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Qualitative Analysis of Distributed Delay Systems: Methodology and Algorithms

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Abstract

This thesis focuses on the problem of *stability* and *robust stability* of a class of linear systems with distributed delays. Our work is motivated by the increasing number of models from biology to communication over network and traffic flow, models that can be included in the general class which is the subject of our study. We develop two complementary approaches, geometric and algebraic, respectively. Such approaches allow deriving the *stability regions* in the parameter space defined by the pair (mean delay, gap), where the gap is mainly a propagation delay. In other words, we obtain *necessary and sufficient conditions* for stability of the systems belonging to considered class.

Throughout the thesis various applications are presented. First, we point out some *qualitative properties* concerning the stability of some models arising in biology and in communication over networks. Next, we develop a method to study a recent model of traffic flow dynamic that include a memory effect. Illustrative numerical examples complete the presentation.

Résumé

Dans cette thèse, on considère la problématique de la *stabilité* et de la *stabilité robuste* d'une classe de *systèmes linéaires/* à *retards distribués.* Notre travail est motivé par le nombre croissant de modéles en allant de la biologie vers les réseaux de communication et de transport, modèles qui peuvent être inclus dans la classe générale considerée dans la thèse. Nous développons deux approches complèmentaires, géométriques et algébrique. Ces approches nous permettent de dériver les *régions de stabilité* dans l'espace de paramètres défini par (délai moyen, retard de propagation). En d'autres termes, nous obtenons des *conditions nécessaires et suffisantes* pour la stabilité des systèmes appartenant à la classe considérée.

Dans la thèse, diverses applications sont présentées. D'abord, nous précisons quelques *propriétés qualitatives* au sujet de la stabilité de quelques modèles en biologie et en communication dans des réseaux. Ensuite, nous développons une méthode afin d'étudier un modèle récent de la dynamique du réseau de transport qui inclus un effet de mémoire. Des exemples numériques illustratifs complètent la présentation.

Prefaţă

Această teză tratează problema *stabilităţii* şi a *robusteţii* stabilităţii pentru o clasă de sisteme dinamice cu *întârzieri distribuite*. Studiul nostru este motivat de numărul tot mai mare de modele (provenite din biologie, comunicaţii în reţea, reţele de transport, etc) ce pot fi incluse in clasa considerată în teză. Pentru a studia stabilitatea sistemelor din clasa considerată dezvoltăm două abordări (una bazată pe niste interpretări geometrice şi cealaltă pe o analiză matriceală) care ne permit să obţinem *regiunile de stabilitate* în spaţiul definit de perechea (întârziere medie, întârziere de propagare). Cu alte cuvinte, exprimăm *condiţii necesare şi suficiente* pentru stabilitatea sistemelor ce aparţin clasei considerate.

Pe parcursul tezei prezentăm diverse aplicaţii în diferite domenii de cercetare. Mai întâi punem în evidenţă diferite *aspecte calitative* ce privesc stabilitatea unor modele provenite din biologie şi comunicaţii în reţea. Apoi, dezvoltăm o metoda pentru studiul unui model de dinamică în reţele de transport, recent apărut. O mulţime de exemple numerice ilustrează diferitele proprietăţi obţinute pe parcursul tezei.

Acknowledgement

With the research for this PhD thesis finished, it is the moment to close the loop with a short paragraph devoted to thank several people who contributed to the thesis.

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Thesis Synthesis (in French)

Dans cette partie, nous faisons une courte présentation de la thèse, dans laquelle seront synthétisés les principaux rsultats. L'analyse développée dans ce mémoire est motivée par l'intérêt accru pour les systèmes dynamiques à retards distribués rencontrés dans des applications à commencer par les dynamiques de populations en biologie et jusqu'aux réseaux de communication.

0.1 Introduction

L'analyse de la stabilité des équations différentielles à retards a commencé dans les années 50, et l'une des premières approches est présentée par Krasovskii [77], qui généralise la deuxième méthode de Lyapunov. Ensuite, des nombreux auteurs ont développé différentes méthodes et ont posé différents problèmes concernant l'analyse de la stabilité des *équations différentielles avec un argument retardé*. En ce qui concerne les travaux de recherche dans le *domaine fréquence*, des résultats fondamentaux ont été obtenus par Pontryagin [124]. Chebotarev a publié également quelques travaux concernant l'approche Routh-Hurwitz pour les quasi-polynômes (par exemple [27]).

A partir des années 90, des nombreux critères dans le domaine de fréquence ont été proposés, critères prenant en considération des *aspects numériques* (calcul des bornes ou des marges des retards), mais aussi la *robustesse de la stabilité*. Malgré le très grand nombre des travaux qui traitent les systmes dynamiques avec argument retardé, nous mentionnons ici seulement quelques monographies [56, 126, 75, 59, 107, 54] qui, ǹotre avis, permettent d'avoir un tour d'horizon des principaux concepts, méthodes et idées apparues jusqu'à présent. Parmi ces monographies on distingue [56] par la qualité et la profondeur des idées présentées. Nous précisons que ce monographie contient une multitude d'idées intéressantes qui n'ont pas encore été suffisamment exploitées dans la littérature.

Même si la littérature concernant les stabilité des systèmes dynamiques avec argument retardé est vaste, il existe seulement deux approches principales :

- *l'étude dans le domaine fréquence* (systèmes linéaires) - Sans aucune perte de généralité, on considère: les tests analytiques qui étendent la méthode de Hurwitz aux systèmes dynamiques avec retards, les généralisations de méthodes de location des racines ($\mathcal{D}$ - décomposition et $\mathcal{T}$ - décomposition), les tests de stabilité basés sur l'intégration sur un contour et finalement, les procédures basées sur l'étude du spectre de l'opérateur associé au système linéaire considéré.

- *l'étude dans le domaine temps* (systèmes linéaires ou non linéaires) - les généralisations de la deuxième méthode de Lyapunov et les méthodes basées sur le principe de la comparaison, peuvent être mentionnés ici. Même si le principe de comparaison a été initialement développé dans le *domaine temps*, des idées similaires peuvent être appliquées aussi dans le *domaine fréquence*.

Dans cette thèse, nous développons des *méthodes* dans le *domaine fréquence* pour l'analyse de la stabilité d'une classe de systèmes dynamiques linéaires incluant des retards distribués. Les résultats décrits ici sont la suite de certaines études présentées dans des publications de l'auteur écritess en collaboration avec K. Gu, W. Michiels et S.-I. Niculescu [99, 100, 101, 102]. Le choix de la classe de systèmes dynamiques linéaires étudiés est motivé par les modèles existants en sciences appliquées (biologie [35, 119, 24], technologie [153, 130, 143], etc.). *Le principal objectif de cette thèse est de décrire des méthodes intuitives et faciles à mettre en* {*oe*uvre pour l'obtention des zones de stabilité de systèmes dynamiques linéaires (avec retards distribués) qui appartiennent à la classe étudiée.

0.2 Les préliminaires

Le but de ce paragraphe est de présenter des concepts et des notions de base, nécessaires dans le développement de nos résultats. Plus précisément, nous présentons brièvement les résultats classiques de la théorie des équations différentielles fonctionnelles concernant l'analyse de la stabilité. Les résultats proposés sont détaillés dans des monographies comme, par exemple, [15, 59, 107] ou bien [54].

Par la suite, nous notons $C([a,b],\mathbb{R}^n)$ *l'espace Banach des fonctions continues* définies sur l'intervalle $[a,b]$ avec des valeurs dans $\mathbb{R}^n$ munie avec la *topologie de la convergence uniforme.* Si $[a,b] = [-\tau, 0]$, alors la notation sera simplifiée de la manière suivante $C = C([-\tau,0],\mathbb{R}^n)$. Pour la norme d'un élément $\phi \in C$ nous utiliserons la notation $\| \phi \| = \sup\limits_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Si $\sigma \in \mathbb{R}$, $A > 0$, $x \in C([-\sigma-\tau,\sigma+A],\mathbb{R}^n)$ et $t \in [\sigma,\sigma+A]$ on définit la fonction continue $x_t \in C$, $x_t(\theta) = x(t+\theta)$.

Définition 1 *Pour chaque $f : \mathbb{R} \times C \mapsto \mathbb{R}^n$, on rappelle les notions suivantes:*

- *la forme générale d'une équation différentielle fonctionnelle de type retardée (RFDE) est*
$$\dot{x}(t) = f(t, x_t) \tag{1}$$
où $\dot{x}$ est la dérivée à droite de la fonction x.

- *Une fonction x est nommée <u>solution</u> de l'équation antérieure sur l'intervalle $[\sigma-\tau,\sigma+A]$ s'il existe $\sigma \in \mathbb{R}$ et $A > 0$ ainsi que $x \in C([-\sigma-\tau,\sigma+A],\mathbb{R}^n)$, $(t,x_t) \in \mathbb{R}\times C$ et $x(t)$ satisfait l'équation donnée pour $t \in [\sigma,\sigma+A]$.*

- *Etant donnés $\sigma \in \mathbb{R}$, $\phi \in C$ on dit que $x(\sigma,\phi,f)$ est une <u>solution de l'équation du point 1 avec la valeur initiale ϕ en σ</u> (ou une <u>solution à travers (σ,ϕ)</u>) s'il existe $A > 0$ ainsi que $x(\sigma,\phi,f)$ est une solution de l'équation sur l'intervalle $[\sigma-\tau,\sigma+A]$ et $x_\sigma(\sigma,\phi,f) = \phi$.*

L'existence et l'unicité des solutions pour RFDE sont données par le théorème suivant:

Théorème 1 *Soit Ω un ensemble ouvert en $\mathbb{R}\times C$, $f : \Omega \mapsto \mathbb{R}^n$ une fonction continue, et $f(t,\phi)$ Lipchitzienne en variable ϕ sur chaque ensemble compact inclus dans Ω. Si $(\sigma,\phi) \in \Omega$, alors il existe une unique solution à travers (σ,ϕ) pour toute équations de type RFDE.*

Nous remarquons que la propriété de la fonction f d'être Lipchitzienne est nécessaire seulement pour l'unicité, l'existence est garantie aussi sans cette condition.

Définition 2 *Soit $f(t,0) = 0, \forall t \in \mathbb{R}$. On dit que la solution $x = 0$ de l'équation (1) est:*

- *stable si pour tout $\sigma \in \mathbb{R}$, $\epsilon > 0$ il existe $\delta = \delta(\epsilon, \sigma)$ ainsi que $\phi \in \mathcal{B}(0, \delta)$ implique $x_t(\sigma, \phi) \in \mathcal{B}(0, \epsilon)$ pour $t \geq \sigma$.*

- *uniforme stable si c'est stable et δ est indépendant de σ.*

- *asymptotique stable si c'est stable et s'il existe $b_0 = b_0(\sigma) > 0$ ainsi que $\phi \in \mathcal{B}(0, b_0)$ implique $x(\sigma, \phi)(t) \underset{t\to\infty}{\longrightarrow} 0$*

- *uniforme asymptotique stable si c'est uniforme stable et il existe $b_0 > 0$ ainsi que pour tout $\eta > 0$, il existe $t_0(\eta)$ ainsi que $\phi \in \mathcal{B}(0, b_0)$ implique $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$ pour $t \geq \sigma + t_0(\eta)$ et pour tout $\sigma \in \mathbb{R}$.*

- *exponentiel stable s'il existe $B > 0$ et $\alpha > 0$ ainsi que pour toutes les conditions initiale $\phi \in C$, $\parallel \phi \parallel < v$, la solution satisfait l'inégalité:*

$$\parallel x(\sigma, \phi)(t) \parallel \leq B e^{-\alpha(t-\sigma)} \parallel \phi \parallel .$$

Si $y(t)$ est une solution d'une RFDE de la forme $\dot{x}(t) = f(t, x_t)$, alors on dit que y est stable si la solution $z = 0$ de l'équation

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t)$$

est stable. Les autres concepts se définissent par similarité. On rappelle que, pour les systèmes linéaires tous les types de stabilité définis antérieurement sont équivalents [75]. Les systèmes traités dans cette thèse sont *linéaires*, donc l'équivalence antérieure peut être utilisée. Plus précisément, suite à la présentation des différents modèles qui apparaissent en biologie, communications en réseaux et réseaux de transport, on déduit *la nécessité de l'étude de classe de systèmes dynamiques décrite par l'équation caractéristique:*

$$D(s, T, \tau) = P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau} = 0, \tag{2}$$

où P et Q sont des polynômes qui satisfont les propriétés suivantes: $\deg P \geq \deg Q$, $P(0) + Q(0) \neq 0$ et P, Q n'ont pas des facteurs communs.

0.3 L'approche géométrique

Le but principal de cette partie, présentée dans les chapitres 3 et 4 de cette thèse, est d'obtenir les *régions de stabilité* des systèmes dynamiques de la classe considérée. Notre étude commence à partir des quelques interprétations géométriques des équations caractéristiques pour les systèmes dynamiques avec un ou *deux retards discrets* (ou ponctuels). Après la présentation des quelque uns de ces résultats dans les premières sections du chapitre 3, nous développons dans la dernière section des algorithmes qui peuvent être utilisés pour le traitement des différents cas *particuliers* et *dégénérés*. L'un des cas particuliers de systèmes avec deux retards discrets est relatif à la méthode de contrôle prédictif proposé par Smith [121]. De même, certains cas dégénérés mis en évidence (mais non-traités) dans [55] sont discutés.

La principale densité de probabilité utilisée dans cette thèse est donnée par *la distribution gamma*. Le choix de cette distribution n'est pas au hasard, parce que *le comportement stochastique* de plusieurs modèles est décrit dans la littérature en utilisant une telle densité. On constate également que certains systèmes particuliers simples, qui peuvent être inclus dans la classe considérée dans cette thèse, ont été déjà étudiés dans le passé [20]. A la différence de [20] où sont présentés seulement des conditions suffisantes pour la stabilité des systèmes scalaires de la forme:

$$\dot{x}(t) = -\alpha x(t) - \beta \int_0^\infty x(t-\tau) g(\tau) \mathrm{d}\tau,$$

la méthode que nous avons dveloppé dans cette thèse permet l'obtention des conditions *nécessaires et suffisantes* pour la stabilité des systèmes d'une classe plus générale décrite par l'équation caractéristique (2).

Dans le chapitre 4 nous mettons en évidence une interprétation géométrique de l'équation caractéristique qui décrit la classe des systèmes dynamiques avec des *retards distribués*. Cette interprétation géométrique nous permet d'obtenir l'ensemble Ω des fréquences correspondantes aux *points de passage* des solutions de l'équation caractéristique du sémi plan droit du plan complexe vers celui de gauche ou vice-versa.

Proposition 1 *Pour toutes $\omega > 0$, l'équivalence suivante est satisfaite:*

$$\omega \in \Omega \Leftrightarrow 0 < |P(j\omega)| \leq |Q(j\omega)|.$$

De plus, les valeurs T, τ correspondant à la fréquence ω peuvent être calculées en utilisant les formules:

$$\begin{aligned} T &= \frac{1}{\omega}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right)^{1/2}, \\ \tau &= \tau_m = \frac{1}{\omega}(\angle Q(j\omega) - \angle P(j\omega) - n\arctan(\omega T) \\ &\qquad +\pi + 2m\pi), m = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Les couples (T, τ) définis ci-dessus génèrent les courbes qui séparent les zones avec un nombre constant de racines instables (*courbes de stabilité*). La méthode qui nous permet de déterminer la direction dans laquelle passent les racines l'axe imaginaire lorsqu'on traverse une courbe de stabilité est basée sur le théorème des fonctions implicites.

En utilisant les notations du chapitre 4 la *direction de passage* peut être caractérisée ainsi:

Proposition 2 *Soit $\omega \in \Omega$ et (T, τ) le couple correspondant à ω sur la courbe de stabilité. On suppose que $j\omega$ est une solution simple de l'équation (2) et $D(j\omega', T, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (T, τ) n'est pas un point d'intersection à deux courbes de stabilité différentes ou a deux sections différentes de la même courbe). Ainsi, un couple de solutions appartenant à l'équation (2) traverse l'axe imaginaire vers le demi-plan droit, par $s = \pm j\omega$, si $R_2 I_1 - R_1 I_2 > 0$. Le passage est vers le demi-plan gauche si l'inégalité est inversée.*

Le chapitre 4 se termine avec des études de *robustesse de la stabilité* par rapport aux paramètres et par rapport au retard. De même, on a mis en évidence et on a analysé certains cas dégénérés. Tous les résultats et les conclusions sont accompagnés de quelques exemples illustratifs.

0.4 L'approche algébrique

Cette partie est composée des chapitres 5 et 6. La méthode algébrique obtenue dans le chapitre 5 peut être vue comme une *approche complémentaire* de la procédure présentée dans le chapitre 4. En utilisant une technique basée sur l'analyse matricielle, on divise le plan complexe en bandes verticales où le système sans retard de propagation ("gap" en Anglais) a une nombre constant de *racines instables*. Plus précisément, en utilisant les notations du chapitre 5, on obtient le résultat suivant:

Proposition 3 *Soit* $0 < \lambda_1 < \lambda_2 < \ldots \lambda_h$*, les valeurs propres réelles et positives du faisceau matriciel* $\Sigma(\lambda) = (\lambda U + V)$*. Alors, le système décrit par l'équation caractéristique (2) ne peut pas être stable pour* $T = \lambda_i$*,* $i = 1, 2, \ldots h$*. De plus, si pour* $T = T^* \in (\lambda_i, \lambda_{i+1})$*, le système a* r *racines instables (*$0 \leq r \leq n + n_p$*), alors le système conserve* r *racines instables pour toutes les valeurs* $T \in (\lambda_i, \lambda_{i+1})$*. Le même résultat reste valable pour les intervalles* $(0, \lambda_1)$ *et* (λ_h, ∞)*.*

Puis, nous avons développé une méthode permettant de trouver le nombre de passages de l'axe imaginaire lorsque la valeur du retard de propagation ("gap") τ augmente. Par conséquent, cette méthode nous permet de récupérer complètement les *régions de stabilité* du système.

Le chapitre 6 étend la méthode algébrique, obtenue dans le chapitre 5, à des classes de systèmes plus générales. Dans le premier paragraphe, on adapte la technique pour l'analyse d'un système avec des *retards commensurables* (on dit que deux retards sont commensurables si leur rapport est un nombre rationel) provenant des réseaux de transport. Puis, nous considérons une classe de systèmes plus générale qui inclut le modèle du premier paragraphe. Pour analyser cette classe, nous adaptons, d'une part, la méthode de Walton-Marshal [159] de réduction du nombre de retards proportionné et, d'autre part, l'algorithme présenté dans le chapitre 5.

La dernière partie de la thèse inclut des conclusions et des perspectives.

Thesis Synthesis (in Romanian)

În această parte facem o scurtă prezentare a tezei, în care sintetizăm principalele rezultate. Analiza dezvoltată în această lucrare este modivată de numărul tot mai mare de modele provenite din biologie şi comunicaţii prin reţele, care sunt descrise de sisteme dinamice cu întârzieri distribuite.

0.5 Introducere

Analiza stabilităţii ecuaţiilor diferenţiale cu întârzieri a început în anii 50, iar una din primele abordări este prezentată de Krasovskii [77], care generalizează metoda a doua a lui Lyapunov. Apoi, mulţi autori au dezvoltat diferite metode şi au pus diferite probleme cu privire la analiza stabilităţii *ecuaţiilor diferenţiale cu argument întârziat*. Cu privire la studiul în *domeniul frecvenţă*, rezultate fundamentale au fost obţinute de Pontryagin [124]. Chebotarev a publicat de asemenea câteva lucrări dedicate problemelor Routh-Hurwitz pentru quasipolinoame (de exemplu [27]).

Începând cu anii 90 au apărut multe criterii în domeniul frecvenţă, care iau în consideraţie *aspecte computaţionale* şi de *robusteţe a stabilităţii*. Oricum, există un număr imens de lucrări ce tratează systemele dinamice cu argument întârziat şi pentru a avea o bună perspectivă asupra domeniului menţionăm aici doar câteva monografii [56, 126, 75, 59, 107, 54] care conţin principalele concepte şi metode apărute până în prezent. Dintre aceste monografii se distinge [56] prin calitatea ideilor prezentate. Mentionăm că această monografie conţine o mulţime de idei interesante care nu au fost incă suficient exploatate în literatură.

Deşi literatura cu privire la stabilitatea sistemelor dinamice cu argument întârziat este vastă, există doar două abordări principale:

- *Studiul în domeniul frecvenţă* - aici sunt incluse: testele analitice care extind metoda lui Hurwitz la systeme dinamice cu întârzieri, generalizări a metodelor de locaţie a rădăcinilor ($\mathcal{D}$-descompunere şi $\mathcal{T}$-descompunere), teste de stabilitate bazate pe integrarea pe contur, şi proceduri bazate pe studiul spectrului operatorului asociat sistemului liniar considerat.

- *Studiul în domeniul timp* - generalizări ale metodei a doua a lui Lyapunov şi metode bazate pe principiul comparaţiei, pot fi menţionate aici. Deşi principiul comparaţiei a fost dezvoltat iniţial în *domeniul timp*, idei similare pot fi găsite şi în *domeniul frecvenţă*.

În această lucrare dezvoltăm metode în *domeniul frecvenţă*, pentru analiza stabilităţii unei clase de sisteme dinamice liniare cu întârzieri distribuite. Rezultatele descrise aici sunt urmarea unor studii prezentate în unele lucrări ale autorului scrise în colaborare cu Gu, Michiels şi Niculescu [99, 100, 101, 102]. Alegerea clasei de sisteme dinamice liniare studiate este motivată de modele existente în ştiinţele aplicate (biologie [35, 119, 24], inginerie [153, 130, 143], etc). *Principalul obiectiv al tezei este să descrie metode intuitive şi uşor de implementat pentru obţinerea zonelor de stabilitate ale sistemelor dinamice liniare (cu întârzieri distribuite) ce aparţin clasei studiate.*

0.6 Preliminarii

Scopul acestei sectiuni este prezentarea conceptelor şi noţiunilor de bază, necesare în dezvoltarea rezultatelor noastre. Mai precis, vom prezenta pe scurt rezultatele clasice ale teoriei ecuaţiilor diferenţiale funcţionale ce privesc analiza stabilităţii. Rezultatele propuse sunt detaliate în monografii ca [15, 59, 107] ori [54].
În continuare vom nota cu $C([a,b],\mathbb{R}^n)$ *spaţiul Banach al funcţiilor continue* definite pe intervalul $[a,b]$ cu valori în $\mathbb{R}^n$ dotat cu *topologia convergenţei uniforme*. Dacă $[a,b] = [-\tau, 0]$ atunci notaţia va fi simplificată după cum urmează $C = C([-\tau,0],\mathbb{R}^n)$. Pentru norma unui element $\phi \in C$ vom folosi notaţia $\| \phi \| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. Dacă $\sigma \in \mathbb{R}$, $A > 0$, $x \in C([-\sigma-\tau, \sigma+A],\mathbb{R}^n)$, şi $t \in [\sigma, \sigma + A]$ definim funcţia continuă $x_t \in C$, $x_t(\theta) = x(t+\theta)$.

Definiţia 1 *Pentru orice $f : \mathbb{R} \times C \mapsto \mathbb{R}^n$, reamintim următoarele noţiuni:*

- *Forma generală a unei ecuaţii diferenţiale funcţionale de tip retardat (RFDE) este*
$$\dot{x}(t) = f(t, x_t) \tag{3}$$
unde $\dot{x}$ este derivata la dreapta a funcţiei x.

- *O funcţie x este numită soluţie a ecuaţiei anterioare pe intervalul $[\sigma - \tau, \sigma + A]$ daca există $\sigma \in \mathbb{R}$ şi $A > 0$ astfel încât $x \in C([-\sigma - \tau, \sigma + A], \mathbb{R}^n)$, $(t, x_t) \in \mathbb{R} \times C$ şi $x(t)$ satisface ecuaţia dată pentru $t \in [\sigma, \sigma + A]$.*

- *Fiind date $\sigma \in \mathbb{R}$, $\phi \in C$ spunem că $x(\sigma, \phi, f)$ este o soluţie a ecuatiei de la punctul 1 cu valoarea iniţială ϕ în σ (sau o soluţie prin (σ, ϕ)) dacă există $A > 0$ astfel încât $x(\sigma, \phi, f)$ este o soluţie a ecuaţiei pe intervalul $[\sigma - \tau, \sigma + A]$ şi $x_\sigma(\sigma, \phi, f) = \phi$.*

Existenţa şi unicitatea soluţiilor pentru RFDE sunt date de următoarea Teoremă.

Teorema 1 *Fie Ω o mulţime deschisă în $\mathbb{R} \times C$, $f : \Omega \mapsto \mathbb{R}^n$ o funcţie continuă, şi $f(t, \phi)$ Lipschitziană în variabila ϕ pe orice mulţime compactă inclusă în Ω. Dacă $(\sigma, \phi) \in \Omega$, atunci există o unică soluţie prin (σ, ϕ) pentru orice ecuaţie de tip RFDE.*

Remarcăm ca proprietatea funcţiei f de a fi Lipschitz este necesară numai pentru unicitate, existenţa este garantată şi fără această condiţie .

Definiţia 2 *Fie $f(t, 0) = 0$, $\forall t \in \mathbb{R}$. Spunem că soluţia $x = 0$ a ecuaţiei (3) este:*

- *stabilă dacă pentru orice $\sigma \in \mathbb{R}$, $\epsilon > 0$ există $\delta = \delta(\epsilon, \sigma)$ astfel încât $\phi \in \mathcal{B}(0, \delta)$ implică $x_t(\sigma, \phi) \in \mathcal{B}(0, \epsilon)$ for $t \geq \sigma$.*

- *uniform stabilă dacă este stabilă şi δ este independent de σ.*

- *asimptotic stabilă dacă este stabilă şi există $b_0 = b_0(\sigma) > 0$ astfel încât $\phi \in \mathcal{B}(0, b_0)$ implică $x(\sigma, \phi)(t) \underset{t \to \infty}{\longrightarrow} 0$*

- *uniform asimptotic stabilă dacă este uniform stabilă şi există $b_0 > 0$ astfel încât pentru orice $\eta > 0$, există $t_0(\eta)$ astfel încât $\phi \in \mathcal{B}(0, b_0)$ implică $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$ pentru $t \geq \sigma + t_0(\eta)$ şi orice $\sigma \in \mathbb{R}$.*

- <u>*exponenţial stabilă*</u> *dacă există* $B > 0$ *şi* $\alpha > 0$ *astfel încât pentru orice* <u>*condiţie iniţială*</u> $\phi \in C$, $\| \phi \| < v$, *soluţia satisface inegalitatea;*

$$\| x(\sigma, \phi)(t) \| \leq B e^{-\alpha(t-\sigma)} \| \phi \| .$$

Dacă $y(t)$ este o soluţie a unei RFDE de forma $\dot{x}(t) = f(t, x_t)$ atunci spunem că y este stabilă dacă soluţia $z = 0$ of the equation

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t)$$

este stabilă. Celelalte concepte se definesc în mod similar. Reamintim că, pentru sisteme liniare toate tipurile de stabilitate definite anterior sunt echivalente [75]. Sistemele tratate în această teză sunt *liniare* deci echivalenţa anterioară poate fi utilizată. Mai precis, în urma prezentării diferitelor modele care apar în biologie, comunicaţii în reţea şi reţele de transport, deducem *necesitatea studiului clasei de sisteme dinamice descrisă de ecuatia caracteristică*

$$D(s, T, \tau) = P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau} = 0, \tag{4}$$

unde P şi Q sunt polinoame ce satisfac următoarele proprietăţi: $\mathrm{grad}P \geq \mathrm{grad}Q$, $P(0) + Q(0) \neq 0$ şi P, Q nu au factori comuni.

0.7 Abordarea geometrică

Scopul principal al acestei părţi, prezentată în capitolele 3 şi 4 ale tezei, este de a obţine *regiunile de stabilitate* ale sistemelor dinamice din clasa considerată. Studiul nostru porneşte de la nişte interpretări geometrice ale ecuaţiilor caracteristice pentru sisteme dinamice cu una sau *două întârzieri discrete*. După prezentarea câtorva din aceste rezultate în primele două secţiuni ale Capitolului 3, dezvoltăm în ultima secţiune algoritmi ce pot fi utilizaţi pentru tratarea diferitelor cazuri *particulare* şi *degenerate*. Unul din cazurile particulare de sisteme cu două întârzieri discrete se referă la metoda de control predictiv a lui Smith [121]. De asemenea sunt discutate unele din cazurile degenerate puse în evidenţă dar netratate în [55].

Principala densitate de probabilitate folosită pe parcursul acestei teze este dată de *distribuţia gamma*. Alegerea acestei distribuţii nu este întâmplătoare, intrucât *comportamentul stocastic* al multor modele este descris în literatură folosind densitatea gamma. Remarcăm că unele sisteme particulare simple , ce pot fi incluse în clasa considerată în această teză, au fost deja studiate [20]. Spre deosebire de [20], unde sunt prezentate doar condiţii suficiente pentru stabilitatea sistemelor scalare de forma

$$\dot{x}(t) = -\alpha x(t) - \beta \int_0^\infty x(t-\tau)g(\tau)\mathrm{d}\tau,$$

metoda dezvoltată în teză permite obţinerea condiţiilor *necesare şi suficiente* pentru stabilitatea sistemelor din clasa mult mai generală descrisă de ecuaţia caracteristică (4).

În Capitolul 4 punem în evidenţă o interpretare geometrică a ecuatiei caracteristice ce descrie clasa de sisteme dinamice cu *întârzieri distribuite* considerată. Această interpretare geometrică ne permite să obţinem mulţimea Ω a frecvenţelor corespunzătoare *punctelor de trecere* a soluţiilor ecuaţiei caracteristice din semiplanul drept al planului complex în cel stâng sau invers.

Propoziţia 1 *Oricare ar fi $\omega > 0$, următoarea echivalenţă este satisfăcută:*

$$\omega \in \Omega \Leftrightarrow 0 < |P(j\omega)| \leq |Q(j\omega)|.$$

Mai mult, valorile T, τ corespunzătoare frecvenţei ω pot fi calculate folosind formulele:

$$\begin{aligned} T &= \frac{1}{\omega}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right)^{1/2}, \\ \tau &= \tau_m = \frac{1}{\omega}(\angle Q(j\omega) - \angle P(j\omega) - n\arctan(\omega T) \\ &\qquad +\pi + 2m\pi), m = 0, \pm 1, \pm 2, \dots \end{aligned}$$

Perechile (T, τ) definite mai sus generează curbele ce separă zonele cu nr constant de rădăcini instabile (*curbe de stabilitate*). Metoda care ne permite să determinăm direcţia în care trec rădăcinile axa imaginară atunci când traversăm o curbă de stabilitate, se bazează pe teorema funcţiilor implicite.

Folosind notaţiile din Capitolul 4 *direcţia de traversare* poate fi caracterizată astfel:

Propoziţia 2 *Fie $\omega \in \Omega$ şi (T, τ) perechea corespunzătoare lui ω pe curba de stabilitate. Presupunem că $j\omega$ este o soluţie simplă a ecuaţiei (4) şi $D(j\omega', T, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (T, τ) nu este punct de intersecţie a două curbe de stabilitate diferite sau a două secţiuni diferite ale aceleiaşi curbe). Atunci, o pereche de soluţii a ecuaţiei (4) traversează axa imaginară către semiplanul drept, prin $s = \pm j\omega$, dacă $R_2 I_1 - R_1 I_2 > 0$. Traversarea este către semiplanul stâng dacă inegalitatea este inversată.*

Capitolul 4 se încheie cu nişte studii de *robusteţe a stabilităţii* în raport cu parametrii şi în raport cu întârzierea. De asemenea, sunt puse în evidenţă şi analizate nişte cazuri degenerate. Toate rezultatele şi concluziile sunt însoţite de exemple ilustrative.

0.8 Abordarea algebrică

Această parte este formată din capitolele 5 si 6. Metoda algebrică obţinută în Capitolul 5 poate fi văzută ca o *abordare complementară* a procedurii prezentate în Capitolul 4. Folosind o tehnică bazată pe o analiză matricială, partiţionăm planul complex in benzi verticale unde sistemul fără "gap" are un număr constant de *rădăcini instabile.* Mai precis, utilizând notaţiile din Capitolul 5 obţinem următorul rezultat:

Propoziţia 3 *Fie $0 < \lambda_1 < \lambda_2 < \dots \lambda_h$, cu $h \leq n + n_p$ valorile proprii reale şi pozitive ale fasciculului matriceal $\Sigma(\lambda) = (\lambda U + V)$. Atunci sitemul descris de ecuaţia caracteristică (4) nu poate fi stabil pentru $T = \lambda_i$, $i = 1, 2, \dots h$. Mai mult, dacă pentru $T = T^* \in (\lambda_i, \lambda_{i+1})$ sistemul are r rădăcini instabile $(0 \leq r \leq n + n_p)$, atunci sistemul păstrează r rădăcini instabile pentru orice valoare $T \in (\lambda_i, \lambda_{i+1})$. Acelaşi rezultat rămâne valabil pentru intervalele $(0, \lambda_1)$ şi (λ_h, ∞).*

Apoi, dezvoltăm o metodă pentru a afla numărul de traversări ale axei imaginare când valoarea "gap"-ului τ creşte. Prin urmare, metoda ne permite să recuperăm *zonele de stabilitate* ale sistemului.

Capitolul 6 extinde metoda algebrică, obţinută în Capitolul 5, la clase de sisteme mai generale. În prima secţiune adaptăm tehnica pentru analiza unui sistem cu *întârzieri comensurate* (spunem că două întârzieri sunt comensurate dacă raportul lor este un număr raţional) provenit din reţele de transport. Apoi, considerăm o clasă de sisteme mult mai generală care include modelul din prima secţiune. Pentru analiza acestei clase adaptăm atât metoda Walton-Marshal [159] de reducere a numărului de întârzieri comensurate cât şi algoritmul prezentat în Capitolul 5.

Ultima parte a tezei conţine concluzii şi perspective.

Contents

Part I

Delay Systems and Preliminaries

Chapter 1

Introduction

1.1 Delay models

There are many dynamical processes described by equations including information from the past. When the future of a system depends on the past and present "states" (in the usual sense), we say that we deal with a *time-delay* dynamical system (called also *hereditary* or with *memory*, past actions, *dead time*, or *time-lag*). It is well known that the delays are natural components in many dynamical phenomena in physics [13], population dynamics [79, 89], epidemiology [140], communication [139, 161], economy [92] and engineering [153].

For example, in economic systems, some delay appears naturally between decisions and effects generated by some needed analysis time interval. In population dynamics, delays describe approximately maturation processes. In epidemiological and ecological models the time delay appears as a consequence of the simplification of a more complicated model or is introduced to characterize the result of a bad comprehension of the corresponding evolution. The communication over network is always accompanied by some discrete delays due to the physical distance between the users, and some distributed delays due to the algorithm that manages the network. Finally, the traffic flow dynamics include various discrete and distributed delays caused by mechanical processes or by human driver reactions.

The evolutions of all processes presented above lead us to the systems modelled by delay differential equations with a selective memory (discrete, point or pointwise delays) or not (distributed delays - all the values inside a time interval with finite bounds or not). We notice here also the fact that high-order dynamics can be approximated by delays (in some norm sense)[60]. A similar remark concerns the approximation of partial differential equations (PDE) by delay differential equations (DDE)[57, 127].

The effect of delay on the system's evolution can be very important and in such cases it can not be neglected. In the problem of communication over network the omission of the delay leads to congestions and loss of information. The disregard of delays in the model of anesthesia may cause the dead of the patient. Generally, excluding the delay from models that a priori are inherited with some gaps, we may produce *damages* (bad behaviors) or, in the most happy situation, obtain wrong results.

1.2 Stability analysis and methodologies

The stability analysis of delay differential equations started in the 40's and one of the first approach is represented by the work of Krasovskii, who generalized the second method of Lyapunov [77]. Next, many authors developed various methods and posed various problems related to stability analysis of delay differential equation. Regarding the *frequency-domain* study, Pontryagin [124] obtained some fundamental results and Chebotarev published some papers (see, for instance, [27]) devoted to the Routh-Hurwitz problems of quasi-polynomials. Starting with the 90's appeared many *frequency-domain* criteria taking in account *computational considerations* and robustness. Even ill-possedness with respect to time delays of stabilized systems was considered [39, 40]. In control, Smith [145] proposed an appropriate method to construct a controller for a special class of delay systems, if the time delay value is perfectly known. Nevertheless, there exists a bandwidth "sensitivity" which accompanies the delay in feedback systems as is proved in [45]. Furthermore, there exists control problems with a lack of robustness to small delay [9] and other where the increasing of delay may improve the closed-loop response [138]. Anyway, the literature on time delay systems is vast and to have a good perspective of this domain we mention here just two monographs [75, 107] that contain a collection of concepts and methods. In the last ten years new impulses in research of stability and stabilization of time delay systems have been given by the work of Dambrine [37], Niculescu [106] and Michiels [96].

Since the research (and a literature) on the stability of linear delay differential equations is extensive there are also a lot of methods to study it. Without any loss of generality (as it is pointed out in [96]) there are only two main approaches:

1 *The frequency-domain approach.* This includes analytical tests that extend Hurwitz method to delay differential equations, generalizations of root locus method ($\mathcal{D}$-decomposition and τ-decomposition method), and stability tests based on the contour integration. At this point, we would like to mention *the eigenvalue based approach.* Methods based on finite spectrum assignment and generalizations of the pole placement procedure are the main components of this approach.

2 *The time-domain approach.* Generalizations of Lyapunov's second method and methods based on a comparison principle, can be mentioned here. Although, comparison principles were initially derived in time-domain, similar ideas can be found in frequency-domain.

The time-domain stabilization approach is based on Krasovskii's and Razumikhin's theorems, where practical stability conditions are usually expressed by the solvability of algebraic Ricatti equations (ARE) or the feasibility of linear matrix inequalities (LMIs). For linear systems such constructions are sometimes accompanied by conservatism characterized by a large gap between sufficient and complete necessary and sufficient conditions.

For a guided tour of the eigenvalue based approach, see for instance Michiels thesis [96]. In this thesis the author describe new approaches for stabilization and robust stabilization of linear systems. The problem of stabilizability of nonlinear DDE cascades is also considered.

1.3 Further remarks and interpretations

The methodologies presented above consider the modelling of time delay system as functional differential equations. But, into a mathematical framework, such a system may be described in several ways. For example, we can deal with differential equations on abstract [17] or functional spaces [59], or over rings of operators [73] or more generally, with operators over infinite-dimensional spaces [34]. Although these approaches are general and give interesting characterizations of some structural properties (stabilizability and observability), the corresponding methods are not always easy to apply to specific (stability) analysis problems.

In this thesis we develop some methods, in *frequency-domain*, for the stability analysis of a class of systems with distributed delays. The results derived here follows the studies presented in some of the author's papers co-written with Gu, Niculescu and Michiels (see, for instance, [99, 100, 101, 102] in the list of references). Our approaches concern a general class of linear dynamical systems including distributed delays. The stability analysis of this class of systems is motivated by a lot of models existing in applied sciences (biology, engineering, etc). Throughout the thesis we illustrate our results using a biological model introduced by Cushing [35] and improved by Nisbet and Gurney [119]. However, in order to emphasize various properties we use sometimes more complicated models describing traffic flow dynamics or models encountered in communication over network. We try to exploit always new and realistic models. The main interest of this work is to give intuitive and easy to implement methods, that allow to derive the stability regions of the linear dynamical systems belonging to a general class.

The idea to use some geometric interpretations in stability analysis of the dynamical systems is not new. In fact, a lot of research in the existing literature focused on deriving the stability chart of dynamical systems using $\mathcal{D}$-decomposition (decomposition in the parameter space of the coefficients) or $\mathcal{T}$-decomposition (decomposition in the time-delay parameter space), but many of these papers concern only specific first-order systems (see, for instance, [31, 24, 22]). To the best of the author's knowledge the first work based on the geometric interpretation for a general class of linear dynamical systems (with two discrete delays) is due to Gu *et al* [55].

The algebraic approach developed in the third part of this thesis shows that the delay can be used to stabilize some unstable systems. This observation is very useful in some application (like, for example, the problem of communication over network) where we can artificially introduce delays in the system. Furthermore, a recent model of traffic flow dynamic [143] motivates us to extend this method for a class of systems where the coefficients of the characteristic quasi-polynomial depend on the mean-delay. A stability switch criteria for a dynamical system with one discrete delay that leads to a characteristic quasi-polynomial with coefficients depending on delay, can be found in [19].

Chapter 2

Preliminaries

In the sequel, we recall in the first section some useful notions, and fundamental results and concepts. Very briefly, we present the notions of solutions, stability and some criteria to check the stability in the frequency-domain case. The second section of this chapter presents various models encountered in the literature, models that motivate our study. Although the models come from completely different areas we obtain the same class of mathematical systems that covers all the examples considered. The last section is dedicated to describe the content of this thesis.

2.1 Basic concepts

The aim of this section is to introduce the basic concepts and notions needed in developing our results. Precisely, we shall briefly present the classical results in the theory of functional differential equations including stability analysis. The proposed results can be found in more details in [15, 59, 107] or [54]. In the sequel, we discuss only the *retarded* case. Classification of time-delay systems, and the difference between the classes can be found in [15] (see also [54]).

2.1.1 Linear differential difference equations

Our notation is standard. $\mathbb{R} = (-\infty, \infty)$ denotes the set of real numbers and $\mathbb{R}^n$ is the set of n-dimensional vector space. It is well known that all solutions of the scalar differential equation

$$\dot{x}(t) = ax(t), \quad a \text{ constant} \tag{2.1}$$

are given by $c \cdot \exp(at)$, where c is an arbitrary constant. The same result holds in the case when a is replaced with an $n \times n$ matrix A, and x is an n-vector, of course, with c an n-vector. We remember that, each column of $\exp(At)$ has the form $\sum_{j=1}^{n} p_j(t) \exp \lambda_j t$, where each $p_j(t)$ is an n-vector polynomial in t and each λ_j is an eigenvalue of the matrix A; that is, each λ_j satisfies the characteristic equation

$$\det(\lambda I - A) = 0 \tag{2.2}$$

The coefficients of the polynomials p_j are determined from the generalized eigenvectors of the eigenvalue λ_j.

Next, for a nonhomogeneous equation

$$\dot{x}(t) = Ax(t) + f(t),\, x(0) = c \tag{2.3}$$

where $f : \mathbb{R} \mapsto \mathbb{R}^n$ is a given continuous function, using the *variation of constants* method we obtain the solution

$$x(t) = c \cdot \mathrm{e}^{At} + \int_0^t \mathrm{e}^{A(t-\theta)} f(\theta) \mathrm{d}\theta. \tag{2.4}$$

Scalar case

Using the technique introduced for equation (2.1) we can study more "complicated" equations. As briefly presented in the introduction, the dynamics of many models are given by a retarded differential difference equations or delay differential equations. To get familiar with some important ideas of time delay systems, we shall analyze a simple time-delay system given by

$$\dot{x}(t) = ax(t) + bx(t - \tau) + f(t) \tag{2.5}$$

where a, b and τ are constants, $\tau > 0$ is the time delay. In order to solve the equation from the time instant $t = 0$ we must solve first, the initial-value problem. To compute $\dot{x}(0)$ we clearly need $x(0)$ and $x(-\tau)$. In a similar way, in order to compute $\dot{x}(\xi)$, $\xi \in [0,\tau)$ to advance the solution further, we need $x(\xi)$ and $x(\xi-\tau)$. Therefore, a moment of reflection indicates that in order to have a solution uniquely defined, $x(t)$ must be specified on the entire interval $[-\tau, 0]$. In fact the following statements hold.

Theorem 1 *If ϕ is a given continuous function on $[-\tau, 0]$, then there exists a unique function $x((\phi, f))$ defined on $[-\tau, \infty)$ that coincides with ϕ on $[-\tau, 0]$ and satisfies (2.5) for $t \geq 0$. Of course, at $t = 0$, the derivatives in (2.5) represents the right-hand derivative.*

In fact, once the initial condition $x(t) = \phi(t)$, $t \in [-\tau, 0]$ is given, we can treat (2.5) as an ordinary differential equation to obtain

$$x(t) = \mathrm{e}^{At}\phi(0) + \int_0^t \mathrm{e}^{A(t-\theta)}[Bx(\theta-\tau) + f(\theta)]\mathrm{d}\theta. \tag{2.6}$$

Once $x(t)$ is obtained for $t \in [0, \tau]$, we can calculate $x(t)$ for $t \in [\tau, 2\tau]$. Such a process is known as *the method of steps* [56].
If f is only locally integrable on $\mathbb{R}$, then the theorem still holds. In this case, by a solution we mean a function that satisfies (2.5) almost everywhere.

