Saturated control of consensus value under energy and state constraints in multi-agent systems

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Abstract

This work presents a novel decentralized control strategy with a guaranteed cost for bilinear multi-agent systems subjected to products between the state and the control input, state constraints, and limitations on the amplitude and total energy of the control action, which can prevent the consensus from reaching the desired value. We propose state feedback and switching control laws to deal with these restrictions. The main objectives of the work are twofold: i) to design control laws that ensure stability and guaranteed cost bounds under constraints, and ii) to determine an estimation of the domain of attraction (DOA) as large as possible characterized by a polyhedral and an ellipsoidal invariant region contained in the space defined by the state constraints. We adopt a convex optimization procedure based on linear matrix inequalities (LMI) to address these objectives. An original approach employing Lyapunov sets is proposed to deal with control energy and state constraints, and positive system properties are used to estimate the DOA in only one orthant of state space. Through numerical examples, we demonstrate the effectiveness of the proposed Lyapunov-based approach, showing its ability to handle complex constraints and large networks.

Key words: Multi-agent systems; Consensus; Bilinear systems; Constrained states; Linear matrix inequalities; Guaranteed cost control; Saturated control.

1 Introduction

Multi-agent systems (MAS) consist of several independent but interconnected systems with a specific common goal, while every agent has a local view of the network. The flexibility given by the decentralized coordination control makes MAS popular in several applications like opinion dynamics, robotics, and power grids. The coherent behavior of the MAS is often described in terms of consensus, i.e., the agreement of the autonomous agents on a value of interest (Olfati-Saber & Murray 2004), through their operations and interactions with other agents present in the network. The coordination of MAS can face several challenges, like input saturation and constraints on the state variables and the total control energy.

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Saturation of the control signal is a well-known issue in dynamic systems that has received much attention in the literature (see Tarbouriech et al. (2011) and references therein), in particular in the context of Adaptive Dynamic Programming (ADP) (Dong et al. 2017, Shi & Zhou 2022). Notably, the problem becomes more intricate if the saturation is associated with the consensus problems in MAS. The presence of saturation can yield nonlinear closed-loop dynamics and may also prevent reaching a consensus. Some works dealing with consensus under input saturation consider integrator systems (Yang et al. 2014), event-triggered ADP methods (Shi & Zhou 2022), and only a few are related to the global coordination of high-order agents subject to input saturation (Su et al. 2013, Col et al. 2019). Additionally, control effort may be subject to finite energy, interpreted as a budget constraint in some applications of MAS, such as viral marketing (VM) over social networks (SN). In this context, agents' opinions are influenced by advertiser and individual interactions (Morărescu et al. 2020, Alkhorshid et al. 2022). Several approaches exist to deal with the lack of resources. For example, one can cite using the maximum possible investment as soon as possible to maximize the convergence of the trajectories (Morărescu et al. 2020). Constraints in the states represent practical situations due to physical limitations (e.g., level of tanks) or safety requirements. These additional constraints can induce nonlinear behaviors and make the control protocol particularly difficult to design.

This paper considers the problem of designing a decentralized protocol for the consensus of a class of bilinear agents subject to state constraints and energy and magnitude limitations in the exogenous control action. The protocol design is formulated as a convex optimization problem considering a global cost function as the performance criteria. A fixed topology connects the agents in the network. Since the model representing the agent dynamics is bilinear, the control input is saturated, and the states are constrained, we propose polyhedral and ellipsoidal invariant regions as an estimation of the domain of attraction (DOA) of the origin based on quadratic Lyapunov functions (Genesio et al. 1985). However, as the states are restricted to the positive or negative orthants, obtaining invariant ellipsoidal level sets is a challenging problem (Tarbouriech et al. 2011). Then, we propose LMI conditions exploiting properties of positive systems to maximize the invariant region where the state trajectories must belong. To deal with the state dependency in the design of state feedback controllers, we adopt the parametrization of the states as norm-bounded uncertainties. This approach is more appropriate than the polytopic approximation of the bilinear term (Amato et al. 2009) when dealing with high-order systems (many agents).

The paper presents original control approaches, further extending the preliminary results announced by Alkhorshid et al. (2022) by considering a polyhedral invariant region as an estimation for DOA and a switched control action that mitigates the influence of the bilinear term in the protocol design procedure. Consequently, the estimation for the DOA is improved to be the entire domain of the states, and we can apply the proposed technique to large networks. Furthermore, the switching mechanism is designed considering the energy constraint in the control action that usually prevents the agents from reaching the desired value. The work also presents a contribution in the context of opinion dynamics. The proposed approach is directly applied to this class of dynamics where the states represent opinions (normalized between 0 and 1) as a case study. An external action tries to sway the agreement value toward a desired one by a limited budget/energy continuously spent over time.

Notation. The set of real matrices with dimension $n\times m$ is denoted by $\mathbb{R}^{n\times m}$; if the entries are non-negative (non-positive), the set is denoted by $\mathbb{R}_+^{n\times m}$ ($\mathbb{R}_-^{n\times m}$). For a matrix X, X^T and $X_\perp$ denote the transpose of X and any matrix whose columns form a basis for the null space of X, respectively. $X_{(i)}$ denotes the $i-$th row and $X_{(ij)}$ the entry (i,j) of X. If X is square, X^{-1} denotes the inverse of X; $\text{He}\{X\}$ stands for $X+X^T$; and $X>0$ ($X<0$) indicates that matrix X is positive (negative) definite and $X\succcurlyeq 0$ ($X\preccurlyeq 0$) indicates that all the components of matrix X are nonnegative (nonpositive). For a vector $v\in\mathbb{R}^n$, $\text{diag}(v_1,\ldots,v_n)$ is a diagonal matrix composed with the elements of v, and for matrices X_i, $i=1,\ldots,n$, $\text{diag}(X_1,\ldots,X_n)$ denotes a block diagonal matrix with diagonal blocks $X_1,\ldots,X_n$. The I_n, $0_{m,n}$ and $\mathbb{1}$ denote identity $n\times n$ matrix, null $m\times n$ matrix (or simply I and 0 if no confusion arises) and vector of ones, respectively; the symbols $\star$ and $\otimes$ denote symmetric blocks and Kronecker product respectively.

2 Preliminaries

2.1 Problem formulation

Consider the following continuous-time system

$$\dot{x}(t) = Ax(t) + B(x(t))u(t), \tag{1}$$

where $x\in\mathbb{R}^n$ is the state, $u\in\mathbb{R}^n$ is the control input, $A\in\mathbb{R}^{n\times n}$ is a Metzler matrix, and $B(\cdot):\mathbb{R}^n\to\mathbb{R}^{n\times n}$ being a function defined by $B(x)=\text{diag}(x_1,x_2,\ldots,x_n)$. The external control action is bounded as $u_i\in[-\bar{u},\ \bar{u}]$, $\bar{u}\in(0,1)$ and has a finite amount of energy, and the states are constrained as $x_i\in[-d,\ 1-d]$, $i=1,\ldots,n$, $d\in\{0,1\}$. The parameter d defines the constraint of the states, and the origin is an equilibrium point. As we will discuss further, parameter d can also represent the desired consensus value in the framework of MAS.

