-1)} \end{bmatrix}, \tag{16}$$

for $N > 1$ and $\bar{\Phi}_1(1) \triangleq M_{11}$. For $\delta_d \triangleq \sqrt{1-\delta}$ and $\delta_n \triangleq \sqrt{\delta}$,

$$\bar{\Phi}_2 \triangleq \begin{bmatrix} \mathrm{Her}(G_{N+1}) - Q_{N+1} & \bullet & \bullet & \bullet \\ \delta_d AG_{N+1} & Q_{N+1} & \bullet & \bullet \\ \delta_n AG_{N+1} & 0 & Q_1 & \bullet \\ C_zG_{N+1} & 0 & 0 & I_r \end{bmatrix}, \tag{17}$$

$$\bar{\Phi}_{3i} \triangleq \begin{bmatrix} \mathrm{Her}(H_i) - X & \bullet \\ A^{i-1}H_i & Q_1 \end{bmatrix}, i \in \{1,\ldots,N\}, \tag{18}$$

$$\bar{\Phi}_4 \triangleq \begin{bmatrix} S & \bullet \\ I_n & Q_1 \end{bmatrix}, \tag{19}$$

for G_1, G_{N+1}, H_i, Q_1, Q_{N+1}, X, real square matrices of size $n\times n$, and $Y \in \mathbb{R}^{m\times q}$.

Given the SATI parameters N, δ, and η, the next proposition provides conditions to construct controller (4) so that (12) is MSS and (13) holds.

Proposition 1: Given a full rank matrix T as in (14), $N \in \mathbb{Z}_{>0}$, $\delta \in (0,1]$, $\eta \in [0,1]$ and $S^T = S > 0$, if there exist matrices G_1, G_{N+1}, H_i, $Q_1^T = Q_1 > 0$, $Q_{N+1}^T = Q_{N+1} > 0$, $X^T = X > 0$, and Y such that $\Phi_1(N) > 0$, $\Phi_2 > 0$, $\Phi_{3i} > 0$, for all $i \in \{1,\ldots,N\}$, and $\bar{\Phi}_4 > 0$, then by setting $L = YG_{11}^{-1}$, we have that $L \in \mathbb{L}(N,\eta,\delta)$ and $\mathcal{J}(x_0,\theta_0 = 1, L) \le x_0^TSx_0$ for all $x_0 \in \mathbb{R}^n$. $\square$

Proposition 1 does not solve Problem 1 because we need to fix N and η in (16) and (18). Algorithm 1 overcomes this limitation. After the design is carried out in Step 2 of

Algorithm 1 Network-aware Design Algorithm

1: Choose T, $S = S^T > 0$, and set $N = 1$.
2: Calculate the minimum η through a line search procedure in $\eta \in [0,1]$ such that $\Phi_1(N) > 0$, $\Phi_2 > 0$, $\Phi_{3i} > 0$, for all $i \in \{1,\ldots,N\}$, and $\bar{\Phi}_4 > 0$. **If** there exists a feasible solution, set $\eta_{\min}(N) \leftarrow \eta$ and $L(N) \leftarrow YG_{11}^{-1}$ and **goto** Step 3; **else** set $\eta_{\min}(N) \leftarrow []$ and $L(N) \leftarrow []$ and **goto** Step 4.
3: Calculate the maximum $\hat{\eta} \in [\eta_{\min}(N), 1]$ and minimum $\eta \in [0, \eta_{\min}(N)]$ such that (20)-(23) and $P_1 < S$ hold with $L \leftarrow L(N)$ through bisection algorithms. Store $\eta_{\max}(N) \leftarrow \hat{\eta}$ and $\eta_{\min}(N) \leftarrow \eta$, $N \leftarrow N+1$, and **goto** Step 2.
4: Store the feasible $(N, \eta_{\min}(N), \eta_{\max}(N), L(N))$, $N \in \{1,2,\ldots,N_{\max}\}$, where $N_{\max}$ is the biggest N such that $L(N)$ is non-empty.

Algorithm 1, we look for the minimum values of $\eta_{\min}(N)$ that guarantees stability for the range of cumulative probability $[\eta_{\min}(N), \eta_{\max}(N)]$ in Step 3. By doing so iteratively, we are able to establish a range of N in which the system is guaranteed to be MSS and the bound in (13) is satisfied.

Define the set of all solutions of Problem 1 as $\mathcal{S}(S,\delta) \triangleq \{(N,\eta,L) \in \mathbb{Z}_{>0}\times[0,1]\times\mathbb{R}^{m\times q} : (13) \text{ and } L \in \mathbb{L}(N,\eta,\delta)\}$. We have the following theorem linking Algorithm 1 and Problem 1.