Theorem 2 *If $x(\phi, f)$ is the solution of (2.5) defined by Theorem 1, then the following assertions are valid:*

i) *The solution $x(\phi, f)(t)$ has a continuous first derivative for all $t > 0$ and has a continuous derivative at $t = 0$ if and only if $\phi(\theta)$ has a derivative at $\theta = 0$ with*

$$\dot{\phi}(0) = A\phi(0) + B\phi(-\tau) + f(0). \tag{2.7}$$

If f has derivatives of all orders, then $x(\phi, f)$ becomes smoother with increasing values of t.

ii) *If $B \neq 0$, then $x(\phi, f)$ can be extended as a solution of equation (2.5) on $[-\tau-\epsilon, \infty)$, $0 < \epsilon \leq \tau$, and (2.5) is satisfied. Extension further to the left requires more smoothness of ϕ and f and additional boundary conditions similar to (2.7)*

Due to the smoothing property (i) of Theorem 2 many results from ordinary differential equations are valid for retarded equations.

In the following development we will assume that the function f is exponentially bounded

$$f(t) \le K\mathrm{e}^{ct}, \quad K > 0,\, c \in \mathbb{R}$$

such that the following unilateral Laplace transform exists

$$F(s) = \mathcal{L}[f(t)].$$

In order to use the Laplace transform to solve the equation (2.5), we need some exponential bound of $x(\phi, f)$.

Theorem 3 *Suppose $x(\phi, f)$ is the solution of (2.5) defined by Theorem 1. Then there are positive constants a and b such that*

$$|x(\phi, f)(t)| \le a e^{bt} \left(\| \phi \| + \int_0^t |f(\theta)| d\theta \right), \, t \ge 0 \tag{2.8}$$

where $\| \phi \| = \sup\limits_{-\tau \le \theta \le 0} |\phi(\theta)|$.

Taking Laplace transform of (2.5) under initial condition $x(t) = \phi(t)$, $t \in [-\tau, 0]$, we obtain

$$X(s) = \frac{1}{\Delta(s)} \left[\phi(0) + B \int_{-\tau}^{0} \mathrm{e}^{-s(\theta+\tau)} \phi(\theta) \mathrm{d}\theta + F(s) \right] \tag{2.9}$$

where

$$\Delta(s) = s - A - B\mathrm{e}^{-s\tau} \tag{2.10}$$

is the *characteristic quasi-polynomial* of the system.

Definition 1 *The solution $X(t)$ of $\dot{x}(t) = Ax(t) + Bx(t - \tau)$ with initial condition $X(t) = \begin{cases} 0, & t < 0 \\ 1 & t = 0 \end{cases}$ is called the fundamental solution of (2.5).*

Proposition 1 *The fundamental solution of (2.5) is given by*

$$X(t) = \mathcal{L}^{-1}[1/\Delta(s)] \tag{2.11}$$

Using the convolution theorem, the solution of the nonhomogeneous equation (2.5) can be expressed in terms of fundamental solution as follows:

$$x(\phi, f)(t) = X(t)\phi(0) + B \int_{-\tau}^{0} X(t - \tau - \theta)\phi(\theta)\mathrm{d}\theta + \int_0^t X(t - \theta) f(\theta) \mathrm{d}\theta. \tag{2.12}$$

The equation (2.12) is known as Cauchy formula or *variation-of-constant* formula (see, for instance, Hale's monograph [59]). This formula implies the nontrivial result that the exponential behavior of the solutions of the (2.5) is determined by the characteristic equation as stated by the following result [54].

Theorem 4 *For any $\alpha \in \mathbb{R}$ there are only a finite number of roots of the characteristic equation with real parts greater than α. Let $\alpha_0 = \max\{Re(s) \mid \Delta(s) = 0\}$ and $f \equiv 0$. Then, for any $\alpha > \alpha_0$, there is a constant $K = K(\alpha) > 1$ such that*

$$|x(\phi)(t)| \leq K e^{\alpha t} \parallel \phi \parallel, \quad t \geq 0.$$

In particular, if $\alpha_0 < 0$, then one can choose $\alpha_0 < \alpha < 0$ to obtain the fact that all solutions approach zero exponentially as $t \to \infty$.

General case

Now, we start to discuss the existence and uniqueness of solutions for general differential delay equations and then continue with some remarks regarding the linear autonomous equations.

Throughout this work we deal with equations of the general form

$$\dot{x}(t) = f(x(t), x(t-\tau)), \quad t \geq 0 \tag{2.13}$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n$ is a Lipschitz continuous vector function and the time delay $\tau > 0$ is a fixed real number.

Definition 2 *The solution of (2.13) is defined as a vector-valued functions $x : [-\tau, \infty) \mapsto \mathbb{R}^n$ satisfying (2.13) for positive time.*

In order to have a unique solution for (2.13) we need to solve the initial value problem. Similar to the scalar case, we easily get that the minimal amount of information of initial data is given by a continuous function on the interval $[-\tau, 0]$.

In fact, the method of steps allows defining globally a unique solution $x(\,\cdot\,; \phi)$ if $x(t) = \phi(t)$, $\forall t \in [-\tau, 0]$ where ϕ is a continuous function. Furthermore, we can deduce that the solutions of (2.13) are defined only on $[-\tau, \infty)$. Backward continuation is possible with additional smoothness of the initial function. However, the backward continuation is not necessarily unique [156].

Since our further developments concern mainly linear differential delays equations, we shall linearize the general form and present some analysis of the linear systems.

Definition 3 *A steady state of (2.13) is a solution* $x(t) = \bar{x}$*, where* $\bar{x}$ *is a solution of the algebraic equation* $f(\bar{x}, \bar{x}) = 0$*.*

The behavior of solutions of (2.13) near a steady state $\bar{x}$, is approximated by the behavior of solutions of the linearization around $\bar{x}$. Linearizing around a steady state $\bar{x}$ one obtains a linear delay equation

$$\dot{x}(t) = Ax(t) + Bx(t - \tau), \quad t \geq 0 \tag{2.14}$$

with $A = \dfrac{\partial f}{\partial x_1} f(\bar{x}, \bar{x})$ and $B = \dfrac{\partial f}{\partial x_2} f(\bar{x}, \bar{x})$.
Using again Laplace transform for (2.14) with initial data $x(t) = \phi(t)$, $t \in [-\tau, 0]$ we obtain

$$\Delta(s)\mathcal{L}(x) = \phi(0) + B \int_0^\tau \mathrm{e}^{-st}\phi(t - \tau)\mathrm{d}t \tag{2.15}$$

where $\Delta(s)$ denotes the characteristic matrix of (2.14) and is given by

$$\Delta(s) = sI - A - B\mathrm{e}^{-s\tau}. \tag{2.16}$$

Following the same algorithm as in the scalar case, we want to obtain x using the inverse Laplace transform, the Cauchy theorem, and a residue calculus. The characteristic equation associated with (2.14) is given by

$$\det \Delta(s) = \det[sI - A - B\mathrm{e}^{-s\tau}] = 0. \tag{2.17}$$

Since the characteristic equation is transcendental, it has infinitely many zeros. The basic results concerning the behavior of the roots of (2.14) can be resumed as follows.

Lemma 1 *The roots of the transcendental equation (2.17) have the following properties:*

1) *There exists a real* γ_0 *such that the right half plane* $\{s \in \mathbb{C} \mid Re(s) > \gamma_0\}$ *does not include any roots of (2.17).*

2) *For any real numbers* $\gamma_- < \gamma_+$*, the number of roots of (2.17) in a given strip* $\{s \in \mathbb{C} \mid \gamma_- < Re(s) < \gamma_+\}$ *is finite.*

3) *The roots of (2.17) in the left half plane necessarily satisfy*

$$|Im(s)| \leq Ce^{-Re(s)}, \tag{2.18}$$

where C is a constant determined by A and B.

The asymptotic behavior of the solutions of (2.14) as t tends to infinity is completely "controlled" by the behaviors of the roots of the characteristic equation (2.17).

Theorem 5 *Consider x a solution of (2.14) corresponding to an initial function ϕ. For any $\gamma \in \mathbb{R}$ such that (2.17) has no roots on the line $Re(s) = \gamma$, we have the following asymptotic expansion of the solution*

$$x(t) = \sum_{i=1}^{m} p_i(t)e^{\lambda_i t} + o(e^{\gamma t}), \quad \textit{for } t \to \infty \tag{2.19}$$

where $\lambda_1, \ldots \lambda_m$ are the finitely many roots of (2.17) with real part exceeding γ and where $p_1(t), \ldots, p_m(t)$ are polynomials in t.

Proof. We give here just some ideas, more details can be found in [41]. The proof is based on the representation of the solution x using Laplace transform. Using Lemma 1 and (2.15) we get

$$x(t) = \frac{1}{2\pi j}\int_{\gamma_0-\infty}^{\gamma_0+\infty} \mathrm{e}^{st}\Delta(s)^{-1}[\phi(0) + B\int_0^{\tau} \mathrm{e}^{-st}\phi(t-\tau)\mathrm{d}t]\mathrm{d}s. \tag{2.20}$$

Next idea is to shift the line of integration to the left, while keeping track of the residues corresponding to the singularities of $\Delta(s)^{-1}$ that we pass. ■

Corollary 1 *All solutions of (2.14) converge to zero exponentially as $t \to \infty$ if and only if (2.17) has no solutions in the right half plane $\{s \mid Re(s) \geq 0\}$.*

2.1.2 Functional differential equations

The class of functional differential equations generalizes the differential difference equations discussed in the previous section. The basic theory of existence and uniqueness of the solutions will be briefly presented in this paragraph. The fundamental concepts regarding stability of time-delay systems, are also presented. The standard texts on delay differential and functional differential equations are those by Bellman and Cooke [15], by El'sgol'ts and Norkin [43] and by Hale [59]. Our results (developed in the next chapters) concern the functional differential equations of retarded type. Therefore, we present only the concepts related to this type of functional differential equations.

Let $C([a,b],\mathbb{R}^n)$ be the Banach space of continuous functions mapping the interval $[a,b]$ into $\mathbb{R}^n$ with the topology of uniform convergence. If $[a,b] = [-\tau, 0]$ the corresponding notation is simplified to $C = C([-\tau,0],\mathbb{R}^n)$ and the norm of an element $\phi \in C$ will be denoted by $\| \phi \| = \sup_{-\tau\leq\theta\leq 0} |\phi(\theta)|$. For $\sigma \in \mathbb{R}$, $A > 0$ and any continuous functions $x \in C([-\sigma-\tau,\sigma+A],\mathbb{R}^n)$, and $t \in [\sigma, \sigma + A]$ we define a continuous function $x_t \in C$ by $x_t(\theta) = x(t+\theta)$.

Definition 4 *For any $f : \mathbb{R} \times C \mapsto \mathbb{R}^n$, we consider the following notions:*

- *The general form of a functional differential equation of retarded (RFDE) type is*
$$\dot{x}(t) = f(t, x_t) \tag{2.21}$$
where $\dot{x}$ denotes the right-hand derivative of x.

- *A function x is said to be a solution of (2.21) on $[\sigma-\tau,\sigma+A]$ if there exists $\sigma \in \mathbb{R}$ and $A > 0$ such that $x \in C([-\sigma-\tau,\sigma+A],\mathbb{R}^n)$, $(t, x_t) \in \mathbb{R} \times C$ and $x(t)$ satisfies (2.21) for $t \in [\sigma, \sigma + A]$.*

- *For a given $\sigma \in \mathbb{R}$, $\phi \in C$ we say $x(\sigma,\phi,f)$ is a solution of (2.21) with initial value ϕ at σ or simply a solution through (σ,ϕ) if there exists an $A > 0$ such that $x(\sigma,\phi,f)$ is a solution of (2.21) on $[\sigma-\tau,\sigma+A]$ and $x_\sigma(\sigma,\phi,f) = \phi$.*

We note that equation (2.21) is a very general type of equation. To be more explicit we give some classes of equations that can be expressed by (2.21):

a) Ordinary differential equations ($\tau = 0$)
$$\dot{x}(t) = f(x(t));$$

b) Differential difference equations
$$\dot{x}(t) = f(t, x(t), x(t-\tau_1(t)), \ldots, x(t-\tau_p(t)));$$
with $0 \leq \tau_k(t) \leq \tau$, $k = 1, 2, \ldots, p$

c) Integro-differential equations

$$\dot{x}(t) = \int_{-\tau}^{0} f(t, \theta, x(t+\theta))\mathrm{d}\theta; .$$

The existence and uniqueness of the solutions of RFDEs are given by the following Theorem.

Theorem 6 *Suppose Ω is an open set in $\mathbb{R} \times C$, $f : \Omega \mapsto \mathbb{R}^n$ is continuous, and $f(t, \phi)$ is Lipschitzian in ϕ in each compact set in Ω. If $(\sigma, \phi) \in \Omega$, then there exists a unique solution of (2.21) through (σ, ϕ).*

We note that the Lipschitzian condition is necessary only for uniqueness, and without this condition the existence is guaranteed.

In the sequel, we give the definitions of process, dynamical system and stability.

Definition 5 *Suppose X is a Banach space, $\mathbb{R}_+ = [0, \infty)$, $u : \mathbb{R} \times X \times \mathbb{R}_+ \mapsto X$ is a given mapping and define $U(\sigma, t) : X \mapsto X$ for $\sigma \in \mathbb{R}$, $t \in \mathbb{R}_+$ by $U(\sigma, t)x = u(\sigma, x, t)$. A process on X is a mapping $u : \mathbb{R} \times X \times \mathbb{R}_+ \mapsto X$ satisfying the following properties:*

i) *u is continuous*

ii) *$U(\sigma, t) = I$, the identity*

iii) *$U(\sigma + s, t)U(\sigma, s) = U(\sigma, s + t)$*

A process u is said to be an *ω-periodic process* if there exists an $\omega > 0$ such that $U(\sigma + \omega, t) = U(\sigma, t)$ for all $\sigma \in \mathbb{R}, t \in \mathbb{R}_+$. A process is said to be *autonomous process* or a (*continuous*) *dynamical system* if $U(\sigma, t)$ is independent of σ, that is, the family of transformations $T(t) = U(0, t)$, $t \geq 0$, is a C^0-semigroup:

i) $T(t)x$ is continuous for $(t, x \in \mathbb{R}_+ \times X)$

ii) $U(\sigma, t) = I$, the identity

iii) $T(t + \tau) = T(t)T(\tau), \quad t, \tau \in \mathbb{R}_+$

If $S : X \mapsto X$ is a continuous map, the family $\{S^k,\ k \geq 0\}$ of iterates of S is called a *discrete dynamical system.* In a process, $u(\sigma, x, t)$ can be considered as the state of a system at time $\sigma + t$ if initially the state at time σ was x.

Definition 6 *For a given process u on X and a given $\sigma \in \mathbb{R}$, we say that a set $M \subset \mathbb{R} \times X$ is $\underline{\text{stable at } \sigma}$ if, for any $\epsilon > 0$, there exists a $\delta(\epsilon, \sigma) > 0$ such that* [1] *$(\sigma, x) \in \mathcal{B}(M, \delta(\epsilon, \sigma))$ implies that $(\sigma + t, U(\sigma, t)x) \in \mathcal{B}(M, \epsilon)$ for $t \geq 0$. The set M is said stable if it is stable at σ for all $\sigma \in \mathbb{R}$. The set M is $\underline{\text{unstable}}$ if it is not stable. The set M is $\underline{\text{uniformly stable}}$ if it is stable and the number δ in the definition of stability is independent of σ. The set M is said to be $\underline{\text{asymptotically stable at } \sigma}$ if it is stable at σ and there exists an $\epsilon_0(\sigma)$ such that $(\sigma + t, U(\sigma, t)) \to M$ as $t \to \infty$ for $(\sigma, x) \in \mathcal{B}(M, \epsilon_0(\sigma))$. The set M is said to be $\underline{\text{uniformly asymptotically stable}}$ if it is uniformly stable and there exists an $\epsilon_0 > 0$ such that for any $\eta > 0$, there exists a $t_0(\eta, \epsilon_0)$ having the property that $(\sigma + t, U(\sigma, t)) \in \mathcal{B}(\mathcal{M}, \eta)$ for $t \geq t_0(\eta, \epsilon_0$ and all x such that $(\sigma, x) \in \mathcal{B}(M, \epsilon_0)$.*

Any RFDE generates a process in the following way. Consider $f : \mathbb{R} \times C \mapsto \mathbb{R}^n$ completely continuous (f is continuous and for every bounded set B in $\mathbb{R} \times C$, the closure of $f(B)$ is compact) and $x(\sigma, \phi)$ the solution of (2.21) through (σ, ϕ). If x is uniquely defined for $t \geq \sigma - \tau$, then $x(\sigma, \phi)(t)$ is continuous in σ, ϕ, t (see [59]) for $\sigma \in \mathbb{R}$, $\phi \in C$ and $t \geq \sigma$. Therefore, we get a process on C defining $u(x, \phi, \tau) = x_{\sigma+\tau}(\sigma, \phi)$. This process will be called the *process generated by the RFDE(f)*. The definition of stability of the solution $x = 0$ for the process generated by the RFDE(f) can be restated as follows.

Definition 7 *Suppose $f(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of (2.21) is said to be*

- $\underline{\textit{stable}}$ *if for any $\sigma \in \mathbb{R}$, $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, \sigma)$ such that $\phi \in \mathcal{B}(0, \delta)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \epsilon)$ for $t \geq \sigma$.*
- $\underline{\textit{uniformly stable}}$ *if it is stable and δ is independent of σ*
- $\underline{\textit{asymptotically stable}}$ *if it is stable and there exists a $b_0 = b_0(\sigma) > 0$ such that $\phi \in \mathcal{B}(0, b_0)$ implies $x(\sigma, \phi)(t) \underset{t \to \infty}{\longrightarrow} 0$*
- $\underline{\textit{uniformly asymptotically stable}}$ *if it is uniformly stable and there exists a $b_0 > 0$ such that for every $\eta > 0$, there exists $t_0(\eta)$such that $\phi \in \mathcal{B}(0, b_0)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$ for $t \geq \sigma + t_0(\eta)$ for every $\sigma \in \mathbb{R}$.*

[1] For any set $H \subseteq X$ and $\epsilon > 0$ we denote $\mathcal{B}(H, \epsilon) = \{x \in X \mid dist(H, x) < \epsilon\}$

- <u>*exponentially stable*</u> *if there exists a* $B > 0$ *and an* $\alpha > 0$ *such that for all initial conditions* $\phi \in C$, $\| \phi \| < v$, *the solution satisfies the inequality;*

$$\| x(\sigma, \phi)(t) \| \leq Be^{-\alpha(t-\sigma)} \| \phi \| .$$

If $y(t)$ is any solution of (2.21) then y is said to be stable if the solution $z = 0$ of the equation

$$\dot{z}(t) = f(t, z_t + y_t) - f(t, y_t)$$

is stable. The other concepts are defined in a similar manner. We recall that, if the system is linear the *uniform asymptotic stability* property is equivalent to the *asymptotic stability* or to the *exponential stability* property [75]. The systems treated in this thesis are linear so the previous equivalence holds.

2.1.3 Some frequency-domain stability tests

The purpose of this paragraph is to review some classical tests and methods regarding the stability of some particular classes of time delay systems. Among the RFDE, of particular interest is the case that f is linear and continuous with respect to x_t. This type of systems are called linear systems. The general form of the linear systems is

$$\begin{aligned} \dot{x}(t) = L(t)x_t + h(t), \quad t \geq \sigma \\ x_\sigma = \phi \end{aligned} \tag{2.22}$$

where $h \in \mathcal{L}_1^{loc}([\sigma, \infty), \mathbb{R}^n)$, the space of locally integrable functions, and $L(t) : C \mapsto \mathbb{R}^n$ is a linear operator for a given time t. Therefore, there exists a matrix function $F : \mathbb{R} \times [-\tau, 0] \mapsto \mathbb{R}^{n \times n}$ of bounded variation and $F(t, 0) = 0$ such that

$$L(t)\phi = \int_{-\tau}^{0} \mathrm{d}_\theta[F(t, \theta)]\phi(\theta).$$

Therefore, we can always write a linear RFDE as

$$\dot{x}(t) = \int_{-\tau}^{0} \mathrm{d}_\theta[F(t, \theta)]x(t + \theta) + h(t) \tag{2.23}$$

If the function F in (2.23) is independent of time t, then the system is *linear time-invariant* (LTI) and his general form it can be written as

$$\dot{x}(t) = \int_{-\tau}^{0} \mathrm{d}[F(\theta)]x(t + \theta) + h(t) \tag{2.24}$$

The characteristic equation of the previous system is given by

$$\det|\Delta(s)| = 0, \quad \Delta(s) = sI - \int_{-\tau}^{0} \mathrm{e}^{s\theta}\mathrm{d}F(\theta) \tag{2.25}$$

The basis of frequency-domain stability analysis consists in the following important result.

Theorem 7 *For any real scalar γ, the number of the solutions of the characteristic equation (2.25) with real parts greater than γ is finite. In particular, the multiplicity of any solution is finite. Then $\alpha_0 = \max\{Re(s) \mid \det|\Delta(s)| = 0\}$ is well define. For any $\alpha > \alpha_0$, there exists a $K > 0$ such that the solution of (2.24) with $h(t) = 0$ and $x_0 = \phi$ satisfies*

$$\| x(t) \| \leq K e^{\alpha t} \| \phi \| \tag{2.26}$$

The number α_0 is known as the *stability exponent* for the system. The Theorem above says that the system is stable if and only if its stability exponent is strictly negative. Further remarks and comments on the distribution of zeros of the characteristic equation can be found in [15, 74].
In the sequel we give some methods for determining when the roots of a characteristic equation are in the left half-plane. The classification below follows the one proposed in [113] and [107]. Some criteria for the location of the zeros of characteristic function, with detailed proofs, are given in [51].

Analytical tests

In fact, we present here some criteria that generalize the Hurwitz method to delay systems.

Pontryagin criterion The most general results are due to Pontryagin [124, 125] for the zeros of characteristic equations of the form $P(z, \mathrm{e}^z) = 0$ where $P(x, y)$ is a polynomial in x, y. To obtain the results, he extended the methods used in proving the Routh-Hurwitz criterion for the zeros of a polynomial to be in the left half-plane.

Remark 1 *For a linear delay system with a single or with commensurable delays, the characteristic function associated to the characteristic equation* *in the quasi-polynomial (or exponential polynomial as suggested by Pinney in [123]) can be written as:*

$$\Delta(z) = P(z, e^z) = \sum_{i=0}^{p} \sum_{k=0}^{q} a_{ik} z^i e^{kz} \tag{2.27}$$

Suppose that $P(z, \mathrm{e}^z)$ given in (2.27) has principal term (i.e. $a_{pq} \neq 0$) and suppose $\Delta(i\omega)$ is separated in to its real and imaginary parts, $\Delta(i\omega) = F(\omega) + jG(\omega)$. Then, the main idea behind the Pontryagin criterion can be summarized as follows:

1. If all the roots of P are in $\mathbb{C}_-$ (the left half-plane), then the roots of $F(\omega)$ and $G(\omega)$ are real, simple, alternate, and

$$F'(\omega)G(\omega) - F(\omega)G'(\omega) > 0, \quad \forall \omega \in \mathbb{R} \tag{2.28}$$

2. Conversely, all the roots of P are in $\mathbb{C}_-$ if one of the next conditions is satisfied:

   a) All the roots of $F(\omega)$ and $G(\omega)$ are real, simple, alternate, and the inequality (2.28) is satisfied for at least one $\omega \in \mathbb{R}$.

   b) All the roots of $F(\omega)$ (or $G(\omega)$) are real, simple and for each root the inequality (2.28) holds.

Some comments on the method applied to scalar system are proposed in [75]. The second-order systems with a single delay are completely treated in [21, 64]. application of this technique for constructing PI controllers for stabilizing first-order plants with input delay can be found in [141].

Other criteria The Chebotarev criterion can be seen as the "direct" generalization of the Routh-Hurwitz criterion to the quasi-polynomials in commensurate delay case (for the formulation see, for instance Chebotarev and Meĭman [27] or Răsvan [126]). In order to use this result, we must compute a *large* number of determinants. Therefore, the application of such a criterion is not very practical.

The Yesupovisch-Svirskii criterion (see [147] for the formulation) works perfectly for single pointwise delay system. The idea of this method is to transform the stability analysis to test the sign of some functions with respect to the real axis. Cooke and van den Driessche [32] (Mathematics) or Walton and Marshall [159] (Control) used the similar arguments for characterizing the behavior of the roots of the characteristic equation

$$f(s) + g(s)\mathrm{e}^{-s\tau} = 0.$$

We note that [159] consider f, g polynomials and [32] focusses on analytic functions. In [79], Kuang shows that under appropriate conditions the method can be also applied to neutral systems.

Root locus methods

Various methods to determine the values of the parameters for which the characteristic equation has at least one root on the imaginary axis, are presented below. Since the roots of the corresponding characteristic equation continuously depend on the system's parameters, we can see the cases with roots on imaginary axis as the situations for which the system behavior changes in the sense of increasing or decreasing the number of roots in open right half-plane.

D-decomposition method Neimark [105] developed a method to decompose the parameter space in region with a constant number of unstable roots. Each region is bounded by a hypersurface which corresponds to the case when at least one root lies on the imaginary axis. It is clear that the stability of the system depends on its parameters, i. e. the entries $F(t), h(t)$ and the delays parameters τ_i. Thus, each hypersurface can be seen as a function of τ_i.

In the single-delay case this method allows detecting the particular "delay-independent/delay-dependent" regions. The scalar case can be easily analyzed [107], but for more general systems the method may become more difficult to be applied.

τ-decomposition method The main idea here, is to transform the characteristic quasi-polynomial into the following form:

$$\mathrm{e}^{\tau s} = D_0(s) = \frac{P(s)}{Q(s)},$$

where P and Q are polynomials. Obviously, this method is applied only for a system with a single delay. The results are given analyzing the intersection of the contour $D_0(j\omega)$ with the unit circle in the complex plane. In particular, if there is no intersection with the unit circle, it is easy to conclude that the stability for the case $\tau = 0$ is preserved for all positive values of delay, that is a delay-independent type result. Further comments are presented in [107]. Some remarks on the numerical algorithms can be found in [63]. Note also some similarities between the methods in [32, 159] and the method above.

Argument principle methods We briefly discuss here the extension of some classical methods used in control, as for example Nyquist or Michailov-Leonhard criteria. Let us consider a linear system with a single delay and with a corresponding characteristic equation given by:

$$F(s) = s^n + \sum_{k=1}^{n} f_k(s) s^k, \tag{2.29}$$

where f_k are bounded and F analytical in $Re(s) \geq -\alpha$, for some $\alpha > 0$. Assume that F has no imaginary roots, that is, it can be written as $F(j\omega) = u(\omega) + jv(\omega)$, with u different from 0. In this conditions Michailov criterion says that the system is (uniformly) asymptotically stable if and only if the variation of $\arg[F(j\omega)]$ is $\dfrac{n\pi}{2}$ when ω varies from 0 to ∞. This criterion is difficult to verify in practice. However, if one define

$$I(\omega) = \frac{\mathrm{d}}{\mathrm{d}\omega} \arg[F(j\omega)]$$

the Michailov condition can be expressed as

$$\int_0^\infty I(\omega) = \frac{\pi}{2}. \tag{2.30}$$

This criterion is known as *integral criterion of stability* [75]. Note that the criterion holds for more general distributed delay systems and some simple example in this direction can be found in [74].

2.2 Motivating examples

In order to motivate our study, we discuss some examples encountered in biology, network communication models and traffic flow dynamics. In fact we emphasize several aspects concerning the origin and effects of delays in the models that arise in the specified areas. Also we make some remarks regarding the degree of accuracy of the considered models with respect to the reality.

2.2.1 Biology models

Systems with delays appear naturally in biology as well as in several branches of engineering, and economy, to cite only a few. A nice presentation of some biological models has been proposed by MacDonald [89]. The discussion in this paragraph will include a lot of remarks pointed out in that book. A more recent and appealing presentation can be found in [11].

In most applications in life sciences, a delay is introduced when there are some hidden variables and processes which are not well understood but are known to include a time-lag [31]. The mathematical properties of the delay differential equations justify such methodologies. In some cases, (*e.g.*, simplistic ecological models), it seems that delays have been introduced rather *ad hoc* and sufficiently justified, as it was remarked in [36]. In biological examples, discrete delays can occur, but more often, the delays are distributed (the present state of the system is changing at time t in a stochastic manner given by an appropriate probability density). Each model is characterized by several properties of the system and always we must choose carefully the type of the model according to the behavior of the system. More precisely, we have the following possibilities:

- a model can be either *deterministic* or *stochastic*
- a model can operate in *continuous* or *discrete time*
- the variables of the model can be *continuous* or *discrete*
- a model can be *homogeneous* or *nonhomogeneous*

If the present "state" of the system allows predicting his future behavior, then we have a *deterministic* model. When the present state allows us only to assign probabilities to various outcomes, the model is of *stochastic* type. Obviously, populations are discrete quantities, but in the case of large enough number of individuals we can treat they like a continuous quantity. Some populations have a determined active period (for example, they reproduce only a short time each year), and then, the appropriate model must consider

discrete one-year steps. The opposite situation consists in populations with overlapping generations and no distinct breeding season, and in this case it seems that a continuous time treatment works better. Finally, in the experimental study, for example, we may employ a stirred vessel, or an unstirred vessel respectively. In the first case, we have the same properties in each point of the vessel and the model is considered *homogeneous*. In the second case, the properties in a particular point are determined by its position in the vessel. Therefore, the appropriate variable is $c(x, y, z, t)$, the concentration at a particular point in the vessel and the detailed deterministic model is expressed using partial differential equations. The easiest way to describe the model is to suppose that the vessel is stratified. The structured model requires boundary conditions to be specified as well as initial values, to give unambiguous predictions. In practice, the model is often simplified and there exists two methods to do that. The first method assume that, for example, a variable $f(x, y, t)$ is replaced by an average over space, $F(t)$, thus we get a homogeneous model. The second method considers the system reduced at one point or a small set of points. In this case the coordinates of these points become the dependent variables $x_i(t)$.
Now, let discuss about a very important choice for a model, the choice related to the deterministic or stochastic behavior. More precisely, the solutions at any time t, for a model that is set up in terms of coupled first order ordinary differential-equations

$$\frac{\mathrm{d}x_k(t)}{\mathrm{d}t} = f_k\left(x_1(t), \ldots, x_n(t)\right), \tag{2.31}$$

is given by the initial (current) values $x_k(0)$. However, in order to predict the future behavior of the system, frequently we need to use some information from the past history of at least one variable x_k. When there is a discrete time laps in the action of x_k on some other variable, we say to have a *discrete delay*. It is possible also a cumulative effect of all the earlier values of x_k, and in this case we speak of a distributed delay. In the first case replacing $f(\ldots, x_k(t), \ldots)$ by $f(\ldots, x_k(t-\tau)\ldots)$ we obtain a *delay-differential equations* (or a differential-difference equation). In the second case we must replace $f(\ldots, x_k(t), \ldots)$ by an integral over earlier values of x_k, with a suitable weighting factor. Obviously, in the second case we obtain an *integro-differential equation*. The weighting function, which is often called *delay kernel*, or *delay distribution*, may or may not incorporate a cut-off at a specific earlier time. In the previous section we saw that each of previous types of equations can be included in the more general class of *functional differential equation*. Most of the biological models proposed in the literature belong to the class of *retarded equations*[15, 89, 52].

Systematic analysis of mathematical models in medicine and biology began with the *epidemiological* studies of Ross [131] in the early years of the 20th century. The work was continued in the 20s and 30s, and a great deal work in population ecology is due to Lotka, Voltera and Kostitzin. Much of this literature has been assembled by Scudo and Ziegler [135]. The need for delays was emphasized first by Lotka [140], who discussed the discrete delays due to the incubation times in the Ross epidemic model, and then by Voltera [158], who pointed out the implications of cumulative effects of the presence of a population on the future prospects for growth both of that population and the population of a predator on it.

In a review of theories of business cycles Tinbergen [152] observed the distinction between the two kind of delay. In a particular model of industrial production, for example, the deliveries of a new product at time t, are proportional to investment orders placed earlier time. These investment orders are placed in order to set up a production run of the new product. So the model employs a discrete delay. The second kind of delay is a cumulative dependence on the value of a time-dependent variable. For example, the total investment tied up in the new product builds up from the time the investment orders are placed through to the time of delivery and beyond. Timbergen [152] used the term *production profile* for the weighting factor in the cumulative effect, which is analogous to the modern term of *delay kernel*.

Discrete Delays: from modelling to simulation/validation

The classic early works that first incorporated feedback loops and time delays into biological models were written with few experimental data at hand and so made the simplest or most convenient assumptions about the form of the delay distributions. Fixed lags served to represent parasite maturation times in the 1923 malaria epidemic model of Sharpe and Lotka [140] and also the population self-regulation process in the 1948 logistic model of Hutchinson [65]. Some population model presented by Gopalsamy [50] has also two discrete delays that enter in the characteristic equation as their sum. A model with two discrete delays can be also found in [88]. In [85], is analyzed a blood cell population model with two delays, but only one enters the characteristic equation, although the other affects the relative phase of the oscillations of the two populations. Mackey and Glass [91] also proposed a model for chronic granulocytic leukemia and cyclical neutropenia. The time lag in this model appears explicitly between initiation of cellular production in the bone marrow and release of mature cells into the blood, and the observed oscillations are linked to increased generation time. Another blood cell population model with the same two delays in self-interaction and inter-species interaction terms, is proposed by MacDonald [87]. The role of time lags in certain dynamical respiratory and hematopoietic diseases was examined by Mackey and Glass [49]. For example, the irregular breathing pattern in adults with Cheyne-Stokes respiration appears to originate in disturbances of the feedback loop between ventilation and arterial carbon dioxide concentration. Ventilation is a sigmoidal function of the concentration, but there is a time lag between oxygenation of blood in the lungs and stimulation of chemoreceptors in the brainstem, and so the dynamical model for the process written by Mackey and Glass reduces to a delay system. This agrees with clinical observations that, in the case of patients with Cheyne-Stokes respiration, the lags in stimulation of the chemoreceptors are often increased. However, the experimental difficulties in measuring the respiratory control parameters make difficult detailed numerical comparison of model predictions with experiment.

Distributed Delays: from modelling to simulation/validation

During the 70s, however, authors such as May [93], MacDonald [86] and Cushing [35] pointed out that in applications to biology the use of distributed delays often leads to models that are more tractable and also more realistic than those with discrete delays. Since then, the study of diverse kinds of distributed delays, typically represented by general probability measures on the time axis, has played an important role in the development of theory. In a variety of biological applications, while a delay may have a precise value for an individual member of a population, whether of animals or of cells, it must have a statistical distribution of values over the whole population. The preponderance of distributed delays in population models thus reflects the inclusion of a statistical feature in otherwise deterministic models.

Insect maturation times

Data on insects maturation times can be incorporated in terms of a delay kernel, which can be experimentally determined with sufficient precision to allow a fit with a discrete gap τ and a displaced distribution

$$h(u) = \begin{cases} 0, & \text{if} \quad 0 < u < \tau \\ f(u-\tau), & \text{if} \quad u > \tau \end{cases} \tag{2.32}$$

where f is a given function (in general the gamma distribution with precised parameters). The distributions are of the time from lying an egg to the appearance of the final adult form. Several examples of distributed delays in insect population data have been studied using the above model. The species are the blowfly *Phoenicia sericata* [8], the dragonflies *Epitheca cynosura* and *Epitheca semiaquea* [16] and the damselfly *Pyrrhosoma nymphula* [81]. For any individual insect, the time in question has an exact value. Over the whole population there is a statistical distribution of these times, which the model interprets in terms of distributed delay. So, in this sense, the deterministic model incorporates a statistical aspect of the data.

In order to incorporate this kind of maturation data in a model one needs to start with an age structured mode, which is necessarily formulated in terms of partial differential equation, and to make sure that this model can reasonably be replaced by one formulated in terms of functional differential equation.

Maturation of blood cells

Certain cell populations, such as that of blood cells in the marrow, can be approximately treated as homogeneous with respect to the space. It is well known that different types of cells interact by a variety of agents. Some of these agents can be defined only by their effects. However, a dynamical model of these populations must either include age structure or allows a delay for the maturation time of the cells. Fortunately, the average maturation times for red cells (erythrocytes) and white cells (granulocytes) are fairly well established at seven days and ten days respectively. So we can assume that each individual cell has a fairly sharply defined maturation time. Since the spread in the population is wide and not well specified in form, often it is used the gamma distribution.

Several models of the blood diseases have been studied in the last 30 years. Among these models, cyclical neutropenia and myeloid leukemia received a lot of attention. In the cyclical neutropenia case, the granulocyte populations falls every three weeks to a very low level. In most of the well documented models, notably [120], this trough in granulocyte numbers is accompanied by a sharp peak in the numbers of another type of white cell, the monocytes. Another models on the same topic were formulated in [85] and [104].

In the myeloid leukemia case, the oscillation period for granulocyte population is less well defined but is around two months. Some models regarding this disease can be found in [25, 87]. Some remarks on the stability crossing curves of some delay models related to immune dynamics in leukemia are discussed in [111]. All the models mentioned here share the following features. The delay is introduced so that a model can be formulated in terms of only two populations (mature granulocytes and mature monocytes), instead of the populations of a number of immature stages in the life history of these cells. The knowledge about the immature stages are insufficient to justify a more elaborate model at present. The necessity of distributed delay is given by the fact that in our opinion the maturation times of individual cells have an appreciable spread (see also [89]).

Gamma Distribution

Cushing [35] studied the population dynamics using a model with *gamma-distributed delay*. The linearization of this model is written as:

$$\dot{x}(t) = -\alpha x(t) + \beta \int_{-\infty}^{t} g(t-\theta)x(\theta)d\theta, \tag{2.33}$$

where the integration kernel of the distributed delay is the *gamma distribution*

$$g(\xi) = \frac{a^{n+1}}{n!}\xi^n e^{-a\xi}. \tag{2.34}$$

A Laplace transform of (2.33), with $g(\xi)$ expression (2.34) yields a parameter-dependent polynomial characteristic equation

$$D(s,\tau_1,n) := (s+\alpha)\left(1 + s\frac{\tau_1}{n+1}\right)^{n+1} - \beta = 0, \tag{2.35}$$

where $\tau_1 = \dfrac{n+1}{a}$ is the *mean delay*. Cooke and Grossman [31] discussed the change of stability of (2.35) when one of the parameters, mean delay value τ_1 or the exponent n, varies while the other is fixed.

Nisbet, and Gurney [119] modified the gamma distribution $g(\xi)$ expressed in (2.34) to the *gamma distribution with a gap*

$$\hat{g}(\xi) = \begin{cases} 0, & \xi < \tau_2 \\ \frac{a^{n+1}}{n!}(\xi - \tau_2)^n e^{-a(\xi-\tau_2)}, & \xi \geq \tau_2, \end{cases} \tag{2.36}$$

to more accurately reflect the reality. See [22, 89] for additional discussions. In this case, a simple computation shows that the *mean delay* is $\tau_1 = \tau_2 + \dfrac{n+1}{a}$. The characteristic equation becomes a parameter-dependent quasipolynomial equation [22, 24]

$$\hat{D}(s, \tau_1, \tau_2, n) := (s + \alpha)\left(1 + s\frac{\tau_1}{n+1}\right)^{n+1} - \beta e^{-s\tau_2} = 0. \tag{2.37}$$

It is interesting to note that some of the earlier results in [31, 22] on stability analysis contain some errors as pointed out by Boese [24].

2.2.2 Network congestion

The problem of controlling objects over communication network produced a new class of dynamical system with specific characteristics: presence of propagation-delays, traffic congestion, lose of information and related consequences. These constraints are very important in the control of rapid processes, and the network characteristics can not be neglected.

The communication channel between the controller and the system to be controlled is often modeled as a transmission line, that is, a physic element inducing some delays that corresponds to the properties of the channel. This description becomes more complex when the network includes multiple users. In this case, the delays also depend on the traffic management. The communication channel used to control a considered system is strongly affected by the presence of other information flows in the network.
A network can be characterized by two very important elements:

- *local management* of the traffic flow (controller/receptor); we take in consideration the process of transformation of the physical activities in information which can be transmitted over the network (and viceversa).
- *global management* of the traffic flow, that is, the algorithm that manage the interaction between different information flow such that the collisions or losses of information (for example) are not possible.

Since the traffic flow depends on the algorithm and its implementation, the dimensions, the frequencies and the priorities of the packages can be modified accordingly to the global management of the network. For example, if the network capacity is exceeded, the specific algorithms may send a message to inform the users that the network is saturated. In this case, the users will adapt their transmission to the present situation.

In the sequel, we discuss some of the existing models existing in the literature. One of the way to mathematically represent the communication of binaries data is based on the model of type Shannon-Weaver [139], with an information corruption of ϵ probability. More precisely, a "0" sent with $1-\epsilon$ probability means that "0" will be received and a "0" sent with ϵ probability means that "1" will be received. Another approach, used in automatic environment, is based on the idea that the network is a source of delay in the communication. In this case one supposes the transmission is faithful and the problem is to give the best model of delay in the corresponding schemes. Our work focusses on the last approach, thus we consider only some models where the network induces some delay in communication. Four elementary models of communication network, where the lag is an essential element, are presented.

Model 1: Ideal transmission line

Here, one considers that the transmission line is uniform, has no losses of data, and the frequency parameters are independent (see for instance [95]). The model can be expressed using the telegraph equation:

$$\begin{aligned} C\frac{\partial v(t,x)}{\partial t} &= -\frac{\partial i(t,x)}{\partial x} \\ \frac{\partial v(t,x)}{\partial x} &= -L\frac{\partial i(t,x)}{\partial t} \end{aligned}$$

where v is the tension, i is the flow, x is the position, d is the total length of the line and, L and C are the inductance and the characteristic capacity per length units of the transmission line. Under this circumstances we have a constant delay $\tau = d\sqrt{L/C}$. A complete analysis of this model can be found in [151].

Model 2: Non-homogeneous transmission line

The external perturbations of the data flow is the main inconvenient of the ideal transmission line. Also, the compression or digital coding are ignored when the network is represented as an ideal transmission line. More complex models have been developed but they are seldom used since their complexity means also a difficult implementation of the control law. Such a model, named non-homogeneous transmission line, can be found in [160]. There, it is pointed out the effect of resonance induced by the variation of some of the parameters.

Model 3: Systems with variable delay
Suppose that the network induces a delay which is time-varying. The delay can be characterized, in this case, either by a distribution of probability or by a Markov chain. The Markov chain approach of this model can be found in [78]. They use this technique to fix some parameters such that one can switch between different delay values.

Model 4: Complex models
The communication network can be considered as a system that operates in different way, accordingly with the instruction given by the global management algorithm. Thus, we may consider different models that correspond to the global state of the network. The analysis of this complex models can be done using a Markov chain with multiple state (see [117, 118]). Intuitively, we can consider that this approach introduces another variable (like impedance) in the state of the network.

The presence of the network in the control loop rises various specific problems:

1. the loss of data, generated by the traffic congestion;
2. the compression/decompression of the information;
3. the quantification and coding of the information;
4. the availability of the network at the given moment;
5. the necessity of a router that can manage the performance of the system;
6. the stabilization of the systems in presence of some delays.

In the sequel, we give a brief analysis of some of these problems. We note that each of these problems are studied in a specific research branch.

Problem 1: The loss of data

The problem of loss of packages is a very important and actual issue, thus it received a lot of attention [155]. The management of waiting information is made by a protocol at the router level. This protocol decides which packages are rejected in the congestion situation. The control of congestion was studied both from computer science perspective [94] and the network stability perspective [62]. We note that for the last approach, a particular form of rejected packages function was considered. If a package can't reach its destination, it can be re-emitted (TCP-Transfer Control Protocol) or can be lost (UDP-User Data Protocol). In the second case, the stability analysis of the system controlled over network must take in consideration the loss of information. In [10] it is used a threshold uncertainty principle to establish the loss rate for which the loop system still remains stable. Another approach proposes a controller based on the information from the past (see [82, 83]). The variety of the methods used to study this problem is completed by [4, 5]. The authors proposed a "game theory" based approach, (uniqueness of Nash equilibrium), to obtain the stability of the system under some appropriate conditions.

Quantification and coding of the information

The quantification and coding problems are connected with the information theory. These issues appear during the process of conversion of a signal in data units that can be sent over network. More exactly, the coding problem is a particular problem of control with a limited information. In the context of communication network, this problem is discussed in [3]. Some results regarding the length of the words of the alphabet and the admissible sampling rate can be found in [162]. The chaotic effects induced by approximation when we use words with finite length are pointed out in [154]. The problems of quantification received a lot of attention in the last period. Thus, we can identify various domains of research connected with this issue. In control theory, [23] treated the problem of stabilization using a quantified control, in order to lead the system towards an invariant model around the equilibrium (see also [122]). The synthesis of quantifiers for systems with sampled data can be found in [67, 69]. A probabilistic approach for the quantified systems is given in [68]. The problem of variable period of sampling combined with a variable delay has a solution that uses LMI (Linear Matrix Inequality)[46, 137]. At last, we mention a method of coder and decoder synthesis for classical and linear stochastic control, proposed in [150].

