Control design with guaranteed cost for synchronization in networks of linear singularly perturbed systems ⋆

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Abstract

This work presents the design of a decentralized control strategy that allows singularly perturbed multi-agent systems to achieve synchronization with global performance guarantees. The study is mainly motivated by the presence of two features that characterize many physical systems. The first is the complexity in terms of interconnected subsystems and the second is that each subsystem involves processes evolving on different time-scales. In the context of interconnected systems, the decentralized control is interesting since it considerably reduces the communication load (and the associated energy) which can be very important when dealing with centralized policies. Therefore, the main difficulty that we have to overcome is that we have to avoid the use of centralized information related to the interconnection network structure. This problem is solved by rewriting the synchronization problem in terms of stabilization of a singularly perturbed uncertain linear system. The singularly perturbed dynamics of subsystems generates theoretical challenges related to the stabilizing controller design but also numerical issues related to the computation of the controller gains. We show that these problems can be solved by decoupling the slow and fast dynamics. Our theoretical developments are illustrated by some numerical examples.

Key words: Multi-agent systems, synchronization; guaranteed cost control; uncertain singularly perturbed systems;

1 Introduction

The main goal of this paper is to design a decentralized control strategy that allows singularly perturbed multi-agent systems to achieve synchronization with global performance guarantees. Decentralized coordination control of multi-agent systems attracted a lot of attention during the last decade. An important feature of this class of large scale systems is the fact that local information plays a key role. In the decentralized control design each system is able to implement and design its own control law without the help of a central entity that requires important amounts of communication and computation. Consequently, decentralized control aims at reducing the communication and computation costs. When these costs are neglected the centralized control strategies generally outperform the decentralized ones. However, energy aware strategies have to take into account the overall cost and should reduce the communication and computation loads (Hassan & Shamma 2016). Therefore, in this paper we design decentralized controllers that provide a guaranteed cost.

Synchronization of singularly perturbed systems is mainly motivated by two features that characterize the nowadays systems. The first one is the complexity in terms of subsystems interconnected together in order to accomplish a global goal while the second is that physical subsystems often involve processes that evolve on different time-scales. Generally these features are tackled independently one from another. Indeed, the multi-agent formalism allows treating problems coming from a wide application domain such as engineering (Bullo et al. 2009), biology (Pavlopoulos et al. 2011), sociology (Hegselmann & Krause 2002, Morărescu & Girard 2011). Consensus and synchronization were mainly studied for linear agents interacting through a directed or undirected graph with a fixed or dynamically changing topology (Jadbabaie et al. 2003, Moreau 2005). However, there are also studies on nonlinear agents such as oscillators dynamics (Steur et al. 2009, Morărescu et al. 2016), nonholonomic robots (Bullo et al. 2009) or general nonlinear systems (Buşoniu & Morărescu 2014).

On the other hand, one can find many applications ranging from biological systems such as gene expression systems (L.Chen & Aihara 2002), neurons behavior (Hodgkin & Huxley 1952) to engineering problems (Malloci et al. 2009) that involve processes evolving on different time-scales. General stability and stabilization of such linear and non-

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linear systems, called singularly perturbed, can be found in (Kokotović et al. 1999, Khalil 2001).

In a preliminary work (Rejeb et al. 2016), we have combined the two features presented above to study the synchronization of singularly perturbed systems. In that work we have designed decentralized controllers able to achieve asymptotically the synchronization goal. Here, we address the more challenging problem of synchronization with global performance guarantees. This problem was also considered separately in the context of multi-agent systems and singularly perturbed systems. For instance, a linear quadratic (LQ)-based optimal linear consensus protocol for multi-vehicle systems with single integrator dynamics was investigated in (Cao & Ren 2010) in both continuous time and discrete time. The authors of (Kim & Mesbahi 2006) considered an iterative algorithm that maximizes the second smallest eigenvalue of a state-dependent graph Laplacian yielding to optimization of the convergence speed toward consensus. A nonlinear distributed coordination law was presented in (Cortés & Bullo 2005) to achieve optimal consensus under a switching directed communication graph.

On the other hand there exist studies that consider linear quadratic optimal control design for linear singularly perturbed systems (Kokotović et al. 1999, Garcia et al. 2002). One of the most common approaches is the time-scale decomposition that leads to decoupled slow and fast subsystems and an appropriate combination of the corresponding results yields an optimal control design for the original system.

The main contribution of our paper is twofold. Firstly, we strengthen the main result in (Rejeb et al. 2016) and correct the proof of (Rejeb et al. 2016, Proposition 4) that contains a flaw in the reasoning since we had implicitly assumed a particular Lyapunov function for the systems under consideration. The corresponding result in this work is Proposition 3 which is instrumental for the design of a decentralized synchronizing controller presented in Theorem 4. Secondly, we go beyond those results by considering the more challenging problem of decentralized guaranteed cost control design. The supplementary difficulty that we face is related to the fact that we have to ensure bounds on a global cost in a decentralized manner *i.e.* by a local design. To the best of the authors' knowledge this is the first attempt of designing guaranteed cost controllers for singularly perturbed multi-agent systems. When we deal with optimal decentralized control, the Riccati equation, which, in the LQ case is the basis for the derivation of the optimal control law, involves the eigenvalues of the graph Laplacian describing the overall network. In order to get rid of this centralized information, instead of looking for an optimal controller, a guaranteed cost controller is designed to ensure a performance level of the closed-loop dynamics. Precisely we consider a multi-agent system under fixed undirected interaction graph. The dynamics of each agent is represented by linear singularly perturbed system. To solve the problem of decentralized guaranteed cost control design, we transform the synchronization problem in an uncertain system stabilization one. The uncertainty comes from the fact that the graph Laplacian eigenvalues are modeled as unknown but bounded uncertain parameters in order to avoid an explicit use of Laplacian's eigenvalues. This is motivated by the fact that the only available graph information consists in its connectivity.

The paper is organized as follows. Section 2 presents some basic definitions and notations in graph theory. In Section 3, we introduce a change of variables that allows us to reformulate the synchronization problem under consideration in terms of simultaneous synchronization of linear singularly perturbed systems. Section 4 provides conditions under which the decentralized synchronization is possible. Moreover, it presents a methodology to design a decentralized controller that achieves the synchronization goal. Section 5 is devoted to the design of the decentralized guaranteed cost control law. Conditions on the gain matrix such that the closed-loop singularly perturbed systems achieve asymptotic synchronization while an upper bound on the performance index is minimized, are expressed through linear matrix inequalities (LMIs). Simulation results are presented in Section 6. The paper ends with some concluding remarks.

