THE GEOMETRY OF STABILITY CROSSING CURVES OF PI CONTROLLERS FOR SISO SYSTEMS I/O DELAYS

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This paper focuses on the stability analysis of the closed-loop single-input-single-output (SISO) system subject to a proportional-integral (PI) controllers in the presence of input or output (I/O) delays. A complete characterization of the *frequency crossing set* consisting of all frequency values where the number of unstable characteristic roots changes is given in the space defined by the controller's parameters. Next, we explicitly derive the PI controllers corresponding to each frequency in the crossing set. The collection of all these points will define the *stability crossing curves.* Such curves make a partition of the PI (controller) parameter-space in regions where the number of unstable characteristic roots remains constant for all the parameters belonging to the corresponding region. Next, we present an explicit *methodology* to compute the *number of unstable roots* in each region. Finally, some illustrative examples complete the presentation.

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Key words: delay, PI controller, SISO systems, stability crossing curves.

1. INTRODUCTION

One of the simplest controllers' scheme largely used in process control is represented by the PI controller (see, for instance, [2, 18, 27, 25] and the references therein). However, as pointed out by [4], a high percentage of PI and PID controllers seem to be tuned badly mainly since the tuning methods are limited to some restricted conditions of the plant. Among the problems we can cite, for instance, neglected parameter uncertainty or modelling errors, including the *delay* presence. Roughly speaking, the *delay* represents one of the easiest way for modelling *transport* and *propagation* phenomena in the dynamics of any interconnection scheme [17, 7] as well as modelling maturation and/or growth in population dynamics [6, 11].

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In this paper, we will address the PI control of SISO systems including input or output (I/O) delays. As discussed in [18], there exists several methods for the controllers construction, and several techniques have been proposed for the analysis of the (asymptotic) stability and of the performances of the corresponding closed-loop schemes. Among them, we can mention the computation of stabilizing PI controller's parameters considered by [24, 26] using a Pontryagin approach. More precisely, [24] addresses the control of first-order system with a time-delay in both cases (stable, and unstable delay-free systems), and [26] deals with some robustness issues in terms of delays for the closed-loop system under the assumption that the delay-free system can be stabilized by a proportional controller. Further discussions can be found in [27, 22].

The aim of this paper is the stability analysis of the closed-loop SISO system with I/O delays subject to a PI-controller by using a *geometric* argument (see, for instance, [8] for the basic ideas). More precisely, we are interested in characterizing the *stability crossing boundaries* (*curves* in our case study) in the parameter-space defined by the controller's parameters. By a *stability crossing boundary*, we understand the set of parameters for which the corresponding characteristic function has at least one zero on the imaginary axis. Such an idea is not new and it sends back to the *D-decomposition* method suggested by Neimark [15] in the 40s (see [12] for further comments) or to the so-called *parameter-space approach* (see, for instance, [1, 3] or [22] and the references therein). However, the novelty of our method consists in analyzing the properties of such stability crossing boundaries leading to a *complete characterization* of such boundaries.

More precisely, the contribution is twofold:

(i) first, the *frequency crossing set* (defined as the collection of all frequencies for which a crossing in the PI-parameter space exists) consists in a *finite number of intervals of finite length* for bounded PI-parameters;

(ii) second, for any *stability crossing curve*, we discuss in detail the smoothness of the boundaries as well as the direction of crossing with respect to the considered parameters.

We believe that the proposed approach is easy to follow, and it offers new insights in the analysis of the corresponding closed-loop system. The proposed results are accompanied by various interpretations and discussions helping in defining an user-friendly *algorithm* for the *computation* and the *characterization of the stability crossing boundaries* in the *PI-parameter space*. It is important to point out that our results also generalize the approach considered in [14], where only the case of proportional controllers subject to input delays was considered. Finally, our approach offers some alternative analysis ways to the approaches proposed by [24, 26] in the literature.

The structure of the paper is as follows: the problem formulation is stated in Section 2. The characterization of the stability crossing curves in the space defined by the PI controllers' parameters is given in Section 3. More precisely, we give an explicit construction of the frequency crossing sets and of the corresponding stability crossing curves. Robust stability with respect to controller parameters deviation is the subject of Section 4. Various illustrative examples are presented in Section 5, and some concluding remarks end the paper.