Problem 4: Resources allocations, bandwidth availability
Almost all the communication networks have more than one user. In this case, the users must share the network resources. Therefore, the problems of resources allocation and band limitations appear naturally. Obviously, the minimum guaranteed bandwidth determines the quality of the service. A global controller (at the router level) ensures the management of the network related to this issue. In discrete mathematics, the resources allocation problem is solved using a linear programming approach. In a Bluetooth network (radio communication at short distance for mobile systems), the analysis of capacity allocation is formulated as a convex optimization problem [146] (a hybrid law of capacity distribution is proposed). For WLAN (Wireless Local Area Network) with an access method through a function of hybrid coordination, the quality of the service, satisfying the IEEE 802.11x norms, is ensured by the method proposed in [53]. The algorithm proposed takes into account the delay induced by the queue and uses a control law by proportional state return. The perturbations were compensated by anticipative actions.

A more general approach considers the control law under communication constraints. The control is made in a distributed manner in the network and an *information complexity* criteria is introduced [149]. The notion of *anytime capacity*, in [133, 134], is used to introduce a new parametric quality of the communication channel, that allows considering the presence of the noise in the corresponding scheme.

Problem 6: Presence of delays
In communication, data transmission is always accompanied by a non-zero time interval between the initiation and the delivery time of a message or a signal. The literature on control of the systems with delays is extensive (see for instance [42, 54, 129, 110]). Here, we are giving just a brief presentation of the methods used in this direction. The major works in classical stability analysis (using Lyapunov-Krasovskii and Lyapunov-Rasumikhin function) consider a constant or bounded delay [76, 113]. We mention that, there is also a stochastic approach for the systems with delay related to communication network [116].

First works in the literature were devoted to some models with con-

stant delay. In this case, the delay can be compensated using a Smith predictor [145] or more generally using a state-predictor with a fixed horizon [90, 80, 7].
More recently, appeared several papers concerning time-varying delay models (the delay depends on the state). Some results on the time-varying delay case, when the open-loop system is stable, can be found in [163, 157]. A more general constructive method for the robust stabilization, based on a descriptive transformation of the model (descriptor model method), is proposed in [47, 48]. An application of this method in the time-varying delay case is presented in [136].

Among the problem related to the communication over network, the teleoperation problem received a lot of attention. In the constant delay case, some solutions based on energetic approach are developed in [6, 114]. The time-varying delay case with the same delay on the both directions is considered in [115] and with different delay on each direction in [84, 18].

The communication over network problem

In this section, we present some considerations on the formulation of the problem of communication over network. Precisely, signal delays due to data transmission are considered, which could occur e.g. by large/long bus system or by the use of the internet (with TCP/IP or UDP as protocols). There are different ways to overcome this problem. One of the strategies is represented by a remote control mechanism (control loop runs locally). This is often used in the engineering education with virtual laboratories. Another approach is to use the communication delays in the definition of the control law. Whereas such an approach seems natural, the problem is largely more complicated since the delays are time-varying, and the time-dependence is very complicated, and it depends on a lot of network parameters, as we have seen in the previous paragraphs (network load, available bandwidth). Thus, the discrete deterministic model used in some works [95, 161, 151] seems to be unrealistic. Roughly speaking, the delay in the network is the sum of a constant component and a *dynamic* (time-varying) component. The constant term depends on the signal propagation time, which is assumed to be the absolute minimum propagation delay value, occurring during the measurement. The varying delay occurs due to data collisions and routing problem on the network and is considered to have a stochastic nature. Therefore, to give a more realistic mathematical model, we need to find a probability density that describes the characteristics of the network. Recently, it was pointed out that the communication delay can be represented by some stochastic gamma-distribution [130]. More explicitly, the overall communication delay in the network is modeled by a gamma-distributed delay with a gap, where the *gap* value corresponds to the minimal *propagation delay* in the network, which is always strictly positive. The stability problem of the closed-loop system in [130] reduces to a parameter-dependent characteristic quasipolynomial equation of the following form,

$$D(s, \tau_1, \tau_2, n) := P(s) \left(1 + s\frac{\tau_1}{n+1}\right)^{n+1} + Q(s)e^{-s\tau_2} = 0, \tag{2.38}$$

where $P(s)$, $Q(s)$ are polynomials. Obviously, the equation (2.37) is a special case of (2.38).

2.2.3 Traffic flow models

It is well known that the traffic dynamics are inherently time delayed because of the limited sensing and acting capabilities of drivers against velocity and position variations. The undesirable effects of a mismanaged traffic flow in social and economical life, made this problem very interesting and challenging for many researchers. Regarding to the traffic flow problem there are two main issues. One of them is to find a model which accurately reflects the reality and the second is the stability analysis of the model found at the previous issue.

Time delay in traffic dynamics appears naturally and according to their origin can be classified in three categories (see, for instance, [142], and the references therein):

* *Physiological time delay* - The origin of this part of time delay is the human operator actions. Obviously there is a delay between the moment when the operator receives the stimulus and the moment he performs the action.

* *Mechanical time delay* - This part of time delay originates from mechanical characteristic of the vehicles and it is independent from human operators. This delay express the time between the action of the human operators and the corresponding response of the vehicle.

* *Delay time of vehicle motion* - It is the period of time necessary to a vehicle to change its velocity to the velocity of the preceding vehicle.

In [12, 13], Bando calls the combination of the first two categories of delay as *delay time of response*.
A simplified model, when multiple vehicles preceding each vehicle are followed by the drivers who are identically under the influence of a single constant time-delay, can be expressed by

$$\dot{x}_n(t) = \sum_{l=1}^{k} \alpha_{n,l}(x_{n+l}(t-\tau) - x_n(t-\tau)) \tag{2.39}$$

where n is the number of vehicles and k represents how many vehicles ahead are followed by the drivers. However, instead of assuming a single time delay in the mathematical model, one can suggest a more realistic model with multiple non-identical time delays for sensing the motion of different vehicles. When we have non-identical time delays for following multiple vehicles, several encountered models in the literature describe the corresponding dynamics of the system under consideration (see [142]).

1. Drivers follow either one or two vehicles ahead of them, by performing only feedback position:

$$\ddot{x}(t) = \sum_{k=1}^{2} \alpha_{n,k}(x_{n+k}(t-\tau_k) - x_n(t-\tau_k));$$

2. Drivers follow either one or two vehicles ahead of them, by performing only velocity feedback:

$$\ddot{x}(t) = \sum_{k=1}^{2} \alpha_{n,k}(\dot{x}_{n+k}(t-\tau_k) - \dot{x}_n(t-\tau_k));$$

3. Drivers follow either one or two vehicles ahead of them, by performing a combination of position and velocity feedback:

$$\ddot{x}(t) = \sum_{k=1}^{2} \alpha_{n,k}(x_{n+k}(t-\tau_k) - x_n(t-\tau_k) + \dot{x}_{n+k}(t-\tau_k) - \dot{x}_n(t-\tau_k)).$$

The control of a human driver is different from an automatic controller, which makes the study of traffic flow dynamics more challenging. For instance, humans retain a short-term memory of the past events and this may affect their control strategy [143]. Therefore, in order to obtain a more realistic model we can extend the previous models by incorporating a general memory effect. Without any loss of generality we can consider the following model:

$$\dot{x}_n(t) = \alpha_n \int_0^\infty f(\theta)(x_{n-1}(t-\theta) - x_n(t-\theta))\mathrm{d}\theta, \tag{2.40}$$

where f is a distribution of delays, which can represent both dead-time and the memory of the past. When the choice of the memory model is represented by a gamma-distribution, and we consider the case of two cars, applying the Laplace transform we get a characteristic equation with general form given by (2.38). More explicitly, the characteristic equation of (2.40) is given by

$$\det[sI - (A_1 + A_2)F(s)] = 0. \tag{2.41}$$

Obviously, F denotes the Laplace transform of f, therefore, $F(s) = \dfrac{\mathrm{e}^{-s\tau}}{(1+sT)^n}$. In the case of two vehicles on a ring the matrix A_1, A_2 are given by

$$A_1 = \begin{pmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & \alpha_1 \\ \alpha_2 & 0 \end{pmatrix}. \tag{2.42}$$

The characteristic equation becomes

$$s(1+sT)^n + (\alpha_1 + \alpha_2)\mathrm{e}^{-s\tau} = 0, \tag{2.43}$$

which is again a particular form of equation (2.38).

2.3 Outline of the thesis

The remaining part of the thesis is structured as follows:

- Chapters 3 and 4 form the second part of the thesis, where we derive the *stability regions* using a geometric approach.
- The third part of the thesis includes Chapters 5 and 6. In this part we develop algebraic procedures for studying the stability of the distributed delay systems. The approach developed in the third part can be seen as a complement of the approach developed in the second part.

- A chapter focusing on the contributions of the thesis and future work ends the thesis.

2.3.1 Geometric approach

The starting point of the geometric approach consists of some nice interpretations given in the case of linear systems with one and two discrete delays. After presenting some of these results in the first two sections of Chapter 3, we develop in the last section appropriate algorithms that can be used in treating various particular and degenerate cases. One of the particular cases of systems with two discrete delays concerns the *Smith Predictor controller*. The problems related to this principle received a lot of attention and there are a lot of methodologies available. For a guided tour see for instance [121]. However, our new approach is very interesting and appealing because of its simplicity and the fact that can be easily implemented. Some discussions regarding the degenerate cases listed in [55] can be also found in Chapter 3.

The main probability distribution considered throughout this thesis is given by the *gamma-distribution* with some *gap*. Is not ad-hoc the decision to study the stability of gamma-distributed delays. In fact the stochastic behavior of many models (or components of the model) is described in the literature using a gamma-distributed kernel. Some sufficient condition for the stability of a system with distributed delay, given by a general density of probability, were obtained for a simple scalar case [20]. Anyway the use of a general probability distribution is more difficult and is not always justified by practical experiments.

In Chapter 4 we give some geometric interpretations that allow to derive the *(zero) crossing (frequencies) set* for a class of systems with gamma-distributed delays. Next, we explain the method to obtain the stability crossing curves in the time-delay parameter space, and a simple procedure to derive the corresponding crossing direction. Furthermore, we discuss the robust stability with respect to the parameters and with respect to delay. The chapter ends with the analysis concerning some degenerate cases. All our results and conclusions are accompanied by various illustrative examples.

2.3.2 Algebraic approach

The algebraic method derived in Chapter 5 can be seen as a complementary approach of the procedure presented in Chapter 4. Using an appropriate procedure (based on matrix analysis) we make a partition of the complex plane in strips where the system without gap has a constant number of unstable roots. Next, we find the crossing set, crossing direction and stability regions. Chapter 6 focuses on extending the methodology to more general classes of systems. In the first section we analyze a model of traffic flow dynamic including memory effect. For this particular type of systems with distributed delay and commensurate discrete delays we are able to adapt the standard procedure described in Chapter 5. In the second section, we consider a general class of linear system including the models in the Chapter 5 and the model presented in the first section of Chapter 6. To analyze this class we adapt and combine reducing procedure (the Walton-Marshall [159]) and the algebraic algorithm presented before. Illustrative examples complete the presentation.

2.3.3 On the methodology

We were concerned in developing *simple* and *appealing methods* for *stability analysis* of a general class of linear dynamical system including delays (distributed or not). Nevertheless, our algorithms are computationally oriented, and some specific Matlab routines expressing the corresponding procedures, are included in Appendix B.

For the sake of clarity, some basic and classical mathematical results used throughout the thesis, can be found in Appendix A.

Part II

Geometric approach

Chapter 3

Geometric ideas

In this chapter we introduce some simple and intuitive ideas related to our subject. First, we discuss a dynamical system with one discrete delay in a geometric setting. Next, we present a method based on some geometric interpretation related to dynamical systems with two delays. Finally, we make the first step in our research adapting the previous results to some degenerate cases.

3.1 Geometry of simple cases: linear systems with one discrete delays

The stability problem for linear system with one discrete delay is by now solved and very well understood. Here, we present just some results and interpretations which open our geometric survey. Consider first the characteristic equation with one discrete delay given by:

$$D(s, \tau) = p_0(s) + p_1(s)\mathrm{e}^{-s\tau} = 0, \quad \tau > 0 \tag{3.1}$$

where $p_0, p_1 : \mathbb{C} \mapsto \mathbb{C}$ are two polynomial functions satisfying the following assumption.

Assumption 1 **a)** $\deg(p_0) > \deg(p_1)$

b) $p_0(0) + p_1(0) \neq 0$

c) *$p_0(s)$ and $p_1(s)$ have no common roots.*

Note that these assumptions make sense with respect to our development and will be discussed in the more general case in the next chapter. However, the assumption a) means that the discussion concern the retarded case, the assumption b) ensure that "0" is not a root of the system for any τ, and c) is introduced in order to simplify the expression (it is clear that if c) is violated then the polynomials p_0, p_1 have some common factors).

The roots s of (3.1) move continuously (with respect to τ) in the complex plane. Therefore, the number of roots in the right half plane can change when τ varies, only if a root passes trough the imaginary axis (in other words, the roots can not jump from left to right, or from right to left without crossing $j\mathbb{R}$. In the sequel we denote Ω the set of all frequencies ω such that $D(j\omega, \tau) = 0$ has at least one solution $\tau > 0$. We note that the symmetry of (3.1) allows us considering in Ω only the positive values of ω. The set Ω, called *crossing set* of the system, can be easily found using the fact that $|e^{j\omega\tau}| = 1$ for any $\omega \in \mathbb{R}$. More exactly, we derive that Ω is the set of all $\omega > 0$ satisfying the following relation:

$$|p_0(j\omega)| = |p_1(j\omega)|. \tag{3.2}$$

Since (3.2) can be written as a polynomial equation in ω^2, the crossing set Ω consists of a finite number of positive real number. Obviously, for a given $\omega \in \Omega$, the delay values corresponding to the crossing are given by:

$$\tau = \tau_k(\omega) = \frac{\angle p_1(j\omega) - \angle p_0(j\omega) + (2k+1)\pi}{\omega}. \tag{3.3}$$

If $p_0(j\omega) \neq 0, \forall\omega \in \mathbb{R}_+$ the crossing set is given by the intersections of $-a(j\omega) = -\dfrac{p_1(j\omega)}{p_0(j\omega)}$ with the unit circle in the complex plane. We note that the ratio curve $-a(j\omega)$ starts from $\left(-\dfrac{p_1(0)}{p_0(0)}, 0\right)$ and ends in $(0,0)$ when ω sweeps the real positive axis from 0 to ∞. Therefore, if $\left|-\dfrac{p_1(0)}{p_0(0)}\right| > 1$ we have at least one crossing frequency ω. Obviously, when $\left|-\dfrac{p_1(0)}{p_0(0)}\right| < 1$ the crossing set can be empty or not. When the crossing set is empty the system is *delay-independent* (stable or unstable). We use the previous geometric interpretation to derive the crossing set for some particular choices of $a(s)$. We present just three cases in order to illustrate the delay-independent, the switch and switch-reversal case (see the figures 3.1 and 3.2).

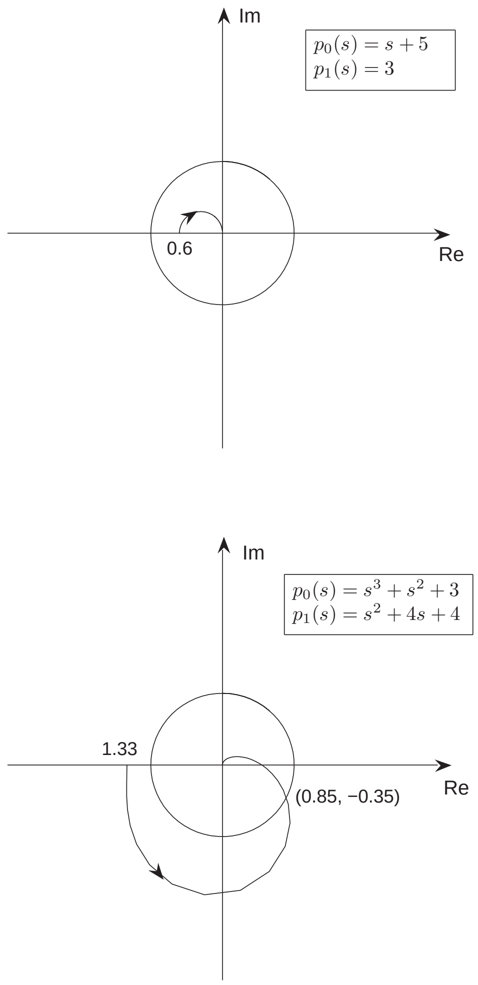


Figure 3.1: Up: No crossing occurs; Down: The crossing set contains one point and the corresponding crossing direction is towards instability.

The same result can be obtained by using various control-based idea and techniques. One of the most used result that can be included in this class is

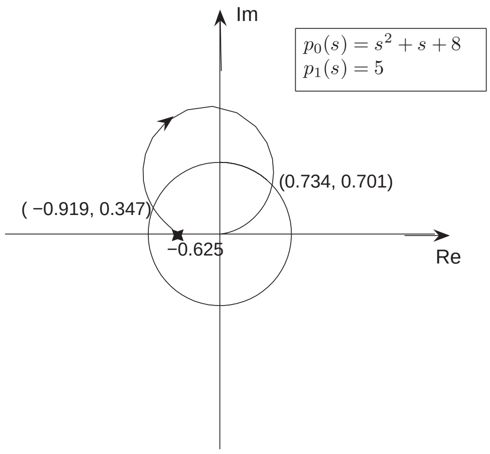


Figure 3.2: The crossing set contains two points, first crossing corresponds to a switch to instability and the second to a reversal one.

due to Tsypkin and consists of a simple frequency-sweeping test for *delay-independent* closed-loop stability. For a system with a single discrete input delay

$$H_0(s) = \frac{p_1(s)}{p_0(s)} \mathrm{e}^{-s\tau},$$

the result can be formulate as follows:

Proposition 2 (Tsypskin) *If $p_0(s)$ is a stable polynomial, then the closed-loop system:*

$$H_b(s) = \frac{p_1(s)e^{-s\tau}}{p_0(s) + p_1(s)e^{-s\tau}}$$

is delay-independent asymptotically stable if and only if:

$$|p_0(j\omega)| > |p_1(j\omega)|, \quad \forall \omega \in \mathbb{R}. \tag{3.4}$$

In other words the Tsypskin's criterion simply says that a SISO system with delay in the input and stable in the open-loop, will be stable in closed-loop for any delay value if an appropriate *frequency-sweeping test* (3.4) holds. The result is quite intuitive and its proof is based on Rouche's theorem (see

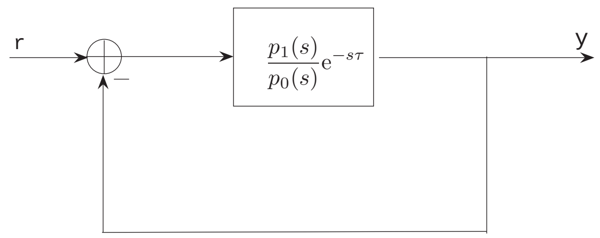


Figure 3.3: The closed-loop of $H_0(s)$.

Appendix A for Rouche's theorem). More precisely, we have to analyze the solutions of the characteristic equation

$$p_0(s) + p_1(s)\mathrm{e}^{-s\tau} = 0. \tag{3.5}$$

Since p_0 is Hurwitz, it follows that $p_0(s) \neq 0$, $\forall s \in \mathbb{C}_+$. Therefore, we can define the analytic function

$$a : \mathbb{C}_+ \mapsto \mathbb{C}, \quad a(s) = \frac{p_1(s)}{p_0(s)}\mathrm{e}^{-s\tau},$$

and using *maximum modulus principle* we get

$$\sup_{s\in\overline{\mathbb{C}}_+} \left|\frac{p_1(s)}{p_0(s)}\mathrm{e}^{-s\tau}\right| = \sup_{\omega\in\mathbb{R}} \left|\frac{p_1(j\omega)}{p_0(j\omega)}\mathrm{e}^{-j\omega\tau}\right| < 1.$$

That means $p_0(s) + p_1(s)\mathrm{e}^{-s\tau} = 0$ has no solutions in the right-half plane. For a general form of $a(s)$, the direction of crossing for a given frequency ω can be studied by computing the *sign* of $\left.\frac{\mathrm{d}s}{\mathrm{d}\tau}\right|_{s=j\omega}$.

Proposition 3 *Under the assumption of a simple root on the imaginary axis if sgn Re* $\left.\frac{ds}{d\tau}\right|_{s=j\omega} > 0$ *then the crossing is from stability to instability and the crossing is from instability to stability if the inequality is reversed.*

Proof. Straightforward computations show us that:

$$\text{sgn } Re \left.\frac{\mathrm{d}s}{\mathrm{d}\tau}\right|_{s=j\omega} = \text{sgn } Im \frac{a'(j\omega)}{a(j\omega)} \tag{3.6}$$

■

Remark 2 *The previous proposition characterizes the crossing direction when $s = j\omega$ is a simple root of $D(s,\tau) = 0$. When $s = j\omega$ has the multiplicity larger than 1 we can use a method based on perturbation theory [29].*

The following theorem characterizes the behavior of the roots for a given multiple root $s = j\omega^*$.

Theorem 8 *Let $j\omega^*$ be a repeated zero of $D(s,\tau^*)$ with multiplicity m. Then for any τ sufficiently close to τ^* but $\tau > \tau^*$, the zeros corresponding to $j\omega^*$ can be expanded in a Puiseaux series*

$$j\omega^* + m! \left| \frac{-j\omega^* a(j\omega^*)}{\sum_{k=0}^{m} \mathbf{C}_m^k \,(\tau^*)^{m-k}\, a^{(k)}(j\omega^*)} \right|^{\frac{1}{m}} e^{j\frac{(2h+1)\pi+\theta}{m}} (\tau-\tau^*)^{\frac{1}{m}} + \ldots \tag{3.7}$$

$$h = 0, 1, \ldots, m-1$$

where $\theta \in [0, 2\pi]$ is the phase angle of

$$\frac{-j\omega^* a(j\omega^*)}{\sum_{k=0}^{m} \mathbf{C}_m^k \,(\tau^*)^{m-k}\, a^{(k)}(j\omega^*)}.$$

Hence, for τ sufficiently close to τ^* but $\tau > \tau^*$, the number of critical zeros entering in the right half plane (or vice versa) can be determined by the condition

$$cos\left(\frac{(2h+1)\pi+\theta}{m}\right) > 0 \ \ (< 0), \quad h = 0, 1, \ldots, m-1.$$

We note that, for $m = 1$ the Theorem 8 leads us to the Proposition 3.

3.2 Linear systems with two discrete delays

As mentioned in the introduction, the stability of a general linear systems with delays is completely determined by the zeros of its characteristic quasipolynomial. Therefore, in order to analyze the general linear system with two delays

$$\sum_{l=0}^{2}\sum_{k=0}^{n} p_{lk}\frac{\mathrm{d}x(t-\tau_l)}{\mathrm{d}t^k}, \quad \tau_0 = 0, \quad p_{lk} \in \mathbb{R} \tag{3.8}$$

we can start by considering the following quasipolynomial:

$$D(s,\tau_1,\tau_2) = p_0(s) + p_1(s)\mathrm{e}^{-s\tau_1} + p_2(s)\mathrm{e}^{-s\tau_2}, \tag{3.9}$$

where $p_l(s) = \sum_{k=0}^{n} p_{lk}s^k$.

There are two methods to study the change of stability systems when τ_1 and τ_2 vary. Both of these algorithms lead us to the stability crossing curves of the system. One of them use a technique based on the Rekasius transformation [128] and is presented in [144] and the other, propose a geometric interpretation that allows finding the crossing set, i.e. the set of all the frequencies corresponding to all the points in the stability crossing curves. The second approach, presented in [55], is the starting point of the work presented in this part.

The assumptions made in the single discrete delay case can be formulated in the two discrete delays case as follows:

Assumption 2 **a)** $\deg(p_0) > \max\{\deg(p_1), \deg(p_2)\}$ *(retarded case)*

b) $p_0(0)+p_1(0)+p_2(0) \neq 0$ *("0" is not a solution of D for any pair (τ_1,τ_2))*

c) $p_0(s)$, $p_1(s)$ *and* $p_2(s)$ *have no common roots (in order to simplify the expression).*

We note that under the assumption

$$\textbf{d)} \quad \lim_{s\to\infty}\left(\left|\frac{p_1(s)}{p_0(s)}\right| + \left|\frac{p_1(s)}{p_0(s)}\right|\right) < 1$$

the analysis can be extended to the neutral case [55].

Remark 3 . *If the system is of retarded type then the assumption (d) is automatically satisfied since its left hand is zero. For neutral systems, let* $c_k = \lim_{s\to\infty}\frac{p_k(s)}{p_0(s)}$, $k = 1,2$. *Then, it is well known that the stability of D is possible only if the difference equation*

$$x(t) + c_1x(t-\tau_1) - c_2x(t-\tau_2) = 0 \tag{3.10}$$

is exponentially stable. Assumption (d) guarantees the stability of (3.10).

Considering that p_0 has no roots on the imaginary axis, the stability analysis of the characteristic equation $D(s) = 0$ can be reduced to the analysis of the following equation:

$$a(s, \tau_1, \tau_2) = 1 + a_1(s)\mathrm{e}^{-s\tau_1} + a_2(s)\mathrm{e}^{-s\tau_2} = 0, \quad a_k(s) = \frac{p_k(s)}{p_0(s)},\ k = 1, 2. \tag{3.11}$$

Then, the form equation (3.11), allows rewriting the condition on the crossing existence as a geometric problem in a triangle. More precisely, for $s = j\omega$, (3.11) it is considered as the sum of three vectors in the complex plane, with the magnitudes 1, $|a_1(j\omega)|$ and $|a_2(j\omega)|$. Furthermore, if we adjust the values of τ_1 and τ_2, we may arbitrarily adjust the directions of the vectors represented by the second and third terms. In other words, the equation (3.11) means that if we put the previous three vectors head to tail, they form a triangle. From geometric point of view: a triangle can be formed by three

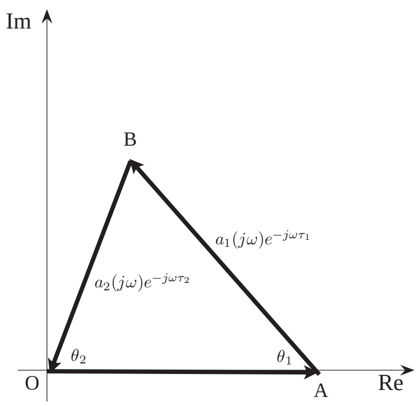


Figure 3.4: Triangle formed by 1, $a_1(j\omega)e^{-j\omega\tau_1}$ and $a_2(j\omega)e^{-j\omega\tau_2}$

line segments with arbitrary orientation if and only if the length of any one side not exceed the sum of the other two sides. Thus, we can characterize the crossing set Ω (i.e. the set of all frequencies $\omega > 0$ such that $D(j\omega, \tau_1, \tau_2 = 0)$ has at least one solution $(\tau_1, \tau_2) \in \mathbb{R}_+^2$) by the following three inequalities:

$$|a_1(j\omega)| + |a_2(j\omega)| \geq 1 \tag{3.12}$$

$$-1 \leq |a_1(j\omega)| + |a_2(j\omega)| \leq 1. \tag{3.13}$$

As (τ_1, τ_2) continuously varies in $\mathbb{R}_+^2$ the number of zeros of $D(s)$ in right half plane can change only if a zero appears on (and cross) the imaginary axis. Therefore, the *stability crossing curves* can be defined as the set $\mathcal{T}$ of all pairs (τ_1, τ_2) such that $D(s, \tau_1, \tau_2) = 0$ has at least one solution on imaginary axis. Precisely, for $\omega \in \Omega$ the set $\mathcal{T}$ is defined by $D(j\omega, \tau_1, \tau_2) = 0$ using the implicit function theorem (see the Appendix A for a precise statement). However, easy computations show us that for any $\omega \in \Omega$ the corresponding pairs $(\tau_1, \tau_2) \in \mathcal{T}$ are defined by:

$$\begin{aligned} \tau_1 &= \tau_1^{u\pm}(\omega) = \frac{\angle a_1(j\omega) + (2u-1)\pi + \theta_1}{\omega} \geq 0, \qquad (3.14) \\ & \quad u = u_0^{\pm}, u_0^{\pm}+1, u_0^{\pm}+2, \ldots \\ \tau_2 &= \tau_2^{v\pm}(\omega) = \frac{\angle a_2(j\omega) + (2u-1)\pi + \theta_2}{\omega} \geq 0, \qquad (3.15) \\ & \quad v = v_0^{\pm}, v_0^{\pm}+1, v_0^{\pm}+2, \ldots \end{aligned}$$

where $\theta_1, \theta_2 \in [0, \pi]$ are the internal angles of the triangle in Fig.3.4 and can be calculated by the law of cosine as

$$\begin{aligned} \theta_1 &= \cos^{-1}\left(\frac{1 + |a_1(j\omega)|^2 - |a_2(j\omega)|^2}{2|a_1(j\omega)|}\right) \\ \theta_2 &= \cos^{-1}\left(\frac{1 + |a_2(j\omega)|^2 - |a_1(j\omega)|^2}{2|a_2(j\omega)|}\right) \end{aligned}$$

and $u_0^{\pm}, v_0^{\pm}$ are the smallest possible integers such that the corresponding τ_1 and τ_2 are nonnegative. Using the implicit function theorem we can study the smoothness of the crossing curves and the corresponding crossing direction. More details about these properties will be given in the following section.

It is also interesting to remark that some degenerate cases of the system characterized by the equation (3.11) were pointed out in [55]:

1. $p_0(j\omega) = 0$, $|p_1(j\omega)| = |p_2(j\omega)| \neq 0$;
2. $p_1(j\omega) = 0$, $|p_0(j\omega)| = |p_2(j\omega)| \neq 0$;
3. $p_2(j\omega) = 0$, $|p_0(j\omega)| = |p_1(j\omega)| \neq 0$;
4. $D(j\omega, \tau_1, \tau_2) = 0$, $D'(j\omega, \tau_1, \tau_2) = 0$ (multiple solutions case);

5. $|a_1(j\omega)| + |a_2(j\omega)| = 1,\ \dfrac{\mathrm{d}}{\mathrm{d}\omega}\left(|a_1(j\omega)| + |a_2(j\omega)|\right) = 0$

6. $|a_1(j\omega)| - |a_2(j\omega)| = 1,\ \dfrac{\mathrm{d}}{\mathrm{d}\omega}\left(|a_1(j\omega)| - |a_2(j\omega)|\right) = 0$

A rigorous stability analysis of these degenerate cases doesn't exist in the literature. Further remarks regarding these cases will be given in the following section.

3.3 Particular, and singular cases

This section is devoted to stability analysis of some particular and degenerate cases of system included in the class of linear systems with two discrete delays studied in the previous section.

3.3.1 Smith Predictor principle and related results

An immediate particular case of system with two delays is encountered in Smith predictor controller. More precisely, we are interested in investigating the effects of delay values on the stability regions of the system whose dynamics are described by the following characteristic equation:

$$P(s) + Q(s)\mathrm{e}^{-s\tau_1} - Q(s)\mathrm{e}^{-s(\tau_1+\delta)} = 0. \tag{3.16}$$

Let $\mathcal{G}$ be the set of all pairs $(x, y) \in \mathbb{R}_+^2$ such that $x < y$. It is obvious that replacing $\tau_2 = \tau_1 + \delta$ and taking $(\tau_1, \tau_2) \in \mathcal{G}$ we can consider the following equivalent equation:

$$D(s, \tau_1, \tau_2) = P(s) + Q(s)\mathrm{e}^{-s\tau_1} - Q(s)\mathrm{e}^{-s\tau_2} = 0. \tag{3.17}$$

More explicitly, we study the change of number of zeros of (3.17) on $\mathbb{C}_+$ as the delays (τ_1, τ_2) vary on $\mathcal{G}$. In this case the assumption 1 can be expressed as follows:

The polynomials P, and Q satisfy the following conditions:

Assumption 3 *The polynomials P, and Q satisfy the following conditions:*

(i) $\deg(Q) \leq \deg(P)$;

(ii) $P(0) \neq 0$*;*

(iii) $P(s)$ *and* $Q(s)$ *do not have common zeros;*

(iv) P *and* Q *are such that:*

$$\lim_{s\to\infty}\left|\frac{Q(s)}{P(s)}\right| < \frac{1}{2}. \tag{3.18}$$

Remark 4 . *The assumption 3 above rewrites the conditions from the regular case (proposed in the previous section) to the singular case under consideration.*

Next, we present how we find the crossing points in this particular case. In order to do that, we remember our definition and we adapt them to the new conditions. Let $\mathcal{T}$ denote the set of all points of $(\tau_1, \tau_2) \in \mathcal{G}$ such that $D(s)$ has at least one zero on the imaginary axis. Any $(\tau_1, \tau_2) \in \mathcal{T}$ is known as a crossing point. The set $\mathcal{T}$, which is the collection of all crossing points, is known as the stability crossing curves. Let $\mathcal{T}_\omega$ denote the set of all $(\tau_1, \tau_2) \in \mathcal{G}$ such that $D(s)$ has at least one zero for $s = j\omega$. Let Ω the set of all ω for which there exists a pair (τ_1, τ_2) such that $D(j\omega, \tau_1, \tau_2) = 0$. We will refer to Ω as the *crossing set*. Obviously

$$\mathcal{T} = \{\mathcal{T}_\omega | \omega \in \Omega\}. \tag{3.19}$$

Next, for the clarity of the presentation we will split our discussion in two parts. First we will consider only the case which satisfy the following nondegeneracy condition,

Assumption 4

$$P(j\omega) \cdot Q(j\omega) \neq 0 \text{ for all } \omega \in \Omega \tag{3.20}$$

and then we will discuss what happens in other cases.
In the sequel we consider

$$h(s) = \frac{Q(s)}{P(s)} \tag{3.21}$$

and

$$H(s) = 1 + h(s)e^{-s\tau_1} - h(s)e^{-s\tau_2}. \tag{3.22}$$

For given τ_1 and τ_2, as long as assumption 4 is satisfied, $D(s)$ and $H(s)$ share all the zeros in a neighborhood of the imaginary axis. Therefore, we may obtain all the crossing points and direction of crossing using $H(s) = 0$ instead of $D(s) = 0$. We may also consider the three terms in $H(s)$ as three vectors in the complex plane, with the magnitudes 1, $|h(s)|$ and $|h(s)|$ respectively. So when we adjust the values of τ_1 and τ_2 in fact we adjust the directions of the vectors represented by the second and the third terms. Equation (3.22) means that if we put the first two vectors had to tail then we get the third vector. In other words they form an isosceles triangle. This allows us giving the following condition.

Proposition 4 *For some* $(\tau_1, \tau_2) \in \mathcal{G}$*,* $H(s)$ *has an imaginary zero* $s = j\omega$*,* $\omega \neq 0$ *if and only if*

$$|h(j\omega)| \geq \frac{1}{2}. \tag{3.23}$$

Proof: The relation (3.23) is obvious from the geometric point of view: a triangle can be formed by three line segments with arbitrary orientation if and only if the length of any one side does not exceed the sum of the other two sides. In the case of an isosceles triangle the condition becomes: the sum of the equal sides exceed the other side. Notice also that $\angle[h(s)e^{-j\omega\tau_l}]$, $l = 1, 2$ can assume any value by adjusting τ_l, $l = 1, 2$. ■

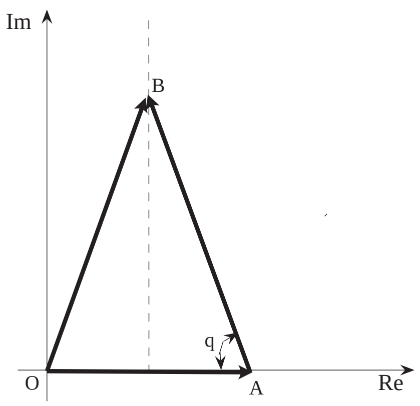


Figure 3.5: Triangle formed by 1, $h(s)e^{-s\tau_1}$ and $h(s)e^{-s\tau_2}$

Due to the symmetry and assumption 3 we only need to consider positive ω. So Ω is the set of all $\omega > 0$ which satisfy (3.23). Also, for a given $\omega \in \Omega$ we may find all the pairs (τ_1, τ_2) satisfying $H(j\omega) = 0$ as follows:

$$\tau_1 = \tau_1^{u\pm}(\omega) = \frac{\angle h(j\omega) + (2u-1)\pi \pm q}{\omega}, \tag{3.24}$$
$$u = u_0^{\pm}, u_0^{\pm}+1, u_0^{\pm}+2, ...$$
$$\tau_2 = \tau_2^{v\pm}(\omega) = \frac{\angle h(j\omega) + 2v\pi \mp q}{\omega}, \tag{3.25}$$
$$v = v_0^{\pm}, v_0^{\pm}+1, v_0^{\pm}+2, ...$$

where $q \in [0, \pi]$ is internal angle of triangle in Figure 3.5 which can be calculated by the cosine law as

$$q(j\omega) = \cos^{-1}\left(\frac{1}{2|h(\omega)|}\right) \tag{3.26}$$

and u_0^+, u_0^- are the smallest integers (may be dependent on ω) such that the corresponding values $\tau_1^{u_0^+ +}$, $\tau_1^{u_0^- -}$ are nonnegative, and v_0^+ and v_0^- are integers dependent on u such that $\tau_2^{v_0^+ +} \geq \tau_1^{u+}$, $\tau_2^{v_0^- -} > \tau_1^{u-}$ are satisfied. The position in Figure 3.5 corresponds to $(\tau_1^{u+}, \tau_2^{v+})$ and the mirror image about the real axis corresponds to $(\tau_1^{u-}, \tau_2^{v-})$. If we define $\mathcal{T}^+_{\omega,u,v}$ and $\mathcal{T}^-_{\omega,u,v}$ as the singletons $(\tau_1^{u+}(\omega), \tau_2^{v+}(\omega))$ and $(\tau_1^{u-}(\omega), \tau_2^{v-}(\omega))$ respectively, then we can characterize $\mathcal{T}_\omega$ as follows:

$$\mathcal{T}_\omega = \left(\bigcup_{u \geq u_0^+, v \geq v_0^+} \mathcal{T}^+_{\omega,u,v}\right) \bigcup \left(\bigcup_{u \geq u_0^-, v \geq v_0^-} \mathcal{T}^-_{\omega,u,v}\right) \tag{3.27}$$

Proposition 5 *The crossing set Ω consists of a finite number of intervals of finite length including the cases which may violate assumption 4.*

Proof: First, one can observe easily that the number of points in Ω violating (3.20) is finite. So, we only need to show that the set of all points satisfying (3.23) consists of a finite number of intervals of finite length. This can be proved like the first statement of proposition 3. In what follows we will denote these intervals as Ω_1, $\Omega_2, ..., \Omega_N$ and without any loss of generality we may suppose that the intervals are ordered such that for any $\omega_1 \in \Omega_{k_1}$, $\omega_2 \in \Omega_{k_2}$, $k_1 < k_2$ we have $\omega_1 < \omega_2$. ■

Remark 5 . *If (3.23) is satisfied for $\omega = 0$ and sufficiently small positive value of ω then we will take 0 the left end of Ω_1. Considering ω_1^r the right end of Ω_1, according to assumption 2 we get $\Omega_1 = (0, \omega_1^r]$, so $0 \notin \Omega$.*

We will not restrict $\angle h(j\omega)$ to be within the 2π range but make it a continuous function of ω within each Ω_k. Thus, for each fixed u, v and k, (3.24) and (3.25) are continuous curves denoted as $\mathcal{T}_{u,v}^{k+}$ respectively $\mathcal{T}_{u,v}^{k-}$. We should keep in mind that, for some u, v and k, part or entire curve $\mathcal{T}_{u,v}^{k+}$ (respectively $\mathcal{T}_{u,v}^{k-}$) may be outside of the range $\mathcal{G}$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_k may be expressed as

$$\begin{aligned} \mathcal{T}^k &= \bigcup_{u=-\infty}^{\infty} \bigcup_{v=-\infty}^{\infty} \left[\left(\mathcal{T}_{u,v}^{k+} \cup \mathcal{T}_{u,v}^{k-} \right) \cap \mathcal{G} \right] \\ &= \bigcup_{\omega \in \Omega_k} \mathcal{T}_\omega \end{aligned} \tag{3.28}$$

Obviously

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k \tag{3.29}$$

Our previous discussions allow us to say that the ends of Ω_k must be in one of the following situation:

Type 1. It satisfies the equation $|h(x)| = \dfrac{1}{2}$.

Type 2. It equals 0.

If one end of Ω_k is of type 1 then $q = 0$ and $\mathcal{T}_{u,v}^{k+}$ is connected with $\mathcal{T}_{u,v}^{k-}$ at this end. So, if both ends of Ω_k are of type 1 we get $\mathcal{T}^k$ is a series of closed curves.
Obviously just the left end of Ω_1 can be 0. In this case, as $\omega \to 0$, both τ_1 and τ_2 approach ∞. In fact $\mathcal{T}_{u,v}^{1+}$ and $\mathcal{T}_{u,v}^{1-}$ approach ∞ with asymptotes with slopes of

$$m_{u,v}^{\pm} = \frac{\tau_2^{v\pm}}{\tau_1^{u\pm}} = \frac{\angle h(0) + 2v\pi \mp q(0)}{\angle h(0) + (2u-1)\pi \pm q(0} \tag{3.30}$$

where $q(0)$ is evaluated using (3.26)

In the sequel we adopt the notations of the previous sections. So, an interval is of type 1,1 if both his end are of type 1, and Ω_1 is of type 2,1 if his left end is 0. Therefore, the crossing set Ω consists in a finite number of intervals of type 1,1, and eventually the first interval is of type 2,1. It is obvious that $\mathcal{T}^k$ consists in a series of curves belonging to one of the following categories:

A) A series of closed curves (Ω_k is of type 11)

B) A series of open ended curves with both ends approaching ∞ (Ω_k is of type 21)

We continue this section with some illustrative examples regarding the above discussion and characterization.

Example 1 **(type 1,1)** *Consider a system with*

$$h(s) = \frac{4s+1}{4(s^2+s+1)} \tag{3.31}$$

Figure 3.6 (left) plots $2|h(j\omega)|$ against ω. The crossing set can be easily computed using (3.23) and also can be identified from the figure 3.6 (left), it contains only one interval: $\Omega_1 = [0.39, 2.21]$

As an illustration of a series of *closed curves* we examine $\mathcal{T}^k$ corresponding to Ω_k of type 1,1. In this case, for a given u and v such that $\tau_2^{u\pm} > \tau_1^{u\pm} > 0$, we get $\mathcal{T}_{u,v}^{k+}$ and $\mathcal{T}_{u,v}^{k-}$ are connected on the both ends to form a closed curve. As u and v vary, we obtain a series of deformed versions of such closed curves situated above the first bisector. A suggestive image of a series of *closed curves* is given in Figure 3.6 (right) which shows $\mathcal{T}$ of the system described in (3.31).

Example 2 **(type 2,1)** *Consider a system with*

$$h(s) = \frac{s+\sqrt{2}}{2s^3+s^2+8s+1} \tag{3.32}$$

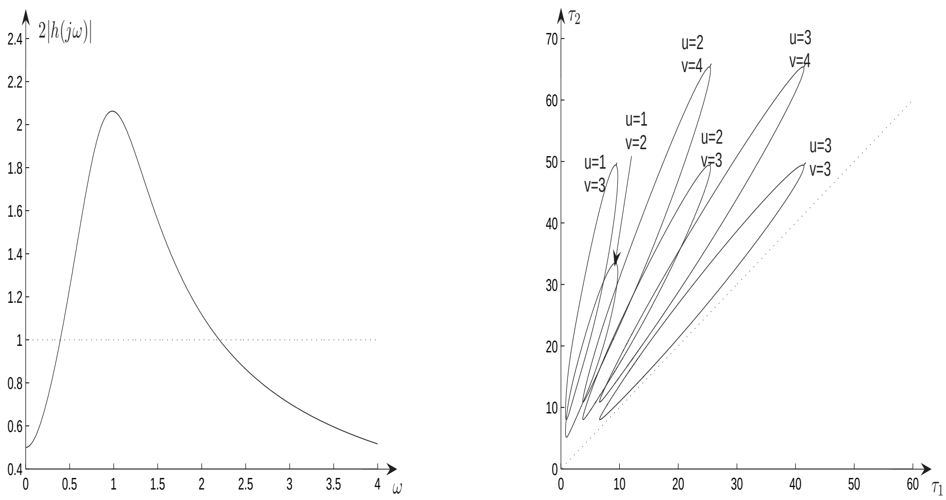


Figure 3.6: The crossing set for the system (3.31) can be identified to the left and some crossing curves of this system are plotted to the right

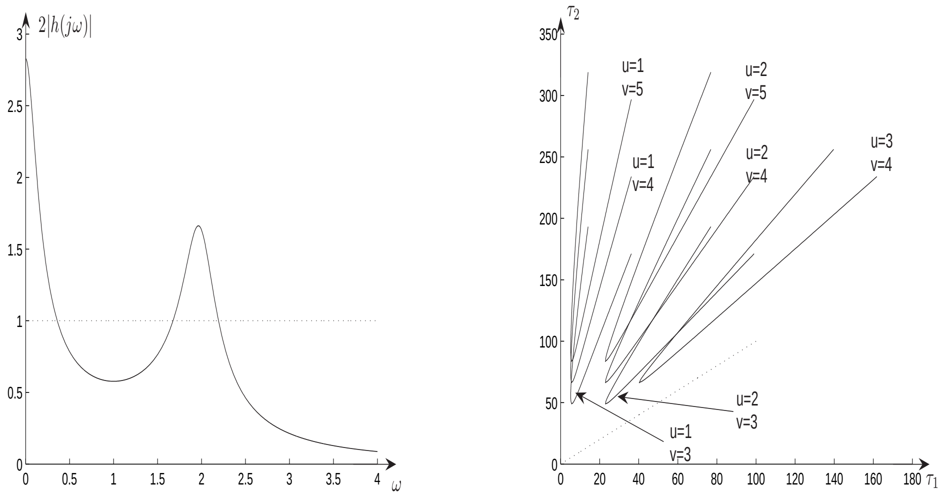


Figure 3.7: The crossing set for the system (3.32) can be identified to the left and some crossing curves of this system are plotted to the right

Figure 3.7 (left) plots $|h(j\omega)|$ against ω. The crossing set Ω can be easily identified from the Figure 3.7, it contains two intervals $\Omega_1 = (0, 0.364]$ of type 2,1 and $\Omega_2 = [1.673, 2.198]$ of type 1,1

In the following, we consider $\Omega_k = [\omega_k^l, \omega_k^r]$. Obviously, the interval Ω_1 is open to the left if its left end is 0. To illustrate the case of *open ended curves* we consider $\mathcal{T}^1$ corresponding to Ω_1 of type 2,1. In this case, $\Omega_1 = (0, \omega_1^r]$ and for a given u and v, $\mathcal{T}_{u,v}^{1+}$ and $\mathcal{T}_{u,v}^{1-}$ are connected at ω_1^r. The other end of $\mathcal{T}_{u,v}^{1-}$ extends to infinity with asymptotes with the slope $m_{u,v}^-$, and the other end of $\mathcal{T}_{u,v}^{1+}$ extends to infinity with asymptotes with the slope $m_{u,v}^+$. Again, as u and v vary, we obtain a series of deformed versions of such open ended curves situated above the first bisector. Evidently, the slope is changing for different u and v. We can see a series of *open ended curves* in Figure 3.7 (right).