Notation: The following standard notation is used throughout the paper. $\mathbb{R}$ is the set of real numbers, $\|x\|$ is the Euclidean norm of the vector x and $\otimes$ denotes the Kronecker product of two matrices. We also denote by $I_n \in \mathbb{R}^{n\times n}$ the identity matrix of size n and by $\mathbf{1}_n, \mathbf{0}_n \in \mathbb{R}^n$ the column vector whose components are all 1 and 0, respectively. By $\mathbf{0}_{n\times m} \in \mathbb{R}^{n\times m}$ we denote the matrix whose all components are 0. Given a matrix $A \in \mathbb{R}^{n\times n}$ and $A > \mathbf{0}$ ($A \geq \mathbf{0}$) means that A is positive (semi-) definite. The transpose of A is denoted by $A^\top$. We denote $diag(A_1, \ldots, A_n)$ the block diagonal matrix having the matrices A_1 to A_n on the diagonal and 0 everywhere else.

2 Preliminaries and problem formulation

We consider a network of n identical singularly perturbed linear systems. For any $i = 1, \ldots, n$, the i^{th} system at time t is characterized by the state $(x_i(t), z_i(t)) \in \mathbb{R}^{n_x+n_z}$ and a small $\varepsilon > 0$ such that its dynamics is given by:

$$\begin{cases} \dot{x}_i(t) = A_{11}x_i(t) + A_{12}z_i(t) + B_1u_i(t) \\ \varepsilon\dot{z}_i(t) = A_{21}x_i(t) + A_{22}z_i(t) + B_2u_i(t) \end{cases}, \quad (1)$$

where $u_i \in \mathbb{R}^m$ is the control input and

$$A_{11} \in \mathbb{R}^{n_x\times n_x},\ A_{12} \in \mathbb{R}^{n_x\times n_z},\ B_1 \in \mathbb{R}^{n_x\times m},$$
$$A_{21} \in \mathbb{R}^{n_z\times n_x},\ A_{22} \in \mathbb{R}^{n_z\times n_z},\ B_2 \in \mathbb{R}^{n_z\times m}$$

such that $rank(B_1) = rank(B_2) = m$.

Assumption 1 *The matrix A_{22} is invertible.*

The previous assumption is standard in singular perturbation theory (see (Kokotović et al. 1999)) but it is not verified for the case of simple integrators which are standard in multi-agent systems. Nevertheless, our analysis applies for a wide range of systems that have an internal dynamics. With the network of n systems we associate a graph $\mathcal{G}$ which is a couple $(\mathcal{V}, \mathcal{E})$. Here, $\mathcal{V} = \{1, \dots, n\}$ represents the vertex set and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the edge set. In the sequel we suppose that the graph is undirected meaning that $(i, j) \in \mathcal{E} \Leftrightarrow (j, i) \in \mathcal{E}$. We also assume that $\mathcal{G}$ has no self-loop (i.e. $\forall\, i = 1, \dots, n$ one has $(i, i) \notin \mathcal{E}$). A weighted adjacency matrix associated with $\mathcal{G}$ is $G = [g_{ij}] \in \mathbb{R}^{n\times n}$ such that $g_{ij} = g_{ji} > 0$ if $(i, j) \in \mathcal{E}$ and $g_{ij} = 0$ otherwise. The corresponding weighted Laplacian matrix is $L = [l_{ij}] \in \mathbb{R}^{n\times n}$ defined by $\begin{cases} l_{ii} = \sum_{j=1}^{n} g_{ij},\ \forall i = 1, \dots, n \\ l_{ij} = -g_{ij} \text{ if } i \neq j \end{cases}$. By definition L is symmetric and all of its rows sums are zero.

Definition 1 *A* **path of length** *p in the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is an union of edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \dots, p-1\}$. The node j is* **connected** *with node i in $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if there exists at least a path in $\mathcal{G}$ from i to j (i.e. $i_1 = i$ and $j_p = j$). A* **connected graph** *is such that any of its two distinct elements are connected.*

Throughout the rest of the paper the following hypothesis holds.

Assumption 2 *The undirected graph $\mathcal{G}$ is connected and all the non-zero weights $g_{i,j} \neq 0$ of the associated weighted Laplacian matrix are within the interval $[g_m, g_M]$ with $g_M \geqslant g_m > 0$.*

This hypothesis is used to get consensus for any initial conditions under consideration and to uniformly lower-bound the second smallest eigenvalue of the graph $\mathcal{G}$ as in the next Remark.

Remark 1 (Basic properties of the Laplacian matrix) *(Godsil & Royle 2001) Let an undirected graph $\mathcal{G}$ that satisfies Assumption 2 and let $0 = \lambda_1 < \lambda_2 \leqslant \dots \leqslant \lambda_n$ be the eigenvalues of the corresponding Laplacian matrix L. A rough lower-bound on λ_2, independent of $\mathcal{G}$, is $\lambda^* = \frac{g_m^2}{2(n-1)n^2}$ (see (Friedland & Nabben 2002) for details). Therefore, one has*

$$\lambda^* < \lambda_2 \leqslant \dots \leqslant \lambda_n < n \cdot g_M \triangleq \lambda^\circ.$$

It is worth noting that there exists an orthonormal matrix $T \in \mathbb{R}^{n\times n}$ (i.e. $TT^\top = T^\top T = I_n$) such that

$$TLT^\top = D = diag(\lambda_1, \lambda_2, \dots, \lambda_n)$$

Definition 2 *The n singularly perturbed systems defined by* (1) *achieve asymptotic synchronization using local information if there exists a state feedback controller of the form*

$$u_i(t) = K_1 \sum_{j=1}^{n} g_{ij}(x_i(t) - x_j(t)) + K_2 \sum_{j=1}^{n} g_{ij}(z_i(t) - z_j(t)) \\ K_1 \in \mathbb{R}^{m\times n_x},\ K_2 \in \mathbb{R}^{m\times n_z} \tag{2}$$

such that

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \textit{ and } \lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0.$$

Remark 2 *It is noteworthy that the state feedback controller in Definition 2 uses local information since it requires only the state of the neighbors (notice that $g_{ij} = 0$ if j is not a neighbor of i). Notice also that we could consider different controller gains for each agent but, as we shall show in the sequel, the agents will solve an identical control design problem and consequently they should find similar gains. We will show that this constraint of common gains can be satisfied in practice.*

Let $x(t) = (x_1(t)^\top, \dots, x_n(t)^\top)^\top \in \mathbb{R}^{n\cdot n_x}$ and $z(t) = (z_1(t)^\top, \dots, z_n(t)^\top)^\top \in \mathbb{R}^{n\cdot n_z}$ be the vectors collecting the individual states $x_i(t)$ and $z_i(t)$ of the n agents. Let also $u(t) \in \mathbb{R}^{n\cdot m}$ be the vector collecting the individual controllers $u_i(t)$, $i = 1, \dots, n$ (i.e $u(t) = (u_1(t)^\top, \dots, u_n(t)^\top)^\top$). Finally, let us consider the following global cost associated with synchronization of the dynamics in (1):

$$J = \int_0^\infty x(t)^\top \big(L \otimes I_{n_x}\big)x(t) + z(t)^\top \big(L \otimes I_{n_x}\big)z(t) \\ + u(t)^\top \big(I_n \otimes R\big)u(t)dt \tag{3}$$

where $R \in \mathbb{R}^{m\times m}$ is a positive definite matrix that penalizes the control effort required for synchronization.