2. PROBLEM FORMULATION

Consider the following class of *strictly proper* SISO open-loop systems subject to an I/O delay represented by the transfer function

$$H_{yu}(s) = \frac{P(s)}{Q(s)}\mathrm{e}^{-s\tau} = c^T(sI_n - A)^{-1}b\mathrm{e}^{-s\tau}, \tag{1}$$

where (A, b, c^T) is a state-space representation of the open-loop system. In this paper, the controller is represented by a classical PI controller $K(s)$ of the the form

$$K(s) = k\left(1 + \frac{T}{s}\right) = k_p + \frac{k_i}{s}. \tag{2}$$

It is noteworthy that in some situations, the existence of a time-delay in the actuating input may induce instability or poor performance for the closed-loop scheme. At the same time, there exists situation when the presence of an appropriate time-delay in the actuating input may have the opposite effect (*stabilizing effect*, see, e.g., [12] and the references therein). However, many problems in process control engineering involve time-delays, thus they cannot be neglected. Under the above considerations, the stability of the closed-loop system is given by the location of the zeros (known as characteristic roots) of the meromorphic function

$$H(s, k, T, \tau) = Q(s) + P(s)\left(k_p + \frac{k_i}{s}\right)\mathrm{e}^{-s\tau}, \tag{3}$$

with an infinite (countable) number of zeros [9].

The problem considered in this paper can be defined as follows:

Problem 1. Find explicit conditions on the parameters pair (k_p, k_i), such that the closed-loop system (3) is asymptotically stable.

In order to simplify the presentation and without any loss of generality, we make the following natural assumption.

Assumption 1. The polynomials $P(s)$ and $Q(s)$ in (3) do not have common zeros.

If this assumption is not satisfied, $P(s)$ and $Q(s)$ have a common factor $c(s) \neq$ constant. Simplifying by $c(s)$ we get a system described by (3) which satisfies the condition above.

Throughout the paper the following standard notation will be adopted: $\mathbb{C}$ (RHP, LHP) is the set of complex numbers (with strictly positive, and strictly negative real parts), and $j = \sqrt{-1}$. For $z \in \mathbb{C}$, $\angle(z) \in (-\pi\ \pi]$, $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ define the argument, the real part and the imaginary part of z. $\mathbb{R}$ $(\mathbb{R}^+, \mathbb{R}^-)$ denotes the set of real numbers (larger or equal to zero, smaller or equal to zero). $\mathbb{N}$ is the set of natural numbers, including zero and $\mathbb{Z}$ the set of integers.

3. STABILITY IN (k_p, k_i) PARAMETER SPACE

In the sequel, we study the behavior of the system for a fixed delay value τ. More precisely, for a given $\tau = \tau^*$ we search the crossing frequencies ω and the corresponding *crossing points* in the parameter space (k_p, k_i) defined by the PI control law such that $H(j\omega, k_p, k_i, \tau^*) = 0$.

According to the continuity of zeros with respect to the parameters (see, e.g., [12] and the references therein), the number of roots in the RHP can change only when some zeros appear and cross the imaginary axis. Thus, it is natural to consider the *frequency crossing set* Ω consisting of all real positive ω such that there exist at least a pair (k_p, k_i) for which

$$H(j\omega, k_p, k_i, \tau^*) := Q(j\omega) + P(j\omega)\left(k_p - j\frac{k_i}{\omega}\right)\mathrm{e}^{-j\omega\tau} = 0. \tag{4}$$

Remark 1. Using the conjugate of a complex number we get

$$H(j\omega, k_p, k_i, \tau) = 0 \Leftrightarrow H(-j\omega, k_p, k_i, \tau) = 0.$$

Therefore, it is natural to consider only positive frequencies, that is $\Omega \subset (0, \infty)$.

3.1. Stability crossing curves

Considering that the set Ω is known, we can easily derive all the crossing points in the parameter space (k_p, k_i).