Next, for a given k we will discuss the smoothness of the curves in $\mathcal{T}^k$ and thus $\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k$. In this case, for a given $\omega \in \Omega_k$ we have:

$$\begin{aligned} R_0 &= Re\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} && (3.33)\\ &= \frac{1}{\omega} Re\left\{\left[h'(j\omega) - \tau_1 h(j\omega)\right] e^{-j\omega\tau_1} \left[\tau_2 h(j\omega) - h'(j\omega)\right] e^{-j\omega\tau_2}\right\}, \\ I_0 &= Im\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} && (3.34)\\ &= \frac{1}{\omega} Im\left\{\left[h'(j\omega) - \tau_1 h(j\omega)\right] e^{-j\omega\tau_1} + \left[\tau_2 h(j\omega) - h'(j\omega)\right] e^{-j\omega\tau_2}\right\}, \end{aligned}$$

and

$$R_l = Re\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Re\left(h(j\omega)e^{-j\omega\tau_l}\right),\ l = 1,2 \quad (3.35)$$

$$I_l = Im\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Im\left(h(j\omega)e^{-j\omega\tau_l}\right),\ l = 1,2. \quad (3.36)$$

Then, since $D(s,\tau_1,\tau_2)$ is an analytic function of s, τ_1 and τ_2, the implicit function theorem indicates that the tangent of $\mathcal{T}^k$ can be expressed as

$$\begin{pmatrix} \frac{\mathrm{d}T}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \quad (3.37)$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{3.38}$$

It follows that $\mathcal{T}_k$ is smooth everywhere except possibly at the points where either (3.38) is not satisfied, or when

$$\frac{\mathrm{d}T}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{3.39}$$

From the above discussions, we can conclude the following proposition.

Proposition 6 *The curve T^k is smooth everywhere except possibly at the points corresponding to $s = j\omega$ a multiple solution of (3.17).*

Proof: If (3.39) is satisfied then $s = j\omega$ is a multiple solution of (3.17). Condition (3.38) is equivalent with

$$|h(j\omega)|^2 Im\left(\mathrm{e}^{j\omega(\tau_2 - \tau_1)}\right) \neq 0 \tag{3.40}$$

For all $\omega \in \Omega$ we have $|h(j\omega)| \geq \dfrac{1}{2}$ and

$$\omega(\tau_2 - \tau_1) = (2v - 2u + 1)\pi \mp 2q$$

Therefore (3.38) is not satisfied if and only if $2q = l\pi, l \in \mathbb{Z}$. Obviously $q \in \left(0, \dfrac{\pi}{2}\right)$ and (4.21) holds for all $\omega \in \Omega_k$ ■

The same technique and arguments used in the previous section allow us to arrive at the following proposition.

Proposition 7 *Let $\omega \in (\omega_k^l, \omega_k^r)$ and $(\tau_1, \tau_2) \in T^k$ such that $j\omega$ is a simple solution of (3.17) and $D(j\omega', \tau_1, \tau_2) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (τ_1, τ_2) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then a pair of solutions of (3.17) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Neutral system example

The first example considers a system of neutral type treated in [97], but using a different approach:

$$P(s) = (k_1k_2 + 1)s + (a + k_1), \quad Q(s) = k_1(k_2s + 1).$$

The authors of [97] assume $a > 0$ and $(a + k_1)/(k_1k_2 + 1) > 0$, which guarantees internal stability of the closed-loop system. The so-called "practical stability" criterion is given by assumption (3 iv.) which simply states $\left|\dfrac{k_1k_2}{1 + k_1k_2}\right| < \dfrac{1}{2} \Leftrightarrow -1/3 < k_1k_2 < 1$. For $a = 1$, $k_1 = 2$, $k_2 = 1/4$ we get $\Omega = (0, 2.37]$, and, in conclusion, Ω consists of one interval of type 21. Correspondingly, $\mathcal{T}$ consists of a series of open ended curves with both ends approaching infinity, conclusion which is the same to the one in [97].

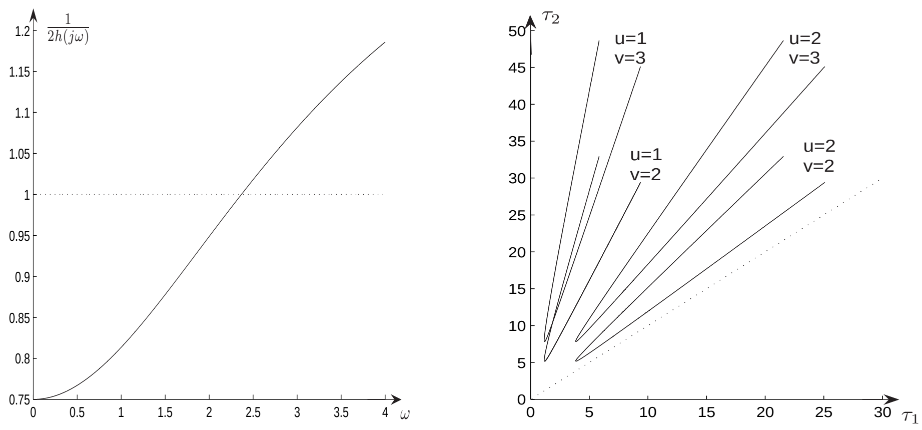


Figure 3.8: Left: The crossing set for the above system Right: Some crossing curves of this system

Smith predictor in virtual environments

The second example refers to Smith predictor in virtual environments and is presented in [30]. The corresponding system is given by:

$$P(s) = (s^2 + 2s + 2)^2, \quad Q(s) = -(2s + 2)^2.$$

Obviously, the system is of retarded type, so the practical stability conditions mentioned above are automatically satisfied. The crossing set Ω consists of one interval of type 21, $\Omega = (0, 2.9]$. This means that the crossing curves have the shape as presented in figure below. Again, we obtain the same crossing curves and stability regions.

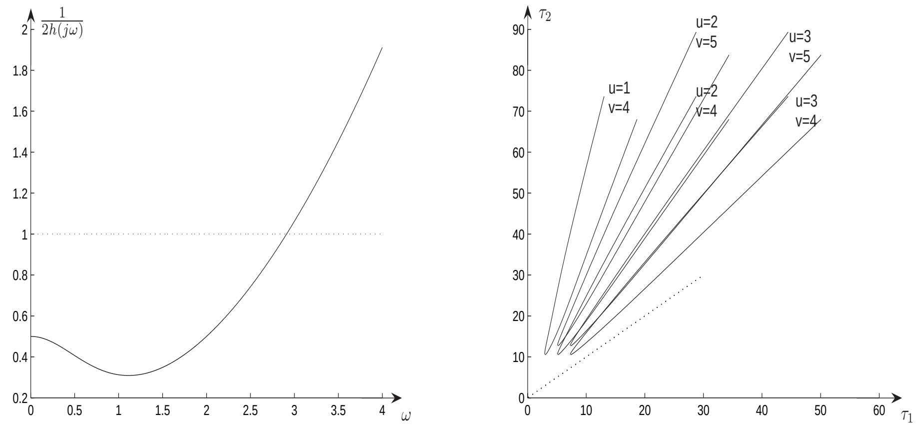


Figure 3.9: Left: The crossing set for the above system Right: Some crossing curves of this system

3.3.2 Degenerate cases of the linear systems with two discrete delays

In the sequel, we discuss some aspects regarding the stability analysis of the systems included in the degenerate cases listed in the previous section.

1. If p_0 has roots on imaginary axis the crossing curves of the system can be derived as follows. Consider

$$\Gamma = \{\omega > 0 \mid p_0(j\omega) = 0, \ |p_1(j\omega)| = |p_2(j\omega)|\}.$$

The crossing set is obtained as the union of Γ and the set Ω' of all frequencies ω satisfying the equations 3.12 and 3.13. Instead of a set of discrete points (τ_1, τ_2) corresponding to $\omega^* \in \Gamma$ we obtain a set of parallel straight lines of slope 1 given by:

$$\angle p_1(j\omega) - \omega\tau_1 + 2u\pi = \angle p_2(j\omega) - \omega\tau_2 + 2v\pi + \pi \tag{3.41}$$

for u, v integers. Obviously, in order to obtain all the crossing curves of the system we have to add this set of straight lines to the set of crossing curves corresponding to Ω'.

Example 3 *Consider*

$$p_1(s) = s + 2, \quad p_2(s) = s - 2, \quad p_0 = s^2 + 2. \tag{3.42}$$

In this case Ω *consist of two intervals:* $\Omega_1 = (0, \sqrt{2})$ *and* $(\sqrt{2}, 3.047]$. *We mention that 3.047 is an end point of type 3. The end point corresponding to* $\sqrt{2}$ *is replaced with a straight line of slope 1 as we pointed out above. More precisely, some crossing curves corresponding to* $\omega_2 = (\sqrt{2}, 3.047]$ *can be seen in figure 3.10.*

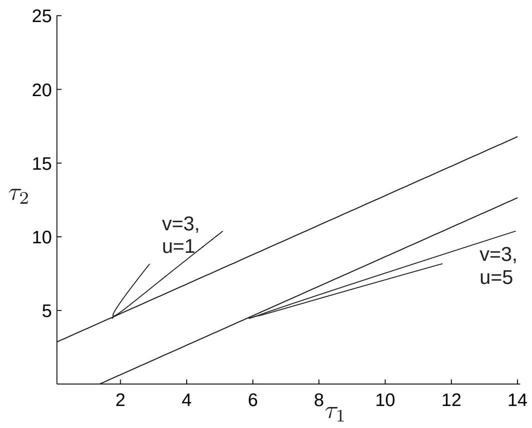


Figure 3.10: Some crossing curves of the system 3.42, corresponding to Ω_2

2. If p_1 (or p_2) has roots on the imaginary axis, the corresponding crossing points represent bifurcation points. In these points the crossing curves corresponding to Ω intersect some horizontal (vertical) lines given by:

$$\tau_2 = \frac{\angle p_2(j\omega^*) - \angle p_0(j\omega^*) + (2k+1)\pi}{\omega^*}$$

$$\left(\tau_1 = \frac{\angle p_1(j\omega^*) - \angle p_0(j\omega^*) + (2k+1)\pi}{\omega^*}, \text{ respectively} \right)$$

Proposition 8 *Let $j\omega^*$ be a solution of p_1 situated on the imaginary axis. Then, increasing τ_2, a pair of solution of (4.35) cross the imaginary axis to the left (right) at $\pm j\omega^*$ if $\frac{d}{d\omega}|a_2(j\omega^*)| < 0$ (> 0). If $j\omega^*$ is a solution of p_2 situated on the imaginary axis, the crossing direction is given by $\frac{d}{d\omega}|a_1(j\omega^*)|$.*

Example 4 *Consider the system given by:*

$$p_1(s) = 3s + 6, \quad p_2(s) = s^2 + 2, \quad p_0 = s^3 - 2s^2 + 5s + 2. \tag{3.43}$$

Then, p_2 has two roots $s = \pm j\sqrt{2}$ situated on the imaginary axis, and $|p_0(j\sqrt{2})| = |p_1(j\sqrt{2})| = 54$. The crossing set is given by $\Omega_1 = [1.067, \sqrt{2})$ of type 11 and $\Omega_2 = (\sqrt{2}, 2.236]$ of type 13. We note that the end points corresponding to $\sqrt{2}$ are replaced by a vertical line. The crossing curves of the system can be seen in the figure 3.11.

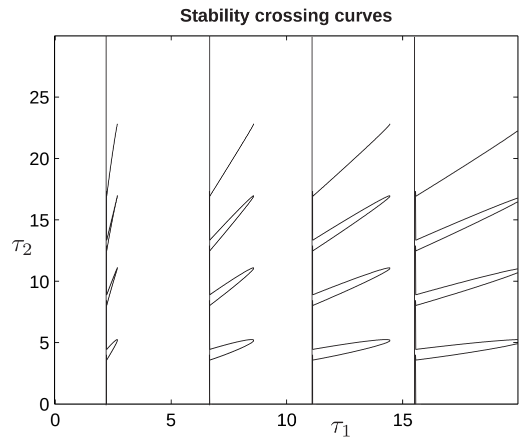


Figure 3.11: Some crossing curves of the system 3.43, corresponding to Ω_1

3. In the case $D(j\omega, \tau_1, \tau_2) = 0$, $D'(j\omega, \tau_1, \tau_2) = 0$, the crossing set is given by the equations (3.12) and (3.13). In order to derive the crossing direction corresponding to the value $j\omega^*$ we can use the following proposition.

Proposition 9 *Let $j\omega^*$ be a repeated zero of $D(s,T,\tau)$ with multiplicity m and let τ^* one of the corresponding value of the gap. For any τ sufficiently close to τ^* but $\tau > \tau^*$, the characteristic zeros corresponding to $j\omega^*$ can be expanded by the Puiseaux series*

$$\begin{aligned} j\omega^* \quad &+ \quad m! \left| \frac{\left.\frac{dD(j\omega^*, e^{-j\omega^*\tau})}{d\tau}\right|_{\tau=\tau^*}}{\left.\frac{d^m D(j\omega^*, e^{-j\omega^*\tau^*})}{ds^m}\right|_{s=j\omega^*}} \right|^{\frac{1}{m}} e^{j\frac{(2k+1)\pi+\theta}{m}} (\tau-\tau^*)^{\frac{1}{m}} \\ &+ \quad \ldots, \quad k = 0, 1, \ldots, m-1 \end{aligned}$$

Hence, for τ sufficiently close to τ^* but $\tau > \tau^*$, the number of critical zeros entering the right-half plane (or vice versa) can be determined by the condition

$$\cos\left(\frac{(2k+1)\pi+\theta}{m}\right) > 0 \ (< 0), \quad k = 0, 1, \ldots, m-1. \tag{3.44}$$

4. The last two cases listed at the end of section 3.2 concern a specific behavior of the end points. In these situations the number of equations exceeds the number of variables (we have 2 equations and one variable ω). They typically represent bifurcation points. If the system depends on one parameter in addition to the two delays, then we generically should expect these degenerate points to appear, and the geometry of $\mathcal{T}$ changes as the parameter passes through these points.

3.4 Concluding remarks

This chapter was devoted to introduce some geometric interpretations related to linear systems in presence of one or two discrete delays. More precisely, we presented a geometric method to derive the crossing set, the crossing curves and the direction of crossing for a class of linear systems with one or two discrete delays. Next, we presented some results regarding the stability analysis of particular and degenerate cases. The analysis of degenerate cases is far to be complete. Illustrative examples are included throughout the presentation of this chapter.

Chapter 4

Distributed delays and related problems

The stability of dynamical systems in the presence of time-delay is a problem of recurring interest (see, for instance, [59, 79, 54, 107], and the references therein). The presence of a time delay may induce instabilities, and complex behaviors. The problem becomes even more difficult when the delays are *distributed.* Systems with distributed delays are present in many scientific disciplines such as physiology, population dynamics, and engineering.

This chapter focuses on the characterization of the stability crossing curves of the systems

$$D(s, T, \tau) = P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau} = 0$$

considered in section §2.3. More explicitly, we compute the crossing set, which consists of all frequencies corresponding to all points on the stability crossing curves, and we give their complete classification. Furthermore, the directions in which the zeros cross the imaginary axis are explicitly derived. We present also some illustrative examples and we end the chapter with some particular cases.

4.1 Introductory remarks

As we can see in the "Motivating examples" section there are many systems in different areas that can be treated by using a model with the characteristic equation given by

$$P(s)\left(1+s\frac{\tau_1}{n+1}\right)^{n+1}+Q(s)\mathrm{e}^{-s\tau_2}=0.$$

To understand how the previous characteristic equation arises from a model containing a gamma distribution law with a gap, is sufficiently to see the way the equation (2.37) is derived. A natural question is: why the analysis is made with respect to τ_1 and τ_2 instead of a and τ_2. To answer, we point out that we introduced the parameter τ_1 in order to simplify the presentation. On the other hand, the parameter a can be easily derived when τ_1 and τ_2 are known.

In this chapter, we will study the stability of the equation (2.38) as the parameters τ_1 and τ_2 vary. Specifically, we will describe the stability crossing curves, i.e., the set of parameters such that there is at least one pair of characteristic roots on the imaginary axis. Such stability crossing curves divide the parameter space $\mathbb{R}_+^2$ into different regions. Within each such a region, the number of characteristic roots on the right hand complex plane is fixed. This naturally describes the regions of parameters where the system is stable.

It should be noted that there have been numerous works in the literature to describe the stability regions in the parameter space, known as stability charts [147, 148]. These descriptions are typically valid for one specific system excepting that the parameters are allowed to vary. In a recent paper, Gu, *et al* [55] gave a characterization of the stability crossing curves for systems with two discrete delays as the parameters. One significant difference of [55] as compared to the stability charts is the fact that such a characterization applies to any systems within the class, i.e., any system with two delays. The current chapter follows the line of [55] since our conclusion is valid for any system of the form (2.38).

4.2 Basic ideas, and assumptions

As mentioned in the previous paragraph, our main interest is to analyze the effects of the gap, and mean delay values on the stability regions of the general characteristic equation (2.38). Consider now the following system, whose dynamics are described by the following characteristic equation:

$$D(s, T, \tau) = P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau} = 0, \tag{4.1}$$

where the two parameters T and τ are nonnegative. We will try to describe the *stability crossing curves*, which is the set of (T, τ) such that the equation (4.1) has solutions on the imaginary axis. We will denote the stability crossing curves as $\mathcal{T}$. As the parameters (T, τ) cross the stability crossing curves, some characteristic roots cross the imaginary axis. Therefore, the number of roots on the right half complex plane are different on the two sides of the crossing curves, from which, we may describe the parameter regions of (T, τ) in $\mathbb{R}_+^2$ for the system to be stable.

Another related useful concept is the *crossing set* Ω, which is defined as the collection of all $\omega > 0$ such that there exists a parameter pair (T, τ) such that $D(j\omega, T, \tau) = 0$. In other words, as the parameters T and τ vary, the characteristic roots may cross the imaginary axis at $j\omega$ if and only if $\omega \in \Omega$.

We will restrict our discussions on the systems that satisfy the following assumptions.

Assumption 5 **i)** $\deg(Q) < \deg(P)$;

ii) $P(0) + Q(0) \neq 0$;

iii) *$P(s)$ and $Q(s)$ do not have common zeros;*

iv) *If $P(s) = p$, $Q(s) = q$, where p and q are constant real, then $|p| \neq |q|$;*

v) *$P(0) \neq 0$, $|P(0)| \neq |Q(0)|$;*

vi) *$P'(j\omega) \neq 0$ whenever $P(j\omega) = 0$.*

Remark 6 *Assumption (i) to (iii) above rewrites the corresponding assumption from the previous chapter. Assumptions (iv) to (vi) are made to exclude some rare singular cases in order to simplify presentation.*

Notice that we have restricted any element ω of the crossing set Ω to satisfy $\omega > 0$. Indeed, the discussion of $\omega < 0$ is redundant in view of the fact that $D(-j\omega, T, \tau)$ is the complex conjugate of $D(j\omega, T, \tau)$, and therefore, $D(-j\omega, T, \tau) = 0$ if and only if $D(j\omega, T, \tau) = 0$. Also, $\omega = 0$ is never an element of Ω in view of assumption (ii).

4.3 Geometric characterization and classification

4.3.1 Crossing set and stability crossing curves

In [58], the authors introduced the notion of *hyperbolicity* for linear delay system. More explicitly, the characteristic equation (4.1) is said to be *hyperbolic* at some point (T_0, τ_0) if no root of the characteristic equation lies on the imaginary axis for $T = T_0$, and $\tau = \tau_0$.

Using the assumption 5, and the hyperbolicity notion introduced above, we have the following simple result:

Proposition 10 *The system (4.1) is hyperbolic for all* $(T, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ *if and only if:*

$$|P(j\omega)| > |Q(j\omega)|, \qquad \forall \omega \in \mathbb{R}_+^*. \tag{4.2}$$

Proof: "$\Leftarrow$" It is clear that:

$$|\ (1 + j\omega T)^n P(j\omega))\ | \geq |\ P(j\omega)\ |,$$

for all pairs $(\omega, T) \in \mathbb{R} \times \mathbb{R}_+$. Next, using (4.2), it follows:

$$|\ (1 + j\omega T)^n P(j\omega))\ | > |\ Q(j\omega)\ |,$$

for all $(\omega, T) \in \mathbb{R} \times \mathbb{R}_+$. In conclusion, the modulus equation associated to (4.1) cannot have any solution $j\omega$, with $\omega \in \mathbf{R}^*$, for all $(T, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$, fact which is equivalent to say that the corresponding characteristic equation has no roots on the imaginary axis, excepting eventually the origin.

Let us consider the case at the origin now. Using a simple continuity argument, (4.2) leads to the inequality $|P(0)| \geq |Q(0)|$, and thus the only case that one needs to consider is $|P(0)| = |Q(0)|$, case which is excluded by Assumption V. In conclusion the hyperbolicity property follows.

"$\Rightarrow$" The argument can be simply done by contradiction, and it is omitted. The proof is completed. ■

Remark 7 *The proposition above gives a simple* frequency-sweeping *characterization of the so-called* delay-independent hyperbolicity *property. Further discussions on this topics can be found in [107]. In the case when the system free of delays is asymptotically stable, then the result above gives a very simple condition of* delay-independent *stability (see also [54], and the references therein).*

Proposition 11 *Given any $\omega > 0$, $\omega \in \Omega$ if and only if it satisfies*

$$0 < |P(j\omega)| \leq |Q(j\omega)|, \tag{4.3}$$

and all the corresponding T, τ can be calculated by

$$T = \frac{1}{\omega}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right)^{1/2}, \tag{4.4}$$

$$\begin{aligned} \tau = \tau_m &= \frac{1}{\omega}(\angle Q(j\omega) - \angle P(j\omega) - n\arctan(\omega T) \\ &\quad +\pi + m2\pi), \\ &\quad m = 0, \pm 1, \pm 2, ... \end{aligned} \tag{4.5}$$

Proof. For necessity of (4.3), let $\omega \in \Omega$, and apply modulus to (4.1), we obtain

$$|(1 + j\omega T)^n|\,|P(j\omega)| = |Q(j\omega)|\,. \tag{4.6}$$

This implies

$$|P(j\omega)| \leq |Q(j\omega)|,$$

because

$$|(1 + j\omega T)^n| \geq 1.$$

In addition, $|P(j\omega)| > 0$ is also necessary. Otherwise, $P(j\omega) = 0$, which implies $Q(j\omega) = 0$ in view of (4.6). But this violates Assumption III.

For sufficiency of (4.3), we only need to recognize that T and τ given by (4.4) and (4.5) make $s = j\omega$ a solution of (4.1).

It is also easy to see by direct solution that T and τ given by (4.4) and (4.5) are all the solutions. ■

There are only a finite number of solutions to each of the following two equations

$$P(j\omega) = 0, \tag{4.7}$$

and

$$|P(j\omega)| = |Q(j\omega)|, \tag{4.8}$$

because P and Q are both polynomials satisfying Assumptions I to IV. Therefore, Ω, which is the collection of ω satisfying (4.3), consists of a finite number of intervals. Denote these intervals as Ω_1, Ω_2, ..., Ω_N. Then

$$\Omega = \bigcup_{k=1}^{N} \Omega_k.$$

Without loss of generality, we may order these intervals from left to right, i.e., for any $\omega_1 \in \Omega_{k_1}$, $\omega_2 \in \Omega_{k_2}$, $k_1 < k_2$, we have $\omega_1 < \omega_2$.

Separation principle

In this paragraph we present an idea that will allow to suggest some simple geometric approach for the stability analysis by enabling the separation of parameters.
The characteristic equation (4.1) can be written as:

$$\Phi(s,\tau) \cdot \Psi(s,T) = -1. \tag{4.9}$$

where

$$\Phi(s,\tau) = \frac{P(s)}{Q(s)} \mathrm{e}^{s\tau} \tag{4.10}$$

and

$$\Psi(s,T) = \frac{1}{(1+sT)^n} \tag{4.11}$$

Using the fact that $|\Psi(s,T)| < 1$ the previous equation means that s can be a root of $\Phi(s,\tau) \cdot \Psi(s,T) = -1$ if and only if $|\Phi(s,\tau)| > 1$. In other words, we are able to reduce the analysis of (4.1) to the stability analysis of some interconnection scheme (4.9) of the blocs $\Phi(s,\tau)$ and $\Psi(s,T)$.

Geometric interpretation of the crossing set

In order to give a geometric interpretation that allows deriving the crossing set Ω, for $s = j\omega$, we rewrite (4.1) as

$$\left(-\frac{Q(j\omega)}{P(j\omega)}\right)^{1/n} \mathrm{e}^{-j\omega\tau/n} = 1 + j\omega T \tag{4.12}$$

The equation (4.12) can be interpreted as the intersection between a circle with radius $\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{1/n}$ and a vertical line passing trough the point $(1,0)$, in the complex plane. Therefore, the characterization of Ω can be easily derived from the following figure:

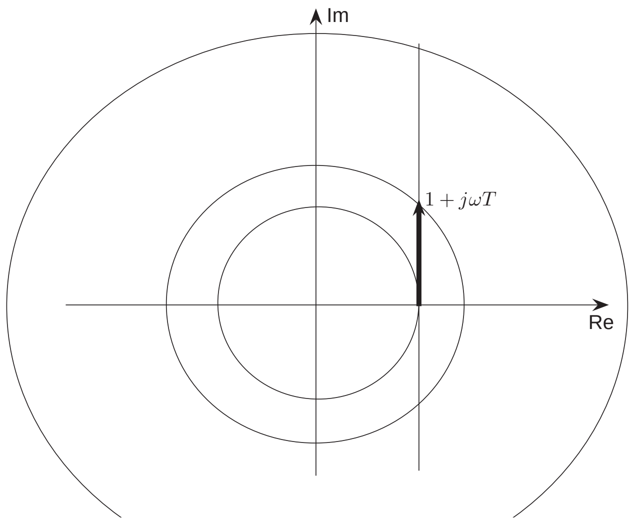


Figure 4.1: The intersection is possible only if the radius $\left|\frac{Q(j\omega)}{P(j\omega)}\right|$, is larger than 1. The extreme cases for intersection are given by $\left|\frac{Q(j\omega)}{P(j\omega)}\right| = 1$ or $\left|\frac{Q(j\omega)}{P(j\omega)}\right| \to \infty \Leftrightarrow P(j\omega) \to 0$

We will not restrict $\angle Q(j\omega)$ and $\angle P(j\omega)$ to a 2π range. Rather, we allow them to vary continuously within each interval Ω_k. Thus, for each fixed m, (4.4) and (4.5) represent a continuous curve. We denote such a curve as $\mathcal{T}_m^k$. Therefore, corresponding to a given interval Ω_k, we have an infinite number of continuous stability crossing curves $\mathcal{T}_m^k$, $m = 0, \pm 1, \pm 2, ...$ It should be noted that, for some m, part or the entire curve may be outside of the range $\mathbb{R}_+^2$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_k may be expressed as

$$\mathcal{T}^k = \bigcup_{m=-\infty}^{+\infty} \left(\mathcal{T}_m^k \bigcap \mathbb{R}_+^2 \right).$$

Obviously,

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k.$$

4.3.2 Classification of stability crossing curves

We use the same notations and terminology as in the previous chapter. Let the left and right end points of interval Ω_k be denoted as ω_k^l and ω_k^r, respectively. Due to assumptions (iv) and (v), it is not difficult to see that each end point ω_k^l or ω_k^r must belong to one, and only one, of the following three types.

Type 1. It satisfies the equation (4.8).

Type 2. It satisfies the equation (4.7).

Type 3. It equals 0.

Denote an end point as ω_0, which may be either a left end or a right end of an interval Ω_k. Then the corresponding points in $\mathcal{T}_k^m$ may be described as follows.

If ω_0 is of type 1, then $T = 0$. In other words, $\mathcal{T}_k^m$ intersects the τ-axis at $\omega = \omega_0$.

If ω_0 is of type 2, then as $\omega \to \omega_0$, $T \to \infty$ and

$$\tau \to \frac{1}{\omega_0} \left(\angle Q(j\omega_0) - \lim_{\omega \to \omega_0} \angle P(j\omega) - \frac{n\pi}{2} + \pi + m2\pi \right). \tag{4.13}$$

Obviously,

$$\lim_{\omega \to \omega_0} \angle P(j\omega) = \angle \left[\frac{d}{d\omega} P(j\omega) \right]_{\omega \to \omega_0} \tag{4.14}$$

if ω_0 is the left end point ω_k^l of Ω_k, and

$$\lim_{\omega\to\omega_0} \angle P(j\omega) = \angle\left[\frac{d}{d\omega}P(j\omega)\right]_{\omega\to\omega_0} + \pi \tag{4.15}$$

if ω_0 is the right end point ω_k^r of Ω_k. In other words, $\mathcal{T}_k^m$ approaches a horizontal line.

Obviously, only ω_1^l may be of type 3. Due to the nonsingularity assumptions, if $\omega_1^l = 0$, we must have $0 < |P(0)| < |Q(0)|$. In this case, as $\omega \to 0$, both T and τ approach ∞. In fact, (T, τ) approaches a straight line with slope

$$\tau/T \to \frac{(\angle Q(0) - \angle P(0) - n \arctan\alpha + \pi + m2\pi)}{\alpha}, \tag{4.16}$$

where

$$\alpha = \left(\left|\frac{Q(0)}{P(0)}\right|^{2/n} - 1\right)^{1/2}.$$

We say an interval Ω_k is of type lr if its left end is of type l and its right end is of type r. We may accordingly divide these intervals into the following 6 types.

Type 11. In this case, $\mathcal{T}_k^m$ starts at a point on the τ-axis, and ends at another point on the τ-axis.

Type 12. In this case, $\mathcal{T}_k^m$ starts at a point on the τ-axis, and the other end approaches ∞ along a horizontal line.

Type 21. This is the reverse of type 12; $\mathcal{T}_k^m$ starts at ∞ along a horizontal line, and ends at the τ-axis.

Type 22. In this case, both ends of $\mathcal{T}_k^m$ approaches horizontal lines.

Type 31. In this case, $\mathcal{T}_k^m$ begins at ∞ with an asymptote of slope expressed in (4.16). The other end is at the τ-axis.

Type 32. In this case, $\mathcal{T}_k^m$ again begins at ∞ with an asymptote of slope expressed in (4.16). The other end approaches ∞ along a horizontal line.

In the sequel, we present a number of examples to illustrate some cases discussed above.

Example 5 (**Type 11**) *Let $n = 1$, $P(s) = s^2 + 3s + 2$ and $Q(s) = \sqrt{10}s$. Figure 4.2 (left) plots $\dfrac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω. From the plot, it can be seen that the crossing set Ω contains only one interval $\Omega = \Omega_1 = [1, 2]$ of type 11. Correspondingly, the stability crossing curves $\mathcal{T}$ is shown in Figure 4.2 (right), which consists of a series of curves with both ends on the τ-axis.*

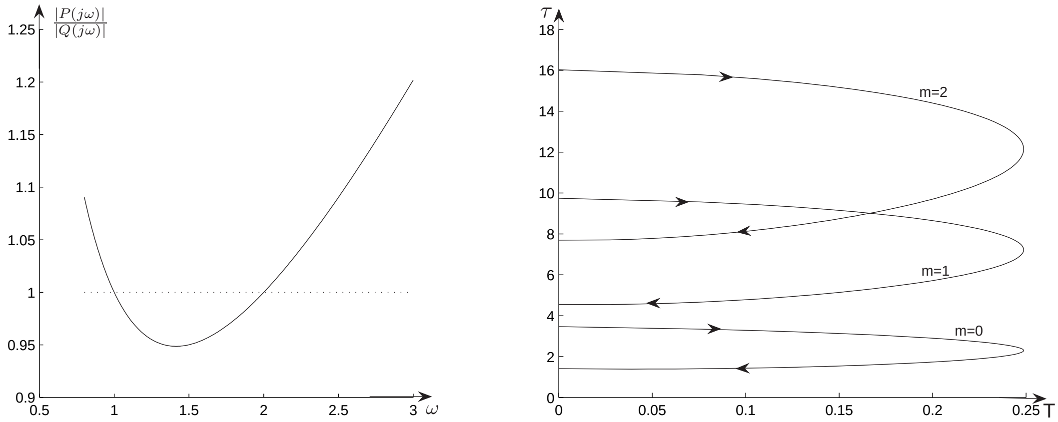


Figure 4.2: Left: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω. Right: $\mathcal{T}_1^m$, for $m = 0, 1, 2$.

Example 6 (**Type 31**) *Consider a system with $n = 1$,*

$$P(s) = s + 3 \text{ and } Q(s) = 5 \tag{4.17}$$

Figure 4.3 (left) plots $\dfrac{|P(j\omega)|}{|Q(j\omega)|}$ against ω. The crossing set Ω contains one interval $\Omega = (0, 4]$ of type 31. The corresponding crossing curves is shown in Figure 4.3 (right). It begins at infinity with an asymptote of slope calculated by (4.16) as

$$\tau/T = 1.6606 + 4.7124m,$$

and ends on the τ-axis.

Example 7 (**Type 12 and 21**) *Figure 4.4 (left) plots $\dfrac{|P(j\omega)|}{|Q(j\omega)|}$ against ω with $n = 1$,*

$$P(s) = s^2 + 2 \text{ and } Q(s) = s. \tag{4.18}$$

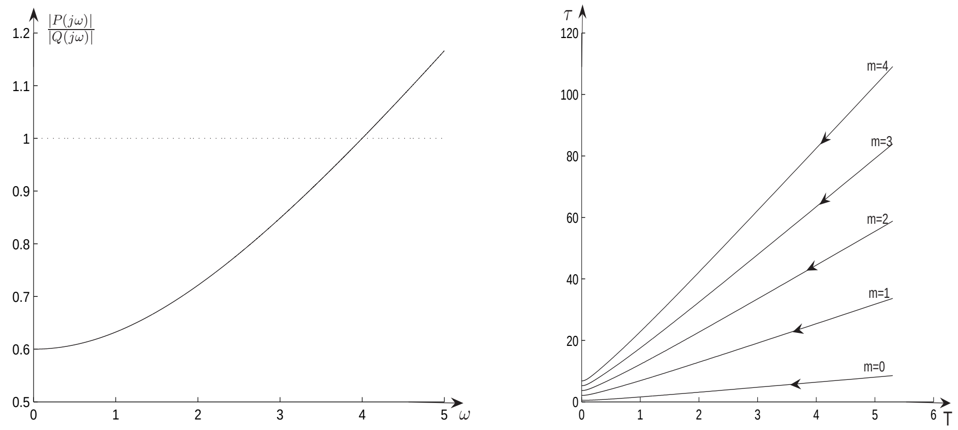


Figure 4.3: Left: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω. Right: $\mathcal{T}_1^m$ for $m = 0, 1, 2, 3, 4$.

In this case Ω contains two intervals $\Omega_1 = [1, \sqrt{2})$ of type 12 and $\Omega_2 = (\sqrt{2}, 2]$ of type 21. The stability crossing curves consist of two series of curves: the series corresponding to Ω_1 start on the τ-axis and approach infinity along the horizontal direction; the other series corresponding to Ω_2, start at infinity along the horizontal direction, and end on the τ-axis. Notice also that for the same m, the asymptotes for $k = 1$ and $k = 2$ as $\omega \to \sqrt{2}$ has a difference of $\pi/\sqrt{2} \approx 2.22$, consistent to (4.13) to (4.15)

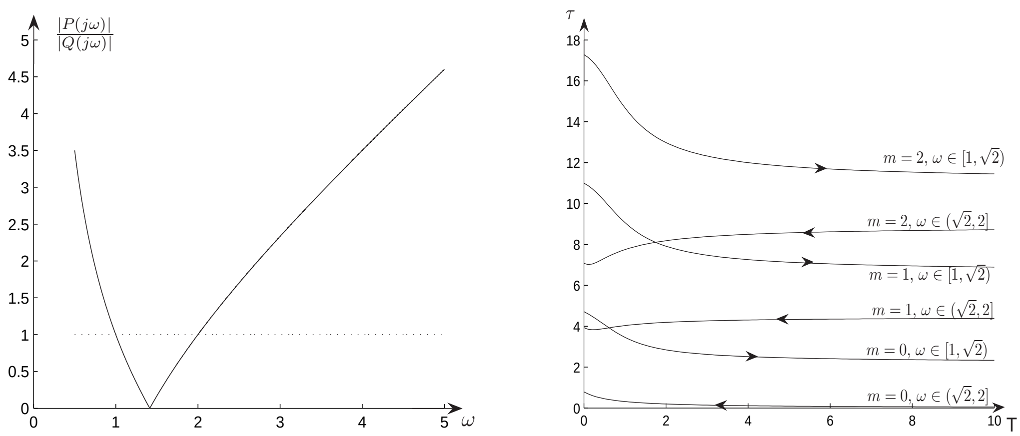


Figure 4.4: Left: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω. Right: $\mathcal{T}_k^m$ for $m = 0, 1, 2$ and $k = 1, 2$.

Example 8 (**Type 22** and **32**) *Figure 4.5 (left) plots* $\dfrac{|P(j\omega)|}{|Q(j\omega)|}$ *against* ω *with* $n = 1$,

$$P(s) = s^4 + 3s^2 + 2 \text{ and } Q(s) = s + 4. \tag{4.19}$$

In this case Ω *contains three intervals:* $\Omega_1 = (0, 1)$ *of type 32,* $\Omega_2 = (1, \sqrt{2})$ *of type 22*$\Omega_3 = (\sqrt{2}, 1.91]$ *of type 21. The stability crossing curves consist of three series of curves. Since type 21 has already been shown in Example 3 above, here we will only show the two series corresponding to* Ω_2 *and* Ω_1*. The series corresponding to* Ω_2 *of type 22 is shown in Figure 4.5 (right). We can see that both ends approach infinity along the horizontal direction. The series corresponding to* Ω_1 *of type 32 is shown in Figure 4.6 (left). The curves start from infinite in directions that can be calculated by (4.16), and end at infinity along the horizontal direction.*

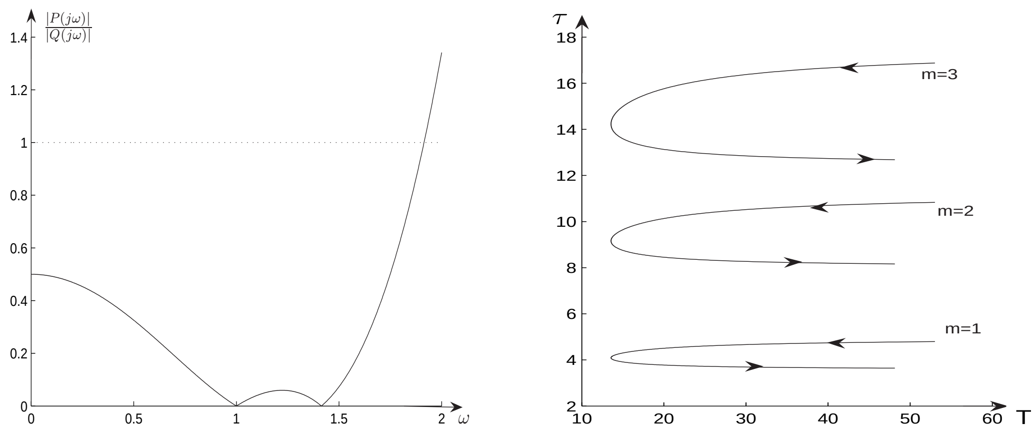


Figure 4.5: Left: $\frac{|P(j\omega)|}{|Q(j\omega)|}$ versus ω. Right: $\mathcal{T}_2^m$, $m = 1, 2, 3$.

In Figure 4.6 (right) we identify the stability regions in the corresponding delay parameters space. The method used to derive the stability regions is described in details in the next paragraphs.

4.3.3 Tangents and smoothness

For a given k we will discuss the smoothness of the curves in $\mathcal{T}_k^m$ and thus $\mathcal{T} = \bigcup_{k=1}^{N} \bigcup_{m=-\infty}^{+\infty} \left(\mathcal{T}_k^m \bigcap \mathbb{R}_+^2\right)$. In this part we use an approach based on the

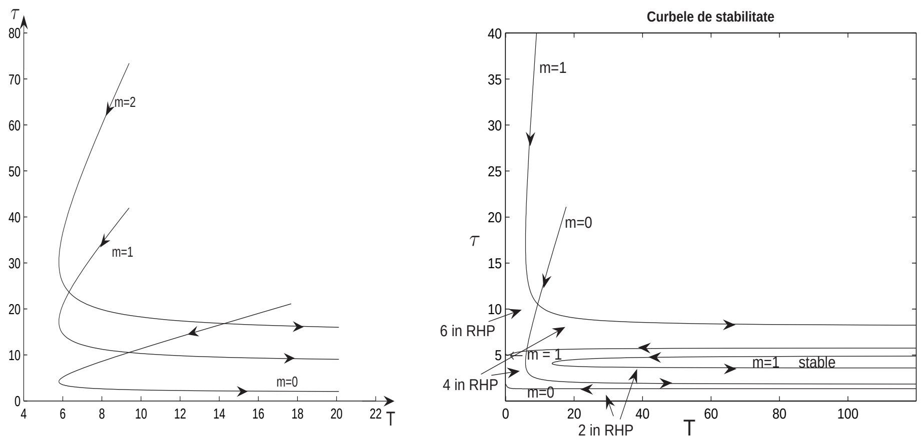


Figure 4.6: Left:$\mathcal{T}_2^m$, $m = 0, 1, 2$. Right: Stability regions for the system 4.19

implicit function theorem. For this purpose we consider T and τ as implicit functions of $s = j\omega$ defined by (4.1). For a given m and k, as $s = j\omega$ moves along the imaginary axis with $\omega \in \Omega_k$, $(T, \tau) = (T(\omega), \tau(\omega))$ moves along $\mathcal{T}_k^m$. For a given $\omega \in \Omega_k$, let

$$\begin{aligned} R_0 &= Re\left(\frac{j}{s}\frac{\partial D(s,T,\tau)}{\partial s}\right)_{s=j\omega} \\ &= \frac{1}{\omega} Re\left\{[nTP(j\omega) + (1+j\omega T)P'(j\omega)]\right. \\ &\quad \left. \cdot (1+j\omega T)^{n-1} + (Q'(j\omega) - \tau Q(j\omega))\mathrm{e}^{-j\omega\tau}\right\}, \end{aligned}$$

$$\begin{aligned} I_0 &= Im\left(\frac{j}{s}\frac{\partial D(s,T,\tau)}{\partial s}\right)_{s=j\omega} \\ &= \frac{1}{\omega} Im\left\{[nTP(j\omega) + (1+j\omega T)P'(j\omega)]\right. \\ &\quad \left. \cdot (1+j\omega T)^{n-1} + (Q'(j\omega) - \tau Q(j\omega))\mathrm{e}^{-j\omega\tau}\right\}, \end{aligned}$$

$$\begin{aligned} R_1 &= Re\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial T}\right)_{s=j\omega} \\ &= Re\left(n(1+j\omega T)^{n-1}P(j\omega)\right), \end{aligned}$$

$$
\begin{aligned}
I_1 &= Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial T}\right)_{s=j\omega} \\
&= Im\left(n(1+j\omega T)^{n-1}P(j\omega)\right),
\end{aligned}
$$

$$
R_2 = Re\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Re\left(Q(j\omega)\mathrm{e}^{-j\omega\tau}\right),
$$

$$
I_2 = Im\left(\frac{1}{s}\frac{\partial D(s,T,\tau)}{\partial \tau}\right)_{s=j\omega} = -Im\left(Q(j\omega)\mathrm{e}^{-j\omega\tau}\right).
$$

Then, since $D(s,T,\tau)$ is an analytic function of s,T and τ, the implicit function theorem indicates that the tangent of $\mathcal{T}_k^m$ can be expressed as

$$
\begin{aligned}
\begin{pmatrix} \frac{\mathrm{d}T}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} &= \begin{pmatrix} R_1 & R_2 \\ I_1 & I_2 \end{pmatrix}^{-1} \begin{pmatrix} R_0 \\ I_0 \end{pmatrix} \\
&= \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix},
\end{aligned}
\tag{4.20}
$$

provided that

$$
R_1 I_2 - R_2 I_1 \neq 0. \tag{4.21}
$$

It follows that $\mathcal{T}_k$ is smooth everywhere except possibly at the points where either

$$
R_1 I_2 - R_2 I_1 = 0, \tag{4.22}
$$

or when

$$
\frac{\mathrm{d}T}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{4.23}
$$

From the above discussions, we can conclude the following Proposition.

