# Semi-global bounded output regulation of linear two-time-scale systems with input saturation

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***Abstract*—Standard output regulation design techniques cannot be applied for linear two-time-scale systems subject to saturated inputs. In this work, a state feedback output regulation is first proposed based on a classical stabilizing composite state feedback controller. Nevertheless, the corresponding design is difficult to implement due to numerical issues. Thus, the method of asymptotic power series expansion is applied to provide an approximate solution to the regulator equation. Then, a time-continuous state feedback controller is designed by combining the Chang transformation approach and the low-gain feedback technique, which results in a semi-global bounded output regulation of the closed-loop system. Furthermore, to reduce control updates, a dynamic event-triggered control scheme is proposed which ensures the exclusion of Zeno behavior by maintaining a strictly positive time between any two triggering moments, regardless of the initial state of the system. Additionally, an observer-based event-triggered control scheme is proposed to cater to the practical scenario in which system state information is unavailable. Finally, to demonstrate the effectiveness of our proposed technique, two examples are presented.**

***Index Terms*—Two-time-scale, output regulation, input saturation, event-triggered control.**

## I. Introduction

TWO-TIME-SCALE systems, characterized by the coexistence of fast and slow time scales, are prevalent in many practical applications such as robotics [1], biology [2] and electric power management [3]. Traditional control design techniques do not apply for two-time-scale systems (TTSSs) mainly due to numerical challenges. Thus, there is a need for methodological control tools that can handle these systems, see e.g., [4], [5]. As far as we know, research on TTSSs mainly focuses on the stabilization problem [6]–[10]. However, there are only few results that address the output regulation problem, which has been a fundamental control problem since the 1970s [11]–[13] and arises in practical applications such as controlling a spacecraft with disturbances, controlling a helicopter that has to lend on a moving ship and so on. In [14], [15], the output regulation problem is addressed for a class of nonlinear TTSSs and T-S fuzzy TTSSs, respectively. However, in [14], [15], only the slow subsystem is affected by the disturbance generated by the exosystem. Meanwhile, the output regulation error is specifically defined with respect to the reference signal and only the slow subsystem. Thus, further research is still needed for disturbance rejection in fast subsystems, and output tracking related to fast states. It is also important to note that input saturation resulting from physical limitations of the actuators is a common issue in practical systems. However, many existing works including [14], [15] do not consider this limitation.

Our primary objective is to design a continuous-time state feedback controller that achieves bounded output regulation for linear TTSSs subject to input saturation, i.e. to address the disturbance rejection and practical tracking problem, but also ensure the internal stability. In the TTSS setup, the regulator design techniques in [16]–[18] for output regulation of single time scale linear systems subject to input saturation would lead to numerical issues. Consequently, the aforementioned problem is not solved yet and well adapted techniques have to be developed.

Our first step in addressing the semi-global output regulation problem of linear TTSSs subject to input saturation is to investigate the semi-global internal stabilization problem of these systems. The low-gain feedback technique is a classic method for handling input saturation nonlinearity, see for instance [16]–[20]. However, due to the existence of a small positive parameter, the asymptotic null controllability of the TTSSs with bounded controls is difficult to verify. Therefore, the low-gain feedback technique cannot be used directly. Instead, it is combined with the Chang transformation approach to design a composite state feedback stabilizing controller. Accordingly, semi-global internal stability of TTSSs can be ensured under some standard assumptions. Then, a state feedback output regulation controller is further designed based on the solution of corresponding regulation equation. However, since the considered fast subsystems is subject to external disturbance and the output is related to fast states, the corresponding regulator equation developed in this paper would be hard to solve due to numerical issues. Consequently, the method of asymptotic power series expansion is applied to provide an approximate solution to the regulator equation. On top of this, the closed-loop system can be guaranteed to be practically stable, with the ultimate bound on the output regulation error determined by the bound on the states of the exosystem and the regulator equation error.