Observe that (1) is a constrained bilinear system with a decoupled control input and normalized states. For $d=0$, the states are constrained to $[0,1]^n$, and the system is positive. Many applications employ normalized states, and one can cite, for instance, the per-unit system in power systems, among many others. We can also consider the case where the states are restricted to belong to the negative orthant by adopting $d=1$. The bilinear product between the states and the control input can be found in many physical phenomena, such as biological processes. Decoupled control actions are mainly found in decentralized or distributed control schemes where each control algorithm acts locally (Ge et al. 2017). Although most of the techniques presented in the paper can be adapted for a more general class of bilinear systems, we will explore the properties of positive systems and decoupled control actions to deal with the state constraint. In the following, we describe a multi-agent system as the main motivation for studying (1).

Consider a set of n systems interconnected over a network represented by the graph $\mathcal{G}(\mathcal{V},\mathcal{E},\mathcal{A})$. We associate a vertex in $\mathcal{V}=\{v_1,\ldots,v_n\}$ to each system. The interconnection between systems i and j is represented by the edge (v_i,v_j) whose weight is the component a_{ij} of the adjacency matrix $\mathcal{A}$. Thus, $\mathcal{A}=[a_{ij}]$ with $a_{ij}>0$ if

$(v_i, v_j) \in \mathcal{E}$ and $a_{ij} = 0$ otherwise. The Laplacian matrix associated with the graph $\mathcal{G}(\mathcal{V}, \mathcal{E}, \mathcal{A})$ is defined by $L = [l_{ij}]$ with $l_{ii} := \sum_{j=1}^{n} a_{ij}$ and $l_{ij} := -a_{ij}$, for $i \neq j$. A neighbor of node v_i is every node v_j for which $a_{ij} > 0$; and the neighborhood of node v_i is described by the set $\mathcal{N}_i := \{v_j \in \mathcal{V} : (v_j, v_i) \in \mathcal{E}\}$.

Assumption 1 *We consider the graph $\mathcal{G}$ as weakly connected, i.e., it contains at least one directed spanning tree.*

Lemma 2 *Under Assumption 1, the Laplacian matrix L has a simple eigenvalue equal to 0 associated with the right eigenvector $\mathbb{1} \in \mathbb{R}^n$, meaning $L\mathbb{1} = 0$. The other $n - 1$ eigenvalues of matrix L have positive real parts.*

In the sequel, the interconnected systems are referred to as agents. We define the state of agent i by a scalar value ξ_i normalized between 0 and 1. The dynamics in the network evolve with respect to two different aspects: an external control action that aims to sway the network consensus toward a desired value $d \in \{0, 1\}$, and the interactions between neighboring agents. The following dynamic model characterizes the evolution of the states through time

$$\dot{\xi}_i(t) = \sum_{j=1}^{n} a_{ij}(\xi_j(t) - \xi_i(t)) + (\xi_i(t) - d)u_i(t), \quad (2)$$

$\forall i \in \{1, \ldots, n\}$, where $u_i(t) \in [-\bar{u}, \bar{u}]$, $\bar{u} \in (0, 1)$, is a bounded external control action with finite amount of energy and $\xi \in [0, 1]^n$. The first part of the expression (2) represents the disagreement between the states of neighboring agents considering their respective connection weight. The second part considers the disagreement between the state of an agent and the desired goal d considering a local control input. In this context, if the agents are far from the desired value d, the external action has a stronger effect on the dynamics. For instance, in opinion dynamics, this term expresses an increasing resistance of individuals while approaching the advertised state (Morărescu et al. 2020). Suppose $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))^T$ and $u(t) = (u_1(t), \ldots, u_n(t))^T$ are the vectors collecting agents' states and control inputs, respectively. The collective dynamics of the system is expressed by

$$\dot{\xi}(t) = -L\xi(t) + B(\xi(t) - \mathbb{1}d)u(t), \quad (3)$$

where $B(\cdot)$ is a function as defined in (1). Let us define $x_i(t) = \xi_i(t) - d$, $i = 1, \ldots, n$. Thanks to Lemma 2, system (3) is rewritten as (1) with $A = -L$:

$$\dot{x}(t) = -Lx(t) + B(x(t))u(t), \quad (4)$$

where $x(t) \in \mathcal{X}$, $\mathcal{X} = \{x \in \mathbb{R}^n : x_i \in [-d,\ 1 - d]\}$. The following problem describes the main challenge we aim to tackle in this work.

Problem 3 *Design the control input u considering to the following optimization problem:*

$$\min_{u(t)} J_x, \quad J_x = \int_0^{\infty} x(t)^T R x(t)\, dt \quad (5)$$

subject to

$$x(t) \in \mathcal{X} \quad (6)$$
$$|u_i(t)| \leq \bar{u} \quad (7)$$
$$J_u = \int_0^{\infty} u(t)^T Q u(t)\, dt \leq \mu \quad (8)$$

where J_x is the global cost associated with (4) and considered as performance criteria, J_u is the total energy of u limited by μ, and R and Q are positive definite matrices used to balance the agent's convergence and energy required for synchronization, respectively.

The afforded energy μ in (8) gives an upper bound on the total energy cost we accept to pay to change the final agreement value in the network toward the desired value d. Note that functions J_x and J_u defined by $\int_0^{\infty} z(t)^T z(t)\, dt$ and $\int_0^{\infty} y(t)^T y(t)\, dt$, can be rewritten in a more convenient form for design conditions as

$$z(t) = R^{\frac{1}{2}} x(t), \quad y(t) = Q^{\frac{1}{2}} u(t). \quad (9)$$

Remark 4 *It is possible to consider that only a subset of agents can access to the desired consensus value d, which can be seen as a virtual leader reference. In this case, (2) becomes $\dot{\xi}_i(t) = \sum_{j=1}^{n} a_{ij}(\xi_j(t) - \xi_i(t)) + g_i(\xi_i(t) - d)u_i(t)$, where $g_i \geq 0$, $i = 1, \ldots, n$, are pinning gains such that if there is path between the leader (reference value) and the i-th agent, $g_i > 0$, otherwise $g_i = 0$. The design conditions can be adjusted by replacing $B(x)$ by $\mathcal{G}B(x)$ in (4), with $\mathcal{G} = diag(g_1, \ldots, g_n)$, as long as at least one gain g_i is different from zero.*

Remark 5 *Problem 3 with respect to (2) can model a problem of opinion dynamics in social networks where the edges represent the interactions between individuals and the state assigned to each node is a normalized opinion. The control action corresponds to a campaign of viral marketing, which, in practical applications, is limited by an available budget μ. Moreover, d embodies the desired goal of the campaign (for instance, consume/ not consume). Hence, a campaign aims to persuade a network toward a desired opinion minimizing (5) with respect to (6) using a bounded and limited control effort (7)–(8).*