Proposition 12 *The curve T_k^m is smooth everywhere except possibly at the points corresponding to $s = j\omega$ in either of the following two cases:*

1) $s = j\omega$ is a multiple solution of (4.1);

2) ω is a type 1 end point of Ω_k.

Proof. From the above discussion, we only need to show that (4.22) or (4.61) can be satisfied only in the above two cases.

If (4.61) is satisfied then, in view of (4.20), $R_0 = I_0 = 0$, which implies

$$\frac{\partial D}{\partial s} = 0.$$

This, together with $D = 0$, means that $s = j\omega$ is a multiple solution of (4.1) in case 1) above.

If Condition (4.22) is satisfied, then

$$\frac{I_1}{R_1} = \frac{I_2}{R_2},$$

or

$$\angle\left(n(1 + j\omega T)^{n-1} P(j\omega)\right) = \angle\left(-Q(j\omega)\mathrm{e}^{-j\omega\tau}\right).$$

But (4.1) implies

$$\angle\left((1 + j\omega T)^n P(j\omega)\right) = \angle\left(-Q(j\omega)\mathrm{e}^{-j\omega\tau}\right).$$

Therefore, $\angle(1 + j\omega T) = 0$, which in turn means $T = 0$. From this, we can conclude $|P(j\omega)| = |Q(j\omega)|$, and ω is a type 1 end point of Ω_k. ■

4.3.4 Direction of crossing

Next we will discuss the direction in which the solutions of (4.1) cross the imaginary axis as (T, τ) deviates from the curve $\mathcal{T}_k^m$. We will call the direction of the curve that corresponds to increasing ω the *positive direction.* We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider T and τ as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_k^m$ along the positive direction is $\left(\frac{\partial T}{\partial \omega}, \frac{\partial \tau}{\partial \omega}\right)$, the normal to $\mathcal{T}_k^m$ pointing to the left hand side of positive direction is $\left(-\frac{\partial \tau}{\partial \omega}, \frac{\partial T}{\partial \omega}\right)$. Corresponding to a pair of complex conjugate solutions of (4.1) crossing the imaginary axis along the horizontal direction, (T, τ) moves along the direction $\left(\frac{\partial T}{\partial \sigma}, \frac{\partial \tau}{\partial \sigma}\right)$. So, as (T, τ) crosses the stability crossing curves from the right hand side to the left hand side, a pair of complex conjugate solutions of (4.1) cross the imaginary axis to the right half plane, if

$$\left(\frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma}\right)_{s=j\omega} > 0, \tag{4.24}$$

i.e. the region on the left of $\mathcal{T}_k^m$ gains two solutions on the right half plane. If the inequality (4.24) is reversed then the region on the left of $\mathcal{T}_k^m$ loses two right half plane solutions. Similar to (4.20) we can express

$$\left(\begin{array}{c} \frac{\mathrm{d}T}{\mathrm{d}\sigma} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} \end{array}\right)_{s=j\omega} = \frac{1}{R_1 I_2 - R_2 I_1}\left(\begin{array}{c} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{array}\right). \tag{4.25}$$

Using this we arrive at the following Proposition.

Proposition 13 *Let $\omega \in (\omega_k^l, \omega_k^r)$ and $(T,\tau) \in T_k$ such that $j\omega$ is a simple solution of (4.1) and $D(j\omega', T, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (T,τ) is not an intersection point of two curves or different sections of a single curve of T). Then a pair of solutions of (4.1) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Proof. Direct computation shows that

$$\left(\frac{\partial T}{\partial \omega}\frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega}\frac{\partial T}{\partial \sigma}\right)_{s=j\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_1 I_2 - R_2 I_1)^2}$$

Therefore (4.24) can be written as $R_2 I_1 - R_1 I_2 > 0$.

4.4 Illustrative Examples

In order to illustrate the cases presented in the previous sections, we shall consider two examples: the linearized Cushing equation with a gap, and a second-order example. In the sequel we find the crossing set, we draw the crossing curves and the stability regions for the system to be considered. To complete this section we give some robustness stability issues. More precisely, we will study what happens with a characteristic root of some second-order system when we modify the polynomial P by a "small" term.

4.4.1 Linearized Cushing equation with a gap

In this example, we apply the above method for the Cushing linearized equation $(s+a)(1+sT)^n + b\mathrm{e}^{-s\tau} = 0$. First it is easy to remark that the only interesting case is $|a| < |b|$. All the other cases do not present any stability switch since the crossing set Ω is empty.

If $|a| < |b|$ then $\Omega = (0, \sqrt{b^2 - a^2}\,]$ and the corresponding pairs (T, τ) are given by:

$$T = \frac{1}{\omega}\left[\left(\frac{b^2}{\omega^2 + a^2}\right)^{1/n} - 1\right]^{1/2}, \quad \tau_i = \frac{1}{\omega}\left[\arg\left(\frac{-b}{(a + j\omega)(1 + j\omega T)^n}\right) + 2i\pi\right]$$

According with the Proposition 12 we get:

$$\lim_{\omega \to \sqrt{b^2 - a^2}} T = 0, \quad \lim_{\omega \to 0} T = \infty, \quad \lim_{\omega \to 0} \tau_i = \infty$$

and

$$\lim_{\omega \to \sqrt{b^2 - a^2}} \tau_i = \frac{1}{\sqrt{b^2 - a^2}}\left(2i\pi + \arg\frac{-b}{a} - \arctan\frac{\sqrt{b^2 - a^2}}{a}\right).$$

Also the slopes of the corresponding asymptotes are given by

$$\lim_{\omega \to 0} \frac{\tau}{T} = \frac{-n\arctan\left[\left(\dfrac{b^2}{a^2}\right)^{1/n} - 1\right]^{1/2} + \arg\dfrac{-b}{a} + 2i\pi}{\left[\left(\dfrac{b^2}{a^2}\right)^{1/n} - 1\right]^{1/2}}$$

The following pictures plots τ_k, $k \in \{0, 1, 2, 3, 4\}$ against T in the case $n = 1$ and $n = 4$ for $a = 3$ and $b = 5$. We can easily see that $\tau_{k+1}(\omega) > \tau_k(\omega)$, $\forall k > i_0$ and $\omega \in \Omega$.

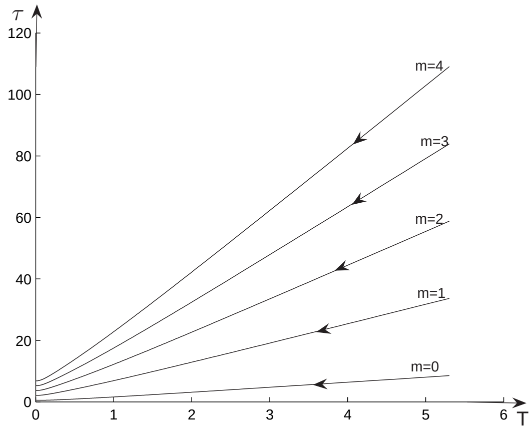


Figure 4.7: τ_k, $k \in \{0, 1, 2, 3, 4\}$ versus T when $n = 1$

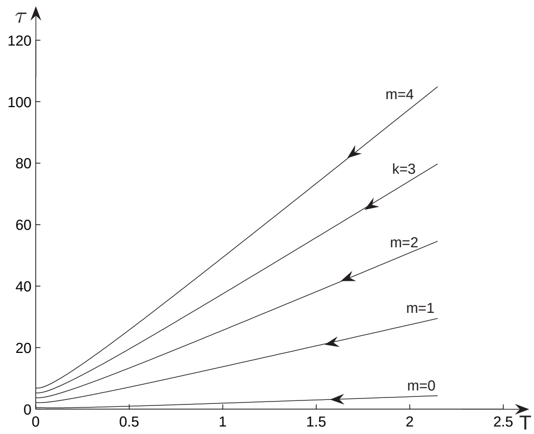


Figure 4.8: τ_k, $k \in \{0, 1, 2, 3, 4\}$ versus T when $n = 4$

Proposition 14 *For the previous system all the crossing directions of the characteristic roots are towards instability.*

Proof We can easily compute

$$\left.\frac{\mathrm{d}s}{\mathrm{d}\tau}\right|_{s=j\omega} = \frac{j\omega b \mathrm{e}^{-j\omega\tau}}{(1 + j\omega T)^n + nT(j\omega + a)(1 + j\omega T)^{n-1} - b\tau \mathrm{e}^{-j\omega\tau}} \tag{4.26}$$

and then

$$\operatorname{sgn}\ \mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1}\Bigg|_{s=j\omega} = sgn\left(\frac{\omega}{a^2+\omega^2}+\frac{n\omega T^2}{1+\omega^2T^2}\right) > 0 \tag{4.27}$$

and the proof is complete. ■

So, after the first cross the stability is lost and never regained. Therefore, for all n, we have only one stability region delimited by $\mathcal{T}^0$.

4.4.2 Second-order time-delay linear system with a gap

Consider the following second order system: $Q(s) = s$ and $P(s) = s^2 + 2$. Using (4.3) we compute the crossing set $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = [1, \sqrt{2})$ is of type 12, and $\Omega_2 = (\sqrt{2}, 2]$ is of type 21. Simple computations show that

$$T = \frac{1}{\omega}\sqrt{\left(\frac{\omega^2}{(2-\omega^2)^2}\right)^{1/n} - 1}$$

and

$$\tau_m = \frac{1}{\omega}\left(\arg\frac{j\omega}{(\omega^2-2)(1+j\omega T)^n} + 2m\pi\right).$$

According to the result of the Proposition 12 we have $\lim\limits_{\omega\to 1} T = 0$, $\lim\limits_{\omega\to 2} T = 0$, $\lim\limits_{\omega\to\sqrt{2}} T = \infty$, $\lim\limits_{\omega\to 1}\tau_m = -\dfrac{\pi}{2} + 2m\pi$, $\lim\limits_{\omega\to 2}\tau_m = \dfrac{\pi}{4} + m\pi$ and

$$\begin{aligned}
\lim_{\omega\to\sqrt{2}+0} \tau_m &= \frac{[2k-(n-1)/2]\pi}{\sqrt{2}} \\
\lim_{\omega\to\sqrt{2}-0} \tau_k &= \frac{[2k-(n+1)/2]\pi}{\sqrt{2}}
\end{aligned}$$

In the sequel, RHP will denote the right half of the complex plane.

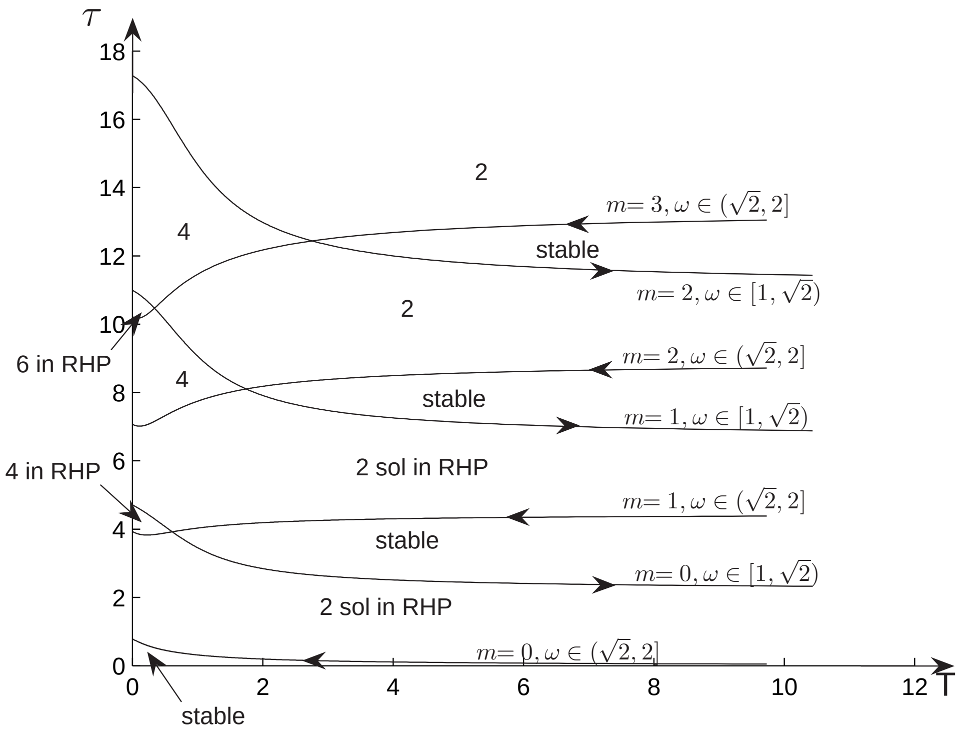


Figure 4.9: τ_m, $m \in \{0, 1, 2, 3\}$ versus T when $n = 1$

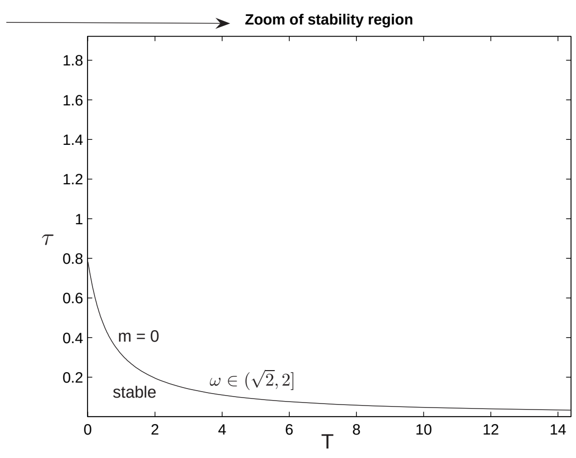


Figure 4.10: First stability region

Proposition 15 *For the system presented in this case the crossing direction of the characteristic equation is towards stability for $\omega < \sqrt{2}$ and towards instability if $\omega > \sqrt{2}$. Therefore one can obtain h stability regions, where h is the first integer with $\min\limits_{\omega<\sqrt{2}} \tau_h(\omega) \geq \max\limits_{\omega>\sqrt{2}} \tau_{h+1}(\omega)$.*

Proof Straight computations show us that

$$\left.\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1}\right|_{s=j\omega} = -\frac{2}{2-\omega^2} + \frac{nTj}{\omega(1+j\omega T)} - \frac{1}{\omega^2} + \frac{j\tau}{\omega} \tag{4.28}$$

$$\Rightarrow \operatorname{sgn} \operatorname{Re} \left.\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1}\right|_{s=j\omega} = \operatorname{sgn} \frac{-2\omega^4T^2 - \omega^2 - 2}{2-\omega^2} \tag{4.29}$$

so, $\operatorname{sgn} \operatorname{Re} \left.\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1}\right|_{s=j\omega} < 0$ if $\omega < \sqrt{2}$ and the inequality is reversed if $\omega > \sqrt{2}$. So, for a fixed T, when τ rise, each crossing of a branch that corresponds to $\omega > \sqrt{2}$ increase the number of zeros in RHP and each crossing of a branch that corresponds to $\omega < \sqrt{2}$ decrease the number of zeros in RHP. Also we can use the following result:

$$\begin{aligned} R_2I_1 - R_1I_2 &= n(1+\omega^2T^2)^{n-1}(2-\omega^2)^2 Im(1-j\omega T) \\ &= -n\omega T(1+\omega^2T^2)^{n-1}(2-\omega^2)^2 < 0 \end{aligned}$$

Using proposition 13, the previous inequality says that the left hand side of all crossing curves has two fewer solution than the right hand side of the same curve. We illustrate this result in figure 4.9.
The last part of the proposition states that we can have one switch to stability if between two instability crossings we have at least one stability crossing.
■

4.5 Robustness stability issues

4.5.1 Stability radius deviation

We consider now the problem of the computation of the maximum delay deviation without changing the number of unstable roots of our system. More precisely, given the delay T_0, τ_0, such that the system with the characteristic quasi-polynomial

$$D(s, T_0, \tau_0) = P(s)(1 + sT_0)^n + Q(s)\mathrm{e}^{-s\tau_0}$$

is stable or unstable (with a prescribed number of unstable roots), find the maximum deviation d such that for any $T \geq 0$, $\tau \geq 0$ the system with characteristic quasi-polynomial

$$D(s, T, \tau) = P(s)(1 + sT)^n + Q(s)\mathrm{e}^{-s\tau}$$

is stable/unstable as long as

$$\sqrt{(T - T_0)^2 + (\tau - \tau_0)^2} < d$$

Remark 8 *In the case of $T_0 = \tau_0 = 0$ and system free of delay stable, we have the problem of finding the minimum delay to destabilize a stable system without delay. Such a minimum delay value is called also delay margin.*

We can reformulate the problem to find the distance between (T_0, τ_0) and $\mathcal{T}$. In what follows we will show that there exists a point $(T, \tau) \in \mathcal{T}$ where the distance is reached. Therefore, we have a minimum problem not an infimum problem. According to this remark, we will use in the sequel, <u>minimum</u> instead of <u>infimum</u> and we will prove that our notation makes sense a little later. Using

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}_k,$$

where

$$\mathcal{T}_k = \bigcup_{m=-\infty}^{+\infty} \left(\mathcal{T}_k^m \cap \mathbb{R}_+^2\right),$$

we obtain

$$d = \min\{d_k^m \mid m, k \text{ integers } \},$$

where

$$d_k^m = \min\left\{\sqrt{(T - T_0)^2 + (\tau - \tau_0)^2} \mid (T, \tau) \in \left(\mathcal{T}_k^m \cap \mathbb{R}_+^2\right)\right\}$$

Proposition 16 *If $\mathcal{T}_k^m$ is smooth, the distance between (T_0, τ_0) and $\mathcal{T}_k^m \cap \mathbb{R}_+^2$ is obtained in one of the following points:*

a) *The point (T, τ) in the $\mathcal{T}_k^m \cap \mathbb{R}_+^2$ where the tangent of $\mathcal{T}_k^m$ is orthogonal to the vector $(T - T_0, \tau - \tau_0)$;*

b) *The intersection of $\mathcal{T}_k^m$ with OT axis;*

c) *The intersection of $\mathcal{T}_k^m$ with $O\tau$ axis.*

We note that there exists only a finite number of points in the previous three items, so, the use of minimum in definition of d_k^m is justified. On the other hand the points in the last two items are independent of (T_0, τ_0) and can be easily found. Therefore, in order to compute d_k^m we need to identify the points in item a). Using the definition of orthogonality and the expression of the tangent (4.20) we get that the points in item a) must satisfy the following relations:

$$f(\omega) = (T - T_0)(R_0 I_2 - R_2 I_0) + (\tau - \tau_0)(R_0 I_1 - R_1 I_0) = 0, \tag{4.30}$$

and

$$T \geq 0, \quad \tau \geq 0.$$

We can identify the solutions of (4.30) as the points where the sign of $f(\omega)$ is changing. These points are revealed when ω sweeps the interval Ω_k.

From proposition 4 and the characterization of the degenerate points we can conclude that the degenerate points belong to one of the three cases mentioned above.

Remark 9 *In order to prove that the use of the minimum in the definition of d is coherent we must show that only a finite number of d_k^m is really important in our computation. Obviously, in our definition we have an infinite number of d_k^m due to an infinite number of integers m. Therefore, we need to show that one can consider only a finite number of m.*

First we define:

$$\begin{aligned}
\angle Q_{min}^k &= \min_{\omega \in \Omega_k} \angle Q(j\omega), \quad \angle Q_{max}^k = \max_{\omega \in \Omega_k} \angle Q(j\omega) \\
\angle P_{min}^k &= \min_{\omega \in \Omega_k} \angle P(j\omega), \quad \angle P_{max}^k = \max_{\omega \in \Omega_k} \angle P(j\omega)
\end{aligned}$$

Then, a bound of $\mathcal{T}_k^m$ on $O\tau$ axis can be easily found to be:

$$\begin{aligned}
\tau_{min} &= \frac{1}{\omega_k^r}(\angle Q_{min}^k - \angle P_{max}^k - \frac{n\pi}{2} + (2m+1)\pi) \\
\tau_{max} &= \frac{1}{\omega_k^l}(\angle Q_{max}^k - \angle P_{min}^k + (2m+1)\pi)
\end{aligned}$$

Since $\tau_{max} \geq \tau \geq 0$ we can conclude that there exists some integer m_1 such that for $m < m_1$, $\mathcal{T}_k^m$ does not have to be considered in searching for d since it is outside $\mathbb{R}_+^2$. Also, if we know already the distance from (T_0, τ_0) to a point $(T, \tau) \in \mathcal{T}$ is d_0, then $d \leq d_0$. We search a value m_2 such that for $\mathcal{T}_k^{m_2}$ we obtain $\tau_{min} - \tau_0 \geq d_0$ and thus we can eliminate $\mathcal{T}_k^m$ with $m \geq m_2$.

Numerical example

To have a better perspective of our results, we present a numerical example. Our aim here is simply to show how the algorithm works.
Consider again the second-order system with $Q(s) = s$ and $P(s) = s^2+2$. For $T_0 = \tau_0 = 0$ the characteristic equation becomes $s^2 + s + 2 = 0$. Clearly, the last equation has two complex solution in the left half plane, so the system without delay is stable. We want to find the minimum delay to destabilize the system. To reach our goal we must compute the points in the three items introduced in the previous section.

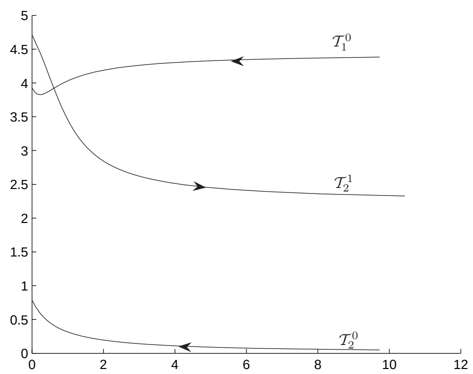


Figure 4.11: : $\mathcal{T}_k^m$, $k \in \{1, 2\}, m \in \{0, 1\}$

The points in item **b)** represents the intersection with OT axis. Therefore, we must solve the equation:

$$\tau = 0 \Leftrightarrow \frac{\pi}{2} + \arctan(\omega T) + r\pi = 0$$

where r is even for $\omega < \sqrt{2}$ and r is odd for $\omega > \sqrt{2}$. Straightforward computation show that the previous equation has no finite solution ($\tau \rightarrow$

$0 \Leftrightarrow T \to \infty$). We can conclude that the points in item **b)**, in this case, are not relevant for the minimum delay deviation problem.
The points in item **c)** represents the intersection with $O\tau$ axis.

$$\begin{aligned} T = 0 \quad &\Leftrightarrow \quad \frac{\omega^2}{(2-\omega^2)^2} - 1 = 0 \\ &\Leftrightarrow \quad (\omega^2 - 1)(\omega^2 - 4) = 0 \Leftrightarrow \omega \in \{1, 2\} \end{aligned}$$

We can compute very easy

$$\tau(1) = \frac{3\pi}{2} + 2m\pi$$

and

$$\tau(2) = \frac{\pi}{2} + 2m\pi$$

So the closest point in item **c)** is $\left(0, \frac{\pi}{2}\right)$, and the distance from (T_0, τ_0) is evidently $\frac{\pi}{2}$.
To find the point in item **a)** we have to solve the equation [4.30]. It is obvious from the figure 4.11 that we need only the point in item **a)** situated on $\mathcal{T}_2^0$. Plotting f against ω (see the figure 4.12)we can identify the point in **a)** as the point on $O\omega$ axis.

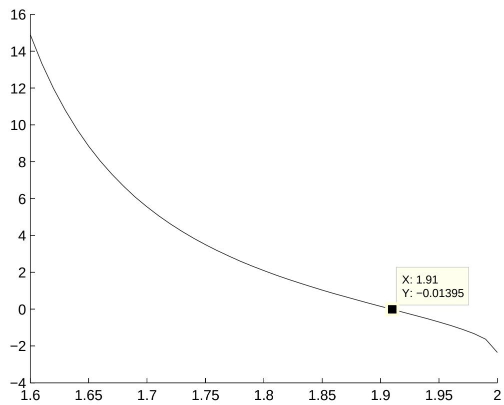


Figure 4.12: : f versus ω

More exactly, we get $\omega \simeq 1.91$, $T \simeq 0.306$ and $\tau \simeq 0.545$. So the minimal distance from $(0,0)$ to a point in **a)** is approximately 0.625. In conclusion, the minimum delay to destabilize the system without delay, is approximately 0.625.

4.5.2 Parametric robustness

In this paragraph we consider the system given by $P(s) = s^2+\alpha^2, \quad Q(s) = k$ and we are concerned by the behavior of a characteristic root when we add a small term $(2\epsilon\alpha s)$ to P. More exactly, we consider $s_{ref} = j\omega_{ref}$ an imaginary characteristic root and we want to see the condition under which it moves to the right or to the left half plain. In order to do this we will use an approach based on Taylor expansion theorem. So, in the sequel, we consider that s_{ref} is a root of

$$(s^2+\alpha^2)(1+sT)^n + k\mathrm{e}^{-s\tau} = 0. \tag{4.31}$$

For ϵ sufficiently small we can consider that a solution of

$$(s^2+2\epsilon\alpha s+\alpha^2)(1+sT)^n + k\mathrm{e}^{-s\tau} = 0. \tag{4.32}$$

is $s_{ref} + \epsilon s_{1,ref}$. This will lead us at the following relation:

$$\begin{gathered}((s_{ref}+\epsilon s_{1,ref})^2 + 2\epsilon\alpha(s_{ref}+\epsilon s_{1,ref}) + \alpha^2)\left[1+(s_{ref}+\epsilon s_{1,ref})T\right]^n \\ +k\mathrm{e}^{-(s_{ref}+\epsilon s_{1,ref})\tau} = 0\end{gathered}$$

or equivalently

$$\begin{aligned}&[s_{ref}^2+\alpha^2+\epsilon(2s_{ref}s_{1,ref}+2\alpha s_{ref})+\epsilon^2(s_{1,ref}^2+2\alpha s_{1,ref})](1+ \\ &+s_{ref}T+\epsilon s_{1,ref}T)^n = -k\mathrm{e}^{-s_{ref}}\left(1-\epsilon s_{1,ref}\tau+\tfrac{\epsilon^2 s_{1,ref}^2\tau^2}{2}+...\right).\end{aligned}$$

Identifying the free terms and coefficients of ϵ in the last equation we get respectively:

$$(s_{ref}^2+\alpha^2)(1+s_{ref}T)^n + k\mathrm{e}^{-s_{ref}\tau} = 0, \tag{4.33}$$

which is evidently true, and

$$\begin{gathered}(2s_{ref}s_{1,ref}+2\alpha s_{ref})(1+s_{ref}T)^n + ns_{1,ref}T(s_{ref}^2+\alpha^2)(1+s_{ref}T)^{n-1} \\ = k\mathrm{e}^{-s_{ref}}s_{1,ref}\tau\end{gathered}$$

Using (4.33), the last equation can pe expressed as follows

$$(2s_{ref}s_{1,ref} + 2\alpha s_{ref})(1 + s_{ref}T)^n + ns_{1,ref}T(s_{ref}^2 + \alpha^2)(1 + s_{ref}T)^{n-1}$$
$$= -s_{1,ref}\tau(s_{ref}^2 + \alpha^2)(1 + s_{ref}T)^n \Leftrightarrow$$

$$(2s_{ref}s_{1,ref} + 2\alpha s_{ref})(1 + s_{ref}T) + ns_{1,ref}T(s_{ref}^2 + \alpha^2)$$
$$= -s_{1,ref}\tau(s_{ref}^2 + \alpha^2)(1 + s_{ref}T) \Leftrightarrow$$

$$s_{1,ref} = \frac{2\alpha s_{ref}(1 + s_{ref}T)}{2s_{ref}(1 + s_{ref}T) + nT(s_{ref}^2 + \alpha^2) + \tau(s_{ref}^2 + \alpha^2)(1 + s_{ref}T)}$$

We remember here that $s_{ref} = j\omega_{ref}$. Therefore, adding the small term $2\epsilon\alpha s$ to P, the root will cross to the right half plane (instability) if $Re(s_{1,ref}) > 0$ and cross to the left half plane (stability) if the $Re(s_{1,ref}) < 0$. So, we are interested in finding the sign of $Re(s_{1,ref})$. It is clear that $\mathrm{sgn}Re(s_{1,ref}) = \mathrm{sgn}Re(s_{1,ref})^{-1}$ and straightforward computation lead to the following:

$$\begin{aligned} \mathrm{sgn}Re(s_{1,ref}) &= \mathrm{sgn}\frac{1}{\alpha} \cdot \mathrm{sgn}\left(-1 + \frac{nT^2(\alpha^2 - \omega_{ref}^2)}{2 + 2\omega_{ref}^2 T^2}\right) \\ &= \mathrm{sgn}\alpha \cdot \mathrm{sgn}(-2 - (n+2)\omega_{ref}^2 T^2 + nT^2\alpha^2) \\ &= -\mathrm{sgn}\alpha \cdot \mathrm{sgn}(2 + (n+2)\omega_{ref}^2 T^2 - nT^2\alpha^2) \qquad (4.34) \end{aligned}$$

To conclude this paragraph we summarize the previous computations and remarks in the next theorem.

Theorem 9 *The behavior of an imaginary characteristic root of (4.31) when we add a small term to P is given by the following statement.*

A) *If $T \le \sqrt{\dfrac{2}{n\alpha^2}}$ then $\mathrm{sgn}Re(s_{1,ref}) = -\mathrm{sgn}\alpha$ so the root cross towards instability if $\alpha < 0$ and the root cross towards stability if $\alpha > 0$.*

B) *If $T > \sqrt{\dfrac{2}{n\alpha^2}}$ we have two possibilities*

* *if $\omega_{ref} < \dfrac{\sqrt{nT^2\alpha^2 - 2}}{T\sqrt{n+2}}$ then $\mathrm{sgn}Re(s_{1,ref}) = \mathrm{sgn}\alpha$*

* *if* $\omega_{ref} > \dfrac{\sqrt{nT^2\alpha^2 - 2}}{T\sqrt{n+2}}$ *then* $\text{sgn}Re(s_{1,ref}) = -\text{sgn}\alpha$

The procedure above can be also used for general systems if we find a method to express in a simple form the value of $s_{1,ref}$. More exactly, assuming that $s_{ref} = j\omega_{ref}$ is a root situated on the imaginary axis, we can study its behavior when we modify the polynomial P with a small term. Since the roots of the characteristic quasipolynomial depends continuously on its coefficients, a small modification of the coefficients will imply a small modification of the root $(s_{ref} + \epsilon s_{1,ref})$. Furthermore, the tendency towards stability/instability is given by the sign of the real part of $s_{1,ref}$.

Remark 10 *In order to compute sgnRe$(s_{1,ref})$ we can use either the method based on Taylor expansion presented above, or the classical method that consist of computing* $\dfrac{ds}{d\tau}$

4.6 Further remarks, and limit cases

The main interest of this section is twofold: first, to show the coherence of our result with respect to the existent results, and second, to see that this method can be used to study the stability of other systems. To be more exact, first part of this section is devoted to prove that from our results we can obtain the characterization of a special case of a linear system with two discrete delays. The second part of the section presents how we can use our geometric approach to study the stability of the output feedback stabilization problem by using delays in the corresponding control law.

4.6.1 Two delays versus delay with a gap

The general linear scalar time-delay systems of arbitrary order with two delays was studied by Gu, Niculescu and Chen in [55]. It is obvious that when $n \to \infty$ we obtain the limit case consisting in a system with two delays. In the sequel we show that our results give, in this special case of a system with two delays, the same stability crossing curves like those obtained by Gu *et* *al* in [55].

Regarding to the system described by

$$P(s)\mathrm{e}^{-s\tau_1} + Q(s)\mathrm{e}^{-s\tau_2} = 0, \tag{4.35}$$

under the following appropriate assumptions (which represents nothing else that the assumption 2 from Chapter 3)

i) $P(0) + Q(0) \neq 0$;

ii) The polynomials $P(s)$ and $Q(s)$ do not have any common zeros;

Gu *et al* [55] derived the following results:

* The crossing set Ω consist in a finite number of points. More precisely, Ω is the set of all $\omega \neq 0$ which are the roots of the equation

$$|P(j\omega)| = |Q(j\omega)|. \tag{4.36}$$

* $\mathcal{T}_\omega$ consists of the solutions in $\mathbb{R}^2_+$ of

$$\arg P(j\omega) - \omega\tau_1 = \arg Q(j\omega) - \omega\tau_2 + (2l+1)\pi \tag{4.37}$$

where $l \in \mathbb{Z}$ such that τ_1 and τ_2 are simultaneously positive. So, $\mathcal{T}_\omega$ consists of an infinite number of straight lines of slope 1 of equal distances.

Next, we consider the time-delay system with a gap whose dynamics is given by the following characteristic equation:

$$P(s)\left(1 - s\frac{\tau_1}{n}\right)^n + Q(s)\mathrm{e}^{-s\tau_2} = 0 \tag{4.38}$$

We suppose that polynomials P and Q satisfy the assumption (i). In order to be in the same condition with the case of the system with two delays, we also presume that $|P(0)| > |Q(0)|$ and $P(j\omega) \neq 0$, $\forall \omega \in \mathbb{R}$.

Remark 11 *It is clear that:*

- *The equation (4.38) is just another form of the equation (4.1);*
- *When $n \to \infty$ in (4.38) we get the equation (4.35).*

Straightforward computations show that in our case of time-delay system with a gap, the crossing points $(\tau_1^{(n)}, \tau_2^{(n)})$ corresponding to $\omega^* \in \Omega$ are given by

$$\begin{aligned}
\tau_1^{(n)} &= \frac{n}{\omega_*}\left(\left|\frac{Q(j\omega_*)}{P(j\omega_*)}\right|^{\frac{2}{n}} - 1\right)^{\frac{1}{2}}, \qquad (4.39)\\
\tau_2^{(n)} &= \frac{1}{\omega_*}\left(\arg Q(j\omega_*) - \arg P(j\omega_*) + n \arctan\frac{\omega_*\tau_1^{(n)}}{n} + (2l+1)\pi\right),
\end{aligned}$$

where $l \in \mathbb{Z}$ such that $\tau_2^{(n)} > 0$. Like we already showed, in the first section of this chapter, the set Ω consists in a finite number of interval of finite length. Although Ω doesn't change when n changes, we will show that if $n \to \infty$ the crossing curves of (4.38) approach the crossing curves of (4.35). In this sense, we have the following results:

Theorem 10 *The following statements are true:*

1. *The slopes of the tangents to the crossing curves of the system described by (4.38) are 1;*
2. *For $\tau_1^{(n)}$ defined in (4.39) we have*
$$\lim_{n\to\infty}\max_{\omega\in\Omega_k}\tau_1^{(n)}(\omega) = \infty; \qquad (4.40)$$
3. *Each crossing curve has only one turning point with respect to $\tau_1^{(n)}$.*

Proof 1) Let ω^* be and end of one interval of Ω. The equation of the tangent in a point $\omega \in \Omega$ is given by:

$$\frac{\mathrm{d}\tau_2^{(n)}}{\mathrm{d}\tau_1^{(n)}} = \frac{I_0R_1 - R_0I_1}{R_0I_2 - I_0R_2} = -\frac{Im[(R_0+jI_0)(R_1-jI_1)]}{Im[(R_0+jI_0)(R_2-jI_2)]} \qquad (4.41)$$

Since

$$\begin{aligned}
R_1 + jI_1 &= -P(j\omega)\left(1 - j\omega\frac{\tau_1^{(n)}}{n}\right)^{n-1},\\
R_2 + jI_2 &= P(j\omega)\left(1 - j\omega\frac{\tau_1^{(n)}}{n}\right)^{n},
\end{aligned}$$

and

$$\lim_{\omega \to \omega*} \tau_1^{(n)} = 0 \tag{4.42}$$

replacing in (4.41) and passing to the limit we get

$$\lim_{\omega \to \omega*} \frac{\mathrm{d}\tau_2^{(n)}}{\mathrm{d}\tau_1^{(n)}} = 1 \tag{4.43}$$

2) Let $\tau_1^{(n)}(\omega_{M_n})$ be the maximum value of $\tau_1^{(n)}$ on Ω_k. In this case, obviously $\left|\dfrac{Q(j\omega_{M_n})}{P(j\omega_{M_n})}\right| > 1$ and $\tau_1^{(n)}(\omega_{M_n}) \neq 0$. Therefore, it is clear that

$$\begin{aligned}
\lim_{n\to\infty} \tau_1^{(n)}(\omega_{M_n}) &= \lim_{n\to\infty} \frac{n}{\omega_{M_n}} \left(\left| \frac{Q(j\omega_{M_n})}{P(j\omega_{M_n})} \right|^{\frac{2}{n}} - 1 \right)^{\frac{1}{2}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2n}}{\omega_{M_n}} \left[\frac{n}{2} \left(\left| \frac{Q(j\omega_{M_n})}{P(j\omega_{M_n})} \right|^{\frac{2}{n}} - 1 \right) \right]^{\frac{1}{2}} \\
&= \lim_{n\to\infty} \frac{\sqrt{2n}}{\omega_{M_n}} \left[\lim_{n\to\infty} \frac{n}{2} \left(\left| \frac{Q(j\omega_{M_n})}{P(j\omega_{M_n})} \right|^{\frac{2}{n}} - 1 \right) \right]^{\frac{1}{2}} \\
&= \infty \cdot \left[\ln \left| \frac{Q(j\omega_{M_n})}{P(j\omega_{M_n})} \right|^{\frac{2}{n}} \right]^{\frac{1}{2}} = \infty
\end{aligned}$$

3) A turning point with respect of $\tau_1^{(n)}$ is characterized by a vertical tangent. So, we want to prove that there is only one positive value of $\tau_1^{(n)}$ which satisfies the equation $I_0R_2 - R_0I_2 = 0$. We can rewrite the equation as follows:

$$Im\left(\frac{Q'(j\omega)}{Q(j\omega)} - \frac{P'(j\omega)}{P(j\omega)} + \frac{n\tau_1^{(n)}}{n - j\omega\tau_1^{(n)}} \right) = 0 \Leftrightarrow$$

$$Im\left(\frac{Q'(j\omega)}{Q(j\omega)} - \frac{P'(j\omega)}{P(j\omega)} \right) + \frac{n\omega\left(\tau_1^{(n)}\right)^2}{n^2 + \omega^2\left(\tau_1^{(n)}\right)^2} = 0$$

So, we get a second order equation in $\tau_1^{(n)}$ with only one positive solution. ■

Remark 12 *Summarizing the previous theorem we can conclude that each crossing curve of (4.38) tends to split in two straight lines of slope 1. Each of this straight lines passes through an end point of initial curve.*

To understand better we illustrate the remark above in the figure 4.13. To

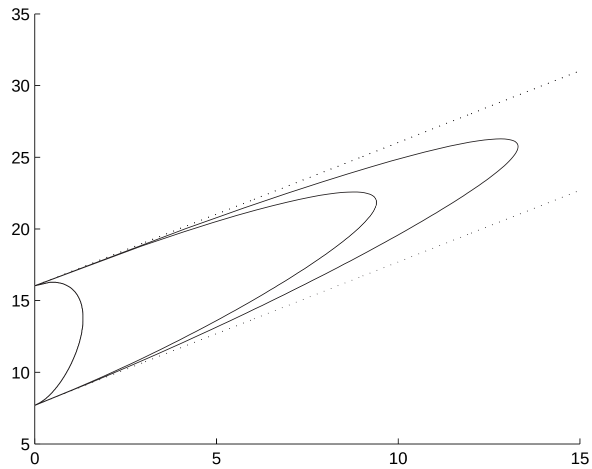


Figure 4.13: Each curve in $(T)^k$ tends to split in two straight lines passing through the ends of the curve

reach our goal we only need to prove that the straight lines presented in the previous remark are the crossing curves for (4.35). It is clear that $\tau_1^{(n)}$ varies between 0 and $\tau_1^{(n)}(\omega_{M_n})$. We already showed that $\tau_1^{(n)}(\omega_{M_n}) \underset{n\to\infty}{\longrightarrow} \infty$. So, when $n \to \infty$ we get $\tau_1 \in [0, \infty)$ and it rest to prove the next proposition.

Proposition 17 *Using the notations introduced above, the following relation holds:*

$$\tau_2^{(n)} \underset{n\to\infty}{\longrightarrow} \tau_2.$$

Proof. Passing to the limit when $n \to \infty$ in the second relation of (4.39) we get (4.37). Straightforward computations. ∎

4.6.2 Delayed output feedback

We have included the study of this type of linear system because as we will see in the sequel we get a characteristic equation, with two parameters that resembles to (4.1). The existence of a time delay at the actuating input in a feedback control systems may induce instability or poor performance for the closed-loop schemes as pointed out by [54, 107] (and the references therein). In the same time, there exists situations in which the presence of appropriate delay in the actuating input may also stabilize the unstable delay free feedback system as suggested by [112] in the oscillator case.

Consider the following class of strictly proper SISO open-loop systems:

$$\frac{P(s)}{Q(s)} = c^T(sI_n - A)^{-1}b \tag{4.44}$$

where (A, b, c^T) is a state-space representation of the open-loop system, with the controller law:

$$u(t) = -ky(t-\tau). \tag{4.45}$$

We are interested to find the pair (k, τ) such that the controller (4.45) stabilize the SISO system (4.44). Such a problem proved its interest in the case of congestion controllers in high-speed networks [70, 108]. Starting from (4.44) with the controller law given by (4.45) we find the characteristic equation of the closed-loop expressed as follows:

$$Q(s) + kP(s)\mathrm{e}^{-s\tau} = 0. \tag{4.46}$$

The aim of section is twofold: first, to understand the underlying *stability mechanism* in presence of delays; second, to give a complementary approach for the characterization of all stabilizing pair, to the algebraic one proposed by Niculescu *et al* in [112].

In this case, we are interested in finding the stability regions, in the k, τ - parameter space, of the system whose dynamics are described by the following characteristic equation:

$$D(s, k, \tau) = Q(s) + kP(s)\mathrm{e}^{-s\tau} = 0 \tag{4.47}$$

with polynomials P and Q satisfying the following assumptions:

Assumption 6 i) $\deg(Q) \geq \deg(P)$*;*

ii) *$P(s)$ and $Q(s)$ do not have common zeros;*

iii) *$P'(j\omega) \neq 0$ whenever $P(j\omega) = 0$.*

The first assumption is needed in order to ensure that for a fixed value of k, the real part of any characteristic roots is bounded to the right. This assumption implies that k will have some limited domain $|k| \leq k_{max}$ since larger gain values will induce instability for infinitesimal delay values. The second assumption is already discussed in the previous sections.

Our description will mainly follow the algorithm presented in [55, 99] and based on some simple geometric interpretation of the characteristic equation in the parameter space. First at all we present some necessary considerations proposed by Niculescu *et al* in [112] using a continuity principle argument for the dependence of the roots of the characteristic equation with respect to some real parameter.Introduce now the following Hurwitz matrix associated to some polynomial $A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$:

$$H(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \dots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \dots & a_{2n_a-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{4.48}$$

where the coefficients a_l are assumed to be zero ($a_l = 0$), for all $l > n_a$. We consider $H(Q)$, $H(P) \in \mathbb{R}^{n \times n}$ where $\deg(Q) = n > m = \deg(P)$, the coefficients $q_l = 0$ for all $l > n$ and the coefficients $p_l = 0$ for all $l > m$. We consider also, the set of root of $D(s, k, 0)$ located in the closed right half plane, denoted by $\mathcal{U}$. With these notations, the following result is a simplified form of Lemma 2 in [112] which is a slight modification, and generalization of Theorem 2.1 by Chen [28]:

Lemma 2 *Let $\lambda_1 < \lambda_2 < ... < \lambda_h$, with $h \leq n$ be the real eigenvalues of the matrix pencil $\Sigma(\lambda) = \lambda H(P) + H(Q)$ inside the interval $(-k_{max}, k_{max})$.*

Then, $\mathrm{card}(\mathcal{U})$ *remains constant as k varies within each interval $(\lambda_i, \lambda_{i+1})$. The same holds for the intervals $(-k_{max}, \lambda_1)$ and (λ_h, k_{max}).*

We also note that the lemma above gives a simple method to compute $\mathrm{card}(\mathcal{U})$ by computing the generalized eigenvalues of the matrix pencil $\Sigma(\lambda)$.