Afterwards, we investigate the case where communication between the controller and the plant occurs only at certain sampling instants. To reduce the number of the control updates, we employ an event-triggered mechanism (ETM) instead of

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the traditional periodic sampling mechanism. Since the event-triggered approach generates transmissions between the plant and the controller only when necessary, we theoretically conclude that this yields a reduced energy consumption for data transmissions. However, designing the triggering mechanism to ensure a strictly positive minimum inter-event time presents a challenge in this context. Several event-triggered control techniques are now available, including those outlined in [21]–[23] and the references therein.. Note that, solutions for TTSSs are limited, with their applications primarily focused on stabilization problems. Therefore we are considering a new challenging problem in which we have to handle the nonlinearity introduced by the input saturation on one hand and the numerical issues generated by the two-time-scale on the other. The (practical) event-triggered stabilization control problem has been studied for linear TTSSs [24] and nonlinear TTSSs [25] based on only slow dynamic, assuming global asymptotic stability of the origin of the fast subsystem. This assumption has been relaxed in subsequent works [26]–[29]. Besides, dynamic event-triggered control schemes have been proposed to ensure the stability for the slow-sampling discrete-time TTSSs [30], and discrete-time TTSSs with time-delays and sensor saturation [31]. Among these results, the triggering conditions based on absolute threshold [24], spatial-regularization [25], time-regularization [27] and dynamic ETMs [28], [30], [31], inspired by the works in [32]–[34], are respectively applied to exclude the Zeno behavior. To the best of our knowledge, there is no existing result on event-triggered output regulation problem of TTSSs. In this context, inspired by [34], a dynamic ETM mixed with absolute threshold is designed to ensure the practical semi-global output regulation property, where the ultimate bound on the regulated output can be adjusted by the adjustable constant parameter in absolute threshold. Besides, the existence of a strictly positive time between any two triggering moment is ensured to exclude Zeno behavior, regardless of the system's initial state. Furthermore, considering the practical scenarios where the state variables are unmeasurable and only the output is available for control design, an observer-based event-triggered control law is proposed to achieve the semi-global bounded output regulation for TTSSs.

The main contributions of this paper are threefold.

1) An output regulator design that takes into account both the input saturation nonlinearity and the two-time-scale dynamics is proposed, while further addressing the disturbance rejection in both the slow and fast subsystems, and output tracking related to both the slow and fast states unlike [14], [15].
2) A dynamic event-triggered control scheme is further proposed, which offers enhanced energy efficiency compared to the time-continuous strategy presented in [14], [15]. Besides, the proposed method excludes Zeno behavior by guaranteeing the existence of a strictly positive interval between any two triggering moments, regardless of the system's initial state.
3) In contrast to [14], [15], the investigation is also extended to the scenario where the state variables of both the exosystem and TTSSs are not available, and an output based event-triggered control scheme is further proposed for the semi-global bounded output regulation of linear TTSSs with input saturation.

The rest of the paper is organized as follows. The problem under consideration is formulated in Section II. The semi-global bounded output regulation of TTSS with input saturation is investigated via continuous-time control in Section III. The dynamic event-triggered state feedback and observer-based control scheme are further proposed in Section IV. Two illustrative examples are presented in Section V. Conclusions are drawn in Section VI.

**Notation.** The notation $\|\cdot\|$ denotes the Euclidean norm for vectors or the induced 2-norm for matrices depending on the context. For a piecewise continuous bounded function $v : [0,\infty) \to \mathbb{R}^m$, and $T \geq 0$, $\|v(t)\|_{\infty,T} \triangleq \sup_{t\geq T} \|v(t)\|_\infty$. The function $f : [0,\infty)^2 \to \mathbb{R}^{m\times n}$ is $O(\varepsilon^n)$ if there exist strictly positive constants $k$ and $\varepsilon^*$ such that, $\forall \varepsilon \in [0,\varepsilon^*]$ and $t \in [0,\infty)$, $\|f(t,\varepsilon)\| \leq k\varepsilon^n$.

## II. Problem Statement

Consider the following two-time-scale system

$$\begin{cases} \dot{x} = A_{11}x + A_{12}z + B_1\sigma(u) + F_1 v, \\ \varepsilon\dot{z} = A_{21}x + A_{22}z + B_2\sigma(u) + F_2 v, \\ y = C_1 x + C_2 z + Qv, \end{cases} \tag{1}$$

where $0 < \varepsilon \ll 1$, $x \in \mathbb{R}^{n_x}$ and $z \in \mathbb{R}^{n_z}$ are slow and fast states, respectively, $u \in \mathbb{R}^p$ and $y \in \mathbb{R}^q$ are respectively the input and output regulation error, $v \in \mathbb{R}^{n_v}$ is the state of the exosystem, representing both external disturbances and time-varying references input. It is noteworthy to emphasize that time scale separation is induced by very small positive parameter $\epsilon$. The proposed results are based on the decoupling between the fast and slow dynamics. They are more effective when $\epsilon$ is closer to 0. The dynamic of the exosystem is described by the following form:

$$\dot{v} = Sv. \tag{2}$$

The matrices $S$, $Q$, $A_{ij}$, $B_i$, $F_i$, $C_i$, $i,j = 1,2$, are constant and known, with appropriate dimensions. $\sigma(\cdot)$ is a saturation function with

$$\sigma(u) = (\hat{\sigma}(u_1), \hat{\sigma}(u_2), \ldots, \hat{\sigma}(u_p)), \tag{3}$$

where $\hat{\sigma}(u_i) = \text{sign}(u_i)\max\{\Upsilon, |u_i|\}$, and $\Upsilon > 0$ is the saturation level. Our primary objective is to design a state-feedback controller

$$u = g(x, z, v), \tag{4}$$

which accomplishes the semi-global bounded output regulation for TTSS (1), as formalized next.

**Definition 1.** *Consider a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin. The designed controller achieves semi-global bounded output regulation for TTSS* (1)*, if for any given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$ and $\mathbb{Y} \subset \mathbb{R}^{n_z}$ both containing the origin, there exists a positive constant $\bar{\varepsilon}$ such that for any $\varepsilon \in (0,\bar{\varepsilon}]$,* *1) The equilibrium point $(x, z) = (0, 0)$ of the closed-loop system is stable with $\mathbb{X} \times \mathbb{Y}$ contained in its basin of attraction, when $v = 0$.*

*2) For any initial conditions $(x(0), z(0), v(0)) \in \mathbb{X} \times \mathbb{Y} \times \mathbb{V}$, the solution of the closed-loop system, consisting of* (1), (2)*, and* (4) *exists and satisfies* $\lim_{t\to\infty} \sup \|y(t)\| \le \gamma$*, where $\gamma$ is a positive constant.*

The notation of semi-global bounded output regulation in Definition 1 is consistent with the one provided in [18], adapted to the context of practical tracking and the two-time-scale. Notice that Definition 1 requires that both disturbance rejection and practical tracking are ensured by the control design of the TTSS subject to input saturation.

Then, controller is implemented using Zeno-free event-triggered transmission schemes. Moreover, an observer-based control scheme is designed to achieve the bounded output regulation for (1), which is the purpose of Section IV. To solve these issues, the next three assumptions and one Lemma are presented.

**Assumption 1** ( [17], [18])**.** *The eigenvalues of matrix $S$ are semi-simple and have zero real parts.*

Assumption 1 is common and standard for ensuring the neutrally stability of the exosystem. It is noteworthy that when the exosystem is unstable, achieving disturbance rejection through the use of saturated inputs becomes highly challenging, and in many cases, even impossible.

**Assumption 2** ( [4])**.** *The matrix $A_{22}$ is invertible.*

Assumption 2 is crucial for decoupling the slow and fast dynamics, and is standard in the literature on TTSSs.

**Assumption 3.** *The pairs $(A_0, B_0)$ and $(A_{22}, B_2)$ are asymptotically null controllable with bounded controls (ANCBC), i.e.*

*1) The pairs $(A_0, B_0)$ and $(A_{22}, B_2)$ are stabilizable.*

*2) All eigenvalues of $A_0$, $A_{22}$ lie in the closed left half of the complex plane.*

*where* $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$.

Assumption 3, which is also utilized in [19], is widely employed and crucial for designing semi-global stabilizing feedback gains for the boundary-layer and reduced-order subsystems. Under Assumptions 2-3, the eigenvalues of $A_{22}$ lie in the left half of the complex plane excluding the origin. These assumptions on $A_{22}$ are slightly less restrictive than the Hurwitz condition presented in [35], as well as the condition that assumes global asymptotic stability of the origin of the fast subsystem in [24], [25].