Theorem 1: Consider that Algorithm 1 yields feasible $(N, \eta_{\min}(N), \eta_{\max}(N), L(N))$, $N \in \{1,\ldots,N_{\max}\}$. Then $(N,\eta,L(N)) \in \mathcal{S}(S,\delta)$ for all $N \in \{1,\ldots,N_{\max}\}$, where $\eta \in [\eta_{\min}(N), \eta_{\max}(N)]$. $\square$

Note that the result in Theorem 1 follows from Proposition 1 (Step 2) and from the SATI analysis conditions in Proposition 2 presented in the appendix (Step 3) that guarantees the MSS of (12) and the bound (13).

B. LQ Control

An interesting application of Theorem 1 occurs when $y = x$, so that the controller given by (4) can be rewritten in a state feedback form. Define $\mathcal{J}_{\mathrm{LQR}}(x_0) \triangleq \sum_{k=0}^{\infty}[x(k)^TQx(k) + u(k)^TRu(k)]$ and $u(k) = Lx(k)$, where $Q^T = Q \ge 0$ and $R^T = R > 0$ are given matrices. Then $\mathcal{J}^*_{\mathrm{LQR}}(x_0) =$

$x_0^T P_{\mathrm{LQR}} x_0$, where $P_{\mathrm{LQR}}^T = P_{\mathrm{LQR}} \geq 0$ is the solution of the discrete-time algebraic Riccati equation with $Q = C_z^T C_z$, $R = D_z^T D_z$, and $C_z^T D_z = 0$. Assuming that $P_{\mathrm{LQR}} > 0$, if we set $S = \mu P_{\mathrm{LQR}}$ in (13) with $\mu > 1$ a design parameter, we are able to provide a sufficient design condition in which we do not degrade the original LQ cost $\mathcal{J}_{\mathrm{LQR}}^*$ by more than the factor μ. This is formally stated in the next corollary.

Corollary 1: Consider that $C = I_n$ in (1). Given $N \in \mathbb{Z}_{>0}$, $\delta \in (0,1]$, $\eta \in [0,1]$, and $\mu > 1$, by setting $S = \mu P_{\mathrm{LQR}} > 0$, in (19), if there exist matrices G_1, G_{N+1}, H_i, $Q_1^T = Q_1 > 0$, $Q_{N+1}^T = Q_{N+1} > 0$, $X^T = X > 0$, and Y such that $\Phi_1(N) > 0$, $\Phi_2 > 0$, $\Phi_{3i} > 0$, $i \in \{1,\ldots,N\}$, and $\bar{\Phi}_4 > 0$, then by setting $L = Y G_{11}^{-1}$, we have that $L \in \mathbb{L}(N,\eta,\delta)$ and $\mathcal{J}(x_0, \theta_0 = 1, L) \leq \mu \mathcal{J}_{\mathrm{LQR}}^*(x_0)$ for all $x_0 \in \mathbb{R}^n$. □

Remark 4: By taking $N = 1$, it directly follows from the definition of η that $e(1) = 1 - \eta$, hence we recover a MJLS without restrictions in the transition probabilities. Furthermore, if $\eta = \delta$ and $\mathrm{Prob}(\theta_0 = 1) = \delta$, then the Markov chain becomes a Bernoulli process (see [12], [24]), and we recover the results in [12] as illustrated in the next section. □

IV. Illustrative Example

We consider the exact discretization of the system of [25] with $\beta(t) = 2$ (author's notation), that is an unstable system, with sampling period of 50 ms, leading to the following system matrices

$$A = \begin{bmatrix} 0.9305 & 0 & 0.1107 \\ 0.0584 & 1.0833 & -0.0153 \\ 0.0142 & 0 & 0.8953 \end{bmatrix},$$
$$B^T = \begin{bmatrix} 0.0217 & 0.0207 & 0.0247 \\ 0.2510 & 0.0011 & 0.0030 \end{bmatrix}.$$