The main goal of this paper is the characterization of the feedback controllers (2) that use local information and asymptotically synchronize the singularly perturbed systems defined by (1) with a global guaranteed cost $\bar{J}$ that will be defined in Section 5 (*i.e.* $J \leqslant \bar{J}$). In other words, we want that the sum of individual control efforts satisfies an upper bound $\bar{J}$ while guaranteeing the synchronization of the n singularly perturbed closed-loop dynamics (1).

3 Change of variables - problem reformulation

Following (Rejeb et al. 2016) we make the change of variable

$$\tilde{x}(t) = \big(T \otimes I_{n_x}\big)x(t), \quad \tilde{z}(t) = \big(T \otimes I_{n_z}\big)z(t) \tag{4}$$

where T is defined in Remark 1. This allows us to equivalently describe the dynamics in individual dynamics as the collective dynamics

$$\begin{cases} \dot{\tilde{x}}(t) = \tilde{A}_{11}\tilde{x}(t) + \tilde{A}_{12}\tilde{z}(t) \\ \varepsilon\dot{\tilde{z}}(t) = \tilde{A}_{21}\tilde{x}(t) + \tilde{A}_{22}\tilde{z}(t) \end{cases} \tag{5}$$

where

$$\tilde{A}_{11} = I_n \otimes A_{11} - (I_n \otimes B_1 K_1)(D \otimes I_{n_x}),$$
$$\tilde{A}_{12} = I_n \otimes A_{12} - (I_n \otimes B_1 K_2)(D \otimes I_{n_z}),$$
$$\tilde{A}_{21} = I_n \otimes A_{21} - (I_n \otimes B_2 K_1)(D \otimes I_{n_x}),$$
$$\tilde{A}_{22} = I_n \otimes A_{22} - (I_n \otimes B_2 K_2)(D \otimes I_{n_z}).$$

The collective control vector can be described by:

$$u(t) = -(I_n \otimes K_1)(L \otimes I_{n_x})x(t) - (I_n \otimes K_2)(L \otimes I_{n_x})z(t) \tag{6}$$

It is noteworthy that (6) highlights that the global control performance is coupled with communication network topology. Obviously, local control efforts for synchronization differs from one agent to another depending on initial conditions and local interconnection structures.

Discussion:

(I) Using the properties of Kronecker product, the closed-loop system (5) can be decoupled in n independent singularly perturbed systems:

$$\begin{cases} \dot{\tilde{x}}_i(t) = (A_{11} - \lambda_i B_1 K_1)\tilde{x}_i(t) + (A_{12} - \lambda_i B_1 K_2)\tilde{z}_i(t) \\ \varepsilon \dot{\tilde{z}}_i(t) = (A_{21} - \lambda_i B_2 K_1)\tilde{x}_i(t) + (A_{22} - \lambda_i B_2 K_2)\tilde{z}_i(t) \\ \qquad i = 1, \ldots, n \end{cases} \tag{7}$$

(II) Let us recall that $\lambda_1 = 0$, which means that the first system ($i = 1$) in (7) is uncontrolled. The asymptotic synchronization problem using a decentralized control law described in (1) and (2) becomes a problem of simultaneous stabilization of systems in (7) for $i = 2, \ldots, n$. As shown in (Rejeb et al. 2016) this is equivalent with

$$\lim_{t\to\infty} \big(D \otimes I_{n_x}\big)\tilde{x}(t) = 0 \text{ and } \lim_{t\to\infty} \big(D \otimes I_{n_z}\big)\tilde{z}(t) = 0$$

but since $D = diag(\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 = 0$ the condition is transformed as

$$\lim_{t\to\infty} \tilde{x}_i(t) = 0 \text{ and } \lim_{t\to\infty} \tilde{z}_i(t) = 0, \; i = 2, \ldots n$$

(III) We emphasize that from the definition of T, the following also hold $x(t) = \big(T^\top \otimes I_{n_x}\big)\tilde{x}(t)$ and $z(t) = \big(T^\top \otimes I_{n_z}\big)\tilde{z}(t)$. Thus, the collective control vector can be rewritten in the $\tilde{x}$,$\tilde{z}$ variables as :

$$\begin{aligned} u(t) &= -(LT^\top \otimes K_1)\tilde{x}(t) - (LT^\top \otimes K_2)\tilde{z}(t) \\ &= -(T^\top D \otimes K_1)\tilde{x}(t) - (T^\top D \otimes K_2)\tilde{z}(t) \\ &= -(T^\top \otimes I_m)\big[(D \otimes K_1)\tilde{x}(t) + (D \otimes K_2)\tilde{z}(t)\big] \end{aligned} \tag{8}$$

(IV) The synchronization manifold depends on the dynamics of $(\tilde{x}(t), \tilde{z}(t))$. Precisely, if the system

$$\begin{cases} \dot{\tilde{x}}_1(t) = A_{11}\tilde{x}_1(t) + A_{12}\tilde{z}_1(t) \\ \varepsilon \dot{\tilde{z}}_1(t) = A_{21}\tilde{x}_1(t) + A_{22}\tilde{z}_1(t) \end{cases} \tag{9}$$

has a stable equilibrium point $(\tilde{x}^*, \tilde{z}^*)$, then all the systems in (1) will asymptotically reach a finite consensus. If (9) is unstable then all the systems in (1) will synchronize on divergent trajectories.

(V) The control design for the systems in (7) is done by following the classical arguments in (Kokotović et al. 1999). Therefore we need the invertibility of $A_{22} - \lambda_i B_2 K_2, \; i = 1, \ldots, n$. While for $i = 1$ the condition is satisfied due to Assumption 1, for the rest of the systems in (5), K_2 has to be chosen such that $A_{22} - \lambda_i B_2 K_2, \; i = 2, \ldots, n$ is invertible.

Following the item (III) in the discussion above we introduce

$$\widetilde{u}(t) = (T \otimes I_m)u(t)$$

and we notice that $\widetilde{u}(t) = (\widetilde{u}_1(t)^\top, \ldots, \widetilde{u}_n(t)^\top)^\top$ with

$$\widetilde{u}_i(t) = -\lambda_i K_1 \tilde{x}_i(t) - \lambda_i K_2 \tilde{z}_i(t), \; \forall i \in 1, \ldots, n. \tag{10}$$

Moreover, it is straightforward that

$$\begin{aligned} u(t)^\top (I_n \otimes R)u(t) &= \widetilde{u}(t)^\top (TT^\top \otimes R)\widetilde{u}(t) \\ = \widetilde{u}(t)^\top (I_n \otimes R)\widetilde{u}(t) &= \sum_{i=1}^n \widetilde{u}_i(t)^\top R \widetilde{u}_i(t) \end{aligned}$$