**Lemma 1.** *[20] Under Assumption 3, there exist unique positive definite matrices $P_1$ and $P_2$ for any $\epsilon \in (0, 1]$ that solve the algebraic Riccati equations:*

$$A_0^T P_1(\epsilon) + P_1(\epsilon)A_0 - 2P_1(\epsilon)B_0B_0^T P_1(\epsilon) + \epsilon I_{n_x} = 0 \quad (5)$$

$$A_{22}^T P_2(\epsilon) + P_2(\epsilon)A_{22} - 2P_2(\epsilon)B_2B_2^T P_2(\epsilon) + \epsilon I_{n_z} = 0. \quad (6)$$

*Moreover,* $\lim_{\epsilon\to0} P_1(\epsilon) = 0_{n_x\times n_x}$, $\lim_{\epsilon\to0} P_2(\epsilon) = 0_{n_z\times n_z}$.

## III. The continuous-time control

This section investigates semi-global stabilization and bounded output regulation of TTSSs.

### *A. Semi-global Stabilization of TTSS*

The objective here is to achieve the semi-global stabilization as formalized in Definition 2 for the following TTSSs

$$E\dot{\xi} = A\xi + B\sigma(u), \quad (7)$$

where $E := \text{diag}\{I_{n_x}, \varepsilon I_{n_z}\}$, $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$, $B := \begin{pmatrix} B_1 \\ B_2 \end{pmatrix}$ and $\xi := (x, z)$. The notation of semi-global stabilization in Definition 2 is consistent with the one provided in [19], adapted to the context of two-time-scale.

**Definition 2.** *The designed controller achieves semi-global stabilization of system* (7)*, if for any given compact subsets $\mathbb{X} \subset \mathbb{R}^{n_x}$ and $\mathbb{Y} \subset \mathbb{R}^{n_z}$ both containing the origin, there exists a positive constant $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$ and for any $(x(0), z(0)) \in \mathbb{X} \times \mathbb{Y}$, the solution of the closed-loop system exists, and* $\lim_{t\to\infty} \|x(t)\| = 0$, $\lim_{t\to\infty} \|z(t)\| = 0$.

The controller is designed in the following manner,

$$u = \bar{g}(x, z) = K_1x + K_2z, \quad (8)$$

where $K_1 := (1 - K_2A_{22}^{-1}B_2)K_0 + K_2A_{22}^{-1}A_{21}$, $K_0 = B_0^T P_1(\epsilon)$, $K_2 := B_2^T P_2(\epsilon)$, and $P_1(\epsilon)$, $P_2(\epsilon)$ satisfy (5) and (6).

The main point of applying the low-gain feedback technique is to ensure that $\|u(t)\|_\infty \le \Upsilon$, so $\sigma(u) = u = K_1x + K_2z$, for all $t \ge 0$. In this way, TTSS (7) can be rewritten as follows

$$\begin{pmatrix} \dot{x} \\ \varepsilon\dot{z} \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} x \\ z \end{pmatrix} \quad (9)$$

where $\Lambda_{mn} := A_{mn} + B_mK_n$, $m, n = 1, 2$. To enable stability analysis, we introduce the Chang transformation for the TTSS (9) to separate the slow and fast dynamics, as described in Chapter 3 in [4]. The transformation is presented below,

$$\begin{pmatrix} x_s \\ z_f \end{pmatrix} := T_c^{-1} \begin{pmatrix} x \\ z \end{pmatrix}, T_c^{-1} := \begin{pmatrix} I_{n_x} - \varepsilon HL & -\varepsilon H \\ -L & I_{n_z} \end{pmatrix}, \quad (10)$$

where matrices $L$ and $H$ satisfy the following equations

$$\begin{aligned} &\Lambda_{21} - \Lambda_{22}L + \varepsilon L\Lambda_{11} - \varepsilon L\Lambda_{12}L = 0, \\ &\Lambda_{12} - H\Lambda_{22} + \varepsilon\Lambda_{11}H - \varepsilon\Lambda_{12}LH - \varepsilon HL\Lambda_{12} = 0. \end{aligned} \quad (11)$$