Let $\mathcal{T}$ denote the set of all $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ such that (4.47) has at least one zero on imaginary axis.

Remark 13 *If ω is a real number and $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ then*

$$Q(-j\omega) + kP(-j\omega)e^{j\omega\tau} = \overline{Q(j\omega) + kP(j\omega)e^{-j\omega\tau}}$$

Therefore, in the remaining part of this section, we only need to consider positive ω. Since the gain k is finite, we can assume k within some finite interval $[\alpha, \beta]$, which contains all generalized eigenvalues λ_i of the matrix pencil $\Sigma(\lambda)$.

Proposition 18 *Given any $\omega > 0$, $\omega \in \Omega$ if and only if it satisfies:*

$$\mid P(j\omega) \mid \ > \ 0, \tag{4.49}$$

and all the corresponding pairs (k, τ) can be calculated by:

$$\begin{aligned} k(\omega) &= \pm \left| \frac{Q(j\omega)}{P(j\omega)} \right|; & (4.50) \\ \tau_m(\omega) &= \frac{1}{\omega} \left(\angle P(j\omega) - \angle Q(j\omega) + (2m + \epsilon_k + 1)\pi \right) & (4.51) \\ & m = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

where $\epsilon_k = \begin{cases} 0 \text{ if } k \geq 0 \\ -1 \text{ if } k < 0 \end{cases}$.

Proof. For the necessity of (4.49), let ω be a crossing frequency in Ω and apply modulus to the closed-loop equation:

$$Q(j\omega) + kP(j\omega)\mathrm{e}^{-j\omega\tau} = 0. \tag{4.52}$$

This implies that

$$|Q(j\omega)| = |k||P(j\omega)| \tag{4.53}$$

is satisfied. It becomes clear that $P(j\omega) > 0$ is necessary. Otherwise, $P(j\omega) = 0$, which implies $Q(j\omega) = 0$ for all the gains k, which contradicts the technical assumption 6 (P and Q do not have common zeros).

For the sufficiency of (4.49), we only need to recognize that the pair (k, τ) given by (4.50)-(4.51) makes $s = j\omega$ a solution of the corresponding characteristic equation of the closed-loop system. ■

Remark 14 (small gain) *Assume now that the open-loop SISO system does not include oscillatory modes, that is $Q(s)$ has no roots on the imaginary axis. Some simple algebraic manipulations prove that for all the gains k satisfying the inequality:*

$$| k | \quad < \quad \frac{1}{\sup_{\omega>0}\left\{\frac{|P(j\omega)|}{|Q(j\omega)|}\right\}}, \tag{4.54}$$

the closed-loop system 4.47) is hyperbolic *(see [58, 107] for further details on such a notion), that is there does not exist any crossing roots on the imaginary axis for all positive delays τ.*

In other words, the closed-loop system is stable (unstable) for all delays value if it is stable (unstable) in the free delays case. Furthermore, the frequency-sweeping test above (4.54) gives a simple way to exclude some k-interval from the beginning, since in such a case crossing roots can not exist.

However, it is important to point out that such a frequency-sweeping test (4.54) losses all its interest if if the polynomial $Q(s)$ has roots on the imaginary axis (the corresponding upper bound becomes 0*), that is in the case of oscillatory systems (such a case will be considered in Section 4: Illustrative examples).*

In these circumstances, we can assume k within some finite interval $[\alpha, \beta] \subset (-k_{max}, k_{max})$, which contains all generalized eigenvalues λ_i of the matrix pencil $\Sigma(\lambda)$, but excluding the k-interval given by (4.54) if the SISO system does not include oscillatory modes. Next, Lemma 1 ensures us that the choice of the interval $[\alpha, \beta]$ includes all the remaining possibilities for the system free of delay. In such a case, define $\ell_l := min\{| \alpha |, | \beta |\} \geq 0$, and $\ell_u := max\{| \alpha |, | \beta |\} < \infty$. Then, there are only a *finite number of solutions* to each of the following three equations:

$$| Q(j\omega) | \quad = \quad \ell_l \mid P(j\omega) \mid, \tag{4.55}$$

$$| Q(j\omega) | \quad = \quad \ell_u \mid P(j\omega) \mid, \tag{4.56}$$

and

$$P(j\omega) \quad = \quad 0, \tag{4.57}$$

because P, and Q are polynomials satisfying the Assumptions 6. Therefore, the *crossing set* Ω will be defined by all the frequencies $\omega > 0$ satisfying *simultaneously* the inequalities:

$$\begin{cases} \ell_l \mid P(j\omega) \mid \leq \mid Q(j\omega) \mid \leq \ell_u \mid P(j\omega) \mid, \\ \mid P(j\omega) \mid > 0. \end{cases} \tag{4.58}$$

In conclusion, due to the form of (4.58), and from the Assumptions 6, the corresponding crossing set Ω *consists of a finite number of intervals.* Denote these intervals as: Ω_1, Ω_2, ..., Ω_N. Then:

$$\Omega = \bigcup_{k=1}^{N} \Omega_k.$$

Remark 15 (strictly proper SISO case) *In the case of a strictly proper SISO system $k_{max} = \infty$ (that is no any constraints on the gain k), we note that for $k \in (\beta, \infty)$ (or $k \in (-\infty, \alpha)$) we can still express Ω as a finite number of intervals, but one of them has an infinite end.*

Remark 16 (Invariance root at the origin) *If $\frac{Q(0)}{P(0)} \in [\alpha, \beta]$, then 0 will be a characteristic root for all τ if $k = \frac{Q(0)}{P(0)}$, since $e^{-s\tau} = 1$ for $s = 0$, independently of the delay value τ. The last remark allows us to eliminate the value $\frac{Q(0)}{P(0)}$ from Ω if it is the case.*

Remark 17 (Crossing characterization) *The frequency-sweeping test (4.58) above gives all the frequency intervals for which crossing roots exist for the corresponding chosen gain interval, but it does not give any information on the* crossing direction.

In other words, such a test does not make any distinction between switches *and* reversals. *Such a problem will be considered in the next paragraphs (Direction of crossing).*

In the sequel, we consider $\Omega_i = [\omega_i^l, \omega_i^r]$, for all $i = 1, 2, \ldots, N$. Without any loss of generality, we can order these intervals from left to right, i.e., for any $\omega_1 \in \Omega_{i_1}$, $\omega_2 \in \Omega_{i_2}$, $i_1 < i_2$, we have $\omega_1 < \omega_2$.

We note that ω_1^l can be 0 and in this case Ω_1 is open to the left.

It is clear that $k(\omega_i^l)$, $k(\omega_i^r) \in \{\alpha, \beta\}$ for all $i = 1...N$ if $\omega_1^l \neq 0$. We will not restrict $\angle Q(j\omega)$ and $\angle P(j\omega)$ to a 2π range. Rather, we allow them to vary continuously within each interval Ω_i. Thus, for each fixed m, (4.50) and (4.51) give us two continuous almost everywhere curves. We can lose the continuity of the curve in the points which correspond to the case $Q(j\omega) = 0$. For example, if $Q(j\omega^*)$ is a real polynomial and its sign is changing at ω^*, then $\angle(Q(j\omega))$ is not continuous in ω^*.

It should be noted that condition (4.50) and k finite, imply $P(j\omega) \neq 0, \quad \forall \omega \in \Omega$. We denote the curves defined by (4.50) and (4.51) with $\mathcal{T}_i^{m\pm}$. Therefore, corresponding to a given interval Ω_i, we have an infinite number of continuous stability crossing curves $\mathcal{T}_i^{m\pm}$, $m = 0, \pm 1, \pm 2, ...$

Finally, it should be noted that, for some m, part or the entire curve may be outside of the range $\mathbb{R} \times \mathbb{R}_+$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_i may be expressed as

$$\mathcal{T}_i = \bigcup_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \right]$$

Obviously,

$$\mathcal{T} = \bigcup_{i=1}^{N} \mathcal{T}_i.$$

Also it is easy to see that, for each Ω_i, we define two curves, one to the right of the $O\tau$ axis and the other to the left. According to the fixed limits α, β of the interval where k varies we can eliminate some of these curves. According to the fixed limits α, β of the interval where k varies we can eliminate some of these curves. The end points of these curves are classified as follows:

Type 1. It satisfies the equation $k(\omega) = \alpha$.

Type 2. It satisfies the equation $k(\omega) = \beta$.

Type 3. It equals 0.

Obviously, only ω_1^l can be of type 3. We note that all the crossing curves are situated in the vertical strip $\mathcal{D}$ between the lines $k = \alpha$ and $k = \beta$. Now, let ω_* be an end point of the interval Ω_i. We have already said that each $\mathcal{T}_i^{m+}$ is an continuous almost everywhere curve, so, $(k(\omega_*), \tau_m(\omega_*))$ is an end point of $\mathcal{T}_i^{m\pm}$, and it can be characterized as follows:

- If ω_* is of type 1, then $k(\omega_*) = \alpha$ and $\tau(\omega_*)$ are finite. More precisely, $\mathcal{T}_i^{m+}$ intersects the vertical line $k = \alpha$, which is the left bound of the strip $\mathcal{D}$.

- If ω_* is of type 2 then $k(\omega_*) = \alpha$ and $\tau(\omega_*)$ are finite. Or, we may say that $\mathcal{T}_i^{m+}$ intersects the vertical line $k(\omega) = \beta$, which is the right bound of the strip $\mathcal{D}$.

- If ω_* is of type 2 then τ approaches ∞ and k approaches $\dfrac{Q(0)}{P(0)}$. In other words, $\mathcal{T}_i^{m+}$ has a vertical asymptote given by $k = \dfrac{Q(0)}{P(0)}$.

Remark 18 *The previous description holds also for* $\mathcal{T}_i^{m-}$*.*

As defined in the previous Chapter we say that an interval Ω_k is of type lr if its left end is of type l and its right end is of type r. We may divide accordingly these intervals into 6 types.

For a given i, we will discuss the smoothness of the curves in $\mathcal{T}_i^{m\pm}$ and thus $\mathcal{T} = \bigcup_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+)\right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+)\right)\right]$. For this purpose, we consider k and τ as implicit functions of $s = j\omega$ defined by (4.47). We define again the corresponding quantities R_0, I_0, R_1, I_1, R_2 and I_2. Then, since $H(s, k, \tau)$ is an analytic function of s, k and τ, using the implicit function theorem we obtain that the tangent of $\mathcal{T}_i^{m\pm}$ can be expressed as

$$\begin{pmatrix} \frac{\mathrm{d}k}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \tag{4.59}$$

provided that

$$R_1 I_2 - R_2 I_1 \neq 0. \tag{4.60}$$

It follows that $\mathcal{T}_i^{m\pm}$ is smooth everywhere except possibly at the points where either (4.60) is not satisfied, or when

$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{4.61}$$

From the above discussions, we can conclude the following Proposition.

Proposition 19 *The curve $T_i^{m\pm}$ is smooth everywhere except possibly at the point corresponding to $s = j\omega$ in any one of the following cases:*

1) $s = j\omega$ is a multiple solution of (4.47), and

2) ω is a solution of $Q(j\omega) = 0 \Leftrightarrow k = 0$.

Proof. If (4.61) is satisfied then $s = j\omega$ is a multiple solution of (4.47). On the other hand, $R_1 I_2 - R_2 I_1 = -k\omega|P(j\omega)|^2$. If $P(j\omega) = 0$ we get $Q(j\omega) = 0$ so, assumption 2 is not satisfied. Therefore, (4.60) is violated if and only if $k = 0$. Obviously, $k = 0$ implies that $Q(j\omega) = 0$. So, we can conclude that (4.60) is violated if and only if ω is a solution of $Q(j\omega) = 0$.

The direction of crossing is established in a same manner as in the previous section. More precisely we arrive to the following proposition.

Proposition 20 *Let $\omega \in (\omega_i^l, \omega_i^r)$ and $(k, \tau) \in \mathcal{T}_i$ such that $j\omega$ is a simple solution of (4.47) and $H(j\omega', k, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (k, τ) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then a pair of solutions of (4.47) will cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Numerical examples

This paragraph is devoted to show the coherence of our study. In this sense, we consider some classical examples in the literature and we compare our conclusions with the existing ones. For the clarity of our presentation, we use some figures for illustration.

Example 9 (Scalar delay system revisited) *Consider the system given by the transfer function*

$$H_{y,u}(s) = \frac{1}{s + a} \tag{4.62}$$

subject to the control law $u(t) = -ky(t - \tau)$ The corresponding characteristic equation can be written as:

$$s + a + ke^{-s\tau} = 0. \tag{4.63}$$

For $a > 0$ it is obvious that either for $k = 0$ or $\tau = 0$, $a + k > 0$, we obtain a stable equation. Using proposition 20 we conclude that all the crossings

are towards instability. Boese [24] considered $k > 0$ and he proved that for $k \leq a$ we get a delay independent stable system and for $k > a$ we have only one stability interval $[0, \tau_0)$, where τ_0 is a decreasing function of k.
Using our method for $a = 3$ we can draw the crossing curves and establish the stability region as we can see in figure 4.14. In this case, we have $\mathrm{card}\,(\mathcal{U}) =$

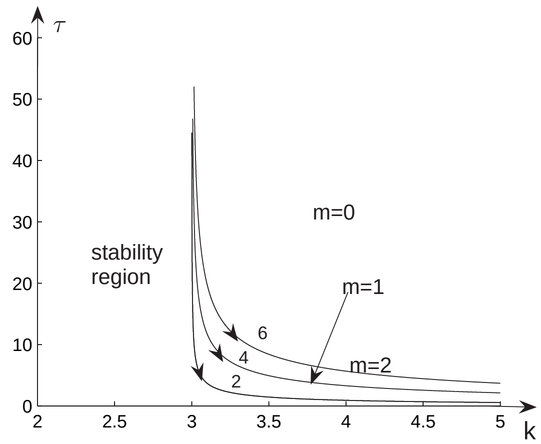


Figure 4.14: $\mathcal{T}_1^{m+}$, $m \in \{0, 1, 2, \}$ for the system (4.63)

$\begin{cases} 0 \text{ if } k > -3 \\ 1 \text{ if } k \leq -3 \end{cases}$ *and for $k \in [-5, 5]$ the crossing set Ω consists in one interval $(0, 4]$ of type 31. Therefore, we obtain one stability interval for $k > 3$, and this interval is $[0, \tau_0)$, where τ_0 is given by:*

$$\tau_0 = \frac{1}{\omega}\left(\pi - \arctan\frac{\omega}{3}\right) = \frac{1}{\sqrt{k^2 - 9}}\left(\pi - \arctan\frac{\sqrt{k^2 - 9}}{3}\right),$$

which is nothing else that the formula given by Boese for the upper bound of the stability interval.
For $a = -3$ (open-loop system unstable) and $k \in [-5, 5]$, once again we derive $\Omega = (0, 4]$ and

$$\mathrm{card}\,(\mathcal{U}) = \begin{cases} 0 \text{ if } k > 3 \\ 1 \text{ if } k \leq 3 \end{cases}.$$

Since all the crossing direction are towards instability we need to plot only the first stability crossing curve. As we can see in figure 4.15 the system quickly become unstable as τ increasing. ■

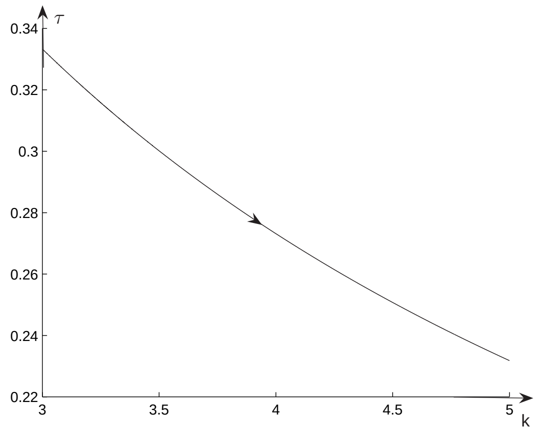


Figure 4.15: $\mathcal{T}_1^{0+}$ for the system (4.63) with $a = -3$

Example 10 A linear oscillator model subject to delayed output. *Consider the transfer function*

$$H_{y,u}(s) = \frac{1}{s^2 + 2} \tag{4.64}$$

subject to the control law $u(t) = -ky(t-\tau)$. *The corresponding characteristic equation is given by:*

$$s^2 + 2 + ke^{-s\tau} = 0. \tag{4.65}$$

For $k \in (-2, 0)$ *the results regarding stability intervals of the systems can be found in [109, 112] and they say that for* $\tau \in \left(0, \dfrac{\pi}{\sqrt{2 + |k|}}\right)$ *the system is stable (see also [1] for a different stability argument). The number of stabilizing delay interval is a decreasing function of* $|k|$.

Our computation in this case point out that for $k \in (-2, 0)$ *the crossing set* Ω *consist in one interval* $(0, 2]$ *of type 32. We note that according to proposition 2 all the crossing curves are not continuous in the points which correspond to* $k = 0$.

Obviously, $\operatorname{card}(\mathcal{U}) = \begin{cases} 1 \text{ if } k < -2 \\ 2 \text{ if } k > -2 \end{cases}$. *Proposition 20 says that for* $k < 0$ *the region on the right hand side of each crossing curve has two fewer unstable roots.*

Remark 19 *If $\omega \in (0, \sqrt{2})$ then $\tau_0(\omega) = 0$ as we can deduce from the computation below:*

$$\tau_0 = \frac{1}{\omega}(\angle(1) - \angle(2-\omega^2) + (\epsilon_k + 1)\pi) = 0, \quad \forall \omega \in (0, \sqrt{2}) \tag{4.66}$$

More precisely, looking at the next figure it is clear that we recover the result proposed in [109, 112]. ■

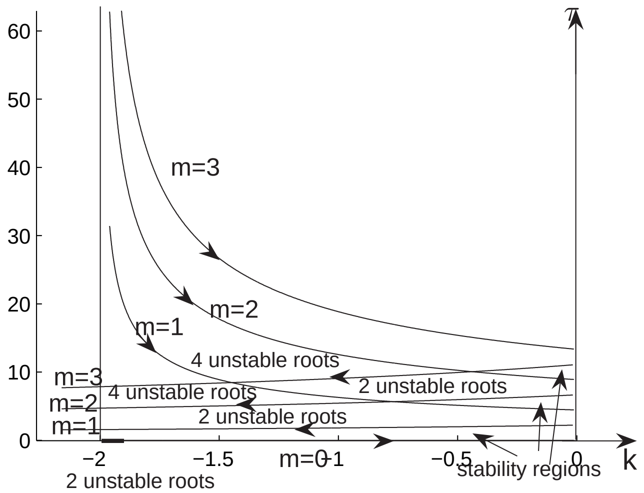


Figure 4.16: τ_i, $m \in \{0, 1, 2, 3\}$ versus k for the system (4.65)

Example 11 (Crossing curves of type 11 and 22) *This example is only to illustrate that it is possible to have all type of curves enumerated in the classification section. In the sequel we present a dynamical system with crossing curves of type 11, 22, 31 and 32.*

Consider the transfer function

$$H_{y,u}(s) = \frac{1}{s^3 - 2s^2 + 9s - 8} \tag{4.67}$$

subject to the control law $u(t) = -ky(t-\tau)$. The corresponding characteristic equation is given by:

$$s^3 - 2s^2 + 9s - 8 + ke^{-s\tau} = 0. \tag{4.68}$$

We note that this system can not be stabilized by any static output feedback free of delay. Straightforward computations show us that

$$\mathrm{card}\,(\mathcal{U}) = \begin{cases} 1 & \textit{if} \quad k < -10 \\ 3 & \textit{if} \quad k \in (-10, 8) \\ 2 & \textit{if} \quad k > 8 \end{cases}.$$

Taking $\alpha = -10$ and $\beta = 10$, we get $\Omega = (0,1] \cup [2,3]$ and $\mathcal{T}_1^{m+}$ is of type 32, $\mathcal{T}_1^{m-}$ is of type 31, $\mathcal{T}_2^{m-}$ is of type 11, $\mathcal{T}_2^{m+}$ is of type 22. We present the last three curves in the following three pictures. Similar conclusions can be derived using the method presented in [112]. ■

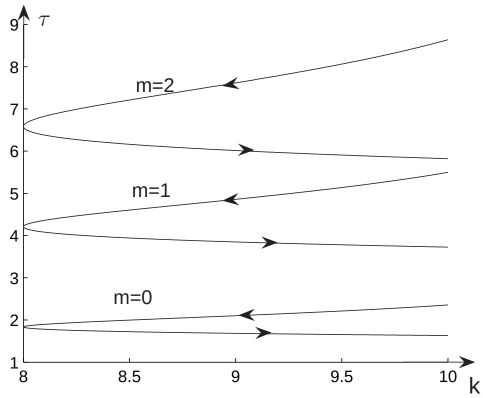


Figure 4.17: $\mathcal{T}_2^{m+}$, $m \in \{0, 1, 2\}$ for the system (4.68)

Example 12 (Sixth-order unstable system) *In this example, we consider a system that can not be stabilized by a static output feedback, but it can be stabilized by a delayed output feedback. This example is borrowed from [112].*

Consider the system:

$$H_{y,u}(s) = \frac{1}{s^6 + p_1 s^5 + p_2 s^4 + p_3 s^3 + p_4 s^2 + p_5 s + p_6} \tag{4.69}$$

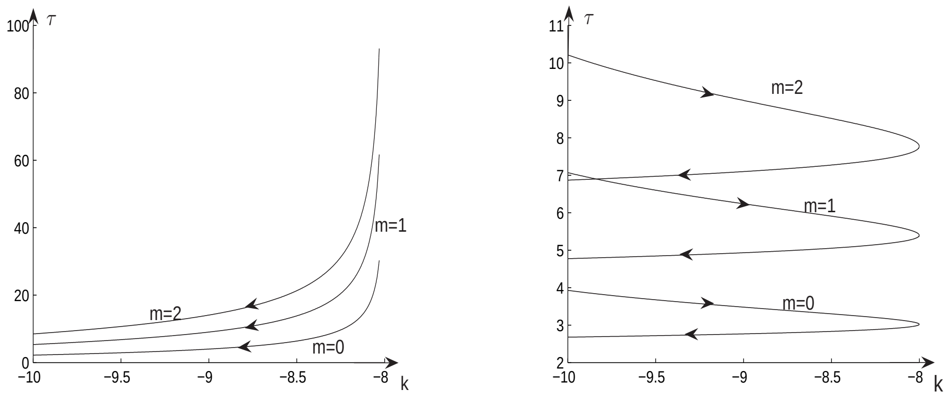


Figure 4.18: $\mathcal{T}_i^{m-}$, $m \in \{0,1,2\}, i \in 1,2$ for the system (4.68)

where

$$p_1 = -6.0000000e-04, \quad p_2 = 1.4081634e+00, \quad p_3 = -5.6326533e-04,$$
$$p_4 = 4.3481891e-01, \quad p_5 = -8,6963771e-05, \quad p_6 = 2.6655565e-02.$$

Using Lemma 1, we obtain:

$$\operatorname{card}(\mathcal{U}) = \begin{cases} 3 & \textit{if} \quad k < -0.0707886, \\ 5 & \textit{if} \quad k \in (-0.0707886; -0.0266556), \\ 6 & \textit{if} \quad (-0.0266556; 0.0120036), \\ 4 & \textit{if} \quad k > 0.0120036. \end{cases}.$$

The stability crossing curves and the first two stability region for $k \in (0, 0.16)$ are plotted in figure 4.20 (see also the graphics of $k = k(\omega)$).

4.7 Concluding remarks

In this chapter, we have explored and presented various problems related to the crossing stability curves. We have characterized the geometry of the stability crossing curves in the parameter space for a class of a distributed delay systems. A robustness analysis (with respect to the parameters; delay radius) is given. Some limit cases and a coherence study are also presented in this chapter.

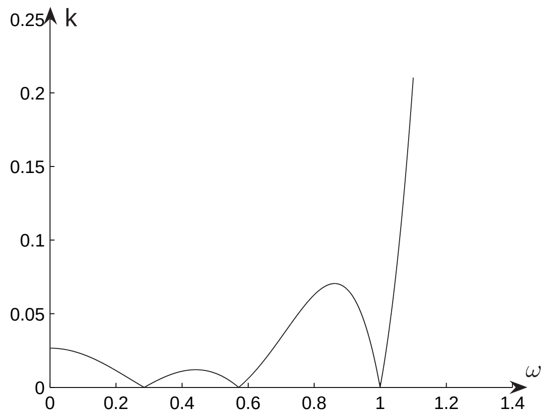


Figure 4.19: The dependence of the gain k as a function of ω for some positive frequencies

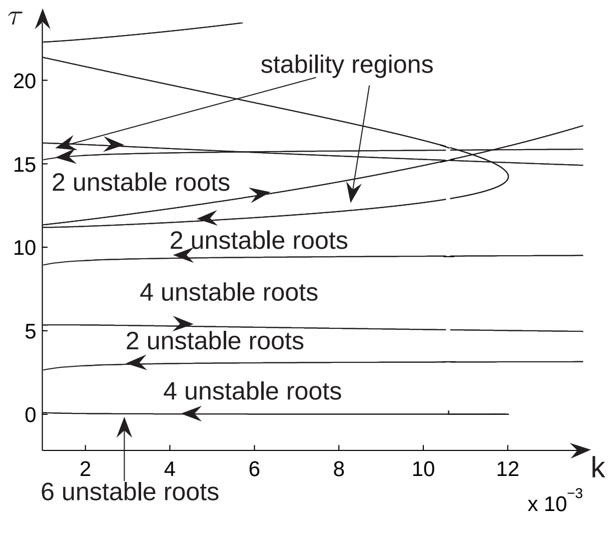


Figure 4.20: Stability crossing curves for the system given by (4.69)

Part III

Algebraic approach

Chapter 5

Distributed delays analysis: algebraic methods

This chapter focuses on the stability of the class of linear systems including gamma-distributed delay with a gap described by (2.38). More precisely, a *complete characterization of stability regions* is given in the corresponding (delay, mean-delay) parameter-space. *Optimal* delay intervals are explicitly computed. The stabilizing/destabilizing delay effect will be explicitly outlined, and discussed. Several illustrative examples complete the chapter.

5.1 Introductory remarks

This chapter can be seen as the "dual" of [99], presented in the previous chapter, where the characterization of the crossing curves was given using some geometric arguments. More precisely, we shall *explicitly* compute all the "points" $\left(\tau, \dfrac{\bar{\tau}}{n+1}\right)$, for which a change of the number of roots in $\mathbb{C}_+$ will take place, and next for *each* mean-delay value interval, an explicit computation of the corresponding (stability) delay interval can be performed.

The interest of the approach is *twofold*:

- First, the computation of the corresponding delay intervals can be performed relatively easily, and the corresponding algorithm includes a finite number of steps. Furthermore, various interesting *instability* cases can be detected, and the underlying ideas can be applied to various other delay analysis problems;

- Second, the propagation delay (gap τ) can be used as a *design parameter* in the case of controlling objects over communication network. Such an idea was already exploited in the context of constant communication delays (see, e.g., [108]), and to the best of the authors' knowledge, there does not exist any extension in the distributed delay case. In other words, the propagation delay can be used to define a so-called "wait-and-act" strategy similar to the one encountered in synchronization, and also mentioned in the case of delayed output feedback stabilization problems [107], *etc*.

The remaining chapter is organized as follows: In Section 2 we briefly present the problem formulation and some prerequisites necessary to develop our (frequency-domain) stability analysis. The main results are presented in Section 3, and illustrative examples are given in Section 4. Some concluding remarks end the chapter. The facts presented in this chapter can be also found in [100]. The approach below is inspired by Niculescu *et al* [112] for a class of SISO systems with one discrete delay.

5.2 Basic ideas, and prerequisites

The problem addressed in the sequel can be resumed as follows: *deriving necessary, and sufficient conditions in terms of* (T, τ) *for guaranteeing the asymptotic stability of (4.1).*

In this sense, the following two quantities will play a major role in the stability study (see also [112] in a different frame):

1) $card(\mathcal{U})$, where $\mathcal{U}$ is the set of roots of $D(s, T, 0) = 0$, situated in the *closed right half plane*, and card$(\cdot)$ denotes the cardinality (number of elements).

2) $card(\mathcal{S})$, where $S = \{\omega > 0 \mid (1 + \omega^2 T^2)^n |P(j\omega)|^2 - |Q(j\omega)|^2 = 0\}$.

Related with the hyperbolicity notion discussed in the previous chapter we remember the following result:

Proposition 21 *The system (4.1) is hyperbolic for all* $(T, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ *if and only if:*

$$|P(j\omega)| > |Q(j\omega)|, \qquad \forall \omega \in \mathbb{R}^*, \tag{5.1}$$

Furthermore, if $card(\mathcal{U}) = 0$ *(*> 0*) for* $T = 0$*, the system is delay-independent stable (unstable).*

Remark 20 *In the stability case, the frequency-sweeping test (5.1) represents a slight modification of the Tsypkin criterion (see, for instance, [107, 54]), and it gives a simple condition for which* $card(\mathcal{S}) = 0$ *for* all *the pairs* (T, τ)*.*

In the sequel, we shall assume that the condition (5.1) in Proposition 21 does *not hold*. If Proposition 21 holds, then we have stability (or instability) for all the pairs (T, τ), etc. In conclusion, the problem of interest is reduced to analyze the cases when *crossing roots exist*.

Without any loss of generality, assume now that $P(0) \neq 0$. If $P(0) = 0$, we get $Q(0) = 0$ from (4.1), which is not possible since it contradicts the Assumption 5(ii). The next step is the characterization of the way the quantities $card(\mathcal{U})$, and $card(\mathcal{S})$ depend on the parameter T if $\tau = 0$.

5.2.1 Stability analysis for the system without the gap

Introduce now the following Hurwitz matrix associated to some polynomial $A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$:

$$H(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \dots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \dots & a_{2n_a-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{5.2}$$

where the coefficients $a_l = 0$, for all $l > n_a$. Next, it is easy to see that $D(s, T, 0)$ can be rewritten as[1]: $D(s, T, 0) = \sum_{k=0}^{n} P_k(s) T^k$, with $P_0(s) =$

[1] We use the following notation $\begin{pmatrix} k \\ n \end{pmatrix} = \dfrac{n!}{k!(n-k)!}$

$P(s) + Q(s)$, $P_1(s) = \begin{pmatrix} 1 \\ n \end{pmatrix} sP(s)$, ..., $P_n(s) = s^n P(s)$ Next introduce the matrix pencil: $\Sigma(\lambda) = \lambda U + V$, with U, V given by:

$$U = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & H(P_n) \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \\ H(P_0) & H(P_1) & \cdots & H(P_{n-1}) \end{bmatrix},$$

where the identity, and the zero-blocks matrices have appropriate dimension, and $H(P_k) \in \mathbb{R}^{(n+n_p)\times(n+n_p)}$ represents the corresponding Hurwitz matrix[2] associated to the polynomial $P_k(s)$ defined above.

The following result gives the characterization of $card(\mathcal{U})$ as a function of T, and represents a generalization of some matrix pencil method proposed by [28] in the context of static output feedback for SISO systems:

Proposition 22 *Let $0 < \lambda_1 < \lambda_2 < \dots \lambda_h$, with $h \leq n + n_p$ be the real eigenvalues of the matrix pencil $\Sigma(\lambda) = (\lambda U + V)$. Then the system (4.1) cannot be stable for any $T = \lambda_i$, $i = 1, 2, \dots h$. Furthermore, if there are r unstable roots ($0 \leq r \leq n + n_p$) for $T = T^*$, $T^* \in (\lambda_i, \lambda_{i+1})$, then, there are r unstable roots for any mean-delay value $T \in (\lambda_i, \lambda_{i+1})$. In other words,* card$(\mathcal{U})$ *remains constant as T varies within each interval $(\lambda_i, \lambda_{i+1})$. The same holds for the intervals $(0, \lambda_1)$ and (λ_h, ∞).*

Proof. First, we need to show that as T varies, there are closed-loop roots on the imaginary axis if and only if $T = \lambda_i$, $i = 1, 2, ..., h$. The proof follows the same step as proposed by Chen in [28], and therefore, will be omitted. ■

Proposition 22 allows studying the behavior of $card(\mathcal{S})$ as a function of T. First we have to compute the positive real eigenvalues of Σ, and then the number of unstable roots inside each interval defined by the corresponding eigenvalues. The characterization is complete when computing $\mathcal{U}$ for intermediate values of T.

[2] The order of P_k is $n_p + k$, for all $k = 0, \dots, n$, and $H(P_k)$ will be constructed as a $(n + n_p) \times (n + n_p)$ matrix by setting the coefficients of high-order terms as zeroes, that is $p_\ell = 0$, for all $\ell > n + k$.

5.2.2 Cardinality of the crossing set

Based on the arguments, assumptions, and remarks above, we have the following result:

Proposition 23 *If the card($\mathcal{S}$) changes at a value T^* then there exists a frequency $\omega^* > 0$ such that for $\omega = \omega^*$ the following relations hold:*

$$F(\omega, T) = (1 + \omega^2 T^2)^n |P(j\omega)|^2 - |Q(j\omega)|^2 = 0 \tag{5.3}$$

and

$$\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right) - T^2\right] = 0 \tag{5.4}$$

Proof. For any T, F cannot have a root ω where $P(j\omega) = 0$, because this would imply that also $Q(j\omega) = 0$. So that the roots of F coincide with the roots of

$$G(\omega, T) = \frac{1}{\omega^2}\left(\left|\frac{Q(j\omega)}{P(j\omega)}\right|^{2/n} - 1\right) - T^2 = 0 \tag{5.5}$$

A change of $card(\mathcal{S})$ at $T = T^*$ implies that $G(\omega, T^*)$ has one root with multiplicity larger than one at some frequency ω^*, i.e.

$$G(\omega^*, T^*) = \frac{\mathrm{d}}{\mathrm{d}\omega}[G(\omega^*, T^*)] = 0.$$

This leads to (5.3) and (5.4). ■

Remark 21 *The equation* $\frac{d}{d\omega}[G(\omega^*, T^*)] = 0$ *has a* finite number *of roots. Thus, the quantity card($\mathcal{S}$) changes for a* finite number *of values of T.*

As in the previous case, the characterization is complete when computing $\mathcal{S}$ for intermediate values of T. We shall see in the next section that $\mathcal{S}$ can be identified with the crossing set.

5.3 Stability analysis for systems with a single discrete delay

For the sake of simplicity, assume that all the roots of F are *simple*. Notice that this condition is satisfied for almost all T. Next, we need to explicitly compute the sensitivity of the roots with respect to the delay parameter τ when crossing the imaginary axis, that is, in other words, the *delay crossing direction*.

5.3.1 Crossing direction

Theorem 11 *The characteristic equation has one root $j\omega$ on the imaginary axis for some τ_0 if and only if $\omega \in \mathcal{S}$. Furthermore, for $\omega \in \mathcal{S}$, the set of corresponding values of τ where card$(\mathcal{U})$ changes is given by*

$$\Gamma_\omega = \left\{ \frac{1}{\omega} \left[\arg \frac{Q(j\omega)}{(1+j\omega T)^n P(j\omega)} + 2k\pi \right] \geq 0, \qquad k \in \mathbb{Z} \right\}. \tag{5.6}$$

When increasing the delay, the corresponding crossing direction of characteristic roots is towards instability (stability) when $F'(\omega) > 0 (< 0)$.

Proof. "$\Leftarrow$" Assume that $j\omega$ is a root of characteristic equation (4.1) then

$$\begin{aligned} & |1+j\omega T|^n |P(j\omega)| = |Q(j\omega)| \\ & \Rightarrow (1+\omega^2T^2)^n |P(j\omega)|^2 - |Q(j\omega)|^2 = 0 \\ & \Rightarrow \omega \in \mathcal{S} \end{aligned}$$

"$\Rightarrow$" Suppose that $F(\omega, T) = 0 \Rightarrow |1+j\omega T|^n \dfrac{|P(j\omega)|}{|Q(j\omega)|} = 1$. So, there exists $\theta \in \mathbb{R}$ s.t. $(1+j\omega T)^n \dfrac{P(j\omega)}{Q(j\omega)} = \mathrm{e}^{-j\theta} \Rightarrow \exists \tau = \dfrac{\theta}{\omega}$ s.t. $j\omega$ is a root of (4.1).
If $j\omega$ is a root of (4.1) then easy computations show that Γ_ω is given by (5.6). In order to establish the corresponding crossing direction we compute the real part of the derivative of s with respect to τ. So, deriving (4.1) with respect to τ we get:

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{sQ(s)\mathrm{e}^{-s\tau}}{(1+sT)^n P'(s) + nT(1+sT)^{n-1}P(s) + Q'(s)\mathrm{e}^{-s\tau} - \tau Q(s)\mathrm{e}^{-s\tau}}$$

and

$$\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1} = \frac{(1+sT)^n P'(s)}{sQ(s)\mathrm{e}^{-s\tau}} + \frac{nT(1+sT)^{n-1}P(s)}{sQ(s)\mathrm{e}^{-s\tau}} + \frac{Q'(s)\mathrm{e}^{-s\tau}}{sQ(s)\mathrm{e}^{-s\tau}} - \frac{\tau}{s}$$

then

$$\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)^{-1} = \mathrm{Re}\left(-\frac{P'(j\omega)}{j\omega P(j\omega)} - \frac{nT}{j\omega(1+j\omega T)} + \frac{Q'(j\omega)}{j\omega Q(j\omega)}\right) =$$

$$= \frac{1}{\omega}\mathrm{Im}\left(-\frac{P'(j\omega)\overline{P(j\omega)}|1+j\omega T|^{2n}}{|P(j\omega)|^2|1+j\omega T|^{2n}} - \frac{nT(1-j\omega T)|P(j\omega)|^2|1+j\omega T|^{2n-2}}{|1+j\omega T|^{2n}}\right)$$
$$+ \frac{1}{\omega}\mathrm{Im}\left(\frac{Q'(j\omega)\overline{Q(j\omega)}}{|Q(j\omega)|^2}\right) = \frac{1}{\omega|Q(j\omega)|^2}\mathrm{Im}\left(-P'(j\omega)\overline{P(j\omega)}|1+j\omega^2T^2|^n\right)$$
$$+\frac{1}{\omega|Q(j\omega)|^2}\mathrm{Im}\left(nT(1-j\omega T)|P(j\omega)|^2|1+j\omega^2T^2|^{n-1} - Q'(j\omega)\overline{Q(j\omega)}\right) \quad (5.7)$$

The roots will cross the imaginary axis towards stability (instability) if $\mathrm{sgn}\left[\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\right] < 0$ (> 0). Taking $Q(j\omega) = Q_R(\omega) + jQ_I(\omega)$ and $P(j\omega) = P_R(\omega) + jP_I(\omega)$, (5.7) leads to:

$$\mathrm{sgn}\left[\mathrm{Re}\left(\frac{\mathrm{d}s}{\mathrm{d}\tau}\right)\right] = -\frac{1}{\omega|Q(j\omega)|^2}\mathrm{sgn}\left\{\left(-P_R'(\omega)P_R(\omega) - P_I'(\omega)P_I(\omega)\right)(1+\omega^2T^2)^n\right.$$
$$\left.-n\omega T^2(P_R^2(\omega) + P_I^2(\omega))(1+\omega^2T^2)^{n-1} + Q_R'(\omega)Q_R(\omega) + Q_I'(\omega)Q_I(\omega)\right\}$$
$$= \mathrm{sgn}\left\{\left(P_R'(\omega)P_R(\omega) + P_I'(\omega)P_I(\omega)\right)(1+\omega^2T^2)^n +\right.$$
$$\left.+ n\omega T^2(P_R^2(\omega) + P_I^2(\omega))(1+\omega^2T^2)^{n-1} - [Q_R'(\omega)Q_R(\omega) + Q_I'(\omega)Q_I(\omega)]\right\}$$
$$= \mathrm{sgn}F'(\omega, T) \quad (5.8)$$

condition which simply says that the sign of F' will give the crossing direction. ∎

The above theorem combined with the continuous dependence of the characteristic roots with respect to the delay, allows to say that $\Gamma = \bigcup_{\omega\in\mathcal{S}} \Gamma_\omega$ makes a partitions of the τ-delay space ($\mathbb{R}_+$) into intervals in which the *number of roots in the open right half plane is constant.* Such an argument will be used in developing our stability region characterization.

5.3.2 Small delays

First, assume that the system free of delays is asymptotically stable ($\tau, T = 0$), that is $card(\mathcal{U}) = 0$ with $T = 0$, and that the frequency-sweeping condition (5.1) does not hold. Then Theorem 11, combined with the Propositions 22, and 23 give a simple way to compute the first delay-intervals guaranteeing stability:

Proposition 24 *Under the assumption $card(\mathcal{U}) = 0$ for the system free of delays, the system (4.1) is asymptotically stable for all the pairs (T, τ), with $0 \leq T < T^*$, where T^* is the smallest positive generalized eigenvalue of Σ, and $\tau \in [0, \tau^*)$, where τ^* is given by:*

$$\tau^* = \min_{\omega \in \mathcal{S}(T)} \{\Gamma_\omega(T)\} \tag{5.9}$$

as a function of T, for all $T \in [0, T^*)$.

In other words, Proposition 24 defines the explicit dependence of the stability boundary in (T, τ) space bounded by the corresponding OT, and $O\tau$-axis, and by the curve $\tau(T)$, defined as a function of T, for all $T \in [0, T^*)$. The case $T = 0$ gives the standard first delay-interval bound (see, e.g. [107]). Using the terminology of [54], we derive the corresponding *delay margins* in $OT\tau$ parameter-space.

5.3.3 Delay-induced stability/instability

Assume now that the system free-of-delays ($\tau = 0$, $T = 0$) is *unstable*. We start by presenting various cases in which the *gap*, seen as a free-parameter cannot have a stabilizing effect. The proof ideas follow closer the approach in Niculescu *et al* [112]. However, in order to have self-contained results, we will detail them below. We have the following results:

Proposition 25 *If the $card(\mathcal{U})$ is an odd number then the stability of the system cannot be obtain increasing the time delay τ.*

Proof. Assume that the closed-loop system is asymptotically stable for some delay value τ_s. Because the number of roots in the right half plane changes from odd to even when increasing the delay τ from 0 to τ_s, a characteristic root at *zero* must occur for some $\tau_0 \in [0, \tau_s]$. But $D(0, T, \tau_0) = 0$ implies

$H(0, T, \tau) = 0$, $\forall \tau \geq 0$ which contradicts the asymptotic stability at $\tau = \tau_s$. ■

Proposition 26 *If* $card(\mathcal{S}) \in \{0, 1\}$ *then the stability of the system cannot be obtain increasing the time delay* τ.

Proof. When $card(\mathcal{S}) = 0$ from Theorem 11 follows that characteristic roots cannot cross the imaginary axis and the system maintain his nature (instability persist when varying the delay τ). When $card(\mathcal{S}) = 1$ there is only one crossing frequency ω^*. Because $\lim\limits_{\omega \to \infty} F(\omega) = \infty$ and $F(\omega) \neq 0$, $\forall \omega \neq \omega^*$, the sign of its derivative in ω^* must be "+". Therefore we have one crossing and the crossing direction is towards instability. ■

The first case, when the delay gap τ may induce stability in the system by increasing its value appears when $card(\mathcal{S}) \in \{2, 3\}$. More precisely, we have the following result:

Proposition 27 *If* $card(\mathcal{S}) \in \{2, 3\}$ *then the stability of the system can be obtain increasing the time delay* τ*, if and only if:*
1. $card(\mathcal{U}) = 2$
2. $\tau_- < \tau+$*, where* $\begin{cases} \tau_- = \min \bigcup\limits_{\omega \in \mathcal{S},\, F'(\omega) < 0} \Gamma_\omega \\ \tau_+ = \min \bigcup\limits_{\omega \in \mathcal{S},\, F'(\omega) > 0} \Gamma_\omega \setminus \{0\} \end{cases}$
In this case, for all delay values $\tau \in (\tau_-, \tau_+)$ *the system is stable.*

Proof."$\Rightarrow$" First we consider the case where $card(\mathcal{S}) = 2$. If $S = \{\omega_1, \omega_2\}$,with $\omega_1 > \omega_2$, then $F'(\omega_1) > 0$ and $F'(\omega_2) < 0$. The set of delay values Γ_{ω_1}, where the roots cross the imaginary axis to the right, consists of numbers equally spaced by $\frac{2\pi}{\omega_1}$. Crossing towards stability occur for delay values of the set Γ_{ω_2}, which are equally spaced by $\frac{2\pi}{\omega_2} > \frac{2\pi}{\omega_1}$. As a consequence, between two stability crossing, an instability crossing must occur, i.e. the number of unstable roots in the closed right half plane cannot be reduced with more than two by increasing the delay τ. Therefore the stability can be obtain if and only if $card(\mathcal{U}) = 2$ and the first crossing is towards stability, mathematically expressed by $\tau_- < \tau+$.