Remark 6 *Note that (8) might hamper x from reaching the origin (or, equivalently, $\xi_i(t)$ from reaching the desired value d), then J_x becomes infinity. For the cases where we want to converge as close as possible to 0 in a limited amount of time T, we replace J_x in (5) by $J_x = \int_0^T x(t)^T R x(t) dt$. In the absence of external action,* *the system dynamics become $\dot{x}(t) = -Lx(t)$, which has a global exponentially stable attractor set $x_i = x_j, \forall i, j$. Therefore, when the external control action is active for a period T such that the transient response of the closed-loop system vanishes, $x_i(T)$ is a good approximation of $\lim_{t\to\infty} x_i(t)$.*

2.2 Representation of the bilinear terms and saturation model

The amplitude bound restriction in (7) is represented in the system dynamics using the decentralized saturation function $u_{(\ell)} = \text{sat}(v_{(\ell)}) = \text{sign}(v_{(\ell)}) \min(|v_{(\ell)}|, \bar{u})$, $\ell = 1, \ldots, n$, where v represents the unbounded control input to be designed. Thus, system (4) with $u = \text{sat}(v)$ is rewritten with respect to the decentralized dead-zone nonlinearity $\psi(v) = v - sat(v)$ as

$$\dot{x}(t) = -Lx(t) + B(x(t))v(t) - B(x(t))\psi(v(t)). \quad (10)$$

To assess the stability of the closed-loop system under the saturation of the control signal, the following lemma is applied to allow the use of a sector condition valid in a compact set of the state space (Gomes da Silva Jr. & Tarbouriech 2005).

Lemma 7 *Let $G \in \mathbb{R}^{n\times n}$ be a matrix related to the region*

$$\Pi = \{x \in \mathbb{R}^n : |v_i - G_i x| \leq \bar{u},\ i = 1, \ldots, n\}. \quad (11)$$

If $x \in \Pi$, the following condition holds

$$\psi(v)^T T(\psi(v) - Gx) \leq 0 \quad (12)$$

for any diagonal and positive definite matrix $T \in \mathbb{R}^{n\times n}$.

Let us present the term $B(x)$, for $x \in \mathcal{X}$, as the following norm-bounded uncertainty model

$$B(x) = B_0 + B_1\Delta(t), \quad (13)$$

where $B_0 = (0.5-d)I, B_1 = 0.5I, \Delta(t) = \text{diag}(\delta_1(t), \ldots, \delta_n(t)) \in \mathbb{R}^{n\times n}$, and $\delta_i(t)$ is a bounded Lebesgue measurable uncertainty associated with the set $\mathcal{D} = \{\delta \in \mathbb{R} :\ \delta^T\delta \leq 1\}$. We will rewrite Problem 3 in the framework of positive systems to establish an invariant region where the initial conditions will belong. First, we present some basic results associated with positive systems and some instrumental lemmas useful for further development.

Definition 1 (Kaczorek (2002)) *System $\dot{x}(t) = Ax(t)$ is called positive if for any $x(0) \in \mathbb{R}^n_+$, one has $x(t) \in \mathbb{R}^n_+$ for $t \geq 0$.*

Lemma 8 (Kaczorek (2002)) *System $\dot{x}(t) = Ax(t)$ is positive if and only if A is a Metzler matrix (i.e., $\forall i \neq j : A_{(ij)} \geq 0$).*

Lemma 9 (Petersen (1987)) *Consider $G = G^T \in \mathbb{R}^{n\times n}$, $M \in \mathbb{R}^{n\times p}$, and $N \in \mathbb{R}^{q\times n}$ as pre-defined matrices. For all $\Delta \in \mathbb{R}^{p\times q}$ such that $\Delta^T\Delta \leq I$, the inequality*

$$G + M\Delta N + N^T\Delta^T M^T \leq 0$$

holds if and only if there exists a scalar value $\lambda > 0$ such that

$$G + \lambda MM^T + \frac{1}{\lambda}N^T N \leq 0.$$

Lemma 10 (Boyd et al. (1994)) *Consider $w \in \mathbb{R}^n$, $M = M^T \in \mathbb{R}^{n\times n}$,$B \in \mathbb{R}^{m\times n}$ with $rank(B) < n$ and $B_\perp$ a basis of B nullspace ($BB_\perp = 0$). The following statements are equivalent:*
(i) $w^T Mw < 0, \forall w \neq 0, Bw = 0$.
(ii) $B_\perp^T MB_\perp < 0$.
(iii) $\exists N \in \mathbb{R}^{n\times m} : M + NB + B^T N^T < 0$.

Lemma 11 (Bitsoris (1991)) *The polyhedral set $\{x \in \mathbb{R}^n :\ \Omega x \preccurlyeq \mathbb{1}\}$, with $\Omega \in \mathbb{R}^{m\times n}$, $rank(\Omega) = m$, $\mathbb{1} \in \mathbb{R}^m$, is a positively invariant set of system $\dot{x}(t) = Ax(t)$ if and only if there exist a Metzler matrix $H \in \mathbb{R}^{m\times m}$ such that $\Omega A = H\Omega$ and $H\mathbb{1} \preccurlyeq 0$.*

3 Main results

This section presents two approaches for solving Problem 3: state feedback and on-off control laws. Due to the bilinear nature of the dynamics, the results hold locally, and we provide an estimate of the region of stability of the closed-loop system. We propose methods to compute ellipsoidal and polyhedral estimates of the DOA as large as possible.

3.1 Consensus based on state feedback controllers

In this section, we discuss a solution to Problem 3 using the following state feedback control law in (10)

$$v(t) = Kx(t), \quad K = \text{diag}(k_1, \ldots, k_n) \in \mathbb{R}^{n\times n}, \quad (14)$$

where $k_i \in \mathbb{R}$, $i = 1, \ldots, n$, are gains to be designed. A diagonal structure for K is assumed to assure the local action of the control law, i.e., agents do not have access to the state of their neighbors. This scenario is more challenging than considering the availability of the neighboring states. Considering (9), (10) and (14), the closed-loop system is given by

$$\dot{x}(t) = (-L + (B_0 + B_1\Delta(t))K)x(t) \\ - (B_0 + B_1\Delta(t))\psi(Kx(t)) \quad (15a)$$
$$z(t) = R^{\frac{1}{2}}x(t) \quad (15b)$$
$$y(t) = Q^{\frac{1}{2}}Kx(t) + Q^{\frac{1}{2}}\psi(Kx(t)). \quad (15c)$$

Remark 12 *Observe that the controllability matrix of $(-L, B(x))$ loses rank for $x = 0$, a stable equilibrium point. Therefore, synthesis conditions based on the closed-loop system* (15) *with $B_0 = (0.5 - d)I$ and $B_1 = 0.5I$ are not feasible for all $\Delta(t) \in \mathcal{D}$. We can circumvent this problem modeling $B(x)$ for the interval $x_i \in [\varepsilon-(1+2\varepsilon)d, 1+\varepsilon-(1+2\varepsilon)d]$, where $\varepsilon > 0$ is an arbitrarily small scalar, yielding $B_0 = (0.5-d+(1-2d)\varepsilon)I$. Therefore, we assess the stability of the closed-loop system* (15) *for $x \in \mathcal{X} \setminus \mathcal{B}_\varepsilon$, where $\mathcal{B}_\varepsilon = \{x \in \mathcal{S} : x^T x \leq \varepsilon\}$. This implies that we can only guarantee the convergence of the system's trajectories to $\mathcal{B}_\varepsilon$. However, as long as ε is sufficiently small, this has no significant practical implications since constraint* (8) *usually prevents x_i from reaching the origin (or, equivalently, $\xi_i(t)$ from reaching d).*