Remark 3 *It is worth noting that using the change of variables* (4) *the global cost J in the left hand side of* (3) *can be rewritten as a sum of individual costs associated with the systems in* (7)*:*

$$\begin{aligned} J = &\int_0^\infty \tilde{x}(t)^\top \big(D \otimes I_{n_x}\big)\tilde{x}(t) + \tilde{z}(t)^\top \big(D \otimes I_{n_x}\big)\tilde{z}(t) \\ &+ \widetilde{u}(t)^\top \big(I_n \otimes R\big)\widetilde{u}(t)dt \\ = &\sum_{i=1}^n \Big(\int_0^\infty \lambda_i \tilde{x}_i(t)^\top \tilde{x}_i(t) + \lambda_i \tilde{z}_i(t)^\top \tilde{z}_i(t) \\ &+ \widetilde{u}_i(t)^\top R \widetilde{u}_i(t)dt \Big) = \sum_{i=1}^n \widetilde{\mathcal{J}}_i \end{aligned} \tag{11}$$

where for all $i = 1, \ldots, n$ one has

$$\widetilde{\mathcal{J}}_i = \int_0^\infty \lambda_i \tilde{x}_i(t)^\top \tilde{x}_i(t) + \lambda_i \tilde{z}_i(t)^\top \tilde{z}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t)dt. \tag{12}$$

4 Decentralized control design for synchronization

In this section we derive conditions under which the decentralized synchronization is feasible (see Assumption 3 and Theorem 4). Moreover, we present a control design procedure yielding common gains K_1 and K_2 in (2) that synchronize the singularly perturbed dynamics (1). The design of the decentralized guaranteed cost controller is postponed to section 5.

Before giving our result, let us introduce some notation that allows at completely decouple the slow and fast dynamics that occur in the overall system (see (Kokotović et al. 1999) for details). We define the reduced-order (slow) systems by:

$$\begin{cases} \dot{\tilde{x}}_{i,s}(t) = A_0\tilde{x}_{i,s}(t) - \lambda_i B_0 \tilde{u}_{i,s}(t), \quad \tilde{x}_{i,s}(0) = \tilde{x}_i(0) \\ \qquad i = 1, \ldots, n \end{cases}$$

where $\tilde{u}_{i,s}(t) = K_0\tilde{x}_{i,s}$ with K_0 to be designed in order to simultaneously stabilize the systems above, $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}$ and $B_0 = B_1 - A_{12}A_{22}^{-1}B_2$. Consequently, one has $\tilde{z}_{i,s}(t) = -A_{22}^{-1}\big(A_{21} - \lambda_i B_2 K_0\big)\tilde{x}_{i,s}(t)$ and the corresponding boundary-layer (fast) systems are

$$\begin{aligned} \varepsilon\dot{\tilde{z}}_{i,f}(t) = \big(A_{22} - \lambda_i B_2 K_2\big)\tilde{z}_{i,f}(t), \\ \tilde{z}_{i,f}(0) = \tilde{z}_i(0) - \tilde{z}_{i,s}(0) \end{aligned}$$

Problem 1: For $A \in \mathbb{R}^{p\times p}, B \in \mathbb{R}^{p\times m}$ and $0 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n$ find $K \in \mathbb{R}^{m\times p}$ such that for $i = 2, \ldots, n$ the matrices $A - \lambda_i BK$ are Hurwitz.

Proposition 3 *If the pair (A, B) is stabilizable then there exists K solving Problem 1.*

Proof. If (A, B) is stabilizable then $(A, \lambda^* B)$ is stabilizable too. This is equivalent with the existence of $P = P^\top > 0$ and $K \in \mathbb{R}^{m\times p}$ such that

$$(A - \lambda^* BK)^\top P + P(A - \lambda^* BK) < 0. \tag{13}$$

Since $P > 0$ one has that P is invertible. Using the notation $S = P^{-1}$ and $W = KS$ the matrix inequality (13) can be equivalently rewritten as an LMI (see (Boyd et al. 1994)):

$$SA^\top + AS - \lambda^*(W^\top B^\top + BW) < 0,$$

which can be also seen as

$$S\frac{A^\top}{\lambda^*} + \frac{A}{\lambda^*}S - (W^\top B^\top + BW) < 0.$$

Following again (Boyd et al. 1994, Section 2.6.2), the last LMI is equivalent with the existence of $\sigma > 0$ such that

$$S\frac{A^\top}{\lambda^*} + \frac{A}{\lambda^*}S - \sigma BB^\top < 0. \tag{14}$$

Consequently, the pair (A, B) is stabilizable is equivalent with the existence of $\sigma > 0$ and $S = S^\top > 0$ satisfying (14). It is noteworthy that a stabilizing feedback for the pair $(A, \lambda^* B)$ is defined by $K = \frac{\sigma}{2}B^\top S^{-1}$. Consequently, (14) implies

$$SA^\top + AS < \lambda^*\sigma BB^\top \leq \lambda_i \sigma BB^\top,\ i = 2, \ldots, n,$$

yielding

$$S\frac{A^\top}{\lambda_i} + \frac{A}{\lambda_i}S - \sigma BB^\top < 0, \quad i = 2, \ldots, n,$$

which means that $K = \frac{\sigma}{2}B^\top S^{-1}$ is a stabilizing feedback for all the pairs $(A, \lambda_i B),\ i = 2, \ldots, n.$

Throughout the rest of the paper, the following assumption is imposed.

Assumption 3 *The pairs (A_0, B_0) and (A_{22}, B_2) are stabilizable.*

Theorem 4 *Under Assumption 3, there exist K_2 and K_0 such that for $i = 2, \ldots, n$ the matrices $A_{22} - \lambda_i B_2 K_2$ and $A_0 - \lambda_i B_0 K_0$ are all Hurwitz. Then, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$ the controllers* (2) *with*

$$K_1 = (I_m - K_2 A_{22}^{-1} B_2)K_0 + K_2 A_{22}^{-1} A_{21}.$$

asymptotically synchronize with local information the systems (1).

Proof. Following (Kokotović et al. 1999), the choice of K_1 and K_2 as in the statement above ensures that for $i = 2, \ldots, n$ the systems in (7) are asymptotically stable. Moreover, for all $\varepsilon \in (0, \varepsilon^*]$ and all $t \geq 0$ one has

$$\begin{aligned} &\tilde{x}_i(t) = \tilde{x}_{i,s}(t) + O(\varepsilon), \\ &\tilde{z}_i(t) = -A_{22}^{-1}\big(A_{21} - \lambda_i B_2 K_0\big)\tilde{x}_{i,s}(t) + \tilde{z}_{i,f}(t) + O(\varepsilon). \end{aligned}$$

We recall here that the asymptotic synchronization is equivalent with

$$\lim_{t\to\infty}\big(L \otimes I_{n_x}\big)x(t) = 0 \text{ and } \lim_{t\to\infty}\big(L \otimes I_{n_z}\big)z(t) = 0$$

which hold true since

$$\big(L \otimes I_{n_x}\big)x(t) = \big(D \otimes I_{n_x}\big)\tilde{x}(t) = \Big[0, \lambda_2\tilde{x}_2, \ldots, \lambda_n\tilde{x}_n\Big]^\top$$

and

$$\big(L \otimes I_{n_z}\big)z(t) = \big(D \otimes I_{n_z}\big)\tilde{z}(t) = \Big[0, \lambda_2\tilde{z}_2, \ldots, \lambda_n\tilde{z}_n\Big]^\top$$

Theorem 4 basically says that in order to asymptotically synchronize systems in (1) we have to separately synchronize the fast and slow dynamics by stabilizing the dynamics of the error between the different systems. Notice that all the agents have to solve two LMIs of type (14). These LMIs have as parameters A, B and λ^* which are identical for all the agents. Consequently, as far as they use a similar LMI solver, the agents will find similar S and σ leading to similar K_0, K_2 and therefore K_1. This justifies the supplementary constraint of identical gains that we imposed in the design of the local controller. Another strategy would be to solve the LMIs once in a central computer and make just the implementation decentralized. In other words the controller gains K_1 and K_2 are provided by the central computer to the agents and the agents use them to build their decentralized controllers based on local information.