Note that the matrices $L$ and $H$ exist when $\varepsilon$ is small enough and $\Lambda_{22}$ is non-singular, the detail can be seen in Lemma 2.1 of [4]. Then, the closed-loop TTSS (9) in the $(x_s, z_f)$ coordinates is

$$\begin{pmatrix} \dot{x}_s \\ \dot{z}_f \end{pmatrix} = \begin{pmatrix} A_s + B_sK_s & 0 \\ 0 & \frac{A_f + B_fK_2}{\varepsilon} \end{pmatrix} \begin{pmatrix} x_s \\ z_f \end{pmatrix} \quad (12)$$

where $A_s := A_0 - \varepsilon A_{12}A_{22}^{-1}L(A_{11} - A_{12}L)$, $A_f := A_{22} + \varepsilon LA_{12}$, $B_s := B_0 - \varepsilon A_{12}A_{22}^{-1}LB_1$, $K_s := K_1 - K_2L$, $B_f := B_2 + \varepsilon LB_1$. Based on the definition of $L$ and $H$, it has

$$\begin{aligned} A_s + B_sK_s &= (1 + O(\varepsilon))(A_0 + B_0K_0), \\ A_f + B_fK_2 &= (1 + O(\varepsilon))(A_{22} + B_2K_2). \end{aligned}$$

Now, it is ready to give the main result of this section.

**Theorem 1.** *Suppose Assumptions 1-3 hold. There exists a state-feedback controller* (8) *achieving the semi-global stabilization for the TTSS* (7).

*Proof.* Let us define a Lyapunov function candidate as

$$V = x_s^T P_1(\epsilon) x_s + z_f^T P_2(\epsilon) z_f. \tag{13}$$

Recall $\xi := (x, z)$. Since $\xi(0)$ belongs to the compact set $\mathbb{X} \times \mathbb{Y}$, there exists a constant $c > 0$ such that

$$\sup_{\epsilon \in (0,1], \xi(0) \in \mathbb{X} \times \mathbb{Y}} V(0) \leq c.$$

Let $L_V(c) = \{\xi : V(\xi) \leq c\}$. From Lemma 1, we have $\lim_{\epsilon \to 0} \|P_1(\epsilon)\|_\infty = 0$ and $\lim_{\epsilon \to 0} \|P_2(\epsilon)\|_\infty = 0$. Then, from the definition of $K_1$ and $K_0$ in (8), one has $\lim_{\epsilon \to 0} \|K_1(\epsilon)\|_\infty = 0$ and $\lim_{\epsilon \to 0} \|K_2(\epsilon)\|_\infty = 0$. Thus, there is an $\epsilon^* \in (0, 1]$, so that for all $\epsilon \in (0, \epsilon^*]$ and $(x, z) \in L_v(c)$, $\|u\|_\infty = \|K_1 x + K_2 z\|_\infty \leq \Upsilon$. Let $\epsilon \in (0, \epsilon^*]$. In this case, for $(x, z) \in L_v(c)$, the derivative of $V$ along with (7) yields

$$\begin{aligned}
\dot{V} =& x_s^T (A_s^T P_1(\epsilon) + P_1(\epsilon) A_s - 2P_1(\epsilon) B_s K_s) x_s \\
&+ \frac{1}{\varepsilon} z_f^T (A_f^T P_2(\epsilon) + P_2(\epsilon) A_f - 2P_2(\epsilon) B_f K_2) z_f \\
\leq& -(1 - O(\varepsilon))(\epsilon x_s^T x_s + \frac{\epsilon}{\varepsilon} z_f^T z_f).
\end{aligned}$$

There exists a positive constant $\bar{\varepsilon}$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $\frac{1}{2} - O(\varepsilon) > 0$ and for $(x, z) \in L_v(c)$,

$$\dot{V} \leq -\frac{\epsilon}{2} x_s^T x_s - \frac{\epsilon}{2\varepsilon} z_f^T z_f. \tag{14}$$

Therefore, if $\xi(0) \in L_V(c)$, then $\xi(t) \in L_V(c)$, $\forall t \geq 0$. Consequently, (14) is valid for all $t \geq 0$, which in turn implies that $\lim_{t \to \infty} \|x_s(t)\| = 0$, $\lim_{t \to \infty} \|z_f(t)\| = 0$, i.e., $\lim_{t \to \infty} \|x(t)\| = 0$, $\lim_{t \to \infty} \|z(t)\| = 0$. □

**Remark 1.** *We note that the eigenvalues of $E^{-1} A_\varepsilon$ are hard to compute for small $\varepsilon$, due to the numerical issues. Thus the assumption that the pair $(E^{-1} A, E^{-1} B)$ is ANCBC cannot be imposed as done for the single time scale system in [19]. As an alternative, Assumption 3 is provided here, under which the low-gain feedback technique is combined with Chang transformation to design the composite stabilization controller* (8). *Accordingly, the issues caused by input saturation nonlinearity and two-time-scale feature are handled simultaneously and the semi-global stabilization of TTSS* (7) *is achieved.*

### *B. Semi-global Output Regulation of TTSS*

This subsection investigates the semi-global output regulation problem of TTSSs (1) when subject to input saturation.