In the following, we propose design conditions based on the generalized sector condition (12), valid in region Π, to deal with the saturation of the control input and the constraint $x \in \mathcal{X}$. To ensure the trajectories of the closed-loop system remain in $\mathcal{X} \cap \Pi$, the following region $\mathcal{S}$ is defined for $d = 0$ such that for all $x(0) \in \mathcal{S}$, $x(t) \in \mathcal{X} \cap \Pi$, $\forall t \geq 0$:

$$\mathcal{S} = \left\{x \in \mathbb{R}^n_+ : x^T W^{-1} x \leq 1,\ W = W^T > 0\right\}, \quad (16)$$

where W is a positive definite matrix to be designed. For $d = 1$, the set $\mathcal{S}$ is defined as (16) with domain $x \in \mathbb{R}^n_-$. Note that the set $\mathcal{S}$ is restricted to the positive ($d = 0$) or negative ($d = 1$) orthants, and therefore it is not a region completely defined by a level set of the Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$. Moreover, conditions that guarantee $\mathcal{S} \subset \mathcal{X}$ are not easy to obtain. We first introduce the following level sets, and then we show that $\mathcal{S}$ is an invariant set and an estimate for the domain of attraction of the closed-loop system:

$$\mathcal{S}_a = \left\{x \in \mathbb{R}^n : x^T W^{-1} x \leq 1,\ W = W^T > 0\right\}, \quad (17)$$
$$\mathcal{X}_a = \left\{x \in \mathbb{R}^n : \Omega x \preccurlyeq \mathbb{1}\right\}, \quad (18)$$

where $\Omega = I_n \otimes [1\ \ -1]^T \in \mathbb{R}^{2n\times n}$ and $\mathcal{X}_a$ is a polyhedral region representing $x_i \in [-1,\ 1]$. Figure 1 depicts the sets Π, $\mathcal{X}$, $\mathcal{X}_a$, $\mathcal{S}$, $\mathcal{S}_a$ and a trajectory of $x(t)$ for $d = 0$.

The approach used to assure $\mathcal{S} \subset \mathcal{X} \cap \Pi$, with $\mathcal{S}$ being an estimation of DOA of the origin for the closed-loop system (15), is to obtain conditions that guarantee

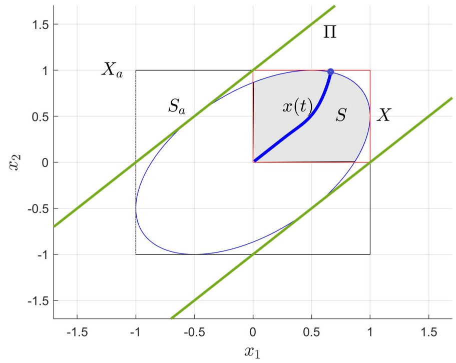


Fig. 1. Sets Π (green box), $\mathcal{X}_a$ (black dashed box), $\mathcal{X}$ (red box), $\mathcal{S}$ (region in gray), $\mathcal{S}_a$ (blue ellipsoid), and a trajectory of $x(t)$ (bold blue line) for $d = 0$.

$\mathcal{S}_a \subset \mathcal{X}_a \cap \Pi$, and next to demonstrate that $\mathcal{S} = \mathcal{S}_a \cap \mathbb{R}^n_+$ is an invariant region. By construction, $-L$ is a Metzler matrix. Thus, the presented approach relies on the property that the closed-loop system (15) remains positive (see Definition 1 and Lemma 8). Constraint (8) may prevent the trajectories belonging to $\mathcal{S}$ from reaching the origin in practical applications. For instance, in opinion dynamics, the bounded budget investment usually dictates the convergence of opinions only to a neighboring value of d. Thus, we establish an invariant region $\mathcal{S}_u \subseteq \mathcal{S}$ such that $\forall x(0) \in \mathcal{S}_u$ the agents' states converge to the origin respecting the constraint (8). Additionally, for initial conditions $x(0) \in \mathcal{S} \setminus \mathcal{S}_u$, we propose a mechanism to restrict the control action when the energy constraint (8) is violated. Problem 3 can be rewritten as follows.

Problem 13 *Design the state feedback gain matrix K such that the closed-loop system* (15) *is asymptotically stable and find*

- *(I) an estimate $\mathcal{S} \subseteq (\mathcal{X} \cap \Pi)$ for the DOA of the origin such that $\forall x(0) \in \mathcal{S}$, the trajectories of* (15) *converge asymptotically to the origin with guaranteed cost for J_x, $\forall \delta_i(t) \in \mathcal{D}$ and $|u_i(t)| \leq \bar{u}$;*
- *(II) an estimate of an invariant region $\mathcal{S}_u \subseteq \mathcal{S}$ such that $\forall x(0) \in \mathcal{S}_u$ the trajectories of* (15) *converge asymptotically to the origin with guaranteed cost $J_u < \mu$, $\forall \delta_i(t) \in \mathcal{D}$ and $|u_i(t)| \leq \bar{u}$;*
- *(III) a mechanism to turn off the control action ($u = 0$) when $J_u \geq \mu$, $\forall x(0) \in \mathcal{S} \setminus \mathcal{S}_u$.*

Following ideas presented in (Morărescu et al. 2020, Proposition 4.1), where the maximum possible investment is used as soon as possible to minimize a cost function related to the convergence of x, we propose a solution to Problem 13 such that, first, we minimize J_x considering constraints (6)-(7). After that, we present conditions to estimate the set $\mathcal{S}_u$ solving Problems 13 (II) and (III).

Remark 14 *Since L is a Metzler matrix, following Lemma 8, the open-loop system $\dot{x}(t) = -Lx(t)$ is* *positive. Furthermore, since $B(x)K$ is diagonal, the closed-loop system without saturation is positive once $-L+B(x)K$ is Metzler. Using, for instance, a polytopic model for the saturation term (Tarbouriech et al. 2011), it is possible to show that the closed-loop dynamic matrix remains Metzler in the presence of saturation.*

Theorem 15 *If there exist diagonal positive definite matrices $W \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{n\times n}$, a diagonal matrix $Z \in \mathbb{R}^{n\times n}$, and a scalar $\lambda > 0$, such that the following inequalities are satisfied*

$$\begin{bmatrix} He\{-LW + B_0 Z\} + \lambda I & \star & \star & \star \\ SB_0^T + Y & -2S & \star & \star \\ W & 0 & -R^{-1} & \star \\ Z & S & 0 & -4\lambda I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} W & \star \\ \Omega_{(i)} W & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, 2n \tag{20}$$

$$\begin{bmatrix} W & \star \\ Z_{(i)} - Y_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \dots, n, \tag{21}$$

then the state feedback gain $K = ZW^{-1}$ makes the closed-loop system (15a) *exponential stable with guaranteed cost $J_x \leq x(0)^T W^{-1} x(0)$, and $\mathcal{S}$ is an invariant set estimate for the DOA of the origin such that $\mathcal{S} \subseteq \mathcal{X} \cap \Pi$.*