Corollary 1 *Let K_0 be designed such that for $i = 2, \ldots, n$ the matrices $A_0 - \lambda_i B_0 K_0$ are Hurwitz. If the matrix A_{22} is Hurwitz the controllers u_i in* (2) *with $K_1 = K_0$ and $K_2 = \mathbf{0}_{m\times n_z}$, asymptotically synchronize the systems* (1).

It is noteworthy that the controller design can be done in a decentralized manner since each agent needs to use only λ^* in order to get K_2 and K_0 respectively.

5 Guaranteed cost control design

In this section, we enforce the state feedback simultaneous stabilization problem (SFSS) by considering the additional constraint that consists in ensuring a guaranteed cost on control effort.

5.1 Guaranteed cost control problem for simultaneous stabilization

Consider the linear singularly perturbed system (7) which is equivalent with :

$$\begin{pmatrix} \dot{\tilde{x}}_i(t) \\ \dot{\tilde{z}}_i(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix} \begin{pmatrix} \tilde{x}_i(t) \\ \tilde{z}_i(t) \end{pmatrix} + \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix} \widetilde{u}_i(t), \quad i = 1, \ldots, n \tag{15}$$

where $\varepsilon > 0$, $\tilde{x}_i(t) \in \mathbb{R}^{n_x}, \tilde{z}_i(t) \in \mathbb{R}^{n_z}$ are the components of the state variables defined in (4) and $\widetilde{u}_i \in \mathbb{R}^m$ is the control input defined in (10).

We recall that the synchronization problem of the systems (1) is translated into a simultaneous stabilization problem of systems (7) or equivalently (15). Furthermore, as shown in (11), the global cost associated with the asymptotic synchronization of the n singularly perturbed systems (1) can be seen as the sum of n individual costs associated with the stabilization of systems (7).

Remark 4 *Since $\lambda_1 = 0$ it is noteworthy that $\widetilde{\mathcal{J}}_1 = 0$.*

In the following, for all $i = 2, \ldots, n$, we rewrite $\widetilde{\mathcal{J}}_i$ more compactly as:

$$\widetilde{\mathcal{J}}_i = \int_0^\infty \big(\lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t)\big) dt, \tag{16}$$

where $\widetilde{\mathbf{x}}_i(t) = \left[\tilde{x}_i(t)^\top, \tilde{z}_i(t)^\top\right]^\top \in \mathbb{R}^{n_{\mathbf{x}} = n_x + n_z}$ and the control input weight matrix $R \in \mathbb{R}^{m\times m}$ is symmetric positive definite.

Definition 5 *We say that β_i is a guaranteed cost for the i^{th} system in* (15) *with the control law $\widetilde{u}_i(t)$ if the value of the cost function* (16) *satisfies the inequality $\widetilde{\mathcal{J}}_i \leqslant \beta_i$.*

If there exists a guaranteed cost $\beta_i > 0$ such that the closed-loop value of the cost function (16) satisfies $\widetilde{\mathcal{J}}_i \leqslant \beta_i$ for all $i = 2, \ldots, n$ then a guaranteed cost $\bar{J} \triangleq (n-1) \max_{i=2,\ldots,n} (\beta_i)$ is ensured for the global control performance required to asymptotically synchronize the collective closed loop dynamics (1).

It is noteworthy that the controller in (10) which are used in (16) requires knowledge on the Laplacian eigenvalues. This means that the designed controller cannot be decentralized although it can be chosen to minimize the global cost function J in (3). However, according to Assumption 2, the only available graph information consists in its connectivity.

5.2 Decentralized guaranteed cost control design

In this subsection, we present the design of a decentralized guaranteed cost control law that simultaneously stabilizes the closed-loop singularly perturbed uncertain systems (15) with respect to an adequate level of performance fixed by an upper bound on the integral cost function (16). Before providing the control design, let us introduce the following partitioned matrices

$$A_\varepsilon = \begin{pmatrix} A_{11} & A_{12} \\ \varepsilon^{-1}A_{21} & \varepsilon^{-1}A_{22} \end{pmatrix}, \quad B_\varepsilon = \begin{pmatrix} B_1 \\ \varepsilon^{-1}B_2 \end{pmatrix}$$

System (15) can be rewritten as follows :

$$\dot{\widetilde{\mathbf{x}}}_i(t) = A_\varepsilon \widetilde{\mathbf{x}}_i(t) + B_\varepsilon \widetilde{u}_i(t), \quad \forall i = 2, \ldots, n \tag{17}$$

with the feedback control law $\widetilde{u}_i = F_i \hat{u}_i$ defined by $\hat{u}_i$ of the form

$$\hat{u}_i(t) = -\mathcal{K}\, \widetilde{\mathbf{x}}_i(t) \tag{18}$$

where $\mathcal{K} = [K_1,\ K_2] \in \mathbb{R}^{m\times n_{\mathbf{x}}}$, and $F_i \in R^{n_{\mathbf{x}}\times n_{\mathbf{x}}}$ defined as

$$F_i = \lambda_i I_{n_{\mathbf{x}}}. \tag{19}$$

From basic properties of the Laplacian matrix (Remark 1), one can conclude that, for a given undirected graph of n vertices, the following holds :

$$(\lambda^*)^2 I_{n_{\mathbf{x}}} \leqslant F_i^\top F_i \leqslant (\lambda^\circ)^2 I_{n_{\mathbf{x}}},\ i = 2, \ldots, n \tag{20}$$

Therefore, a general manner of studying (17) without the knowledge of F_i is to analyse the system

$$\dot{\widetilde{\mathbf{x}}}(t) = A_\varepsilon \widetilde{\mathbf{x}}(t) + B_\varepsilon F \hat{u}(t), \tag{21}$$

with F an uncertain matrix satisfying (20). It is noteworthy that (20) is so called "norm bounded uncertainty" in robust control literature (see (Garcia et al. 1998)). Of course, this approach will introduce a certain degree of conservatism which cannot be avoided when we disregard the network topology.