PROPOSITION 1. *For a given $\tau > 0$ and $\omega \in \Omega$ the corresponding crossing point (k_p, k_i) is given by*

$$k_p = -\mathrm{Re}\left(\frac{Q(j\omega)}{P(j\omega)}\,\mathrm{e}^{j\omega\tau}\right), \tag{5}$$

$$k_i = \omega \cdot \mathrm{Im}\left(\frac{Q(j\omega)}{P(j\omega)}\,\mathrm{e}^{j\omega\tau}\right). \tag{6}$$

Proof. Obviously, Eq. (4) can be rewritten as

$$\frac{Q(j\omega)}{P(j\omega)}\mathrm{e}^{j\omega\tau} + k_p = j\frac{k_i}{\omega}.$$

Since k_p, k_i and ω are real, the previous relation states nothing else that the real part of the left hand side is zero and the imaginary part is k_i/ω. Next, straightforward computations allow deriving (5) and (6). □

Remark 2. For all $\omega \in \Omega$ we have $P(j\omega) \neq 0$. Indeed, it is easy to see that if $\omega \in \Omega$, then there exists at least one pair (k_p, k_i) such that $H(j\omega, k_p, k_i, \tau) = 0$. Therefore, assuming that $P(j\omega) = 0$ we get also $Q(j\omega) = 0$ which contradicts Assumption 1.

Remark 3. It is important to point out that the controller's gains k_p and k_i include *explicitly* delay information. Furthermore, throughout the paper, we assume that the corresponding I/O delay is (perfectly) known, and it is not subject to any uncertainty. The way the delay parameter affects the crossing boundaries can be also analyzed using similar geometric arguments, and, for the sake of brevity, such an aspect is not considered in the paper.

Remark 4. Proposition 1 gives a simple characterization of the crossing points (k_p, k_i) for a given crossing frequency ω. The result above is derived in the spirit of [8]. Similar characterizations can be found in [21, 10, 20].

3.2. **Analytic determination of the crossing set**

In the sequel, we are interested in finding the crossing points (k_p, k_i) such that k_p and k_i are *finite*. This will not restrict the usefulness of the following results since the controller parameters cannot be set to some infinite values in practical situation.

PROPOSITION 2. *Let k_p^* and $k_i^* > 0$ be given. Let $\Omega_{k_p^*, k_i^*}$ denotes the set of all frequencies $\omega > 0$ satisfying equation (4) for at least one pair of (k_p, k_i)*

in the rectangle $|k_p| \leq k_p^*$, $|k_i| \leq k_i^*$. *Then, the set* $\Omega_{k_p^*,k_i^*}$ *consists of a finite number of intervals of finite length.*

Proof. Using the modulus, Eq. (4) becomes

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right|^2 = k_p^2 + \frac{k_i^2}{\omega^2}. \tag{7}$$

Since $\deg(Q) > \deg(P)$ (SISO system strictly proper), when ω tends to infinity the left hand side approaches infinity and the right hand side decreases to k_p^2. In other words, Ω is a bounded set. On the other hand, it is clear that $\omega \in \Omega_{k_p^*,k_i^*}$ implies

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right|^2 \leq \left(k_p^*\right)^2 + \frac{\left(k_i^*\right)^2}{\omega^2}. \tag{8}$$

Solving the polynomial inequality above, we find that ω belongs to $\bigcup_{h=1}^n \Omega_h$ where $\Omega_h = [\omega_h^l, \omega_h^r]$. We note that the first interval might be $\Omega_1 = (0, \omega_1^r]$ and if $\omega_h \neq 0$ is a left or right end of Ω_h then

$$\left|\frac{Q(j\omega_h)}{P(j\omega_h)}\right|^2 = \left(k_p^*\right)^2 + \frac{\left(k_i^*\right)^2}{\omega_h^2}, \quad \forall h = 1, 2, \ldots, n.$$

Thus,

$$\Omega_{k_p^*,k_i^*} \subseteq \bigcup_{h=1}^n \Omega_h.$$

On the other hand, $\omega \in \bigcup_{h=1}^n \Omega_h$ is equivalent to (8) which means that we can find $|k_p| \leq k_p^*$, $|k_i| \leq k_i^*$ satisfying (4). In other words, one gets $\bigcup_{h=1}^n \Omega_h \subseteq \Omega_{k_p^*,k_i^*}$ and therefore,

$$\Omega_{k_p^*,k_i^*} = \bigcup_{h=1}^n \Omega_h. \quad \square$$

Remark 5. To the best of the authors' knowledge, there does not exist any similar result in the open literature giving a *complete characterization* of the *frequency crossing set* for bounded PI-parameters.