PROOF. Consider the quadratic Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$ and the closed-loop system (15). The integral from 0 to ∞ of

$$\dot{V}(t) + z(t)^T z(t) < 0, \quad \forall x \in \mathcal{X}, \tag{22}$$

yields $J_x < V(0)$, where $z(t)$ is defined in (9). Observe that $\dot{V}(t) < -cx(t)^T x(t)$ is equivalent to (22), where c represents the maximum eigenvalue of matrix R. Thus, the exponential stability of the origin is verified. By using Lemma 7, if $\dot{V}(t) + z(t)^T z(t) - 2\psi(v(t))^T T\psi(v(t)) + 2\psi(v(t))^T TGx(t) < 0$, then (22) holds with G the matrix related to the region (11). Considering (13) and (15), this last inequality can be stated as

$$\begin{bmatrix} \mathrm{He}\left\{-W^{-1}L + W^{-1}B(x)K\right\} + R & \star \\ B(x)^T W^{-1} + TG & -2T \end{bmatrix} < 0,$$

and, pre- and post-multiplying it by $\mathrm{diag}(W, T^{-1})$ and its transpose, respectively, one has

$$\begin{bmatrix} \mathrm{He}\{-LW + B(x)Z\} + W^T RW & \star \\ SB(x)^T + Y & -2S \end{bmatrix} < 0,$$

where $Y = GW$, $S = T^{-1}$, and $Z = KW$. Using the Schur complement in the above inequality, one has

$$\begin{bmatrix} \mathrm{He}\{-LW + B(x)Z\} & \star & \star \\ SB(x)^T + Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0.$$

Replacing $B(x)$ in the previous inequality by (13) for all $x \in \mathcal{X}$ yields

$$\begin{bmatrix} \mathrm{He}\{-LW + B_0 Z\} & \star & \star \\ SB_0^T + Y & -2S & \star \\ W & 0 & -R^{-1} \end{bmatrix} + \mathrm{He}\left\{ \begin{bmatrix} \Delta \\ 0 \\ 0 \end{bmatrix} 0.5 \begin{bmatrix} Z & S & 0 \end{bmatrix} \right\} < 0.$$

Applying Lemma 9 for $\Delta = \mathrm{diag}(\delta_1, \dots, \delta_n)$, $\delta_i \in \mathcal{D}$, and the Schur Complement, we recover (19). Note that pre- and post-multiplying (20) by $\mathrm{diag}(W^{-1}, I)$, one has

$$\begin{bmatrix} W^{-1} & \star \\ \Omega_{(i)} & 1 \end{bmatrix} \geq 0$$

which certifies $\mathcal{S}_a \subseteq \mathcal{X}_a$ (Boyd et al. 1994). Finally, by pre-and-post multiplying the inequality (21) with $\mathrm{diag}(W^{-1}, I)$, one has

$$\begin{bmatrix} W^{-1} & \star \\ K_{(i)} - G_{(i)} & \bar{u}^2 \end{bmatrix} \geq 0.$$

Considering the set Π in Lemma 7 and $\mathcal{S}_a$ in (17), the above inequality verifies $\mathcal{S}_a \subseteq \Pi$ (Gomes da Silva Jr. & Tarbouriech 2005) and, consequently, $\mathcal{S} \subseteq \Pi$. Since the closed-loop system (15) is positive (Remark 14) and (19) implies $\dot{V} \leq 0$, $\mathcal{S}_a$ defined in (17) is an invariant region (Boyd et al. 1994) and we can conclude that $\mathcal{S} = \mathcal{S}_a \cap \mathbb{R}_+^n$ is invariant and contractive (Blanchini & Miani 2015). □

Theorem 15 provides a solution for Problem 13 (I), that is, sufficient conditions to guarantee the exponential stability of the closed-loop system (15) and the estimate $\mathcal{S}$ for the DOA such that (6) and (7) hold. Next theorem presents a solution for Problems 13 (II) and (III) taking into account constraint (8) and using the gain K designed in Theorem 15.

Theorem 16 *Let K be a stabilizing state feedback gain for the closed-loop system* (15a)*. If there exist diagonal* *positive definite matrices $P \in \mathbb{R}^{n\times n}$ and $S \in \mathbb{R}^{n\times n}$, a matrix $Y \in \mathbb{R}^{n\times n}$, and a scalar $\lambda > 0$, such that the following inequality is satisfied*

$$\begin{bmatrix} He\{-LP + B_0KP\} + \lambda I & \star & \star & \star \\ SB_0^T + Y & -2S & \star & \star \\ KP & S & -Q^{-1} & \star \\ KP & S & 0 & -4\lambda I \end{bmatrix} < 0, \tag{23}$$

then

- *the closed-loop system* (15a) *has guaranteed cost $J_u \leq x(0)^T P^{-1} x(0)$;*
- *the set of all initial conditions such that $x(t) \to 0$ as $t \to \infty$ satisfying the energy bound $J_u \leq \mu$, is defined by $\mathcal{S}_u = \{x \in \mathbb{R}^n_+ : \ x^T P^{-1} x \leq \mu\}$;*
- *the control mechanism*

$$u(t) = \begin{cases} sat(Kx), & x(t)^T P^{-1} x(t) \geq \\ & \quad x(0)^T P^{-1} x(0) - \mu, \\ 0, & otherwise, \end{cases} \tag{24}$$

ensures $J_u \leq \mu$ for all $x(0) \in \mathcal{S} \setminus \mathcal{S}_u$.

PROOF. Consider $V(t) = x(t)^T P^{-1} x(t)$ as Lyapunov function. Following the same lines of the proof of Theorem 15, condition (23) implies $\dot{V}(t) + y(t)^T y(t) - 2\psi(v(t))^T T(\psi(v(t)) + Gx(t)) < 0$, with $y(t)$ defined in (9), and, by Lemma 7, $\dot{V}(t) + y(t)^T y(t) < 0$. The integral from 0 to ∞ of the last inequality yields $J_u < V(0) = x(0)^T P^{-1} x(0)$. From $J_u < V(0)$, it is evident that for all $x \in \mathcal{S}_u$, $J_u \leq \mu$, the trajectories can reach asymptotically the origin. Finally, if $x(0) \in \mathcal{S} \setminus \mathcal{S}_u$, one cannot ensure $J_u \leq \mu$, and $\int_0^T \dot{V}(t) + y(t)^T y(t)\, dt < 0$ yields $\int_0^T u(t)^T Q u(t)\, dt \leq \mu$, where T is such that $V(T) = V(0) - \mu$. The control mechanism (24) implies that the control action becomes zero when $V(t) = V(T)$, then $\int_T^\infty u(t)^T Q u(t)\, dt = 0$ and, therefore, $J_u \leq \mu$. □

The following remarks present optimization problems to maximize $\mathcal{S}$ and minimize J_x.