Remark 5 *Under Assumption 3, there exists $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*]$, the pair $(A_\varepsilon, B_\varepsilon)$ is stabilizable.*

Theorem 6 *Consider the uncertain system* (21) *and suppose Assumptions 2 and 3 hold. Then, there exists $\varepsilon^* > 0$ such that for each $\varepsilon \in (0, \varepsilon^*]$ the following Riccati equation:*

$$P_\varepsilon A_\varepsilon + A_\varepsilon^\top P_\varepsilon - 2(\lambda^*)^2 P_\varepsilon B_\varepsilon R^{-1} B_\varepsilon^\top P_\varepsilon + \lambda^\circ I_{n_{\mathbf{x}}} = 0 \quad (22)$$

admits a positive definite symmetric solution P_ε. Moreover, the controller

$$\widetilde{u}_i(t) = -\lambda^* \; R^{-1} B_\varepsilon^\top P_\varepsilon \; \widetilde{\mathbf{x}}_i(t), \quad \forall i \in 2, \ldots, n \quad (23)$$

stabilizes (17). *Furthermore, a guaranteed cost $\beta_i = \widetilde{\mathbf{x}}_i(0)^\top P_\varepsilon \widetilde{\mathbf{x}}_i(0)$ is achieved for the i^{th} system in* (17) *using the controller* (23).

Proof. We split the proof into two steps as follows. The first one focuses on the design of a common Lyapunov function for the systems in (17) while the second deals with the design of the decentralized controller ensuring a guaranteed cost.
Step 1. For the singularly perturbed systems (17) we search a common Lyapunov function of the form

$$V(t, \widetilde{\mathbf{x}}_i) = \widetilde{\mathbf{x}}_i(t)^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t)$$

such that:

$$\min_{\widetilde{u}_i} \left(\lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t) + \dot{V}(t, \widetilde{\mathbf{x}}_i) \right) = 0 \quad (24)$$

According to Remark 5, there exist a positive definite symmetric matrix P_ε and a constant $\varepsilon^* > 0$ such that for all $\varepsilon \in (0, \varepsilon^*)$ one has :

$$A_\varepsilon^\top P_\varepsilon + P_\varepsilon A_\varepsilon - 2(\lambda^*)^2 P_\varepsilon B_\varepsilon R^{-1} B_\varepsilon^\top P_\varepsilon + \lambda^\circ I_{n_{\mathbf{x}}} = 0. \quad (25)$$

Using Assumption 2, one has $\lambda^* \leqslant \lambda_i \leq \lambda^\circ$ for all $i = 2, \ldots, n$. Recalling that $F_i = \lambda_i I_{n_x}$ it follows that:

$$(\lambda^*)^2 B_\varepsilon R^{-1} B_\varepsilon^\top \leqslant B_\varepsilon F_i R^{-1} F_i^\top B_\varepsilon^\top$$

Consequently, for any $\varepsilon \in (0, \varepsilon^*)$ and for all $i = 2, \ldots, n$, the positive definite symmetric solution of (25) satisfies the following Riccati inequality:

$$A_\varepsilon^\top P_\varepsilon + P_\varepsilon A_\varepsilon - 2 P_\varepsilon B_\varepsilon F_i R^{-1} F_i^\top B_\varepsilon^\top P_\varepsilon + \lambda_i I_{n_{\mathbf{x}}} \leqslant 0 \quad (26)$$

Therefore, for all $\varepsilon \in (0, \varepsilon^*)$ along the trajectories of (17) one has that

$$\dot{V}(t, \widetilde{\mathbf{x}}_i) \leqslant -\lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) - \widetilde{u}_i(t)^\top R \widetilde{u}_i(t), \quad \forall i = 2, \ldots, n.$$

Let us now find a suitable controller that solves (24). Let the Hamiltonian of the i^{th} system in (17):

$$\begin{aligned} H_i(\widetilde{\mathbf{x}}_i, \widetilde{u}_i, t) &= \lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t) \\ &+ \dot{\widetilde{\mathbf{x}}}_i(t)^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t) + \widetilde{\mathbf{x}}_i(t)^\top P_\varepsilon \dot{\widetilde{\mathbf{x}}}_i(t) \end{aligned}$$

which is equivalent with

$$\begin{aligned} H_i(\widetilde{\mathbf{x}}_i, \widetilde{u}_i, t) &= \lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t) \\ &+ \left[A_\varepsilon \widetilde{\mathbf{x}}_i(t) + B_\varepsilon F_i \widetilde{u}_i(t) \right]^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t) \\ &+ \widetilde{\mathbf{x}}_i(t)^\top P_\varepsilon [A_\varepsilon \widetilde{\mathbf{x}}_i(t) + B_\varepsilon F_i \widetilde{u}_i(t)] \end{aligned} \quad (27)$$

The necessary optimality condition $\nabla_{\widetilde{u}_i} H_i(\widetilde{\mathbf{x}}_i, \widetilde{u}_i, t) = 0$ implies that $2F_i^\top B_\varepsilon^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t) + 2R\widetilde{u}_i(t) = 0$. Consequently, one obtains that an optimal controller for the i^{th} system in (17) is defined by:

$$\widetilde{u}_i(t) = -R^{-1} F_i^\top B_\varepsilon^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t) = -\lambda_i R^{-1} B_\varepsilon^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t) \quad (28)$$

Step 2. It is noteworthy that the controller in (28) is optimal (but not decentralized) for (17) with the cost function (16). To obtain a decentralized controller we relax the optimality requirement and we only ask to ensure a guaranteed cost. In this case the control design is simply obtained as (23):

$$\widetilde{u}_i^*(t) = -\lambda^* R^{-1} B_\varepsilon^\top P_\varepsilon \widetilde{\mathbf{x}}_i(t).$$

The controller (23) ensures that (25) holds and consequently (26) holds as well. Therefore,

$$\begin{aligned} \widetilde{\mathcal{J}}_i &= \int_0^\infty \left(\lambda_i \widetilde{\mathbf{x}}_i(t)^\top \widetilde{\mathbf{x}}_i(t) + \widetilde{u}_i(t)^\top R \widetilde{u}_i(t) \right) dt \\ &\leqslant V(0, \widetilde{\mathbf{x}}_i(0)) = \widetilde{\mathbf{x}}_i(0)^\top P_\varepsilon \widetilde{\mathbf{x}}_i(0), \; \forall i = 2, \ldots, n \end{aligned}$$

where $\widetilde{\mathbf{x}}_i(0) = [\tilde{x}_i(0)^\top, \; \tilde{z}_i(0)^\top]^\top$.

Remark 6 *Although the individual cost functions $\widetilde{\mathcal{J}}_i$ in* (16) *depend on λ_i, the individual guaranteed costs β_i do not. Consequently, the global guaranteed cost $\bar{J}$ do not depend on λ_i. This means that both the decentralized controllers* (23) *and the guaranteed cost are independent of λ_i and they depend only on the network size: n.*

Now our problem is stated as finding the matrix P_ε which through conditions in Theorem 6, determines the solution to the guaranteed cost control problem for the i^{th} singularly perturbed system. Indeed, the Riccati equation (22) is first solved for P_ε and then the guaranteed cost controller gains are obtained by substituting the Riccati solution to (23).