We consider that $C = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}$. In this case, we take matrix $T = I_3$ in (14) so that $CT = \begin{bmatrix} I_2 & 0_{2\times 1} \end{bmatrix}$. We consider that $\delta = 0.6$ and set $S = \mu P_{\mathrm{r}}$ where P_{r} is the solution of the iterative Riccati method presented in [26] (Algorithm B) for network-free linear time-invariant systems, by taking $C_z^T = \begin{bmatrix} C^T & 0_{3\times 2} \end{bmatrix}$ and $D_z^T = \begin{bmatrix} 0_{2\times 2} & I_2 \end{bmatrix}$. We are thus designing controllers for which the quadratic costs in (13) are less than $\mu x_0^T P_{\mathrm{r}} x_0$. We apply Algorithm 1 by initially setting $\mu = 5$ and get $\eta_{\min}(N) = 0$ and $\eta_{\max}(N) = 1$ for $N \in \{1,2\}$. That is, for each $N \in \{1,2\}$, the controllers are able to stabilize the origin of (1) for all possible values of $\eta \in [0,1]$. On the other hand, if we increase μ to 8.0, we get that the origin of the closed-loop system is stabilized for $\eta \in [0,1]$, for $N \in \{1,2,3,4\}$. That is, when we consider $\mu = 8.0$, by allowing for a degradation on the control cost to be bigger, we get a larger set of values of N compared to the case in which $\mu = 5.0$. In particular for $N = 3$, we have the following static output feedback controller

$$L_{\mathrm{SATI}} = \begin{bmatrix} -1.0095 & -3.6727 \\ -3.1819 & -7.9773 \end{bmatrix}.$$

To compare the results of Section III-B with the results in [12] for the zeroing case, we now concentrate on the state feedback case by setting $C = I_3$, $C_z' = \begin{bmatrix} I_3 & 0_{3\times 2} \end{bmatrix}$ and $D_z' = \begin{bmatrix} 0_{2\times 3} & I_2 \end{bmatrix}$. For that, we solve the discrete-time LQR problem in order to get P_{LQR} and set $S = \mu P_{\mathrm{LQR}}$ in (19). We take $\delta = \eta = 0.9$, and construct the controller using Proposition 1 by minimizing μ for $N \in \{1,\ldots,5\}$. We resort to the algorithm presented in [10] that optimizes the communication power to calculate the transition (dropout) probabilities $\bar{e}$. Given $\bar{e} = (e_1,\ldots,e_N)$, we calculate the frequency of successful communication π_1 from the stationary distribution π of the Markov chain for all $N \in \{1,\ldots,5\}$. We show in Table I the dropout probabilities $\bar{e}$ calculated through the conditions presented in [10] and π_1 for $N \in \{1,\ldots,5\}$. For the simulations of the SATI controller, we set the initial

TABLE I

Dropout probabilities $\bar{e}$ and π_1 for $N \in \{1,\ldots,5\}$

N	1	2	3	4	5
$\bar{e}$	(0.1)	$(0.60, 0.17)$	$(1, 0.5, 0.2)$	$(1, 0.8, 0.5, 0.2)$	(*)
π_1	0.90	0.59	0.38	0.30	0.24

(*) $(1, 1, 0.76, 0.51, 0.26)$

distribution of the Markov chain as the stationary distribution π so that $P(\theta(k) = 1) = \pi_1$ for all k. On the other hand, for calculating the controller in [12], we consider that the probability of packet dropout is equal to $\nu = 1 - \pi_1$ with the notation of [12], for π_1 taken from Table I for $N \in \{1,\ldots,5\}$. By doing so we get that the frequency of successful communication is equal for both the SATI and [12] controllers. For each $N \in \{1,\ldots,5\}$, we perform a Monte Carlo simulation of 2000 rounds for the controllers calculated through Proposition 1 and [12], by sampling x_0 from a standard Gaussian distribution, that is, $x_0 \sim \mathcal{N}_4(v, \Sigma)$, for $v = 0_{4\times 1}$ and $\Sigma = I_4$. The quadratic costs $\mathcal{J}(x_0, \theta_0, L)$ obtained in the simulations are shown in Figure 2 against π_1. We note that for $N = 1$ ($\pi_1 = 0.9$), the costs yielded by the

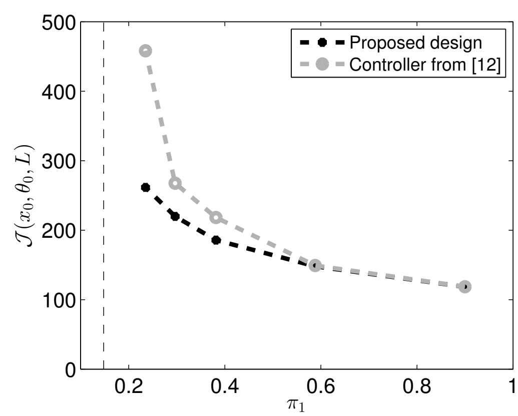

Fig. 2. Control costs for the proposed design and [12] against the frequency of successful communication π_1. The dashed gray line shows the critical loss probability of [12].

controller in [12] and the one obtained through Proposition 1 are very close. However as we decrease the frequency of successful communication, the controllers from Proposition 1 performs better than the ones calculated through [12]. For smaller values of π_1, we note that the costs for [12] start to increase abruptly due to the proximity of ν to the so-called critical loss probability, see [12]. Even though the costs obtained through the proposed technique also rise, the degradation is not so important as the one yielded by the controllers in [12].