If $card(\mathcal{S}) = 3$ and $S = \{\omega_1, \omega_2, \omega_3\}$, with $\omega_1 > \omega_2 > \omega_3$ then $F'(\omega_1) > 0$, $F'(\omega_2) < 0$ and $F'(\omega_3) > 0$. So a careful examination allows us to say that we can use the same argument as in previous case.

"$\Leftarrow$" The condition $\tau_- < \tau+$ implies that the first crossing is towards stability when the delay is increased from zero. Since $card(\mathcal{U}) = 2$, the system is asymptotically stable for any $\tau \in (\tau_-, \tau_+)$. ■

Remark 22 *One can conclude that in the previous case is sufficiently to investigate the first crossing in order to check the stabilizability in the delay. When one determines the stability by numerically computations the Proposition 27 is very useful because we can stop the computations after the first root crossing.*

In the case $card(\mathcal{S}) = 2$, the set of all stabilizing delay values can be expressed analytically:

Corollary 2 *Assume that the following conditions are satisfied*

1. $card(\mathcal{S}) = 2$
2. $card(\mathcal{U}) = 2$
3. $\tau_- < \tau+$

Then all the delay values guaranteing stability are defined by $\tau \in (\underline{\tau}_k, \overline{\tau}_k)$, $k = 0, 1, ..., k_m$, *where*

$$\underline{\tau}_k = \tau_- + \frac{2k\pi}{\omega_-}, \quad \overline{\tau}_k = \tau_+ + \frac{2k\pi}{\omega_+}$$

and k_m *is the largest integer for which* $\underline{\tau}_k < \overline{\tau}_k$, *which can be explicitly expressed as*

$$k_m = \max_{l \in \mathbb{Z}} \left\{ l < \frac{\omega_- \omega_+}{\omega_+ - \omega_-} \cdot \frac{\tau_+ - \tau_-}{2\pi} \right\} \tag{5.10}$$

Based on the results, and the remarks above, we have the following

Proposition 28 *Assume that* $card(\mathcal{S}) = 2p$ *or* $card(\mathcal{S}) = 2p+1$, *with* $p \geq 1$ *and* $card(\mathcal{U}) > 2p$. *Then there does not exist any gap* $\tau > 0$ *such that (4.1) becomes asymptotically stable.*

Proof Let $\mathcal{S} = \{\omega_1, \omega_2, ...\}$ with $\omega_1 > \omega_2 > ....$ Because $\lim_{\omega\to\infty} F(\omega) = \infty$ and the roots of F are simple, the sign of its derivatives in the zeros alternates and $F'(\omega_1) > 0$.
The Proposition 27 state that for a pair (ω_1, ω_2) between two stability crossing an instability crossing must occur. So for $n = 1$ no more than 2 characteristic roots can be shifted to the left hand plane. If $n > 1$ we can use the same argument for the pairs $(\omega_1, \omega_2), (\omega_3, \omega_4), ...$ In the case $card(\mathcal{S}) = 2n$ we get that no more than $2n$ characteristic roots can cross towards stability and the proof is complete. In the case $card(\mathcal{S}) = 2n + 1$ the previous argument remains since $F'(\omega_{2n+1}) > 0$ i.e the first crossing is towards instability. ■
Define now the following quantities:

$$n_+(\tau) = \sum_{\omega\in\mathcal{S}_+,\ F'(\omega)>0} \text{card}\left\{\Gamma_\omega \cap (0, \tau]\right\}, \tag{5.11}$$

$$n_-(\tau) = \sum_{\omega\in\mathcal{S}_+,\ F'(\omega)<0} \text{card}\left\{\Gamma_\omega \cap [0, \tau]\right\}, \tag{5.12}$$

for some positive $\tau > 0$. Furthermore, introduce the sets Γ^+, and Γ^-, which represent a partition of Γ in function of the sign of the derivative F' evaluated at the corresponding crossing frequency, that is:

$$\Gamma^+ = \bigcup_{\omega\in\mathcal{S}_+,\ F'(\omega)>0} \Gamma_\omega \setminus \{0\},$$
$$\Gamma^- = \bigcup_{\omega\in\mathcal{S}_+,\ F'(\omega)<0} \Gamma_\omega.$$

Based on the conditions and the notations above, we conclude with the following result:

Proposition 29 *For a given T the system with characteristic equation (4.1) is asymptotically stable if and only if the following conditions are satisfied:*

1. *$card(\mathcal{U}(T))$ is a strictly positive even integer and the following inequality holds: $card(\mathcal{U}(T)) \leq card(\mathcal{S}(T))$*

2. *there exists at least one gap value $\tau^* \in \Gamma$, such that: $n_-(\tau^*) = n_+(\tau^*) + card(\mathcal{U}(T))$.*

Then all gap values $\tau \in (\tau^*, \tau_+^*)$, with $\tau_+^* = \min\{\Gamma^+ \cap (\tau^*, +\infty)\}$ guarantee the asymptotic stability.

Proof. The condition 1) is clear from Proposition 25 and Proposition 28 and the condition 2) simply characterizes the existence of crossings towards stability such that there are no more unstable roots for $\tau = \tau^* + \epsilon$, for sufficiently small $\epsilon > 0$. Finally, the definition of the gap interval where we have stability follows straightforwardly from the previous notation. $\blacksquare$

The algorithm to find stability pair in the parameter-space (mean delay,gap)

The stability analysis proposed in this chapter is based on two quantity, $card(\mathcal{U})$ and $card(\mathcal{S})$, which depend only on the mean delay T. Both quantity can be efficiently determined as a function of T by numerical computation. The first quantity is given by the generalized eigenvalues of some matrix pencil (Proposition 22) and the second one by the roots of a polynomial (Proposition 23). The main results of the previous sections of this chapter are displayed in Table 1.

	0	1	2	3	4	5	6	7	8	9	$card(\mathcal{S})$
1	/	/	/	/	/	/	/	/	/	/	
2	/	/	$\tau_- < \tau_+$								
3	/	/	/	/	/	/	/	/	/	/	
4	/	/	/	/							
5	/	/	/	/	/	/	/		/	/	
6	/	/	/	/	/	/					
7	/	/	/	/	/	/	/	/	/	/	
8	/	/	/	/	/	/	/	/			
$card(\mathcal{U})$											

Table 5.1: When we use the gap parameter τ to control the stability of our system, the necessary and sufficient conditions are given by Proposition 29. In the special case of $card(\mathcal{U}) = 2$ and $card(\mathcal{S}) \in \{2, 3\}$ the condition written in the table is given by Proposition 27

In the sequel we resume the method to find a stabilizing pair (T, τ) for the system studied in the previous section.

A) In order to select the possible mean delay values that satisfy the first condition of Proposition 29, we need first, the values of $card(\mathcal{S})$ and $card(\mathcal{U})$. These values can be computed using Proposition 23 and Proposition 22, respectively.

B) For a given T satisfying the first condition of Proposition 29, one needs the gap values which satisfy the second condition. In the special case of $card(\mathcal{U}) = 2$ and $card(\mathcal{S}) \in \{2, 3\}$, it is specified in the table that we can use the Proposition 27. So, it is sufficient to know if the first root crossing of the imaginary axis is towards stability as τ increasing from 0. In the general case, the analysis is more complicated and the stabilizing interval is defined by the last part of Proposition 29.

5.4 Illustrative examples

In order to illustrate the method proposed above, we consider several examples. The first example concerns linearized Cushing equation with a gap. Then we apply this algebraic approach to some second-order time-delay system with a gap.

5.4.1 Linearized Cushing equation with a gap

In this example we apply the above method for the Cushing linearized equation

$$(s + a)(1 + sT)^n + b\mathrm{e}^{-s\tau} = 0,\ a > 0,\ b < 0.$$

First it is easy to remark that $(s + a)(1 + sT)^n + b$ has at least one (strictly) unstable root if and only if $a + b \leq 0$ and $card(\mathcal{U}) = 0$ if $a + b > 0$. Consider the case $n = 1$, that is the polynomial $F(\omega, T)$ is given by:

$$\begin{aligned} F(\omega, T) &= (\omega^2 + a^2)(1 + \omega^2 T^2) - b^2 \\ &= \omega^4 T^2 + \omega^2(a^2 T^2 + 1) + a^2 - b^2. \end{aligned} \tag{5.13}$$

For $a^2 - b^2 \geq 0$ $(a + b \neq 0)$ we have $card(\mathcal{S}) = 0$, that is no crossing with respect to the imaginary axis for all T (see Proposition 21), while for

$a^2 - b^2 < 0$ we have $card(\mathcal{S}) = 1$. According to the results of the previous section, the stability of the Cushing equation can be delay-independent stable (unstable), function of the *sign* of $a + b$ for all (T, τ) if $card(\mathcal{S}) = 0$. If the system is not delay-independent stable (unstable), Proposition 24 will give the corresponding delay-intervals for which stability is preserved under the assumption of asymptotic stability for some mean-delay intervals (in T) given by Proposition 22, etc.
It is worth to say here that if one root crosses the imaginary axis than it will cross to the right. So if the system is unstable it remains unstable, and if the system is stable and $card(\mathcal{S}) \neq 0$ than the system will lose the stability once for good.

Proposition 30 *For the Cushing equation with a gap all the crossing directions are towards instability.*

Proof: Straightforward computation show us that

$$F'(\omega) = 2\omega(1 + \omega^2 T^2)^{n-1}(1 + \omega^2 T^2 + nT^2) > 0,$$

which simply says that the crossing direction is towards instability. ■

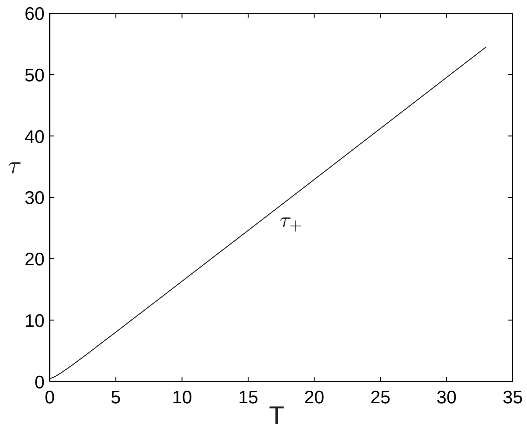


Figure 5.1: In this case all the crossings are towards instability so, τ_- does not exist

5.4.2 Second-order system example

Consider the following second-order system:

$$Q(s) = -s, \quad P(s) = s^2 + 2 \tag{5.14}$$

Simple computations prove that $P(s)(1+sT)+Q(s)$ has two unstable roots. So that $card(\mathcal{U}) = 2$.
The characteristic equation of the closed-loop system is given by

$$(s^2 + 2)(1 + sT) - s\mathrm{e}^{-s\tau} = 0 \tag{5.15}$$

and polynomial $F(\omega, T)$ by

$$\begin{aligned} F(\omega, T) &= (2-\omega^2)^2(1+\omega^2T^2) - \omega^2 \\ &= \omega^6T^2 + \omega^4(1-4T^2) + \omega^2(4T^2-5) + 4. \end{aligned}$$

So we need to find how many positive roots has the following equation:

$$x^3T^2 + x^2(1-4T^2) + x(4T^2-5) + 4 \tag{5.16}$$

First it is easy to see that the previous equation has at least one real negative solution because $x_1x_2x_3 = -\dfrac{4}{T^2} < 0$ (where x_1, x_2, x_3 are the solutions of the equation (5.16)). Computing the discriminant and the Hurwitz determinants of the equation (5.16) we find $card(\mathcal{S}) = \begin{cases} 2 & T > \dfrac{1}{2} \\ 0 & T \leq \dfrac{1}{2} \end{cases}$. According to the result of the previous section a necessary condition for asymptotic stability of the closed-loop system is given by

$$T > \frac{1}{2} \tag{5.17}$$

Furthermore, for T satisfying (5.17) the existence of a stability region in the delay parameter is determined by the condition $\tau_- < \tau_+$.

Summarizing, we have:

Proposition 31 *The system (5.14) is asymptotically stable if and only if* $T > \dfrac{1}{2}$ *and in addition* $\tau_- < \tau_+$, *where:*

$$\tau_- = \min \bigcup_{\omega\in\mathcal{S},\, F'(\omega)<0} \frac{1}{\omega^2 T}, \qquad \tau_+ = \min \bigcup_{\omega\in\mathcal{S},\, F'(\omega)>0} \frac{1}{\omega^2 T}$$

A stability region is defined by the pair (T, τ)*, where* $T > \dfrac{1}{2}$*, and* $\tau \in (\tau_-(T), \tau_+(T))$.

As we can see in figure 5.2, there is no stability region for the system 5.14.

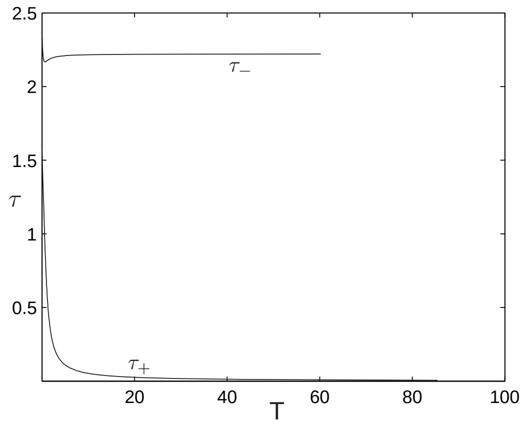


Figure 5.2: Since $\tau_- > \tau+, \forall T > 0$ the system is never stable

5.5 Concluding remarks

This chapter addressed the stability problem of a class of linear systems including distributed delays with a gap. A characterization of stability regions in the (mean-delay,gap) parameter-space has been proposed. Illustrative examples complete the presentation.

Chapter 6

Stability analysis for systems with commensurate delays

In this chapter, we focus on generalizing the results from the previous chapter to some class of systems including commensurate delays. Some classical stability tests for this class of systems can be found in [54, 107]. In this case the model is described by a system of the general form:

$$\dot{x}(t) = A_0 x(t) + \sum_{k=1}^{m} A_k x(t - k\tau), \quad \tau \geq 0 \tag{6.1}$$

where A_0, $A_k \in \mathbb{R}^{n \times n}$. The characteristic equation of such a model is expressed as:

$$\det \left(sI - A_0 - \sum_{k=1}^{m} A_k \mathrm{e}^{-sk\tau} \right). \tag{6.2}$$

Our aim in this chapter is to study more complicated dynamical behaviors under the assumptions of delay distributed kernels. The method developed here is based on the same arguments used in the previous chapter. First, we discuss the stability of a particular model encountered in the traffic flow dynamics. Next, we focuss to the stability analysis of the general case.

6.1 Traffic flow dynamics

One model, often used in the literature, consider the case of multiple vehicles under the influence of a single constant time-delay [26, 61, 132]. This model can be written as

$$\dot{x}_k(t) = \alpha_k(x_{k-1}(t-\tau) - x_k(t-\tau)),\ k = 1, \ldots, p \tag{6.3}$$

where p is the number of considered vehicles and $x_0 = x_n$. The left hand side represents the *acceleration* of the k^{th} vehicle, and the right hand side express the *velocity difference* of consecutive vehicles.
Despite the fact that in some situation one can use the previous model with good results, it is far to be realistic. One of the problem is due to the controller, since the behavior of the human driver can not be reduced to a simple controller law. For instance, humans retain a short-term memory of the past events and this may affect their control strategy [143]. Therefore, in order to obtain a more realistic model we can extend the previous models by incorporating a general memory effect. Without any loss of generality we can consider the following model

$$\dot{x}_k(t) = \alpha_k \int_0^\infty f(\theta)(x_{k-1}(t-\theta) - x_k(t-\theta))\mathrm{d}\theta, \tag{6.4}$$

where f is a distribution of delays, which can represent both dead-time and the memory of the past (such an ideea can be found in Attay *et al* [143]). In order to express realistically a stochastic behavior, one can use the gamma-distribution with a gap. When the choice for the memory model comes from the gamma distributed history (2.36), applying the Laplace transform we get a characteristic equation with general form

$$\det[sI - (A_1 + A_2)F(s)] = 0. \tag{6.5}$$

where F denotes the Laplace transform of f, therefore, $F(s) = \dfrac{\mathrm{e}^{-s\tau}}{(1+sT)^n}$.
Thus, we can rewrite (6.5) as:

$$D(s, T, \tau) = \det[s(1+sT)^n I - A\mathrm{e}^{-s\tau}] = 0 \tag{6.6}$$

where

$$A = \begin{pmatrix} -\alpha_1 & 0 & \ldots & \ldots & \alpha_1 \\ \alpha_2 & -\alpha_2 & 0 & \ldots & 0 \\ 0 & \alpha_3 & -\alpha_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & \alpha_p & -\alpha_p \end{pmatrix} \tag{6.7}$$

Remark 23 *The matrix A defined by 6.7 has always the eigenvalue "0". This corresponds to the situation in which the relative movement of one vehicle to the others is zero (the vehicles are either staying or moving with the same velocity). Obviously, this situation presents no practical interest and will be excluded.*

6.1.1 Stability analysis with respect to the gap

In the sequel, consider the *crossing set* Ω, which is defined as the collection of all $\omega > 0$ such that there exists a parameter pair (T, τ) such that $D(j\omega, T, \tau) = 0$. For a given T we will denote $h(s) = s(1 + sT)^n$ and

$$H : \mathbb{R}_+^* \mapsto \mathbb{R}_+^*, \quad H(\omega) = |h(j\omega)|^2 = \omega^2 \left[(1 + \omega^2 T^2)^n\right].$$

Proposition 32 *The function H has the following properties:*

i) H is continuous and differentiable,

ii) H is monotonic, more precisely $H(\omega_1) > H(\omega_2) \Leftrightarrow \omega_1 > \omega_2$.

Proof. Since H is a polynomial function, it is clear that it is continuous and differentiable. The second property can be derived after simple algebraic manipulation or simply using the fact that the first derivative of H is strictly positive. ■

Simple eigenvalue case

In this paragraph, we consider that the real matrix A has only eigenvalues with multiplicity 1.

Proposition 33 *Consider $\{\mu_k\}_{k=1,p}$ the set of all eigenvalues of the matrix A. For any fixed T the characteristic equation has one root $j\omega$ on the imaginary axis for some τ_0 if and only if $|\omega(1 + j\omega T)^n| = |\mu_k|$. Furthermore, increasing the gap value τ, all the corresponding crossing directions of characteristic roots are towards instability.*

Proof. Obviously, the first part of the proposition needs no arguments. Therefore, let $\omega \in \Omega$, μ an eigenvalue of A and u, v left and right eigenvectors associated with μ, i.e.

$$(h(j\omega)I - A\mathrm{e}^{-j\omega\tau})v = 0, \tag{6.8}$$

$$u^*(h(j\omega)I - A\mathrm{e}^{-j\omega\tau}) = 0 \tag{6.9}$$

where * denote the transposition operation and u, v are viewed as column vectors. In order to derive the crossing direction we must to compute $\mathrm{sgn} Re \left(\frac{\mathrm{d}s}{\mathrm{d}\tau} \right)\Big|_{s=j\omega}$. Consider the equation

$$(h(s)I - A\mathrm{e}^{-s\tau})v = 0.$$

The derivative with respect to τ is given by:

$$\left(h'(s)\frac{\mathrm{d}s}{\mathrm{d}\tau} + A\tau\mathrm{e}^{-s\tau}\frac{\mathrm{d}s}{\mathrm{d}\tau} + sA\mathrm{e}^{-s\tau} \right) v + (h(s)I - A\mathrm{e}^{-s\tau})\frac{\mathrm{d}v}{\mathrm{d}\tau} = 0.$$

Multiplying with u^* to the left and replacing $s = j\omega$ we get

$$\frac{\mathrm{d}s}{\mathrm{d}\tau}\Big|_{s=j\omega} = -\frac{u^* j\omega A v \mathrm{e}^{-j\omega\tau}}{u^*(j\omega I + A\tau\mathrm{e}^{-j\omega\tau})v}.$$

Furthermore,

$$\begin{aligned} Re \left(\frac{\mathrm{d}s}{\mathrm{d}\tau} \right)\Big|_{s=j\omega} &= Re \left(\frac{\mathrm{d}s}{\mathrm{d}\tau} \right)^{-1}\Big|_{s=j\omega} = -Re \left(\frac{u^* I v h'(j\omega)}{u^* A v \mathrm{e}^{-j\omega\tau} j\omega} \right) \\ &= -Im \left(\frac{h'(j\omega)}{h(j\omega)} \right) \end{aligned} \tag{6.10}$$

On the other hand

$$\begin{aligned} H(\omega) &= |h(j\omega)|^2 = h(j\omega)h(-j\omega) \\ H'(\omega) &= j(h'(j\omega)h(-j\omega) - h(j\omega)h'(-j\omega)) \\ \frac{H'(\omega)}{H(\omega)} &= j\frac{h'(j\omega)}{h(j\omega)} - j\frac{h'(-j\omega)}{h(-j\omega)} = -Im \left(\frac{h'(j\omega)}{h(j\omega)} \right) \end{aligned} \tag{6.11}$$

Therefore, $\mathrm{sgn} Re \left(\frac{\mathrm{d}s}{\mathrm{d}\tau} \right)\Big|_{s=j\omega} = \mathrm{sgn} H'(\omega)$. Since $H'(\omega) > 0$ we conclude that all the crossings are towards instability. ■

Repeated eigenvalues case

First, let consider the equation:

$$\det\left(\xi(s;\ T, \tau)I - A\right) = 0$$

This is equivalent with p scalar equations

$$\xi(s;\ T,\tau) - \mu_k = 0, \quad k = 1,\dots,p \tag{6.12}$$

with the complex numbers $\mu_1,\dots,\mu_p$ equal to the eigenvalues of A. Consider also that the matrix A has a repeated eigenvalue with multiplicity $m > 1$. By superimposing solutions of (6.12) for $i = 1,\dots,p$, the eigenvalues of the DDE are obtained. Note that a possible discrepancy between the algebraic and geometric multiplicity of an eigenvalue is determined by properties of ξ and not by properties of the eigenvalues of A.
Given a solution s of (6.12) we get:

$$\frac{\partial s}{\partial \tau} = -\frac{\partial \xi}{\partial \tau} \Big/ \frac{\partial \xi}{\partial s}\,, \tag{6.13}$$

$$\frac{\partial s}{\partial T} = -\frac{\partial \xi}{\partial T} \Big/ \frac{\partial \xi}{\partial s}\,. \tag{6.14}$$

Proposition 34 *If the eigenvalues of A are not distinct, then the following statements hold true:*

1. *The equation (6.6) has only simple or semi-simple eigenvalues.*

2. *The corresponding crossing direction is towards instability*

Proof. 1) In our case,

$$\xi(s;T,\tau) = s(1+sT)^n \mathrm{e}^{s\tau},$$

and thus,

$$\frac{\partial \xi}{\partial s}(s;\ T,\tau) = (1+sT)^n \mathrm{e}^{s\tau} + sn(1+sT)^{n-1}\mathrm{e}^{s\tau} + s\tau(1+sT)^n \mathrm{e}^{s\tau}.$$

Furthermore, for $s = j\omega$ we obtain:

$$\frac{\partial \xi}{\partial s}(j\omega;\ T,\tau) = (1+j\omega T)^{n-1} e^{j\omega\tau}\left(1 + j\omega T + j\omega n + j\omega\tau(1+j\omega T)\right).$$

Since T, n, τ are positive, this expression is nonzero for all $\omega \geq 0$. Thus, if there are eigenvalues on the imaginary axis for some parameters, the multiplicity of the solution of (6.12) is always one.
2) Using (6.13) and the computations above we easily obtain:

$$Re\left(\frac{\partial s}{\partial \tau}\right)^{-1}\Bigg|_{s=j\omega} = \frac{1}{\omega^2} + \frac{n\tau}{1+\omega^2 T^2} > 0 \tag{6.15}$$

■

6.1.2 The methodology to derive the stability region

The previous section allows us to resume the stability analysis in the parameter space of (T, τ) to the following practical steps:

Step 1: Find the value T such that $D(s, T, 0)$ is stable.

Step 2: Compute the minimum value τ corresponding to the first crossing.

Stability analysis for the system without the gap

In order to determine T such that $D(s, T, 0)$ has no solution in the right half plane we can use a method based on matrix pencils arguments. This method was initially proposed in [28] and adapted in [100, 112] to the study of stability analysis of time-delay systems. It is easy to see that $D(s, T, 0)$ can be rewritten as: $D(s, T, 0) = \sum_{k=0}^{np} Q_k(s) T^k$, with $\deg Q_k(s) = k + p$. Since A is a singular matrix ($\det(A) = 0$), all the polynomials Q_k have s as a common factor. Thus, we can simplify by s and obtain another expression $D_1(s, T, 0) = \sum_{k=0}^{np} P_k(s) T^k$, with $\deg P_k(s) = p + k - 1$ (we denoted P_k the previous Q_k simplified by s). Consider the matrix pencil: $\Sigma(\lambda) = (\lambda U + V)$, with U, V constructed as in the previous chapter, where the identity, and the zero-blocks matrices have appropriate dimension, and $H(P_k) \in \mathbb{R}^{((n+1)p-1)\times((n+1)p-1)}$ represents the corresponding Hurwitz matrix[1] associated to the polynomial $P_k(s)$ defined above.
Consider now $\mathcal{U} = \{s \in \mathbb{C}_+ \mid D(s, T, 0) = 0\}$. Using proposition 22 we are able to characterize the behavior of $\mathcal{U}$ with respect to the change of T. Therefore, we have a standard procedure to accomplish the first step of our methodology.

Stability regions in the parameter space of (mean delay, gap)

Assume that the system without gap is asymptotically stable (that is $card(\mathcal{U}_+) = 0$) for any $T \in (0, T^*)$. Increasing the gap value τ, all the corresponding crossing directions of characteristic roots are towards instability.

[1]The order of P_k is $p + k - 1$, for all $k = 0, \ldots, np$, and $H(P_k)$ will be constructed as a $((n+1)p - 1) \times ((n+1)p - 1)$ matrix by setting the coefficients of high-order terms as zeroes, that is $p_\ell = 0$, for all $\ell > p + k - 1$.

More precisely, the system given by (6.6) is asymptotically stable for any $T \in (0, T^*)$ and $\tau \in (0, \tau^*)$, where τ^* will be determined.
Proposition 33 allows to compute all the values $\omega \in \Omega$ corresponding to $T \in (0, T^*)$.

Remark 24 *Combining Proposition 32 and Proposition 33 we deduce that for each eigenvalue μ of A the function $T \to \omega(T)$ is well defined (there is only one value ω for each T).*

Therefore, for a fixed value of T the characteristic equation reduces to an equation with a single variable τ. Using (6.12), some simple algebraic manipulations lead us to the following result.

Proposition 35 *For any $\omega \in \Omega$ the set of corresponding values of τ where the number of unstable roots changes is given by:*

$$\mathcal{T}_\omega = \left\{ \tau \geq 0 \mid \tau = \frac{1}{\omega}[\angle(\mu_k) - \angle j\omega(1 + j\omega T)^n + 2\ell\pi], \quad \ell \in \mathbb{Z} \right\}. \tag{6.16}$$

Since for a fixed $T \in (0, T^*)$ we can get a finite number of values $\omega \in \Omega$, we can denote them by $\omega_1(T), \ldots, \omega_p(T)$. Furthermore, introduce the set $\mathcal{T}_T$ which represents the set of all values τ corresponding to a fixed T, where the number of unstable roots changes:

$$\mathcal{T}_T = \bigcup_{k=1}^{p} \mathcal{T}_{\omega_k(T)} \tag{6.17}$$

Define now the following quantity:

$$\tau^*(T) = \min\{\tau \in \mathcal{T}_T\} \tag{6.18}$$

Summarizing, we obtain the following result which characterize the stability region of the traffic dynamics given by (6.6).

Proposition 36 *Assuming that the system free of delays is stable, the asymptotic stability of the system (6.6) is guaranteed by the following necessary and sufficient conditions:*

1. *card($\mathcal{U}$) $= 0$ that is $T \in (0, T^*]$, where T^* is the smallest positive generalized eigenvalue of Σ.*

2. *$\tau \in (0, \tau^*(T))$ where $\tau^*(T)$ is defined by (6.18).*

6.2 General case

In order to study the general model we first adapt the Walton-Marshall method that allows to reduce the number of commensurate delays. Next, using the appropriated modifications we apply the method presented in the previous chapter to the reduced model.
We consider in this section a class of systems whose dynamics are expressed by the following general characteristic equation:

$$D(s,T,\tau) = p_0(s)(1+sT)^n + \sum_{k=1}^{m} p_k(s)\mathrm{e}^{-sk\tau} = 0. \tag{6.19}$$

for some appropriate pair (T,τ). We will make the usual supplementary *assumptions*: (i) $\deg(p_0) = n_0 > \deg(p_k) = n_k$, $k = 1,m$; (ii) $\sum_{k=0}^{m} p_k(0) \neq 0$; (iii) $p_k(s)$, $k = 0,m$ have no common zeros.

6.2.1 Walton-Marshall reduction method

In the case of multiple commensurate delays, an iterative calculation can be employed to reduce the number of delays [159, 54]. Consider (6.19) is expressed in the more general form:

$$D(s,T,\tau) = p_0(s,T) + \sum_{k=0}^{m} p_k(s)\mathrm{e}^{-sk\tau} = 0. \tag{6.20}$$

Each iteration decrease with one the number of commensurate delays. More precisely, after the first iteration (6.20) is written as

$$D^{(1)}(s,T,\tau) = \sum_{k=0}^{m-1} p_k^{(1)}(s,T)\mathrm{e}^{-sk\tau} = 0 \tag{6.21}$$

where

$$\begin{aligned} p_k^{(1)}(s,T) &= p_0(-s,T)p_k(s) - p_m(s)p_{m-k}(-s), \\ k &= 0,\dots,m-1. \end{aligned} \tag{6.22}$$

We note that

$$\deg_s(p_0^1) = n_0^2 > \deg_s(p_k^1) = \max\{n_0 \cdot n_k,\ n_m \cdot n_{n-k}\},\ k = 1, m-1$$

Consider Ω_k the crossing set of $D^{(k)}(s,T,\tau)$ and Ω the crossing set of $D(s,T,\tau)$.

Proposition 37 *The crossing set Ω_k is a subset of Ω_{k+1}. Furthermore, $\Omega_{k+1} \setminus \Omega_k$ consist in a finite number of real positive numbers ω which are the roots of the following polynomial:*

$$G_k(\omega) = |p_0^{(k)}(j\omega, T)|^2 - |p_{m-k}^{(k)}(j\omega, T)|^2$$

Proof. Since

$$\begin{pmatrix} D^{(k+1)}(s,T,\tau) \\ D^{(k+1)}(-s,T,\tau) \end{pmatrix} = \begin{pmatrix} p_0^{(k)}(s,T) & -p_{m-k}^{(k)}(s,T)\mathrm{e}^{-s(m-k)\tau} \\ -p_{m-k}^{(k)}(-s,T)\mathrm{e}^{s(m-k)\tau} & p_0^{(k)}(-s,T) \end{pmatrix} \cdot \begin{pmatrix} D^{(k)}(s,T,\tau) \\ D^{(k)}(-s,T,\tau) \end{pmatrix},$$

the result is straightforward. ■

For a fixed T, the previous proposition allows to compute the crossing set Ω knowing the crossing set Ω_k of the reduced equation. We will see (Theorem 15) that this is sufficient to develop our stability analysis method.

Next, we focus on stability analysis of the following reduced equation:

$$\begin{aligned} D(s,T,\tau) &= p_0(s,T) + p_1(s,T)\mathrm{e}^{-s\tau} = 0, & (6.23) \\ p_k(s,T) &= \sum_{l=0}^{m_k} a_{kl}(s)T^l,\ k = 0,1. \end{aligned}$$

It is obvious that any system of type (6.19) can be reduced to one of the form (6.23) with $\deg(p_0) > \deg(p_1)$ (we consider the degree with respect to the variable s).

Problem 1 Deriving necessary, and sufficient conditions in terms of (T,τ) for guaranteeing the asymptotic stability of (6.23).

Next, we introduce the appropriate quantities:

1) *card*$(\mathcal{U})$, where $\mathcal{U}$ is the set of roots of $D(s,T,0) = 0$, situated in the *closed right half plane*, and card$(\cdot)$ denotes the cardinality (number of elements).

2) *card*$(\mathcal{S})$, where $S = \{\omega > 0 \mid |p_0(j\omega,T)|^2 - |p_1(j\omega,T)|^2 = 0\}$.

Next, it is easy to see that $D(s,T,0)$ can be rewritten as: $D(s,T,0) = \sum_{k=0}^{m} P_k(s)T^k$, with $P_k(s) = a_{0k}(s) + a_{1k}(s)$ and m represents the biggest power of T. Clearly, $\deg(p_0) \geq \deg(P_k)$, $k = 0, 1, \ldots, n$. We introduce again the matrix pencil: $\Sigma(\lambda) = (\lambda U + V)$, with U, V constructed such as the identity, and the zero-blocks matrices have appropriate dimension, and $H(P_k) \in \mathbb{R}^{n_{p_0} \times n_{p_0}}$ represents the corresponding Hurwitz matrix[2] associated to the polynomial $P_k(s)$ defined above.

Proposition 22 holds again, therefore the behavior of $card(\mathcal{U})$ as a function of T can be done in the same manner as in Chapter 5.

Cardinality of the crossing set

In this paragraph we denote $F(\omega) = |p_0(j\omega, T)|^2 - |p_1(j\omega, T)|^2 = 0$. Obviously, the expression of the polynomial F contains only even powers of ω (in other words F is a polynomial in the variable ω^2). This means that $F(-\omega) = F(\omega)$. Therefore, the number of positive real roots is the half of the total number of real roots of F. In order to compute the cardinality of $\mathcal{S}$ we use the Routh-Sturm algorithm. The notations used in the sequel are rather standard and can be found in [14].

Definition 8 *Let $f(\lambda)$ be a real rational function. Consider α, $\beta \in \mathbb{R} \cup \{\pm\infty\}$ and $N_-^+ =$ the number of jumps of $f(\lambda)$ from $-\infty$ to ∞ (and $N_+^- =$ the number of jumps of $f(\lambda)$ from ∞ to $-\infty$) as λ moves along the real axis between α and β (ignoring any discontinuity at these points).*
The Cauchy index of a real rational function $f(\lambda)$ between the limits α and β is

$$I_\alpha^\beta f(\lambda) = N_-^+ - N_+^- \tag{6.24}$$

An interesting case for our study is obtained when

$$f(\lambda) = \frac{g'(\lambda)}{g(\lambda)} \tag{6.25}$$

where $g(\lambda)$ is a real polynomial.

[2]The order of P_k is smaller (or equal) than n_{p_0}, for all $k = 0, \ldots, n$, and $H(P_k)$ will be constructed as a $n_{p_0} \times n_{p_0}$ matrix by setting the coefficients of high-order terms as zeroes, that is $p_\ell = 0$, for all $\ell > \deg(P_k)$.

Theorem 12 *The number of distinct real roots of the polynomial $g(\lambda)$ in the interval (α, β) equals $I_\alpha^\beta f(\lambda)$ where $f(\lambda) = \dfrac{g'(\lambda)}{g(\lambda)}$.*

The previous theorem reduce the problem of computing the cardinality of $\mathcal{S}$ to the problem of computing the Cauchy index of $\dfrac{F'(\lambda)}{F(\lambda)}$ between $-\infty$ and ∞.

Next, we use Sturm theorem and Routh array to evaluate the index of any rational function of type $\dfrac{g'(\lambda)}{g(\lambda)}$. More precisely, consider $r_{i,k}$ the elements of the Routh array and define the sequence

$$f_0(\lambda), f_1(\lambda), \ldots, f_k(\lambda), \ldots \tag{6.26}$$

where

$$f_0(\lambda) = g(\lambda), \quad f_1(\lambda) = g'(\lambda) \tag{6.27}$$

and

$$f_k(\lambda) = s_k(r_{2k-1,1}\lambda^{n-k} + r_{2k-1,2}\lambda^{n-k-1} + \ldots), \quad n = \deg(g),\, k \geq 2 \tag{6.28}$$

Theorem 13 (Sturm) *Consider f_0 and f_1 two polynomials and α, $\beta \in \mathbb{R} \cup \{\pm\infty\}$. The following relation holds:*

$$I_\alpha^\beta \frac{f_1(\lambda)}{f_0(\lambda)} = V(\alpha) - V(\beta) \tag{6.29}$$

where $V(\theta)$ denotes the number of variations in sign in the sequence (6.26) for a fixed real value $\lambda = \theta$ (any term $f_i(\theta) = 0$ in the sequence is omitted).

Since we are interested to compute $I_{-\infty}^{\infty}\dfrac{F'(\lambda)}{F(\lambda)}$, the number of variation in sign can be computed considering only the dominant term of each f_k. More precisely, we have the following relation:

$$\operatorname{sgn}(f_k(\pm\infty)) = \pm(-1)^{n-k} \cdot \operatorname{sgn}\left(\prod_{i=1}^{2k-1} r_{i,1}\right). \tag{6.30}$$

In the sequel we denote $L_k = \displaystyle\prod_{i=1}^{2k-1} r_{i,1}$. Since the Routh array elements are continuous functions of T, we get L_k are continuous functions of T. Furthermore, $L_k = 0$ reduces to an polynomial equation in T.

Theorem 14 *If $T_1 < T_2 < \ldots < T_N$ is the sequence of all the real positive roots of all L_k, $k = 1, \ldots, n$, then $card(\mathcal{S})$ is constant in each interval (T_k, T_{k+1}).*

We conclude that the characterization of $card(\mathcal{S})$ as a function of T is complete when we solve all the polynomials L_k and compute the value of $card(\mathcal{S})$ for intermediate values of T. We note that for a fixed T we can use again theorem 13 in order to find $card(\mathcal{S})$.

6.2.2 Stability analysis

Next, we assume that all the roots of F are *simple*, this condition is satisfied for almost all T. In the sequel, we explicitly compute the sensitivity of the roots with respect to the delay parameter τ when crossing the imaginary axis, that is, in other words, the *delay crossing direction.*

Theorem 15 *The characteristic equation has a root $j\omega$ on the imaginary axis for some τ_0 if and only if $\omega \in \mathcal{S}$. Furthermore, for $\omega \in \mathcal{S}$, the set of corresponding values of τ where $card(\mathcal{U})$ changes is given by*

$$\mathcal{T}_\omega = \left\{ \frac{1}{\omega} \left[\arg(p_1(j\omega, T)) - \arg(p_0(j\omega, T)) + (2k+1)\pi \right] \geq 0, \quad k \in \mathbb{Z} \right\} \tag{6.31}$$

When increasing the delay, the corresponding crossing direction of characteristic roots is towards instability (stability) when $F'(\omega) > 0 (< 0)$.

The above theorem combined with the continuous dependence of the characteristic roots with respect to the delay, allows to say that $\mathcal{T} = \bigcup_{\omega \in \mathcal{S}} \mathcal{T}_\omega$ makes a partitions of the τ-delay space ($\mathbb{R}_+$) into intervals in which the *number of roots in the open right half plane is constant.* Such an argument will be used in developing our stability region characterization.

Assuming that the system free of delays is asymptotically stable ($\tau, T = 0$), that is $card(\mathcal{U}) = 0$ with $T = 0$, and that the frequency-sweeping condition (5.1) does not hold. Then Theorem 15, combined with the Propositions 22, and 14 give a simple way to compute the first delay-intervals guaranteeing stability:

Proposition 38 *Under the assumption $card(\mathcal{U}) = 0$ for the system free of delays, the system (6.23) is asymptotically stable for all the pairs (T, τ), with*

$0 \leq T < T^*$, where T^* is the smallest positive generalized eigenvalue of Σ, and $\tau \in [0, \tau^*)$, where τ^* is given by:

$$\tau^* = \min_{\omega \in \mathcal{S}(T)} \{\mathcal{T}_\omega(T)\} \tag{6.32}$$

as a function of T, for all $T \in [0, T^*)$.

In other words, Proposition 38 extend the result and the meaning of Proposition 24.

Assuming that the system free-of-delays ($\tau = 0$, $T = 0$) is *unstable* we can extend in the same manner all the results obtained for the stability analysis of the systems with a single discrete delay (section 5.2). We conclude with the result characterizing the asymptotic stability of (6.23):

Proposition 39 *For a given T the system with characteristic equation (6.23) is asymptotically stable if and only if the following conditions are satisfied:*

1. *$card(\mathcal{U}(\mathcal{T}))$ is a strictly positive even integer and the following inequality holds: $card(\mathcal{U}(\mathcal{T})) \leq card(\mathcal{S}(\mathcal{T}))$*

2. *there exists at least one gap value $\tau^* \in \mathcal{T}$, such that: $n_-(\tau^*) = n_+(\tau^*) + card(\mathcal{U}(\mathcal{T}))$.*

Then all gap values $\tau \in (\tau^*, \tau_+^*)$, with $\tau_+^* = \min\{\mathcal{T}^+ \cap (\tau^*, +\infty)\}$ guarantee the asymptotic stability.

6.3 Numerical example

In order to emphasize the properties pointed out in the previous sections we consider some models with a small number of cars on a ring. First, we consider an example with identical behavior of the cars, i.e. $\alpha_k = \alpha$, $\forall k$ (given both by car and driver). We note that this type of model with $\alpha = 2$ (and $\alpha = 1.5$) was also considered in [142]. Next, since the previous model is clear unrealistic, we consider a model with different behaviors of each car, i.e. α_k depends on k.

Example 13 (Model with 3 identical cars) *In this example we consider a conceptual model with $p = 3$ identical cars travelling around a ring. Let us consider $\alpha_k = 2$, $\forall k$ and $n = 1$. Therefore, we get the characteristic equation:*

$$\det[s(1 + sT)^n I - Ae^{-s\tau}] = 0$$

where A is written as

$$A = \begin{pmatrix} -2 & 0 & 2 \\ 2 & -2 & 0 \\ 0 & 2 & -2 \end{pmatrix}$$

Using the algorithm presented above we obtain that $D(s, T, 0) = 0$ has two unstable roots for $T > 1$ and 0 unstable roots for $T \in (0, 1)$. Since we proved that all the crossings are towards instability, we need to plot the stability crossing curves just for $T \in (0, 1)$, the first curve bounds the stability region. The figure 6.1 plots some stability crossing curves corresponding to the non-zero eigenvalues of the matrix A, which are $-3 \pm j\sqrt{3}$. We note that the curves laying under the OT axis present no physically meaning.

Figure 6.2 present the first crossing curves as τ increasing from 0. After this crossing the system becomes more and more unstable.

Example 14 (Model with 4 non-identical cars) *In the sequel, we consider $p = 4$, $n = 2$ and $\alpha_1 = 1$, $\alpha_2 = 7$, $\alpha_3 = 4$ and $\alpha_4 = 5$. The characteristic equation is expressed by (6.6) with A given by:*

$$A = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 7 & -7 & 0 & 0 \\ 0 & 4 & -4 & 0 \\ 0 & 0 & 5 & -5 \end{pmatrix}$$

Applying the method developed in this paper we obtain

$$card(\mathcal{U}_+) = \begin{cases} 0, & if \quad T \in (0,\, 0.135) \\ 2, & if \quad T \in (0.135,\, 0.24) \\ 4, & if \quad T \in (0.24,\, 1.569) \\ 6, & if \quad T > 1.569 \end{cases}$$

The matrix A has three non-zero eigenvalue: -8.3271, $-4.3364 \pm 2.8241j$, and the corresponding crossing curves for $\ell \in \{0, 1, 2, 3\}$ can be seen in figure 6.3. The first stability crossing curves, which separate the stability region from the instability region, can be seen in figure 6.4.

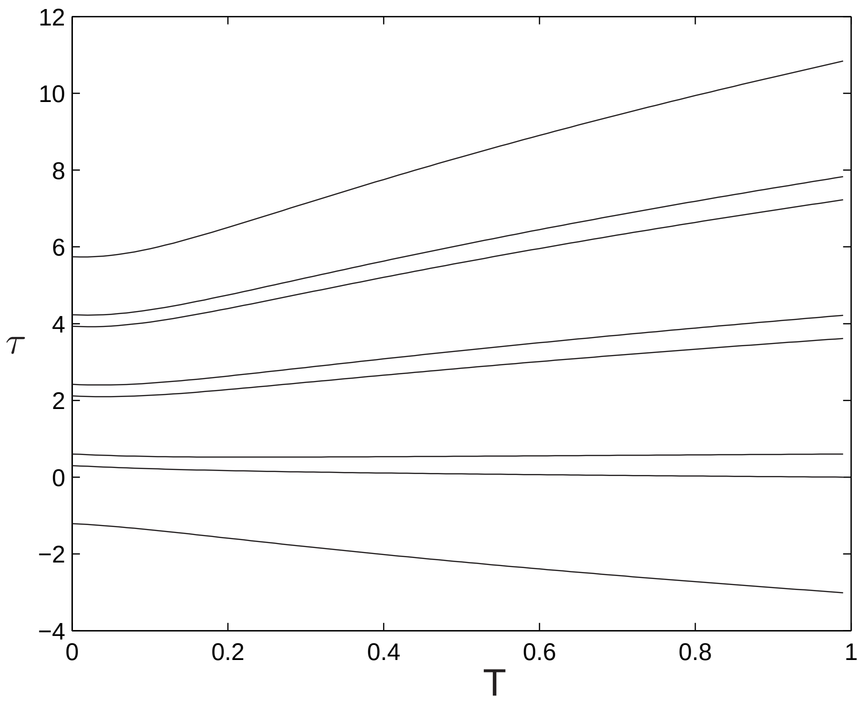


Figure 6.1: τ_ℓ, $\ell \in \{0, 1, 2, 3\}$ versus T

6.4 Concluding remarks

This chapter contains an extension of the algebraic manipulation proposed in Chapter 5. It focusses on the stability problem of a class of linear systems including *distributed delay* with commensurate *gap-values*. First, we present a traffic flow dynamics model in order to motivate our study. After the complete analysis of this model we proceed to the study of a more general class of systems. The Walton-Marshall method was used to simplify the expression of the characteristic equation. In order to find the cardinality of the crossing set we have used an approach based on Routh-Sturm algorithm. A characterization of stability regions in the (mean-delay,gap) parameter-space has been proposed. Illustrative examples are also included.