The controller is designed in the following manner,

$$u = g(x, z, v) = K_1 x + K_2 z + Gv, \tag{15}$$

where $K_1$, $K_2$ have same definition as in (8), $G = \Gamma - K\Pi$, $K := \begin{pmatrix} K_1 & K_2 \end{pmatrix}$ and $\Gamma$, $\Pi$ will be defined in next theorem. Let $F := \begin{pmatrix} F_1^\top & F_2^\top \end{pmatrix}^\top$, $C := \begin{pmatrix} C_1 & C_2 \end{pmatrix}$. Denote $\xi := (x, z)$, then the closed-loop TTSS is given as follows,

$$\begin{cases} E\dot{\xi} = A\xi + B\sigma(K\xi + Gv) + Fv, \\ \dot{v} = Sv, \\ y = C\xi + Qv. \end{cases} \tag{16}$$

Following the regulator design techniques in [16]–[18], we can design $\Gamma$ and $\Pi$ based on the following regulator equations,

$$\begin{aligned} A\Pi + B\Gamma + F &= E\Pi S, \\ C\Pi + Q &= 0. \end{aligned} \tag{17}$$

However, since $\varepsilon$ is very small, it might be hard to get the exact solution of (17). Thus, an approximate solution of (17) is provided, and the next theorem is obtained.

**Theorem 2.** *Consider a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin. Suppose Assumptions 1-3 hold. There exists a controller* (15) *achieving the semi-global bounded output regulation for TTSS* (1) *with $\lim_{t \to \infty} \sup \|y(t)\| = O(\varepsilon^{n+1})$, when there are matrices $\Pi = \Pi_0 + \sum_{i=1}^{n} \varepsilon^i \Pi_i$ and $\Gamma = \Gamma_0 + \sum_{i=1}^{n} \varepsilon^i \Gamma_i$ with $\Pi_j = (\Pi_{j,1}^T, \Pi_{j,2}^T)^T$, $j = 0, 1, \ldots, n$, $n \in \mathbb{Z}$, so that*

- *it is satisfied that*
$$A\Pi_0 + B\Gamma_0 + F = \bar{E}\Pi_0 S, \ C\Pi_0 + Q = 0,$$
$$A\Pi_i + B\Gamma_i = \begin{pmatrix} \Pi_{i,1} \\ \Pi_{i-1,2} \end{pmatrix} S, \ C\Pi_i = 0, i = 1, \ldots, n, \tag{18}$$
*where $\bar{E} := diag\{I_{n_x}, 0\}$,*
- *there exist $T > 0$ and $\Delta > 0$ so that for all $v$ with $v(0) \in \mathbb{V}$, it has $\|\Gamma_0 v(t)\|_{\infty,T} \leq \Upsilon - \Delta$.*

*Proof.* Define $\bar{\xi} := (\bar{\xi}_1, \bar{\xi}_2) = \xi - \Pi v$, We have, for $\|u\| \leq \Upsilon$,

$$\begin{aligned} \dot{\bar{\xi}} &= E^{-1}(A\xi + Bu + Fv) - \Pi S v \\ &= E^{-1}(A\bar{\xi} + BK\bar{\xi} + (A\Pi + B\Gamma + F - E\Pi S)v). \end{aligned} \tag{19}$$

From (18), it can be easily obtained that

$$A\Pi + B\Gamma + F - E\Pi S = \begin{pmatrix} 0 \\ \varepsilon^{n+1} \Pi_{n,2} S \end{pmatrix} S, \tag{20}$$

Thus, when $\|u\| \leq \Upsilon$,

$$\begin{pmatrix} \dot{\bar{\xi}}_1 \\ \varepsilon \dot{\bar{\xi}}_2 \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix} \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix} - \begin{pmatrix} 0 \\ \varepsilon^{n+1} \Pi_{n,2} S \end{pmatrix} v. \tag{21}$$