Finally, if we take $N = 1$ and $\eta = \delta$, which is the case explained in Remark 4, we get by minimizing $\mathrm{tr}(W)$ subject to (16)-(18) and $W > \delta Q_1^{-1} + (1-\delta)Q_2^{-1}$ that $L_{\text{SATI}} \approx L_{[12]}$. Besides, by taking $\delta \to 1$, we get that $L_{\text{SATI}} \approx L_{\text{LQR}}$.

V. Conclusion

We presented a time-triggered network-aware design procedure for linear wireless networked control systems considering the so-called SATI formulation. We provided design conditions for static-output feedback controllers that guarantees mean-square stability of the closed-loop system and an upper bound on a quadratic cost. Further, we performed a comparison on an example with the performance yielded by the SATI controller obtained by our conditions and the optimal Bernoulli state feedback controller.

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Appendix

By defining $Q_N \triangleq \sum_{i=0}^{N-2} (\mathcal{A}_0^T)^i \mathcal{C}_0^T \mathcal{C}_0 \mathcal{A}_0^i$, we introduce the following auxiliary result adapted from [10].

Proposition 2: Given L, $N \in \mathbb{Z}_{\geq 0}$, $\delta \in (0, 1]$ and $\eta \in [0, 1]$, if there exists $P_1 > 0$, $P_{N+1} > 0$, and $V > 0$ such that

$$\begin{aligned} P_1 >& \mathcal{A}_1^T \left[(1-\eta)(\mathcal{A}_0^T)^{N-1} P_{N+1} \mathcal{A}_0^{N-1} + \eta V + Q_N \right] \mathcal{A}_1 \\ &+ \mathcal{C}_1^T \mathcal{C}_1, \end{aligned} \tag{20}$$

$$P_{N+1} > \mathcal{A}_0^T [(1-\delta)P_{N+1} + \delta P_1] \mathcal{A}_0 + \mathcal{C}_0^T \mathcal{C}_0, \tag{21}$$

$$V > (\mathcal{A}_0^T)^{i-1} P_1 \mathcal{A}_0^{i-1}, \tag{22}$$

hold for all $i \in \{1, \ldots, N\}$. Then $L \in \mathbb{L}(N, \eta, \delta)$ and $P_1 \geq P_1^*$ for all $\bar{e}$ such that (10) holds, where $P_1^* \geq 0$ is the solution of $P_1^* = \mathcal{A}_1^T[(1 - e(1)P_1^* + e(1)P_2^*]\mathcal{A}_1 + \mathcal{C}_1^T\mathcal{C}_1$,$P_{N+1}^* = \mathcal{A}_0^T[\delta P_1^* + (1-\delta)P_{N+1}^*]\mathcal{A}_0 + \mathcal{C}_0^T\mathcal{C}_0$, $P_i^* = \mathcal{A}_0^T[(1-e(i)P_1^* + e(i)P_{i+1}^*]\mathcal{A}_0 + \mathcal{C}_0^T\mathcal{C}_0$, for all $i \in \{2, \ldots, N\}$. In particular, if (20)-(22) holds along with

$$\begin{aligned} P_1 >& \mathcal{A}_1^T \left[(1-\hat{\eta})(\mathcal{A}_0^T)^{N-1} P_{N+1} \mathcal{A}_0^{N-1} + \hat{\eta} V \right] \mathcal{A}_1 \\ &+ \mathcal{A}_1^T Q_N \mathcal{A}_1 + \mathcal{C}_1^T \mathcal{C}_1, \end{aligned} \tag{23}$$

with $\hat{\eta} \in [\eta, 1]$, then $L \in \mathbb{L}(N, \bar{\eta}, \delta)$ and $P_1 \geq P_1^*$ for all $\bar{\eta} \in [\eta, \hat{\eta}]$. □

Proof: The first part of Proposition 2 was proved in [10]. The last part follows by taking the convex combination of (20) and (23) with $\bar{\eta} = \alpha\eta + (1-\alpha)\hat{\eta}$, $0 \leq \alpha \leq 1$. ■

Proof of Proposition 1: Given that (16)-(19) are positive definite, we first show that G_{11} is non-singular. Note from (16) that $\mathrm{Her}(TG_1) > Q_1 > 0$ that implies that TG_1 is non-singular. Since T is assumed to be full rank, we get that G_1 is invertible see, for instance, [14]. Due to the lower-triangular block structure of G_1 in (15), then G_{11} is also non-singular.