Remark 17 *To indirectly minimize J_x and maximize the set $\mathcal{S}$ in* (16) *simultaneously, one can maximize the trace of W (see Boyd et al. (1994)). Thus, the following optimization problem is proposed:*

$$\max \quad Trace(W) \tag{25}$$

subject to (19)–(21)*. To achieve a more accurate estimation for the upper bound of cost J_u in Theorem 16, one can solve the optimization problem:*

$$\max \quad Trace(P) \tag{26}$$

subject to (23)*.*

Remark 18 *If the initial conditions $x(0)$ are known, the following optimization problem minimizes J_x:*

$$\min \quad \tau \tag{27}$$

subject to (19)–(21)*, and*

$$\begin{bmatrix} \tau & \star \\ x(0) & W \end{bmatrix} > 0.$$

3.2 Consensus conditions based on on-off control law

In this section, we present an on-off control approach using the control law

$$u_i(t) = \begin{cases} k_i, & t \leq T \\ 0, & t > T, \end{cases} \tag{28}$$

where $|k_i| < \bar{u}$, $i = 1, ..., n$, are gains to be designed, and T is found regarding energy constraint (8) by the relation

$$J_u = \int_0^T \mathbb{1}^T KQK\mathbb{1} dt = \mathbb{1}^T KQK\mathbb{1} T \leq \mu, \tag{29}$$

where $K = \text{diag}(k_1, ..., k_n) \in \mathbb{R}^{n\times n}$. Then, $T = \mu / \mathbb{1}^T KQK\mathbb{1}$ yields the maximum usage of the afforded energy (budget) μ. The closed-loop system with respect to (28) is given by

$$\dot{x}(t) = \begin{cases} -Lx(t) + B(x(t))K\mathbb{1} = (-L + K)x(t), & t \leq T, \\ -Lx(t), & t > T. \end{cases} \tag{30}$$

Control law (28) avoids bilinear terms in the closed-loop system, thus handling control input saturation independently of initial conditions. The following theorems aim to solve Problem 3 by designing (28).

Theorem 19 *If there exist a diagonal positive definite matrix $W \in \mathbb{R}^{n\times n}$ and a diagonal matrix $Z \in \mathbb{R}^{n\times n}$ such that the following inequalities hold*

$$\begin{bmatrix} He\{-LW + Z\} & \star \\ W & -R^{-1} \end{bmatrix} < 0, \tag{31}$$

$$\begin{bmatrix} W & \star \\ \Omega_{(i)} W & 1 \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, n, \tag{32}$$

$$\begin{bmatrix} W_{(ii)} & \star \\ Z_{(ii)} & \bar{u}^2 W_{(ii)} \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, n, \tag{33}$$

then the control law (28) *with* $K = ZW^{-1}$ *guarantees* $\lim_{t\to\infty} x_i(t) = \mathbb{1}^T x(T)/n$*, where* $x(T) = exp((-L + K)T)x(0)$*,* $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$*, for the closed-loop system* (30)*, for all* $x(0) \in \mathcal{S} \subseteq \mathcal{X}$*, with guaranteed cost* $J_u \leq \mu$*.*

PROOF. Considering the quadratic Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$ and the closed-loop system (30), from $\dot{V} + x'Rx < 0$, one has

$$He\{-LW + KW\} + WRW < 0.$$

By applying the Schur complement and considering $Z = KW$, we obtain (31) that guarantees the asymptotic stability of (30) and therefore $\mathcal{S} = \mathcal{S}_a \cap \mathbb{R}^n_+$ is invariant and contractive for $t \leq T$. After $t > T$, the dynamics become $\dot{x}(t) = -Lx(t)$ with initial condition $x(T)$, where $x(T) = exp((-L+K)T)x(0)$. From Olfati-Saber & Murray (2004), one has $\lim_{t\to\infty} x_i(t) = \mathbb{1}^T x(T)/n$, and from (29), $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$ that guarantees $J_u \leq \mu$. Following the same reasoning of (20)-(21), we have that by pre- and post-multiplying (32) by diag(W^{-1}, I) and its transpose, one can obtain the inequality that guarantees $\mathcal{S}_a \subseteq \mathcal{X}_a$. Additionally, if we pre- and post-multiply (33) by diag$(W^{-1}_{(ii)}, I)$, we obtain

$$\begin{bmatrix} W^{-1}_{(ii)} & \star \\ k_i & \bar{u}^2 W_{(ii)} \end{bmatrix} \geq 0$$

with $K = ZW^{-1}$. This is equivalent to $|k_i| \leq \bar{u}$. □

Theorem 19 uses the ellipsoidal level set $\mathcal{S} \subseteq \mathcal{X}$ to obtain an estimation of the DOA. Remarks 17 and 18 can also be applied to enlarge $\mathcal{S}$ and minimize J_x. Although $\forall x \in \mathcal{S}$, $\lim_{t\to\infty} x_i(t) = \mathbb{1}^T x(T)/n \in \mathcal{X}$, one disadvantage of Theorem 20 is that we can assure the set $\mathcal{S}$ is invariant only for $t \leq T$. However, for an adequately large energy constraint μ, we verify in practice that the trajectories remain in $\mathcal{S}$. To improve the estimation of the DOA, we propose conditions to certify region $\mathcal{X}$ as a contractive invariant set such that all initial conditions inside $\mathcal{X}$ converge exponentially to the origin.

Theorem 20 *If there exist a diagonal positive definite matrix* $W \in \mathbb{R}^{n\times n}$*, a Metzler matrix* $H \in \mathbb{R}^{2n\times 2n}$*, a diagonal matrix* $K \in \mathbb{R}^{n\times n}$*, scalars* $\lambda > 0$ *and* $0 < \epsilon \ll 1$*, such that the following inequalities are satisfied*

$$\begin{bmatrix} He\{-LW - W\} & \star & \star \\ \lambda W + K + I & -2\lambda I & \star \\ W & 0 & -R^{-1} \end{bmatrix} < 0, \tag{34}$$

$$\begin{aligned} (1-2d)(\Theta(-L+K) - H\Theta) \preccurlyeq 0, \\ H\mathbb{1} \preccurlyeq 0, \end{aligned} \tag{35}$$

$$\begin{bmatrix} 1 & \star \\ k_i & \bar{u}^2 \end{bmatrix} \geq 0, \quad \forall i = 1, \cdots, n, \tag{36}$$

where $\Theta = I_n \otimes [(1-2d) \ \ (2d-1)\epsilon^{-1}]^T \in \mathbb{R}^{2n\times n}$*, then the state feedback gain* K *ensures* $\lim_{t\to\infty} x_i(t) = \mathbb{1}^T x(T)/n$*, with* $x(T) = exp((-L+K)T)x(0)$*,* $T = \mu/(\mathbb{1}^T KQK\mathbb{1})$*, for the closed-loop system* (30)*, for all* $x(0) \in \mathcal{X}$*, with guaranteed cost* $J_u \leq \mu$*.*