Remark 6. The case of proportional controllers ($k_i = 0$) can be handled by using the geometrical approach proposed by [14] in the (gain, delay)-parameter space (see also [16] for further analytical discussions with respect to the delay parameter).

3.3. **Geometric interpretation of the crossing set**

The analytical representation of the crossing set Ω, given in the previous section, was inspired by a simple geometric interpretation of the equation (4). More precisely, one rewrites (4) as

$$\frac{\omega Q(j\omega)}{P(j\omega)}\mathrm{e}^{j\omega\tau} = -\omega k_p + jk_i, \tag{9}$$

which can be interpreted as the intersection between the circle centered in $(0,0)$ and radius $\left|\frac{\omega Q(j\omega)}{P(j\omega)}\right|$, and the line $-\omega k_p + jk_i$ passing through the point $(0, k_i)$.

It is noteworthy that $\left|\frac{\omega Q(j\omega)}{P(j\omega)}\right|$ is always bigger than k_i and therefore the single limit case appears when

$$\left|\frac{Q(j\omega)}{P(j\omega)}\right|^2 = \left(k_p^*\right)^2 + \frac{(k_i^*)^2}{\omega^2}. \tag{10}$$

3.4. **Smoothness of the crossing curves**

When ω varies within some interval Ω_ℓ satisfying (8), the equations (5) and (6) define a continuous curve. Using the notation introduced in the previous paragraph and the technique developed in [8] and [14], we can easily derive the crossing direction corresponding to this curve.

First, let us denote $\mathcal{T}_\ell$ the curve defined above and consider the decompositions into real and imaginary parts

$$R_0 + jI_0 = j\left.\frac{\partial H(s,k_p,k_i,\tau)}{\partial}\right|_{s=j\omega},$$

$$R_1 + jI_1 = -\left.\frac{\partial H(s,k_p,k_i,\tau)}{\partial k_i}\right|_{s=j\omega},$$

$$R_2 + jI_2 = -\left.\frac{\partial H(s,k_p,k_i,\tau)}{\partial k_p}\right|_{s=j\omega}.$$

Then, since $H(s, k_p, k_i, \tau)$ is an analytic function of s, k_p and k_i, the implicit function theorem indicates that the tangent of $\mathcal{T}_\ell$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}k_p}{\mathrm{d}\omega} \\[2mm] \dfrac{\mathrm{d}k_i}{\mathrm{d}\omega} \end{pmatrix} = \begin{pmatrix} R_2 & R_1 \\ I_2 & I_1 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_1 I_0 - R_0 I_1 \\ R_0 I_2 - R_2 I_0 \end{pmatrix}, \tag{11}$$

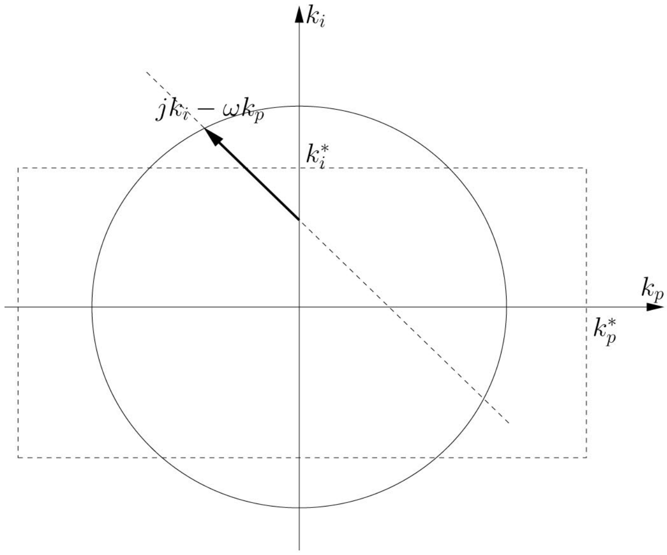

Fig. 1. The geometric interpretation of the crossing set Ω for the system represented by equation (3).