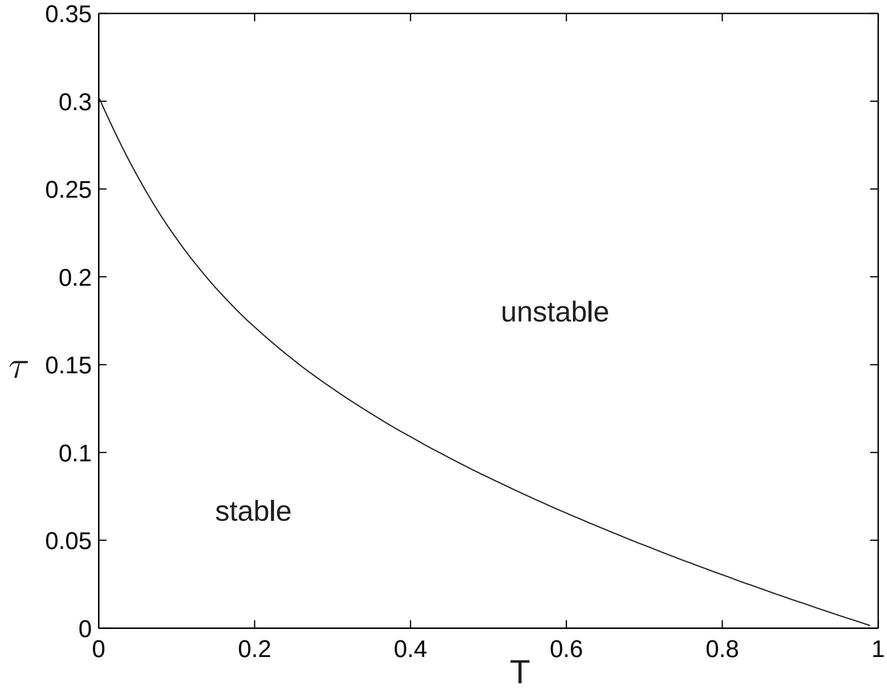


Figure 6.2: τ_0 versus T for $\mu_1 = -3 + j\sqrt{3}$ (the bound of stability region)

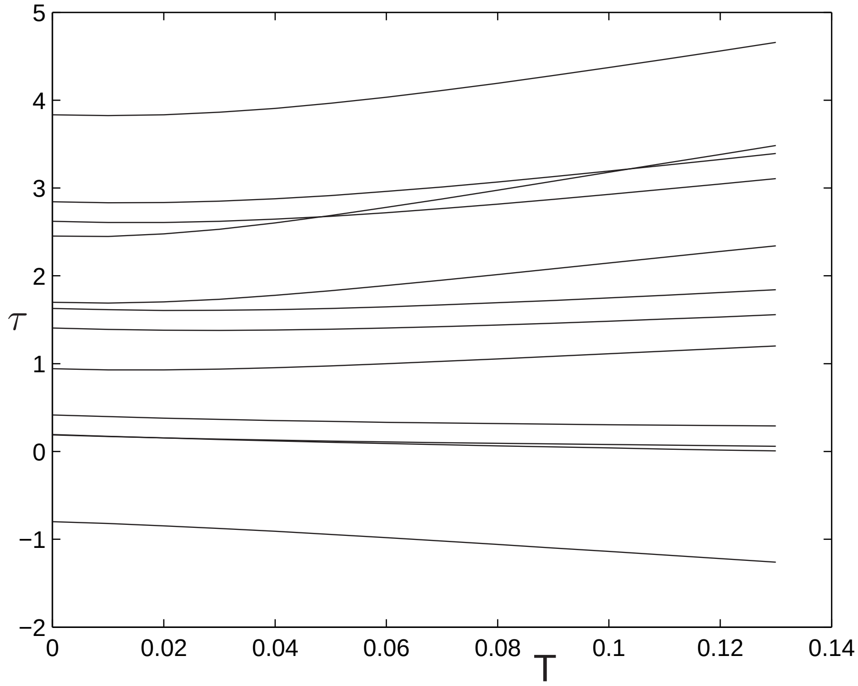


Figure 6.3: τ_ℓ, $\ell \in \{0, 1, 2, 3\}$ versus T

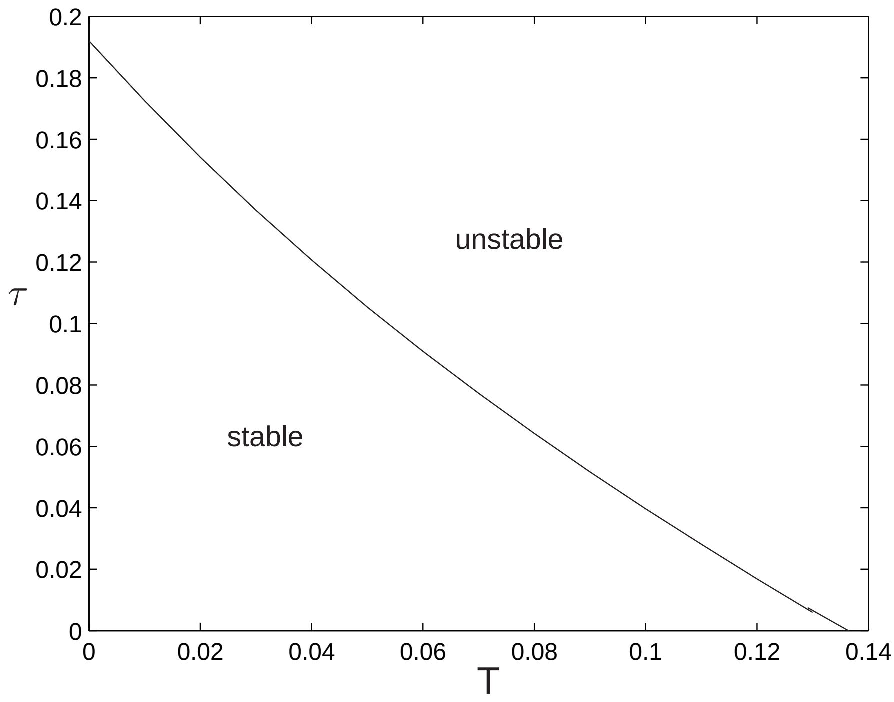


Figure 6.4: τ_0 versus T for $\mu_2 = -4.3364 + 2.8241j$ (the bound of stability region)

Part IV

Further works

Chapter 7

Conclusions and further works

7.1 Final discussions

In this section we point out our contribution of the study of linear dynamical systems in presence of delays. First, we make a brief analysis of our work indicating the main developments presented in this thesis. A list of publications during the thesis completes the discussion.

7.1.1 Contributions

The main interest of the dynamical systems containing delays is twofold: first, the delay systems represent one of the simplest class of infinite dimensional systems, and second, by increasing number of applications in engineering and biology. In the last decades, a lot of research was concentrated in finding simple methods to solve the specific problems whose dynamics are given by delay differential equations. Motivated by some exciting applications in quite distinct research domains (biology, communication over network, traffic flow), in March 2004, we started to work in time-delay systems area. The first aim was to understand the stability mechanism of dynamics behavior of a general class of mathematical equations describing these models (Chapter 2). From the beginning we were interested to find systems that accurately reflect the reality. In biology, the use of distributed delay depicts some stochastic behavior of different components of the model leading to behavior-type models. More recently, it was pointed out that some distributed delays with a gap can be also encountered in the problem of communication over network. Taking into account the memory of the drivers in traffic flow dynamics, we arrive once again to a distributed delay model. Although, we were interested to develop new methods for stability analysis of a class of linear system in presence of distributed delay, the first contribution of this thesis consists in developing some results related to the Smith predictor controller (Chapter 3): regular, and singular cases. If the regular cases have received some attention in the literature, however the analysis of singular cases and related interpretations represents a novelty.

The main contributions of the study of distributed delay systems are presented in Chapter 4, 5 and 6. We note that our algorithms propose simple methods for the analysis of stability regions of the corresponding systems. The method developed in Chapter 4 is based on some geometrical interpretations and its goal is to analyze the stability using the stability crossing curves in the corresponding time-delay parameter space. In the third part of the thesis, containing Chapter 5 and 6, we developed some algebraic techniques based on matrix manipulations. Chapter 5 can be seen as a "dual" of Chapter 4, since it treats the same class of systems using a different approach. Next, the theory is extended for the case of distributed delay systems in presence of commensurate delays. Starting from a traffic flow model we first introduce a direct method to derive the stability region, next, we adapt and combine the Walton-Marshall [159] procedure and the development in Chapter 5 to analyze the stability of a more general class of systems.

Although our approaches are very intuitive, they are computationally oriented. In this context, we note the existence of the DDE-BIFTOOL Matlab-package, developed by a group of researchers from Department of Computer Science of Catholic University of Leuven, which can be used for studying the stability analysis of various delay differential equations. However, in collaboration with Adriana Jianu, student from Craiova University, we prepared a specific and simplified Matlab routine that can be used for our algorithms. I believe that such routines can be integrated in DDE-BIFTOOL for treating particular problems or, if it is not possible, it offers some simple alternative to DDE-BIFTOOL for treating specific problems.

7.1.2 Publications

The publications list during the Ph. D. thesis can be resumed as follows:

1. The geometric approach presented in Chapter 4 was developed in collaboration with one of my promotor Silviu Niculescu and with Keqin Gu:

    - The first results in this direction, named: *Remarks on the stability crossing curves of some distributed delay Systems* were presented by myself in the framework of the "Conference on Differential & Difference Equations and Applications" - Melbourne, Florida, 1-5 August 2005.
    - A final version of our work: *On the Stability Crossing Curves of Some Distributed Delay Systems* was submitted for publication to some journal. (February 2006)

2. In collaboration with Silviu Niculescu, I studied the coherency of our results starting from a classical Biology application: *Old and new in stability analysis of Cushing equation: A geometric perspective*- Physics International Year - 2005 - Bucharest, Romania, 11-13 September 2005 (paper presented by I. C. Morărescu)

3. A variety of application (biology and communication over network) of my work with Silviu Niculescu, Wim Michiels and Keqin Gu, was presented by Silviu Niculescu: *Some simple geometric ideas for the stability analysis of some delay models in biology* - Workshop NSF-CNRS Biology and Control Theory : current challenges 24-25 April, 2006, Toulouse, France

4. The algebraic approach was developed with Silviu Niculescu and Wim Michiels. A part of this work: *Asymptotic stability of some distributed delay systems: An algebraic approach* was presented by myself at the: 13th IFAC Workshop on Control Applications of Optimisation CAO'06, 26-28 April 2006 Paris - Cachan, France.

5. The study on Smith predictor principle is a joint-work with Silviu Niculescu and Keqin Gu.

- Some aspects were presented in Italy - *Some remarks on Smith Predictors: A geometric point of view* - 6th IFAC Workshop on Time Delay Systems, 10-12 July 2006 L'Aquila.
- The complete work:*On the geometry of stability regions of Smith predictors subject to delay uncertainty* was submitted to *IMA Journal of Mathematical Control and Information* (May 2006).

6. The stability of the systems controlled by delayed output feedback was carried out with Silviu Niculescu.

   - Some aspects will be presented in Germany: *Further remarks on stability crossing curvesfor SISO systems controlled by delayed output feedback* - IEEE CCA/CACSD/ISIC, 4-6 October 2006, Munich.
   - The complete work: *Stability crossing curves of SISO systems controlled by delayed output feedback* was submitted to *DCDIS (Dynamics of Continuous, Discrete and Impulsive Systems) Journal*(March 2006).

7. Some part of my thesis, have leaded to the following talks:

   - *Further Remarks on Stability Crossing Curves of Distributed Delay Systems*, Workshop CTS-HYCON, Paris, France,10-12 July 2006.
   - Summer School in Automatics: *Geometric and Algebraic approaches in the Qualitative Analysis of Time Delay systems* - "Politehnica" University of Bucharest, Bucharest, Romania, 23-28 May 2006.

7.2 Research directions

In this section we present some possible future research directions related to the work reported in this thesis.

7.2.1 Other distributions

We note that our developments concern especially models with delays given by a gamma-distributed kernel. It is interesting to develop similar techniques for system including delays given by different probability distributions. Some results in this direction, for first order delay-differential equations (DDEs) can be found in [20], but the characterization is far to be completed. The probability distributions considered are rather standard: Dirac delta function, uniform density, gamma density. However, using a method based on the expectation of the probability density f, the authors of [20] derive sufficient condition for stability of the system

$$\dot{x}(t) = -\alpha x(t) - \beta \int_0^\infty x(t-\tau) f(\tau) \mathrm{d}\tau, \tag{7.1}$$

where α and β are constants such that $\beta > |\alpha|$.
For a general class of linear systems the problem is more exciting and challenging.

In this thesis we considered a gamma distribution

$$g_n(\xi) = \frac{a^{n+1}}{n!} \xi^n e^{-a\xi}.$$

where n is an integer positive number. Although a lot of models can be studied using this probability distribution [2, 88, 130], we note that some models require a gamma-distribution with a rational parameter n. Many results stated in this thesis holds in the case of "rational" gamma-distribution, but a careful analysis and manipulations are necessary.

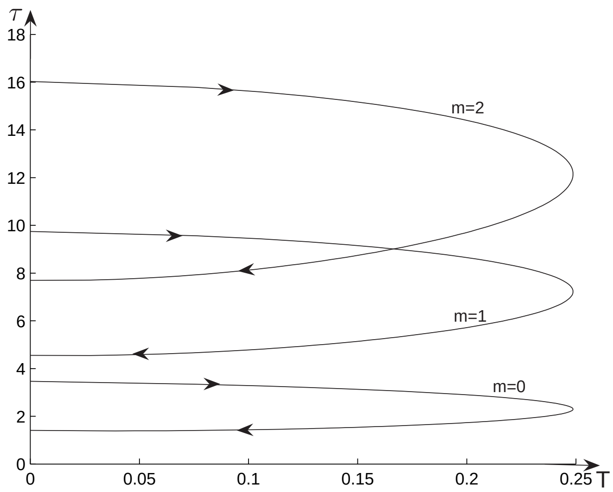
7.2.2 Bifurcation analysis

Throughout this thesis we studied some regular and degenerate cases of models containing discrete and distributed delays. Although the regular cases are completely treated and their behavior is well understood, the analysis of the degenerate cases can be improved and needs further analysis. More precisely, the degenerate cases refer to systems having non-smooth stability crossing curves. The points where the smoothness is lost are typically *bifurcation points.*
The theory of bifurcation applies generally to nonlinear problems, not only when *bifurcation solutions* are *equilibrium solutions* of *evolution problems* like

$$\frac{\mathrm{d}x}{\mathrm{d}t} = F(t, x(t), \mu), \text{ where } -\infty < \mu < \infty \text{ is a parameter,} \tag{7.2}$$

but also in the case of *integral equations*, *nonlinear algebraic* and *functional equations*, *integro-differential* and *functional-differential equations*, especially those of *retarded* type in which the memory effects are important (like in the distributed delay case).

The time-derivative in (7.2) is important in the definition of equilibrium solutions and discussion of their stability [66]. The study of singular points may be connected with stability but the connection is incidental and not intrinsic. Generically speaking the problem of stability depends on whether the system is dissipative or conservative.



We note that it is not necessary for stability crossing curves of functional differential equations to be connected by bifurcation. There are isolated curves that are not connected to other curves through bifurcation. This phenomenon is very common in dynamical problems. However, we are interested to analyze the stability changes around a bifurcation point. *Double-point bifurcation* is the most common form of bifurcation which can occur at a singular point. Other types of bifurcation, *cusp points, triple points*, etc, are less common because they require some relationship between higher-order derivatives of F (for instance, see [66]). Such situations are sometimes called non-generic bifurcation. In the figure 7.1 is easy to identify a *double point* (a singular point through which pass two and only two branches possessing different tangents) and several *regular turning points* (a point at which $\frac{\mathrm{d}T}{\mathrm{d}\tau}$ -

the derivative of $T = T(\tau)$ w.r.t. τ - changes sign).

The elementary theory of singular points of plane curves is discussed in many books on classical analysis (for example, see [33]). To complete the study of bifurcation we shall also need to study the stability of the *bifurcating solutions* [72]. We conclude that the analysis of specific cases pointed out throughout the thesis has to be done carefully in order to see the type of bifurcation points and the behavior of the system around them.

7.2.3 Other class of delay differential equation

As presented in Chapter 6, the algebraic procedure to study the stability region of models described by (4.1) can be extended to more complex models whose dynamics are given by (6.19). The extension of the geometric approach requires an appropriate interpretation or a reduction that keeps the stability crossing curves unchanged. For the particular form of the system (6.4) it is clear that we can easily find the stability crossing curves using the development presented in section 6.1. However, more details and a well formulated algorithm are needed.

Another extension that might be possible is towards the linear neutral systems including distributed delays. Some aspects related to the class of linear neutral systems were considered in Section 3.2 and 3.3. However, we developed our algorithms only for the class of retarded type delay differential equations.

We note that the general methods are not always easy to apply to specific problems. Therefore, in order to study these models, we often need to develop new computationally oriented methods. The aim of this thesis and our future works is to overcome the inconvenient implementation.

Appendix A

Useful results

A.1 Implicit function theorem

In multi-variable calculus of mathematics the implicit function theorem says that for a suitable set of equations, some of the variables are defined as functions of the others. There are some natural limitations on this use of a mathematical relation to define implicit functions, which may be seen in trying to use the unit circle as the graph of a function. Firstly, the projection of the circle onto the x-axis is two-to-one on the interval $(-1, 1)$; this means that y can only be made a local function of x. Further, at the points $(1, 0)$ and $(-1, 0)$, the tangent line to the circle is vertical. This means that y cannot be a differentiable function of x at those points.

The implicit function theorem gets around both these difficulties, which represent the typical obstructions. The implicit function is only locally defined, and points at which the first-order behavior would be problematic are outside the scope of the result.

Theorem 16 *Let $f : \mathbb{R}^{m+n} \mapsto \mathbb{R}^n$ be a continuously differentiable function defined in an open subset of $\mathbb{R}^{m+n}$ which contains the point (a, b), where $a = (x_1, ..., x_m) \in \mathbb{R}^m$ and $b = (y_1, ..., y_n) \in$
$mathbbR^n$, and suppose that f satisfy the following conditions:*

- $f(a, b) = 0$.
- $\left(\left.\dfrac{\partial f_i}{\partial y_k}\right|_{(a,b)} \right)_{1 \leq i \leq n,\ 1 \leq k \leq n}$ *is invertible.*

- $\left(\left.\frac{\partial f_i}{\partial y_k}\right|_{(a,b)} \right)_{1\le i\le n,\ 1\le k\le n}$ *is continuous.*

Then, there exists an $\epsilon > 0$ and an implicit function $\varphi : B(a,\epsilon) \mapsto \mathbb{R}^n$ such that:

- $\varphi(a) = b$*;*
- $f(x,\varphi(x)) = 0, \quad \forall x \in B(a,\epsilon)$*;*
- φ *is a continuous function.*

Furthermore, if f is differentiable (C^k-class respectively) then φ is differentiable (C^k-class respectively) and

$$\frac{d}{dx}\varphi(a) = -\left(\left.\frac{\partial f_i}{\partial y_k}\right|_{(a,b)} \right)^{-1}_{1\le i\le n,\ 1\le k\le n} \cdot \left(\left.\frac{\partial f_i}{\partial x_k}\right|_{(a,b)} \right)_{1\le i\le n,\ 1\le k\le m}$$

A.2 Rouchés theorem

In complex analysis, Rouchés theorem tells us that if the complex-valued functions f and g are holomorphic inside and on some closed contour $\mathcal{C}$, with $|g(z)| < |f(z)|$ on $\mathcal{C}$, then f and $f+g$ have the same number of zeros inside $\mathcal{C}$, where each zero is counted as many times as its multiplicity. This theorem assumes that the contour $\mathcal{C}$ is simple, that is, without self-intersections.
It is possible to provide an informal explanation on why the Rouché's theorem holds.

First we need to rephrase the theorem a little bit. Let $h(z) = f(z)+g(z)$. Notice that f, g holomorphic implies h holomorphic too. Then, with the conditions imposed above, Rouché's theorem says that:

Theorem 17 *If $|f(z)| > |h(z) - f(z)|$ then $f(z)$ and $h(z)$ have the same number of zeros on the interior of $f(z)$.*

Notice that the condition $|f(z)| > |h(z) - f(z)|$ means that for any z, the distance of $f(z)$ to the origin is larger than the length of $h(z) - f(z)$. Informally we can say that the curve $g(z)$ is always closer to the curve $f(z)$ than to the origin.

A.3 Laplace transform

The Laplace transform is an integral transform, perhaps second (only to the Fourier transform) in its utility in solving physical problems. The Laplace transform is particularly useful in solving linear ordinary differential equations. More precisely, this transform is a solution technique which transforms differential equations in the time-domain into algebraic equations in the frequency-domain.

A.3.1 Formal definition

The Laplace transform of a piecewise continuous function $f(t)$, defined for all real numbers $t > 0$ and with an exponential order at infinity (i.e. there exist $\alpha \in \mathbb{R}$ and $M \in \mathbb{R}_+$ such that $|f(t)| < M\mathrm{e}^{\alpha t}$, $\forall t > 0$), is the function $F(s)$, defined by:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{0^-}^{\infty} \mathrm{e}^{-st} f(t)\mathrm{d}t. \tag{A.1}$$

The lower limit of 0^- is short notation to mean $\lim\limits_{\epsilon\to 0, \epsilon>0} -\epsilon$, and assures the inclusion of the entire Dirac delta function $\delta(t)$ at 0 if there is such an impulse in $f(t)$ at 0. The parameter s is in general complex i.e. $s = \sigma + i\omega$. This integral transform has a number of properties that make it useful for analyzing linear dynamical systems. The most significant advantage is that differentiation and integration become multiplication and division, respectively, with s. (This is similar to the way that logarithms change an operation of multiplication of numbers to addition of their logarithms.) This changes integral equations and differential equations to polynomial equations, which are much easier to solve.

A.3.2 Bilateral Laplace transform

When one says "the Laplace transform" without qualification, the unilateral or one-sided transform is normally intended. The Laplace transform can be alternatively defined as the bilateral Laplace transform or two-sided Laplace transform by extending the limits of integration to be the entire real axis. If that is done the common unilateral transform simply becomes a special case of the bilateral transform where the definition of the function being transformed is multiplied by the Heaviside step function.

The bilateral Laplace transform is defined as follows:

$$F(s) = \mathcal{L}\left\{f(t)\right\} = \int_{-\infty}^{+\infty} e^{-st} f(t)\, dt. \tag{A.2}$$

A.4 Riesz representation theorem

Historically, the theorem is often attributed simultaneously to Riesz and Frchet [44]. There are a couple of versions of this theorem. Basically, it says that any bounded linear functional T on the space of compactly supported continuous functions on X is the same as integration against a measure μ,

$$Tf = \int f \mathrm{d}\mu.$$

Here, the integral is the Lebesgue integral.

Because linear functionals form a vector space, and are not *positive*, the measure μ may not be a positive measure. But if the functional T is positive, in the sense that $f \geq 0$ implies that $Tf \geq 0$, then the measure μ is also positive. In the generality of complex linear functionals, the measure μ is a complex measure.

A.4.1 The Hilbert space representation theorem

The Hilbert representation theorem establishes an important connection between a Hilbert space and its dual space: if the ground field is the real numbers, the two are isometrically isomorphic; if the ground field is the complex numbers, the two are isometrically anti-isomorphic. The (anti-) isomorphism is a particular natural one as will be described next.

Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{H}'$ denote its dual space, consisting of all continuous linear operators from $\mathcal{H}$ into the base field $\mathbb{R}$ or $\mathbb{C}$. If x is an element of $\mathcal{H}$, then the function φ_x defined by

$$\varphi_x(y) = (x, y) \quad \forall y \in \mathcal{H}$$

where $(\, , \,)$ denotes the inner product of the Hilbert space, is an element of $\mathcal{H}'$. The Riesz representation theorem states that every element of $\mathcal{H}'$ can be written uniquely as follows:

Theorem 18 *The mapping*

$$\Phi : \mathcal{H} \to \mathcal{H}', \quad \Phi(x) = \varphi_x$$

is an isometric (anti-) isomorphism, meaning that:

- Φ *is bijective.*
- *The norms of* x *and* $\Phi(x)$ *agree:* $||x|| = ||\Phi(x)||$.
- Φ *is additive:* $\Phi(x1 + x2) = \Phi(x1) + \Phi(x2)$.
- *If the base field is* $\mathbb{R}$*, then* $\Phi(\lambda x) = \lambda\Phi(x)$ *for all real numbers* λ.
- *If the base field is* $\mathbb{C}$*, then* $\Phi(\lambda x) = \lambda^*\Phi(x)$ *for all complex numbers* λ*, where* λ^* *denotes the complex conjugation of* λ.

The inverse map of Φ can be described as follows. Given an element φ of $\mathcal{H}'$, the orthogonal complement of the kernel of φ is a one-dimensional subspace of $\mathcal{H}$. Take a non-zero element z in that subspace, and set $x = \varphi(z)/||z||^2 \cdot z$. Then $\Phi(x) = \varphi$.

A.4.2 The representation theorem for linear functionals on $C_c(X)$

The following theorem represents positive linear functionals on $C_c(X)$, the space of continuous complex valued functions of compact support. The Borel sets in the following statement refers to the σ-algebra generated by the open sets.

Definition 9 *A non-negative countably additive Borel measure* μ *on a locally compact Hausdorff space* X *is regular if and only if:*

- $\mu(K) < \infty$ *for every compact* K*;*
- *For every Borel set* E*,*

$$\mu(E) = \inf\{\mu(U) : E \subseteq U, U \text{ open}\}$$

- *The relation*

$$\mu(E) = \sup\{\mu(K) : K \subseteq E, K \text{ compact}\}$$

  holds whenever E is open or when E is Borel and $\mu(E) < \infty$.

Theorem 19 *Let X be a locally compact Hausdorff space. For any positive linear functional ψ on $C_c(X)$, there is a unique Borel regular measure μ on X such that*

$$\psi(f) = \int_X f(x)\, d\mu(x) \quad \forall f \in C_c(X)$$

One approach to measure theory is to start with a Radon measure, defined as a positive linear functional on $C(X)$. This is the way adopted by Bourbaki; it does of course assume that X starts life as a topological space, rather than simply as a set. For locally compact spaces an integration theory is then recovered.

A.4.3 The representation theorem for the dual of $C_0(X)$

The following theorem, also referred to as the Riesz-Markov theorem gives a concrete realization of the dual space of $C_0(X)$, the set of continuous functions on X which vanish at infinity. The Borel sets in the statement of the theorem also refers to the σ-algebra generated by the open sets. This result is similar to the result of the preceding section, but it does not subsume the previous result. See the technical remark below.

If μ is a complex-valued countably additive Borel measure, μ is regular if and only if the non-negative countably additive measure $|\mu|$ is regular as defined above.

Theorem 20 *Let X be a locally compact Hausdorff space. For any continuous linear functional ψ on $C_0(X)$, there is a unique regular countably additive complex Borel measure μ on X such that*

$$\psi(f) = \int_X f(x)\, d\mu(x) \quad \forall f \in C_0(X).$$

The norm of ψ as a linear functional is the total variation of μ, that is

$$\|\psi\| = |\mu|(X).$$

Finally, ψ is positive if and only if the measure μ is non-negative.

Remark 25 *A positive linear functional on $C_c(X)$ may not extend to a bounded linear functional on $C_0(X)$. For this reason the previous results apply to slightly different situations.*

Appendix B

MatLab Routines

In the sequel we present the MatLab routines developed for the algorithms presented in this thesis.

B.1 Smith Predictor principle

```
clear all;

%citesc numaratorul si numitorul lui a1

%Nnum=input('Dati gradul lui numaratorului: ');
%disp('Dati NUMARATORUL lui h:');
%for i=1:Nnum+1
 %fprintf('Dati coeficientul de gradul %d: ',Nnum-i+1)
  %  numh(i)=input('');
%end
%fprintf('\n');
%numh;

%Nden=input('Dati gradul lui numitorului: ');
%disp('Dati NUMITORUL lui h:');
%for i=1:Nden+1
 %fprintf('Dati coeficientul de gradul %d: ',Nden-i+1)
  %  denh(i)=input('');
%end
```

```
%fprintf('\n');
%denh;


%Exemplul 1
numh=[4 1]; denh=[4 4 4];

%Exemplul 2
%numh=[1 sqrt(2)];
%denh=[2 1 8 1];

%Exemplul 3
%numh=[1 2];
%denh=[1 0 2];

Nnum=length(numh); Nden=length(denh);

%afisarea sub forma polinomiala a lui h

syms s w q; disp('Numaratorul lui h este ');%
disp(poly2sym(numh,s)); %
disp('Numitorul lui h este ');
disp(poly2sym(denh,s)); fprintf('\n');


%Impartirea numaratorului lui h in parte reala
%si parte imaginara

numhR=[]; numhI=[]; for i=1:Nnum
 switch (mod(Nnum-i,4))
  case 0
     numhR=[numhR numh(i)];
     numhI=[numhI 0];
  case 1
     numhR=[numhR 0];
     numhI=[numhI numh(i)];
  case 2
     numhR=[numhR -numh(i)];
```

```
     numhI=[numhI 0];
  otherwise
     numhR=[numhR 0];
     numhI=[numhI -numh(i)];
 end
end numhR; numhI;

disp('Real part of the nominator of h ');
numhRP=poly2sym(numhR,w); disp(numhRP);%
disp('Imaginary part of the nominator of h ');
numhIP=poly2sym(numhI,w); disp(numhIP);%
fprintf('\n');

%Impartirea numitorului lui h in parte reala si
%parte imaginara

denhR=[]; denhI=[];%
for i=1:Nden
 switch (mod(Nden-i,4))
   case 0
    denhR=[denhR denh(i)];
    denhI=[denhI 0];
   case 1
    denhR=[denhR 0];
    denhI=[denhI denh(i)];
   case 2
    denhR=[denhR -denh(i)];
    denhI=[denhI 0];
   otherwise
    denhR=[denhR 0];
    denhI=[denhI -denh(i)];
    end
end denhR; denhI;

disp('Real part of the denominator of h ');
denhRP=poly2sym(denhR,w); disp(denhRP); %
disp('Imaginary part of the denominator of h');
denhIP=poly2sym(denhI,w); disp(denhIP); %
```

```
fprintf('\n');


%The end point corresponding to |h|=1/2

A=sym2poly(4*numhRP^2+4*numhIP^2-denhRP^2-denhIP^2);%
A1=roots(A); v1=transp(A1); v1=sort(v1); %
disp('Increasing sequence of the roots:');%
disp(v1); v11=[-inf v1 inf];

C1_interval=[]; for i=1:length(v11)-1
  if v11(i)==-inf && v11(i+1)==inf
    if polyval(A,0)>=0
      C1_interval=v11;
    end
  elseif v11(i)==-inf
    if polyval(A,(v11(i+1)-50))>=0
      C1_interval=[C1_interval v11(i) v11(i+1)];
    end
  elseif v11(i+1)==inf
    if polyval(A,(v11(i)+50))>=0
      C1_interval=[C1_interval v11(i) v11(i+1)];
    end
  elseif polyval(A,(v11(i)+v11(i+1))/2)>=0
      C1_interval=[C1_interval v11(i) v11(i+1)];
  end
end C1_interval

C2_interval=[-inf inf]

int1=[]; for i=1:2:length(C1_interval)-1
    for j=1:2:length(C2_interval)-1
     if C1_interval(i)<=C2_interval(j)
     && C2_interval(j)<=C1_interval(i+1)
     && C1_interval(i)<=C2_interval(j+1)
     && C2_interval(j+1)<=C1_interval(i+1)
 int1=[int1 C2_interval(j) C2_interval(j+1)];%
 disp(1)
```

```
    end

     if C2_interval(j)<C1_interval(i)
     && C1_interval(i)<=C2_interval(j+1)
     && C2_interval(j+1)<=C1_interval(i+1)
 int1=[int1 C1_interval(i) C2_interval(j+1)];%
 disp(2)
   end

    if C1_interval(i)<C2_interval(j)
    && C2_interval(j)<=C1_interval(i+1)
    && C1_interval(i+1)<C2_interval(j+1)
 int1=[int1 C2_interval(j) C1_interval(i+1)];%
 disp(3)
   end

    if C2_interval(j)<C1_interval(i)
    && C1_interval(i)<=C2_interval(j+1)
    && C2_interval(j)<=C1_interval(i+1)
    && C1_interval(i+1)<C2_interval(j+1)
 int1=[int1 C1_interval(i) C1_interval(i+1)];%
 disp(4)
  end
  end
end int1


%crossing set

poz_fin=[]; for i=1:2:length(int1)-1
  if int1(i+1)>0
    if int1(i)<0
      poz_fin=[poz_fin 0 int1(i+1)];
    else
      poz_fin=[poz_fin int1(i) int1(i+1)];
    end
  end
end
```

```
poz_fin

%formez matricea in care am intervalele

x=zeros(length(poz_fin)/2,3); k=1;
 for i=1:2:length(poz_fin)-1
  x(k,1)=poz_fin(i);
  x(k,2)=poz_fin(i+1);
 % display the interval type
  if x(k,1)==0
     x(k,3)= x(k,3);
  elseif find(v1==x(k,1))
     x(k,3)=10;
  end
  if find(v1==x(k,2))
     x(k,3)=x(k,3)+1;
  end
 fprintf('Interval %f %f of type ',x(k,1),x(k,2));
  switch x(k,3)
    case 1
     fprintf('01');
    case 11
     fprintf('11');
  end
    fprintf('\n');
   k=k+1;
end


%tau1 and tau2 computation

for i=1:length(poz_fin)/2
  for u=1:4
   for v=1:4
    tau1_fplus1=[];
    tau2_fplus1=[];
    tau1_fminus1=[];
    tau2_fminus1=[];
```

```
 a=x(i,1);
 b=x(i,2);
 k=0;
 l=0;
 for o=a:(b-a)*0.005:b
 numhRV=polyval(numhR,o);
 numhIV=polyval(numhI,o);
 denhRV=polyval(denhR,o);
 denhIV=polyval(denhI,o);
 q=acos(1/(2*sqrt((numhRV^2+numhIV^2)/
 (denhRV^2+denhIV^2))));
 tau1plus=(angle(complex(numhRV,numhIV)/
 complex(denhRV,denhIV))+(2*u-1)*pi+q)/o;
 tau2plus=(angle(complex(numhRV,numhIV)/
 complex(denhRV,denhIV))+(2*v)*pi-q)/o;
 tau1minus=(angle(complex(numhRV,numhIV)/
 complex(denhRV,denhIV))+(2*u-1)*pi-q)/o;
 tau2minus=(angle(complex(numhRV,numhIV)/
 complex(denhRV,denhIV))+(2*v)*pi+q)/o;
 tau1_fplus1=[tau1_fplus1 tau1plus];
 tau2_fplus1=[tau2_fplus1 tau2plus];
 tau1_fminus1=[tau1_fminus1 tau1minus];
 tau2_fminus1=[tau2_fminus1 tau2minus];

 tau1_fplus=[];
 tau2_fplus=[];
 tau1_fminus=[];
 tau2_fminus=[];
 n=1;
 r=1;
 for m=1:length(tau2_fplus1)
  if tau2_fplus1(m)>tau1_fplus1(n)
   tau1_fplus=[tau1_fplus tau1_fplus1(n)];
   tau2_fplus=[tau2_fplus tau2_fplus1(m)];
  end
 n=n+1;
 k=k+1;
end
```

```
    for q=1:length(tau2_fminus1)
      if tau2_fminus1(q)>tau1_fminus1(r)
       tau1_fminus=[tau1_fminus tau1_fminus1(r)];
       tau2_fminus=[tau2_fminus tau2_fminus1(q)];
      end
     r=r+1;
   end

end

  plot(tau1_fplus,tau2_fplus,'b');
  hold on;
  plot(tau1_fminus,tau2_fminus,'g');
  hold on;
  x1=xlabel('\tau 1','FontWeight','bold');
  y1=ylabel('\tau 2','FontWeight','bold');
title('The crossing curves','FontWeight','bold');

end end
y=[];
 for h=1:tau2_fplus(1)
  y(h)=h;
 end
 plot(y,y,'r')
hold on; pause;%
hold off;
end
pause;
hold off;
```

B.2 Distributed Delay with a gap

```
clear;
%p=[1 0 2];           %tip 12 21
%q=[1 0];

%p=[1 0 3 0 2];       %tip 32 22 21
```

```
%q=[1 4];

%p=[1 3 2];             %tip 11
%q=[sqrt(10) 0];

%p=[1 3];               %tip 31
%q=[5];

n=input('Degree of P: '); for i=1:n+1
 fprintf('The coefficient coresponding %
 to degree %d :',n-i+1)
   p(i)=input('');
end

m=input('Degree of Q: '); for i=1:m+1
 fprintf('The coefficient coresponding %
 to degree %d :',m-i+1)
   q(i)=input('');
end

nP=length(p); nQ=length(q);

%punerea lui P si Q sub forma polinomiala
syms s w; disp('P(s) = ');
disp(poly2sym(p,s)); disp('Q(s)=');
disp(poly2sym(q,s));

%impartirea lui P in Preal si Pimag
preal=[]; pimag=[]; for i=1:nP
    switch (mod(nP-i,4))
        case 0
            preal=[preal p(i)];
            pimag=[pimag 0];
        case 1
            pimag=[pimag p(i)];
            preal=[preal 0];
        case 2
            preal=[preal -p(i)];
```

```
            pimag=[pimag 0];
        otherwise
            pimag=[pimag -p(i)];
            preal=[preal 0];
   end
end

disp('Real part of P(jw)= ');
polPR=poly2sym(preal,w);
disp(polPR);

disp('Imaginary part of P(jw)= ');
polPI=poly2sym(pimag,w);
disp(polPI);

%impartirea lui Q in Q real si Q imaginar
qreal=[]; qimag=[]; for i=1:nQ
    switch (mod(nQ-i,4))
        case 0
            qreal=[qreal q(i)];
            qimag=[qimag 0];
        case 1
            qimag=[qimag q(i)];
            qreal=[qreal 0];
        case 2
            qreal=[qreal -q(i)];
            qimag=[qimag 0];
        otherwise
            qimag=[qimag -q(i)];
            qreal=[qreal 0];
   end
end

disp('Real part of Q(jw)= ');
polQR=poly2sym(qreal,w);
disp(polQR); disp('Imaginary part of Q(jw)= ');
polQI=poly2sym(qimag,w); disp(polQI);
```

```
%calculul radacinilor ec |P(jw)|=|Q(jw)|
R=polPR^2+polPI^2-polQR^2-polQI^2;
polR=sym2poly(R);  %returneaza coeficientii lui R
rad=roots(polR); disp('Roots of P(jw)=Q(jw) are:');
disp(rad');

%eliminarea valorilor imaginare
k=1; for i=1:length(rad)
 if imag(rad(i))==0 %dc exista parte imaginara %
 atunci de elimina
        rad_reale(k)=rad(i);
        k=k+1;
    end
end

rad_sort=sort(rad_reale); %radacinile sortate
disp('The increasing sequence of P(jw)=Q(jw) is'); %
disp(rad_sort);

%radacinile se introduc in matricea x in prima si %
a doua coloana
disp('The intervals given for P(jw)=Q(jw) are'); %
x=[]; k=1; for
i=1:2:length(rad_sort)
  x(k,1)=rad_sort(i);
  x(k,2)=rad_sort(i+1);
  k=k+1;
end disp(x);

%calculul expr |P(jw)|=0
PRV=poly2sym(preal,w) PIV=poly2sym(pimag,w)
if (PIV==0)
    PP=PRV;
else
    PP=PRV^2+PIV^2;
end PV=roots(sym2poly(PP)) PV=PV'; PVS=[];

%eliminarea valorilor imaginare
```

```
k=1;
for i=1:length(PV)
    if (imag(PV(i))==0)
        PVS(k)=PV(i);
        k=k+1;
    end
end PVS=sort(PVS); disp('The roots of P(jw)=0 are');
disp(PVS);


y=[x zeros(length(rad_sort)/2,length(PVS))]; %
for i=1:(length(rad_sort)/2)
    k=2;
    for j=1:length(PVS)
        if x(i,1)<PVS(j)&& PVS(j)<x(i,2)
            y(i,k)=PVS(j);
            k=k+1;
        end
    end
    y(i,k)=x(i,2);
end

%afisarea intervalelor fara eliminarea val negative
x=[]; k=1;
for i=1:length(rad_sort)/2
    for j=1:length(PVS)+1
       if y(i,j+1)~=0
            x(k,1)=y(i,j);
            x(k,2)=y(i,j+1);
            k=k+1;
        end
    end
end
disp('The intervals |P(jw)|<=|Q(jw)|'); disp(x);


disp('crossing set')
z=[]; y=[]; j=1;
```

```
for i=1:k-1
    z(1)=x(i,1);
    z(2)=x(i,2);
    if z(2)>0
        if z(1)<0
            z(1)=0;
            end
        % else dc nu e negativ at afiseaza intervalul
        y(j,1)=z(1);
        y(j,2)=z(2);
        if find(rad_sort==z(1))
            if find(rad_sort==z(2))
                y(j,3)=11;
                disp('Type 11');
            else
                y(j,3)=12;
                disp('Type 12');
            end
        elseif z(1)==0
            if find(PVS==z(2))
                y(j,3)=32;
                disp('Type 32');
            else
               y(j,3)=31;
               disp('Type 31');
            end
            elseif find(PVS==z(1))
                if find(PVS==z(2))
                    y(j,3)=22;
                    disp('Type 22');
                else
                    y(j,3)=21;
                    disp('Type 21');
                end
        end
        j=j+1;
        disp(z);
    end
```

```
end

%afisarea grafica
n=input('n= ');
for k=1:j-1
    a=y(k,1);
    b=y(k,2);
    switch(y(k,3))
        case 11
            b=b+(b-a)*0.01;
        case 12
            b=b-(b-a)*0.1;
        case 21
            a=a+(b-a)*0.1;
            b=b+(b-a)*0.01;
        case 22
            a=a+(b-a)*0.1;
            b=b-(b-a)*0.1;
        case 32
            a=a+(b-a)*0.1;
            b=b-(b-a)*0.1;
        otherwise
            b=b+(b-a)*0.01;
    end
    for m=1:3
        %n=1;
        T1=[];
        tau1=[];
     for o=a:(b-a)*0.01:b
       valPR=polyval(preal,o);
       valPI=polyval(pimag,o);
       valQR=polyval(qreal,o);
       valQI=polyval(qimag,o);
 T=(1/o)*sqrt(((valQR^2+valQI^2)/
 (valPR^2+valPI^2))^(1/n)-1);
 T1=[T1 T];
 tau=(1/o)*(angle(complex(valQR,valQI))-
 angle(complex(valPR,valPI))-n*atan(o*T)+pi+(m-1)*2*pi);
```

```
 tau1=[tau1 tau];
    end
  plot(T1,tau1,'k');
    x1=xlabel('T','FontWeight','bold');
    y1=ylabel('\tau','FontWeight','bold');
title('Stability crossing curves','FontWeight','bold');
%       switch(y(k,3))
%           case 11
%title(['Intervalul [',num2str(y(k,1)),',',
%                     num2str(y(k,2)),']
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%      case 12
%title(['Intervalul [',num2str(y(k,1)),',',
%                     num2str(y(k,2)),')
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%      case 21
%title(['Intervalul (',num2str(y(k,1)),',',
%                     num2str(y(k,2)),']
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%      case 22
%title(['Intervalul (',num2str(y(k,1)),',',
%                     num2str(y(k,2)),')
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%      case 31
%title(['Intervalul (',num2str(y(k,1)),',',
%                     num2str(y(k,2)),']
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%     otherwise
%title(['Intervalul (',num2str(y(k,1)),',',
%                     num2str(y(k,2)),')
% de tipul ',num2str(y(k,3))],'FontWeight','bold');
%    end
 text(T,tau,[' \leftarrow m = ',num2str(m-1)],
 'fontsize',10);
   hold on;
  end
end hold off;
```

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