PROOF. Considering the quadratic Lyapunov function $V(t) = x(t)^T W^{-1} x(t)$ and the closed-loop system (30), from $\dot{V} + x'Rx < 0$, one has $He\{-LW + KW\} + WRW < 0$. By employing the Schur complement, we obtain the left side of the following inequality, one gets

$$\begin{bmatrix} He\{-LW + KW\} & \star \\ W & -R^{-1} \end{bmatrix} = \underbrace{\begin{bmatrix} I & 0 \\ W & 0 \\ 0 & I \end{bmatrix}^T}_{B_\perp^T} \underbrace{\begin{bmatrix} He\{-LW\} & \star & \star \\ K^T & 0 & \star \\ W & 0 & -R^{-1} \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} I & 0 \\ W & 0 \\ 0 & I \end{bmatrix}}_{B_\perp} < 0.$$

Considering the equivalence of conditions (ii) and (iii) in Lemma 10, we have $BB_\perp = 0$ for $B = [W \ -I \ 0]$ and $M = [-I \ \lambda I \ 0]$. Then, the expression $M + NB + B^T N^T$ recovers (34). Note that, (36) is equivalent to $|k_i| \leq \bar{u}$. For the proof of (35) first, define the sets $\mathcal{X}_I = \{x \in \mathbb{R}^n : -I_d x \preccurlyeq 0\}$, $I_d = (1-2d)I$, and $\mathcal{X}_\epsilon = \{x \in \mathbb{R}^n : \Theta x \preccurlyeq \mathbb{1}\}$. Then, one has $\mathcal{X} = \mathcal{X}_\epsilon \cap \mathcal{X}_I = \{x \in \mathbb{R}^n : [\Theta \ \ -I_d]^T x \preccurlyeq [\mathbb{1} \ \ 0]^T\}$. Regarding Lemma 11, the set $\mathcal{X}$ represents a positive ($d = 0$) or negative ($d = 1$) polyhedron invariant set for the closed-loop system (30) in the form $\dot{x}(t) = A_{cl} x(t)$, $A_{cl} = -L + K$, if and only if there exist two Metzler matrices $H_1 \in \mathbb{R}^{2n\times 2n}$ and $H_4 \in \mathbb{R}^{n\times n}$, and two nonnegative matrices $H_2 \in \mathbb{R}^{2n\times n}$ and $H_3 \in \mathbb{R}^{n\times 2n}$, such that

$$\begin{bmatrix} \Theta \\ -I_d \end{bmatrix} A_{cl} = \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} \Theta \\ -I_d \end{bmatrix}, \quad \begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \begin{bmatrix} \mathbb{1} \\ 0 \end{bmatrix} \preccurlyeq 0.$$

Without losing generality, one can replace $H_3 = 0$ and $H_4 = A_{cl}$, yielding

$$\begin{cases} \Theta A_{cl} = H_1\Theta - I_d H_2 \\ H_1\mathbb{1} \preccurlyeq 0 \end{cases} \Rightarrow \begin{cases} I_d(\Theta A_{cl} - H_1\Theta) \preccurlyeq 0 \\ H_1\mathbb{1} \preccurlyeq 0. \end{cases} \tag{37}$$

From (37), we recover (35) for H_1 being Metzler (Du et al. 2020). □

Theorem 20 offers a larger estimation for the DOA compared to Theorems 15–16 since $\mathcal{S} \subseteq \mathcal{X}$. The following remark is presented for minimizing cost J_x in Theorem 20.

Remark 21 *To minimize J_x when the initial conditions $x(0)$ are known, one can solve the following optimization problem*

$$\min \quad \alpha\tau_1 + (1-\alpha)\tau_2, \tag{38}$$

for a given scalar $\alpha \in [0,1]$, with respect to (34)-(36), and

$$\begin{bmatrix} \tau_1 & \star \\ x(0) & W \end{bmatrix} > 0, \quad \begin{bmatrix} \tau_2 & \star \\ K\mathbb{1} & Q^{-1} \end{bmatrix} > 0.$$

Observe that

$$J_x = \int_0^\infty x(t)^T R x(t)\, dt = \underbrace{\int_0^T x(t)^T R x(t)\, dt}_{J_{x_1}} + \underbrace{\lim_{T_f \to \infty} (T_f - \mu/(\mathbb{1}^T K Q K \mathbb{1}))\bar{x}^T R \bar{x}}_{J_{x_2}}$$

with $\bar{x} = x(T) = exp\left(\dfrac{\mu(-L+K)}{\mathbb{1}^T K Q K \mathbb{1}}\right) x(0)$. The minimization of τ_2 implies the minimization of J_{x_2}, that is, $V(T) \approx 0$ with $T = \mu/\mathbb{1}^T K Q K \mathbb{1}$. Additionally, the minimization of τ_1 implies the minimization of J_{x_1}, that can be interpreted to $J_{x_1} \leq x(0)^T W^{-1} x(0)$ according to (34).

4 Numerical examples

In this section, we present numerical experiments to show the effectiveness of the proposed conditions for small and large networks. The algorithms are implemented employing YALMIP (Lofberg 2004) and SeDuMi (Sturm 1999). The comparison of the numerical complexity of the proposed LMIs is verified by the number of LMI rows and scalar variables concerning the network with n agents, as shown in the following table.

	Scalar variables	LMI rows
Theorem 15	n^2+3n+1	$3n^2+7n$
Theorem 16	n^2+2n+1	$4n$
Theorem 19	$2n$	n^2+5n
Theorem 20	$4n^2+2n$	$9n$

Table 1
Numerical complexities of different approaches.

Example 22 (5 agents case) *We consider a connected network with a directed graph $\mathcal{G}$ with $n=5$ agents depicted in Fig. 2. We aim to compare the state feedback control law (14) with the on-off controller (28). We adopt the following specifications for Problem 3: $d=1$, $Q=10^{-1}I$, $\mu=0.86$, and $\bar{u}=0.85$.*

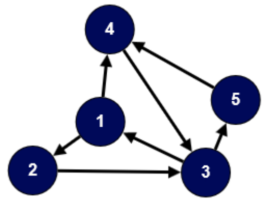
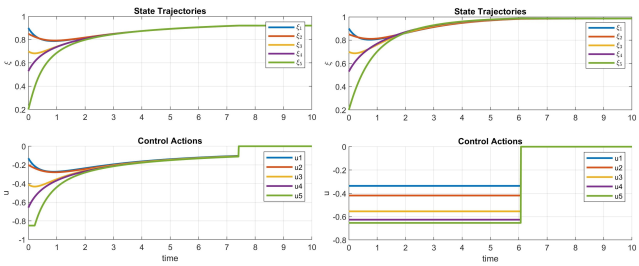


Fig. 2. Network with 5 agents in a connected directed graph for Example 22.