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{12}$$

It follows that $\mathcal{T}_\ell$ is smooth everywhere except possibly at the points where either (12) is not satisfied, or when

$$\frac{\mathrm{d}k_p}{\mathrm{d}\omega} = \frac{\mathrm{d}k_i}{\mathrm{d}\omega} = 0. \tag{13}$$

From the above discussions, we can conclude with

Proposition 3. *The curve $\mathcal{T}_\ell$ is smooth everywhere except possibly at the point corresponding to $s = j\omega$ such that $s = j\omega$ is a multiple solution of* (4).

Proof. If (13) is satisfied then straightforward computations show us that $R_0 = I_0 = 0$. In other words, $s = j\omega$ is a multiple solution of (4).

On the other hand,

$$R_1 I_2 - R_2 I_1 = \frac{1}{\omega}|P(j\omega)|^2.$$

From Remark 2, $P(j\omega) \neq 0$, $\forall \omega \in \Omega$. Thus, $R_1 I_2 - R_2 I_1 \neq 0$, $\forall \omega \in \Omega$ and the proposition is proved. □

3.5. **Direction of crossing**

The next paragraph focuses on the characterization of the crossing direction corresponding to each of the curves defined by (5) and (6) (see, for instance, [13] or [14] for similar results for different problems):

We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider k_p and k_i as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_\ell$ along the positive direction is $\left(\frac{\partial k_p}{\partial \omega}, \frac{\partial k_i}{\partial \omega}\right)$, the normal to $\mathcal{T}_\ell$ pointing to the left hand side of positive direction is $\left(-\frac{\partial k_i}{\partial \omega}, \frac{\partial k_p}{\partial \omega}\right)$. Corresponding to a pair of complex conjugate solutions of (3) crossing the imaginary axis along the horizontal direction, (k_p, k_i) moves along the direction $\left(\frac{\partial k_p}{\partial \sigma}, \frac{\partial k_i}{\partial \sigma}\right)$. So, as (k_p, k_i) crosses the stability crossing curves from the right hand side to the left hand side, a pair of complex conjugate solutions of (4) crosses the imaginary axis to the right half plane, if

$$\left(\frac{\partial k_p}{\partial \omega}\frac{\partial k_i}{\partial \sigma} - \frac{\partial k_i}{\partial \omega}\frac{\partial k_p}{\partial \sigma}\right)_{s=j\omega} > 0, \tag{14}$$

i.e., the region on the left of $\mathcal{T}_\ell$ gains two solutions on the right half plane. If the inequality (14) is reversed, then the region on the left of $\mathcal{T}_\ell$ loses two right half plane solutions. Similarly to (11) we can express

$$\begin{pmatrix} \dfrac{\partial k_p}{\partial \sigma} \\[2mm] \dfrac{\partial k_i}{\partial \sigma} \end{pmatrix}_{s=j\omega} = \begin{pmatrix} R_2 & R_1 \\ I_2 & I_1 \end{pmatrix}^{-1} \begin{pmatrix} I_0 \\ -R_0 \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 R_1 + I_0 I_1 \\ -R_0 R_2 - I_0 I_2 \end{pmatrix}. \tag{15}$$

Using this we arrive at

PROPOSITION 4. *Assume $\omega \in \Omega_\ell$, k_p, k_i satisfy* (5) *and* (6) *respectively, and ω is a simple solution of* (4) *and $H(j\omega', k_p, k_i, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e., (k_p, k_i) is not an intersection point of two curves or different sections of a single curve). Then as (k_p, k_i) moves from the region on the right to the region on the left of the corresponding crossing curve, a pair of solutions of* (3) *crosses the imaginary axis to the right* (*through $s = \pm j\omega$*) *if $R_1 I_2 - R_2 I_1 > 0$. The crossing is to the left if the inequality is reversed.*

Proof. Straightforward computation shows that

$$\left(\frac{\partial k_p}{\partial \omega}\frac{\partial k_i}{\partial \sigma} - \frac{\partial k_i}{\partial \omega}\frac{\partial k_p}{\partial \sigma}\right)_{s=j\omega} = \frac{(R_0^2+I_0^2)(R_1I_2-R_2I_1)}{(R_1I_2-R_2I_1)^2}.$$

Therefore, (14) can be written as $R_1I_2 - R_2I_1 > 0$. □

Remark 7. In the proof of Proposition 3 we have shown that $R_1I_2-R_2I_1$ is always positive. Thus, a system described by a closed-loop characteristic equation of type (3) may have more than one stability region in controller parameter space (k_p, k_i) if one of the following two situations occur:

• it has one or more crossing curves with some turning points (the direction in controller parameter space changes – see for instance example 4);

• it has at least two different crossing curves with opposite direction in (k_p, k_i)-space.