Remark 7 *Note that the obtained cost β_i in Theorem 6 depends on the initial conditions $\tilde{x}_i(0)$ and $\tilde{z}_i(0)$. This dependance can be removed by assuming that initial conditions are zero mean random variables with $E[\widetilde{\mathbf{x}}_i(0)\ \widetilde{\mathbf{x}}_i(0)^\top] = I_{n_{\mathbf{x}}}$.*

5.3 Numerical implementation for decentralized guaranteed cost control design

It is important to stress that some numerical problems may arise in solving equation (22) because of the different magnitudes of its coefficients caused by the small parameter ε. Inspired by (Mukaidani, Xu & Okita 1999) we provide in this subsection an approach that overcomes these computation difficulties. The idea is to transform the full-order Riccati equation (17) depending on ε into slow and fast Riccati equations independent of ε. Doing so, we find a stabilizing solution for the guaranteed cost control problem of singularly perturbed uncertain system (21).
First, let us introduce the following lemma that will be used to solve the algebraic Riccati equation (22).

Lemma 7 *The algebraic Riccati equation* (22) *is equivalent to the following Riccati equation :*

$$PA + A^\top P - 2(\lambda^*)^2 PBR^{-1}B^\top P + \lambda^\circ I_{n_{\mathbf{x}}} = 0 \quad (29)$$

where $P = \Gamma_\varepsilon^{-1} P_\varepsilon$, $A = \Gamma_\varepsilon A_\varepsilon,\ B = \Gamma_\varepsilon B_\varepsilon$,

$$P_\varepsilon = \begin{pmatrix} P_{11} & \varepsilon P_{21}^\top \\ \varepsilon P_{21} & \varepsilon P_{22} \end{pmatrix},\ \textit{such that} \begin{cases} P_{11} = P_{11}^\top \\ P_{22} = P_{22}^\top \end{cases} \quad (30)$$

$$\Gamma_\varepsilon = \begin{pmatrix} I_{n_x} & 0_{n_x,n_z} \\ 0_{n_z,n_x} & \varepsilon I_{n_z} \end{pmatrix},\ \textit{for all}\ \varepsilon \in (0, \varepsilon^*).$$

Further, by making use of relation (30)*, the controller gain in* (23) *becomes*

$$\mathcal{K} = [K_1,\ K_2] = R^{-1}B^\top P. \quad (31)$$

It is worth noting that $P_\varepsilon = \Gamma_\varepsilon^\top P = P^\top \Gamma_\varepsilon$ is symmetric but P is not. The proof of Lemma 7 is omitted since it is similar to the proof of Lemma 3 in (Mukaidani, Xu & Mizukami 1999). In order to avoid this dependence on ε while guaranteeing the existence of a bounded solution P when $\varepsilon = 0$ and its neighborhood, we use a similar reasoning with the one in Theorem 2 in (Mukaidani, Xu & Okita 1999). Doing so, for $\varepsilon = 0$ the Riccati equation (29) becomes:

$$\overline{P}_{11}^\top A_{11} + A_{11}^\top \overline{P}_{11} + \overline{P}_{21}^\top A_{21} + A_{21}^\top \overline{P}_{21} - \overline{P}_{11}\widetilde{S}_{11}\overline{P}_{11} - \overline{P}_{21}^\top \widetilde{S}_{12}^\top \overline{P}_{11} - \overline{P}_{11}\widetilde{S}_{12}\overline{P}_{21} - \overline{P}_{21}^\top \widetilde{S}_{22}\overline{P}_{21} + \lambda^\circ I_{n_x} = 0 \quad (32)$$

$$\overline{P}_{22}^\top A_{21} + A_{12}^\top \overline{P}_{11} + A_{22}^\top \overline{P}_{21} - P_{22}^\top \widetilde{S}_{12}\overline{P}_{11} - \overline{P}_{22}\widetilde{S}_{22}\overline{P}_{21} = 0 \quad (33)$$

$$\overline{P}_{22}^\top A_{22} + A_{22}^\top \overline{P}_{22} - \overline{P}_{22}\widetilde{S}_{22}\overline{P}_{22} + \lambda^\circ I_{n_z} = 0 \quad (34)$$

where $\overline{P}_{11}, \overline{P}_{21}, \overline{P}_{22}$ are the limiting solutions when $\varepsilon \to 0$ and

$$\widetilde{S} = 2(\lambda^*)^2 BR^{-1}B^\top = \begin{pmatrix} \widetilde{S}_{11} & \widetilde{S}_{12} \\ \widetilde{S}_{12}^\top & \widetilde{S}_{22} \end{pmatrix}.$$

Let us define the following set :

$\mathcal{L}_f = \{n > 1,$ such that the Riccati equation (34) has a positive definite solution$\}$,

Note that $A_{22} - \widetilde{S}_{22}\overline{P}_{22}$ is non-singular. Thus, equations (32)-(34) become as follows :

$$\overline{P}_{11}\bar{A}_0 + \bar{A}_0^\top \overline{P}_{11} - \overline{P}_{11}\widetilde{S}_0\overline{P}_{11} + \widetilde{Q}_0 = 0 \quad (35)$$

$$\overline{P}_{21} = -N_2^\top + N_1^\top \overline{P}_{11} \quad (36)$$

$$\overline{P}_{22}A_{22} + A_{22}^\top \overline{P}_{22} - \overline{P}_{22}\widetilde{S}_{22}\overline{P}_{22} + nI_{n_z} = 0 \quad (37)$$

where

$$N_1 = -D_1 D_2^{-1}, \quad N_2 = A_{21}^\top \overline{P}_{22} D_2^{-1}$$
$$\widetilde{Q}_0 = nI_{n_x} - N_2 A_{21} - A_{21}^\top N_2^\top - N_2 \widetilde{S}_{22} N_2^\top$$

and

$$\bar{A}_0 = A_{11} + N_1 A_{21} + \widetilde{S}_{12} N_2^\top + N_1 \widetilde{S}_{22} N_2^\top$$
$$\widetilde{S}_0 = \widetilde{S}_{11} + N_1 \widetilde{S}_{12}^\top + \widetilde{S}_{12} N_1^\top + N_1 \widetilde{S}_{22} N_1^\top$$
$$D_1 = A_{12} - \widetilde{S}_{12}\overline{P}_{22}, \quad D_2 = A_{22} - \widetilde{S}_{22}\overline{P}_{22}.$$

Let us also introduce :

$\mathcal{L}_s = \{n > 1,$ such that the Riccati equation (35) has a positive definite solution$\}$.

Then, we have the following result.

Theorem 8 *For all $n \in \mathcal{L}$, there exists $0 < \bar{\varepsilon} < \varepsilon^*$ such that for all $\varepsilon \in (0, \bar{\varepsilon})$, the generalized algebraic Riccati equation* (29) *has a positive definite solution of the form*

$$P = \begin{pmatrix} \overline{P}_{11} + \mathcal{O}(\varepsilon) & \varepsilon \overline{P}_{12}^\top + \mathcal{O}(\varepsilon^2) \\ \overline{P}_{21} + \mathcal{O}(\varepsilon) & \overline{P}_{22} + \mathcal{O}(\varepsilon) \end{pmatrix}$$

where $\mathcal{L} = \mathcal{L}_s \bigcap \mathcal{L}_f$. Thus, a guaranteed cost controller is given by (23) *with $P_\varepsilon = \Gamma_\varepsilon^\top P$ is a positive definite matrix solving the Riccati equation* (22).