It should be noted that the parameter R is adjusted in each case to enhance the convergence of the trajectories. The state feedback control law (14) is designed with Theorems 15 and 16 with $R = 10^{-1}I$ and $\varepsilon = 0.1$, yielding $K = diag(1.29, 1.34, 1.36, 1.40, 1.40)$. Using Remark 17 in Theorems 15 and 16, we obtain the guaranteed cost of J_u as 0.8849 for $x(0) = (-0.1, -0.15, -0.3, -0.47, -0.8) \in \partial\mathcal{S}$, higher than the specification $\mu = 0.86$, implying that the control action needs to be cut off before reaching the origin. The trajectories of (2), with $\xi(0) = x(0) + \mathbb{1}d$, and the control signal given by (24) of the agents are illustrated in Fig. 3a. Note that the saturation on the control signal is allowed (u_5 saturates in the initial instant due to the constraint $|u_i| < 0.85$), and around $t = 7.41s$, the external action vanishes according to (24). The approximate final consensus value of the agents is $\lim_{t\to\infty} \xi_i(t) \approx 0.9209$, $i = 1, \dots, n$. Furthermore, we



(a) (b)

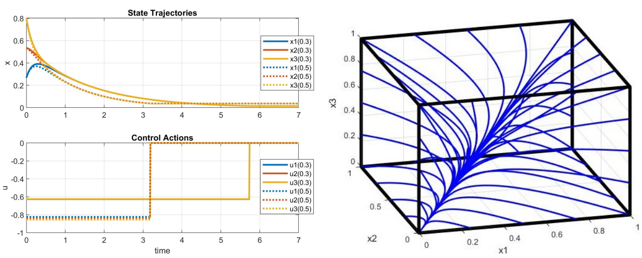
Fig. 3. Trajectories and control inputs of the agents with respect to (3) using (a) state feedback controller with mechanism (24) and (b) Theorem 20 and Remark 21 with $\alpha = 0.3$ for the case $d = 1$.

apply Theorem 20 with Remark 21 considering $R = I$, yielding $K = diag(-0.34, -0.42, -0.55, -0.63, -0.65)$ and $T = 6.08$ for $\alpha = 0.3$. The trajectories are illustrated in Fig. 3b. Table 2 summarizes the results of Example 22 showing the consensus value, that is, $\lim_{t\to\infty} \xi(t)$, obtained from the simulations and the volume of the estimation of the DOA. We observe that Theorem 20 with $\alpha = 0.3$ provided the consensus value closest to $d = 1$ compared to the other results. Moreover, the polyhedral region obtained as an estimation of the DOA by Theorem 20 is larger than the ellipsoidal regions given by Theorems 15 and 19, illustrating the advantage of a polyhedral estimation over ellipsoidal ones when considering states restricted to positive or negative orthants of *the state space.*

Method	$\xi_i(\infty)$	Volume
Th. 15 with Rema. 17	0.9209	0.1645
Th. 19 with Rema. 18	0.9780	0.1645
Th. 20 with Rema. 21 ($\alpha = 0.3$)	0.9857	1

Table 2
Results of Example 22 showing the consensus value $\xi_i(\infty) = \lim_{t\to\infty} \xi_i(t)$, $i = 1, \ldots, n$, obtained from the simulations and the volume of the estimation of DOA obtained by numerical integration.

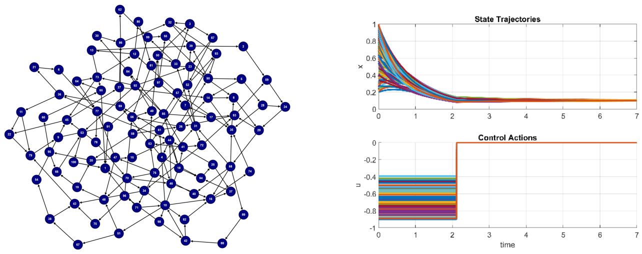
Example 23 (3 agents case) *In this example, we present a comparison with Alkhorshid et al. (2022). Let a connected undirected graph $\mathcal{G}$ from Ben Rejeb et al. (2018), Alkhorshid et al. (2022) describe a communication network with $n = 3$ agents and a Laplacian matrix defined as $L = [3, -1, -2; -1, 3, -2; -2, -2, 4]$, with $Q = 10^{-1}I$, $R = 10^{-1}I$, $d = 0$, $\mu = 0.675$, and $\bar{u} = 0.9$. Theorem 20 is implemented with Remark 21 as the optimization criterion. Using $\alpha = 0.3$ and $\alpha = 0.5$, one has $T = 5.74$ and $T = 3.19$, respectively. As shown in Fig. 4a, using $\alpha = 0.3$ leads to a slower convergence rate and a smaller consensus value (0.01477) than the case $\alpha = 0.5$ (that provides 0.03688). Thus, the parameter α offers a trade-off between faster convergence and a smaller final consensus value. Additionally, Fig. 4b illustrates the trajectories for initial conditions placed on the edge of the polyhedral region $\mathcal{X}$ that is the obtained estimation of DOA, showing asymptotic convergence to the origin. It is noteworthy that the guaranteed cost related to the control action obtained with Theorem 20 and Remark 21 ($J_u = 0.675$) is very close to the real value obtained from the trajectories in Fig. 4a ($J_u = 0.6744$ for $\alpha = 0.3$ and $J_u = 0.6738$ for $\alpha = 0.5$), concluding that Theorem 20 with Remark 21 provides accurate bounds on the costs to guarantee $J_u \leq \mu$. Compared with the results in (Alkhorshid et al. 2022), we can observe that the new approach achieved a better final consensus value and convergence rate, along with a larger estimation of DOA.*



(a) (b)

Fig. 4. Agents' trajectories and respective control inputs for Example 23 with (a) $\alpha = 0.3$ (solid lines) and $\alpha = 0.5$ (dashed lines) and (b) state trajectories for initial conditions on the edge of the polyhedral estimation of DOA (using $\alpha = 0.3$) in Example 23.

Example 24 (100 agents case) *This example illustrates the advantage of Theorem 20 to deal with large networks. The connected direct graph with $n = 100$ agents is depicted in Fig. 5a. Problem 3 is solved considering $d = 0$, $R = I$, $Q = 10^{-1}I$, $\mu = 12$, and $\bar{u} = 0.9$. Theorem 20 certificates the entire domain $\mathcal{X}$ as an estimation of the DOA of the origin. To illustrate the performance, we placed seventy percent of the state's initial conditions between 0.51 to 1 and the rest belonging to the interval $[0.2,\ 0.5]$. In Fig. 5b, the highest value among the agents' states equals 0.1089 as $t \to \infty$, close to the desired value $d = 0$.*



(a) (b)

Fig. 5. (a) Network with 100 agents in a connected directed graph and (b) Agents' trajectories for the controller designed with Theorem 20 for Example 24.

5 Conclusion

The paper provides LMI-based conditions to address the consensus problem of bilinear MAS subject to simultaneous constraints on the states, magnitude, and energy of the control input. Using the notion of polytopic invariant sets, we have maximized the DOA of the closed-loop system. Moreover, we proposed a switching mechanism to circumvent the bilinear product in the closed-loop system, providing suitable conditions to deal with larger networks. Finally, we have employed numerical experiments to illustrate the performance of the proposed results.

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