Any given direction, (d_1, d_2), is to the left-hand side of the curve if its inner product with the left-hand side normal $\left(-\frac{\partial k_i}{\partial \omega}, \frac{\partial k_p}{\partial \omega}\right)$ is positive, i.e.,

$$-d_1\frac{\partial k_i}{\partial \omega} + d_2\frac{\partial k_p}{\partial \omega} > 0, \tag{16}$$

from which we have the following result.

Corollary 1. *Let ω, k_p and k_i satisfy the same condition as Proposition* 4. *Then as (k_p, k_i) crosses the curve along the direction (d_1, d_2), a pair of solutions of* (3) *crosses the imaginary axis to the right if*

$$d_1(R_2I_0 - R_0I_2) + d_2(R_1I_0 - R_0I_1) > 0. \tag{17}$$

The crossing is in the opposite direction if the inequality is reversed.

Proof. Writing out the left-hand side, then (16) becomes

$$\frac{d_1(R_2I_0 - R_0I_2) + d_2(R_1I_0 - R_0I_1)}{R_1I_2 - R_2I_1} > 0. \tag{18}$$

If (d_1, d_2) is in the same side as the left-hand side normal, then, as we move along the (d_1, d_2) direction, the crossing is from the LHP to the RHP if the left-hand sides of (18) and $R_1I_2 - R_2I_1$ have the same sign, i.e., their product is positive. □

Let us note that when s goes to $s + \Delta s$ the gains change from k_i, k_p to $k_i + \Delta k_i, k_p + \Delta k_p$ and the system's transfer function changes from $H_{yu}(s)$ to $H_{yu}(s + \Delta s)$. Then, similar results can be obtained using first order approximations and replacing $\Delta H_{yu}(s) = H_{yu}(s + \Delta s) - H_{yu}(s)$ with $\frac{dH_{yu}}{ds}\Delta s$ (see also [20] for further details).

4. ILLUSTRATIVE EXAMPLES

In this section we present several examples in order to point out on one hand the coherence of our results with respect to other existing developments and on the other hand the usefulness of this methodology in various situations.It is noteworthy that the method requires a small amount of (not complicated) computations.

Example 1 (Scalar system). First, we validate our results by treating an open-loop stable scalar system already studied in the literature (see for instance [23, 19]). More precisely, we consider

$$Q(s) = 4s + 1, \quad P(s) = 1, \tag{19}$$

and we easily find the corresponding closed-loop characteristic equation

$$H(s, k, T, \tau) = 4s + 1 + \left(k_p + \frac{k_i}{s}\right) \mathrm{e}^{-s\tau}.$$

Taking $\tau = 1$ as the authors of [23, 19] and plotting k_i versus k_p we obtain Figure 2.

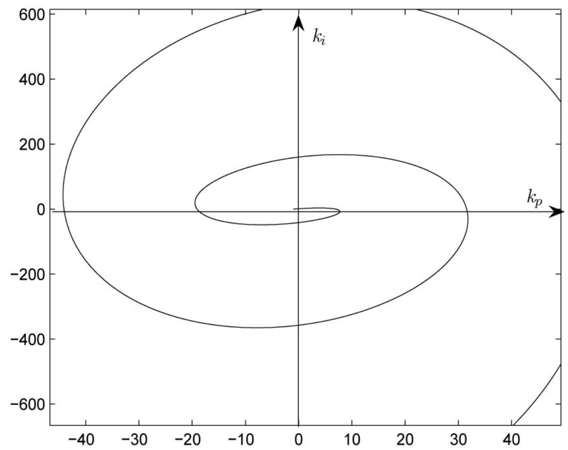

Fig. 2. Stability crossing curve in the (k_p, k_i) space for the system given by (19).