Theorem 8 implies that, if the Riccati equation (29) has a positive matrix solution P then the solution P_ε of (22) will be the smallest upper bound on the criterion (16).
Further, the controller (23) guarantees a simultaneous stabilization of the $n-1$ subsystems (17) with a guaranteed cost of value β_i. Consequently, the n singularly perturbed systems (1) achieve asymptotic synchronization under the state feedback controller (2) with the matrix gain (31). In addition, the global control effort required to achieve the synchronization is upper-bounded by $\bar{J} = (n-1) \max_{i=2,\ldots,n} (\beta_i)$.

6 Numerical examples

In this section, we consider the synchronization of three agents whose dynamics are given by (1) where :

$$A_{11} = \begin{pmatrix} 2.5 & -6 \\ -2 & 2 \end{pmatrix}, \; A_{12} = \begin{pmatrix} 2 & 3 \\ 0 & -2 \end{pmatrix}, \; B_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$A_{21} = \begin{pmatrix} 0.5 & 2 \\ -1 & 1 \end{pmatrix}, \; A_{22} = \begin{pmatrix} -2 & 1 \\ 0 & -1 \end{pmatrix}, \; B_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

To each agent we assign a vector state having 4 components characterized by slow and fast dynamics.
For any agent $i \in \{1, 2, 3\}$, let us denote by $[x_{i,1},\ x_{i,2}]^\top$ and $[z_{i,1},\ z_{i,2}]^\top$ its slow and fast state' components, respectively.

The communication network among the 3 agents is described by an undirected graph $\mathcal{G}$ which is connected and the following Laplacian matrix : $L = \begin{pmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{pmatrix}$.
By choosing $g_m = 1$ and $g_M = 2$, the bounds on the eigenvalues of L are given by $\lambda^* = 0.0278$ and $\lambda^\circ = 6$. In simulation we fix $\varepsilon = 0.01$ and the components of the initial condition are chosen for the 3 agents as $[2.5,\ 2,\ -0.5,\ -1.5]$, $[1.5,\ 1,\ 4,\ -2]$, $[0.5,\ -1,\ 3,\ 1]$. Note that, when rewriting the system dynamics as in (17), one can verify that Assumption 3 holds and the uncertainties are given by $F_{i=2,3} \in \{4 \times I_4,\ 6 \times I_4\}$. We choose $R = 1$ in (16) for the cost function. To apply Theorem 6 one needs to compute the matrix P_ε. In order to do that we first solve (29) which yields the gain matrix

$$\mathcal{K} = [K_1\ K_2] = [38.96\ ,\ -140.43\ ,\ 2.4\ ,\ 8.29].$$

Figures (1) and (2) highlight the simultaneous synchronization of the slow and fast dynamics. Next, we apply the control law in (2) with the obtained gain matrix $\mathcal{K}$ to the two time scale model (1). From Fig 3, it can be observed that the states of the closed-loop systems reach consensus with a global guaranteed cost $\bar{J} = 295.66$.
In order to emphasize the influence of the penalty matrix R, in Fig 4 we plot the control law u for $R = 1$ and $R = 10$. As expected, when R increases the controller gains decrease generating a smaller magnitude of the control input. Solving (29) for $R = 10$ one obtains the controller gain

$$\mathcal{K} = [K_1\ K_2] = [4.83\ ,\ -42.466\ ,\ 0.25\ ,\ 1.44]$$

The systems in (1) will synchronize slower with a guaranteed cost $\bar{J} = 826.375$.

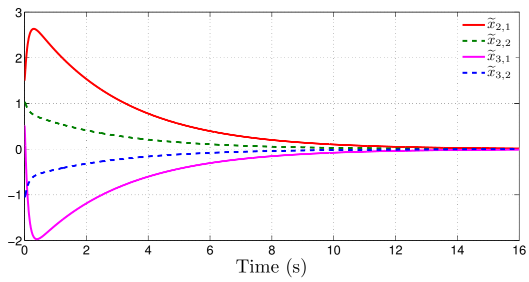

Fig. 1. The trajectories of $\tilde{x}$

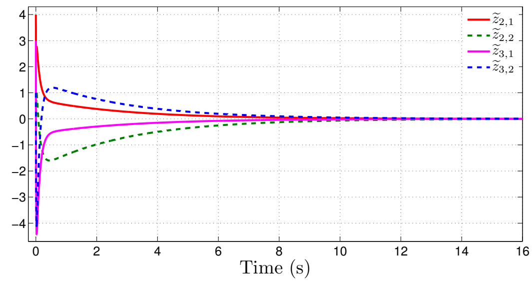

Fig. 2. The trajectories of $\tilde{z}$

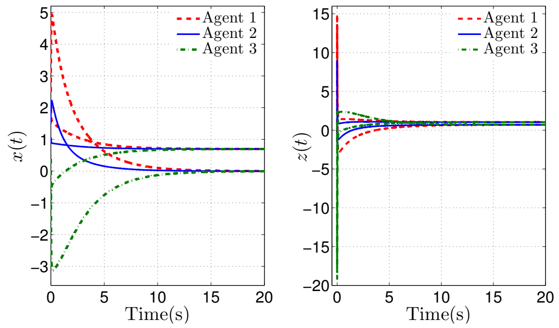

Fig. 3. State trajectories of the system

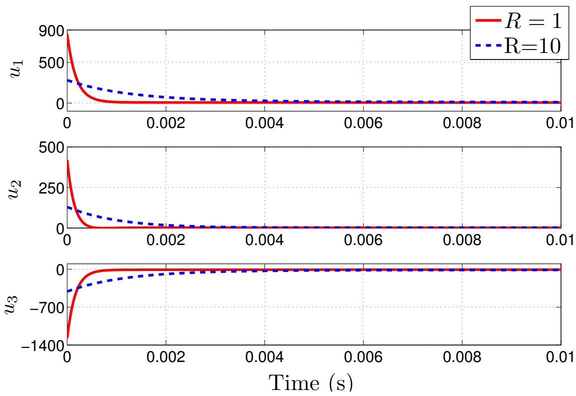

Fig. 4. Control input $u_{i=1,2,3}$ for $R = 1$ in solid line and $R = 10$ in dashed line

7 Conclusion

In this paper we propose a decentralized control strategy that allows singularly perturbed multi-agent systems to achieve synchronization with global performance guarantees. The main difficulty that we have to overcome is that we have to avoid the use of centralized information related to the interconnection network structure. This problem is solved by sequentially rewriting the synchronization problem in terms of simultaneous stabilization of singularly perturbed systems and then as stabilization of a singularly perturbed uncertain linear system. The singularly perturbed dynamics of subsystems generates theoretical challenges related to the stabilizing controller design but also numerical issues related to the computation of the controller gains. We show that these problems can be solved by decoupling the slow and fast dynamics.

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