Concerning (16), recalling the definition of the blocks of G_1 in (15), U in (14), Assumption 1, and that $Y = LG_{11}$, we get that $ATG_1 + BYU = ATG_1 + BL\begin{bmatrix} G_{11} & 0_{q\times n-q}\end{bmatrix} = \mathcal{A}_1 TG_1$, where $\mathcal{A}_1$ is the closed-loop matrix presented in (6). Besides, $C_z TG_1 + D_z YU = C_z TG_1 + D_z L\begin{bmatrix} G_{11} & 0_{q\times n-q}\end{bmatrix} = \mathcal{C}_1 TG_1$, where $\mathcal{C}_1$ is the closed-loop output matrix in (8). Thus, we can rewrite (16), by also recalling that $(TG_1)^T Q_1^{-1} TG_1 \geq \mathrm{Her}(TG_1) - Q_1$ (see, for instance, [27]) as follows

$$\begin{bmatrix} (TG_1)^T Q_1^{-1} TG_1 & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\ \eta_d A^{N-1}\mathcal{A}_1 TG_1 & Q_{N+1} & \bullet & \bullet & \bullet & \bullet & \bullet \\ \eta_n \mathcal{A}_1 TG_1 & 0 & X & \bullet & \bullet & \bullet & \bullet \\ \mathcal{C}_1 TG_1 & 0 & 0 & I & \bullet & \bullet & \bullet \\ C_z\mathcal{A}_1 TG_1 & 0 & 0 & 0 & I & \bullet & \bullet \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \bullet \\ C_z A^{N-2}\mathcal{A}_1 TG_1 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix} > 0.$$

By applying the congruence transformation $\mathbf{diag}((TG_1)^{-1}, I)$ to the resulting inequality, as well as the Schur complement, we get (20) for $P_1 = Q_1^{-1}$, $P_{N+1} = Q_{N+1}^{-1}$, and $V = X^{-1}$. Through (17) and (18), by performing the similar steps applied to (16), we get (21)-(22). Thus, by Proposition 2, we get that $L \in \mathbb{L}(N, \eta, \delta)$ for all x_0 and $\bar{e}$ such that (10) holds. Finally, we get by applying the Schur complement to (19) that $P_1 = Q_1^{-1} < S$. It follows that $\mathcal{J}(x_0, \theta_0 = 1, L) = x_0^T P_1^* x_0 \leq x_0^T P_1 x_0 \leq x_0^T S x_0$ for all $x_0 \in \mathbb{R}^n$. $\square$

Remark 5: Suppose x_0 is a random initial condition with $\mathbf{E}(x_0) = 0$ and $\mathbf{E}(x_0 x_0^T) = \Sigma$, and that $\sigma(0) = 1$. In this case, we proved in Proposition 1 that $\mathcal{J}(x_0, \theta_0 = 1, L) \leq \mathbf{E}(x_0^T P_1 x_0) \leq \mathbf{E}(x_0^T S x_0)$. Note that $\mathbf{E}(x_0^T P_1 x_0) = \mathbf{E}\left(\mathrm{tr}(x_0 x_0^T P_1)\right) = \mathrm{tr}\left(\mathbf{E}(x_0 x_0)^T P_1\right) = \mathrm{tr}(\Sigma P_1)$. Thus, $\mathcal{J}(x_0, \theta_0 = 1, L) \leq \mathrm{tr}(\Sigma P_1) \leq \mathrm{tr}(\Sigma S)$. In particular, if $\Sigma = I_n$, then $\mathcal{J}(x_0, \theta_0 = 1, L) \leq \mathrm{tr}(S)$.

Proof of Corollary 1: Recalling that $\mathcal{J}^*_{\mathrm{LQR}}(x_0) = x_0^T P_{\mathrm{LQR}} x_0$ for any $x_0 \in \mathbb{R}^n$, then by setting $S = \mu P_{\mathrm{LQR}}$ in Proposition 1, we get that $\mathcal{J}(x_0, \theta_0 = 1, L) \leq x_0^T S x_0 = \mu x_0^T P_{\mathrm{LQR}} x_0 = \mu \mathcal{J}^*_{\mathrm{LQR}}(x_0)$ for all $x_0 \in \mathbb{R}^n$. $\square$