Using Proposition 4 we derive that all the crossing direction are *towards instability.*

On the other hand, the characteristic equation is stable only if $k_i > 0$. Therefore, in order to obtain the boundary of the stability region in the (k_p, k_i)

space, we search for the first interval in ω where $k_i > 0$. Explicitly, we solve the inequality

$$\omega \mathrm{Im}\left((4j\omega + 1)\mathrm{e}^{j\omega}\right) > 0,$$

and we get $\omega \in (0, 1.715)$. Using (5) and (6) the boundary of the stability region in the (k_p, k_i) space is plotted in Figure 3.

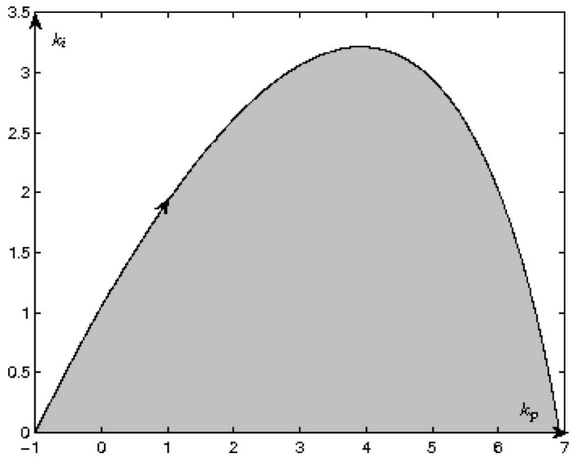

Fig. 3. The boundary of the stability region in the (k_p, k_i) space for the system given by (19).

We note that the same boundary of the stability region has been founded in [23, 19] by using a different argument.

Example 2 (Double integrator subject to input delay). Consider now the case of a double integrator subject to input delay

$$H_{yu}(s) = \frac{\mathrm{e}^{-s\tau}}{s^2},$$

subject to the PI controller

$$K(s) = \left(k_p + \frac{k_i}{s}\right).$$

Then the closed-loop system is

$$s^2 + \left(k_p + \frac{k_i}{s}\right)\mathrm{e}^{-s\tau} = 0. \tag{20}$$

One obtains

$$k_p = \omega^2 \cos(\omega\tau), \quad k_i = -\omega^3 \sin(\omega\tau).$$

Thus, k_i and k_p are even functions of ω. In other words, it is sufficient to plot k_i versus k_p for positive values of ω. We derive again that the number of unstable roots is getting larger when the distance to the origin increases.

All the crossing directions are towards instability. Taking into account that the system in absence of any control is unstable, we conclude that the system can not be stabilized with a PI controller. The crossing curve for the system is plotted in Figure 4.

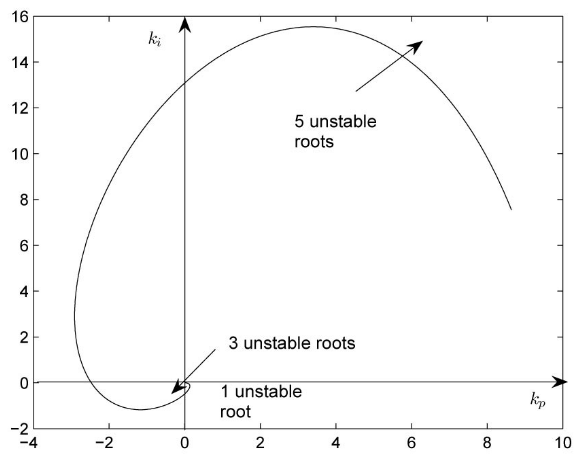

Fig. 4. Stability crossing curve in the (k_p, k_i) space for the system given by (20).

Example 3 (Temperature control of a tank). The dynamics of the temperature of a liquid is given by the transfer function (see [5])

$$H_{yu} = \frac{0.41\,\mathrm{e}^{-50s}}{s(1+50s)}.$$

The output y represents the temperature of the liquid inside a tank which can be controlled using u. Introducing the PI controller

$$K(s) = \left(k_p + \frac{k_i}{s}\right),$$

one obtains the closed-loop system

$$s(1+50s) + 0.41 \cdot \left(k_p + \frac{k_i}{s}\right) \mathrm{e}^{-50s} = 0. \tag{21}$$

All the crossing directions are again towards instability and the border of stability region is illustrated in Figure 5.

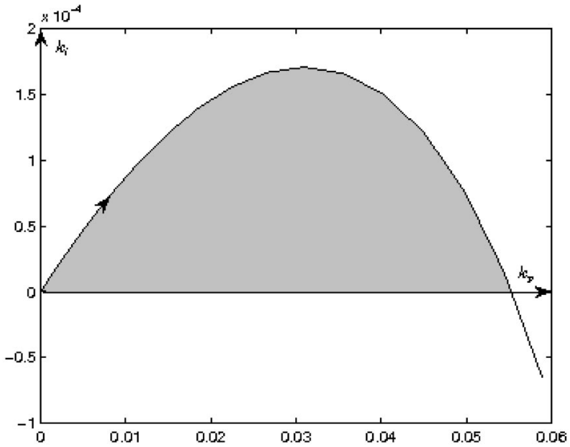

Fig. 5. The boundary of the stability region in the (k_p, k_i) space for the system given by (21).

Example 4 (An academic example). In the sequel we consider the unstable system whose dynamics is expressed by the transfer function ([5])

$$H_{yu} = \frac{(s-1)\mathrm{e}^{-2s}}{s^2 - 0.5s + 0.5}.$$

The characteristic equation in closed-loop by using the PI controller

$$K(s) = \left(k_p + \frac{k_i}{s}\right),$$

is given by

$$s^2 - 0.5s + 0.5 + (s-1)\left(k_p + \frac{k_i}{s}\right)\mathrm{e}^{-2s} = 0. \tag{22}$$

Straightforward computations show that

$$k_p = \frac{(0.5 - 0.5\omega^2)\cos 2\omega + \omega^3 \sin 2\omega}{1+\omega^2},$$
$$k_i = \frac{(0.5 - 0.5\omega^2)\omega \sin 2\omega - \omega^4 \cos 2\omega}{1+\omega^2}.$$

Since k_p and k_i are even functions of ω we only need to consider the case $\omega > 0$. Plotting k_i versus k_p, one obtains the border of stability region as illustrated in Figure 6.

The conclusion in Figure 6 is obtained taking into account the fact that even if we have only one stability crossing curve, it has a turning point. Analyzing the direction of this curve one can see that all the crossings are towards instability except the one concerning the small stability region pointed out on Figure 6.

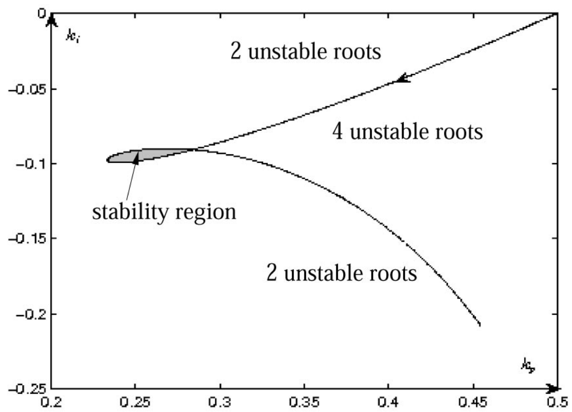

Fig. 6. The boundary of the stability region in the (k_p, k_i) space for the system given by (21).

5. CONCLUDING REMARKS

This paper addresses the geometry of the PI controllers for the stabilization of SISO linear systems with input delay. The proposed analysis is given in the space defined by the controllers' parameters, and it is extremely simple and easy to follow. Several illustrative examples complete the presentation and prove the interest of the method.

More precisely, the procedure can be resumed as follows: we first characterize the *crossing set*, that is the set of all frequencies for which the closed-loop system has at least one characteristic root on the imaginary axis. Such a characterization allows the *explicit computation* of the controller's parameters for which the number of unstable characteristic roots in closed-loop changes. It is important to point out that the controller's gains depend *explicitly* on the delay value. Finally, such computations lead to a *partition* of the controller's parameter-space in several regions where the number of unstable roots remains constant. The procedure above was applied to several illustrative examples taken from the control literature.

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