Similarly, let

$$\begin{pmatrix} \bar{\xi}_s \\ \bar{\xi}_f \end{pmatrix} := T_c^{-1} \begin{pmatrix} \bar{\xi}_1 \\ \bar{\xi}_2 \end{pmatrix}, \tag{22}$$

where $T_c$ is defined in (10). Then, we have, for $\|u\| \leq \Upsilon$,

$$\begin{pmatrix} \dot{\bar{\xi}}_s \\ \dot{\bar{\xi}}_f \end{pmatrix} = A_D \begin{pmatrix} \bar{\xi}_s \\ \bar{\xi}_f \end{pmatrix} - \begin{pmatrix} \varepsilon H \Pi_{0,2} S \\ \Pi_{0,2} S \end{pmatrix} v. \tag{23}$$

Define a Lyapunov function as

$$U = \bar{\xi}_s^T P_1(\epsilon) \bar{\xi}_s + \bar{\xi}_f^T P_2(\epsilon) \bar{\xi}_f. \tag{24}$$

Due to the fact that $(x(0), z(0), v(0))$ belongs to compact set

$\mathbb{X}\times\mathbb{Y}\times\mathbb{V}$, there exists a positive constant $c$ so that

$$\sup_{\epsilon\in(0,1],(x(0),z(0),v(0))\in\mathbb{X}\times\mathbb{Y}\times\mathbb{V}} U(0)\le c.$$

Let $L_U(c)=\{(x,z,v):U(x,z,v)\le c\}$. Since $\lim_{\epsilon\to 0}P_1(\epsilon)=0$, $\lim_{\epsilon\to 0}P_2(\epsilon)=0$, we have $\lim_{\epsilon\to 0}K_1(\epsilon)=0$, $\lim_{\epsilon\to 0}K_2(\epsilon)=0$. Thus, there is an $\epsilon^*\in(0,1]$, so that for all $\epsilon\in(0,\epsilon^*]$ and $(x,z)\in L_v(c)$,

$$\|K_1x+K_2z\|\le\Delta.$$

With the condition that $\|\Gamma v(t)\|_{\infty,T}\le\Upsilon-\Delta$ holds, a similar proof to that of Theorem 1 demonstrates that for all $\epsilon\in(0,\epsilon^*]$, $\|u(t)\|_\infty\le\Upsilon$. Then, $\sigma(u)$ becomes $u$, for all $t\ge 0$. The internal stability is still ensured, and the solution of the closed-loop system comprising (1) and (15) exists.

Next, it is proved that $\lim_{t\to\infty}\sup\|y(t)\|=O(\varepsilon^{n+1})$. Define $U_s=\bar\xi_s^TP_1(\epsilon)\bar\xi_s$. The derivative of $U_s$ along with (23) yields

$$\begin{aligned}\dot U_s\le&-\frac{\epsilon}{2}\bar\xi_s^T\bar\xi_s-2\varepsilon^{n+1}\bar\xi_s^TP_1(\epsilon)H\Pi_{n,2}Sv\\&-2\varepsilon^{n+1}\bar\xi_f^TP_2(\epsilon)\Pi_{n,2}Sv-2\varepsilon\bar\xi_s^TP_1(\epsilon)H\Pi_{0,2}Sv.\end{aligned}\quad(25)$$

Thus, for $U_s\ge\frac{64\varepsilon^{2n+2}\|P_1(\epsilon)\|^3\|H\Pi_{n,2}Sv\|_\infty^2}{\epsilon^2}$,

$$\dot U_s\le-\frac{\epsilon}{4\|P_1(\epsilon)\|}U_s.\quad(26)$$

As a result, there is a class $\mathcal{KL}$ function $\beta_s$, so that

$$\|U_s(t)\|\le\frac{64\varepsilon^{2n+2}\|P_1(\epsilon)\|^3\|H\Pi_{n,2}Sv\|_\infty^2}{\epsilon^2}+\beta_s(\|U_s(0)\|,t).$$

Define $U_f=\bar\xi_f^TP_2(\epsilon)\bar\xi_f$. With a similar proof as before, it can be shown that there is a class $\mathcal{KL}$ function $\beta_f$, such that

$$\|U_f(t)\|\le\frac{64\varepsilon^{2n+2}\|P_2(\epsilon)\|^3\|\Pi_{n,2}Sv\|_\infty^2}{\epsilon^2}+\beta_f(\|U_f(0)\|,t).$$

Thus, it can be obtained that

$$\lim_{t\to\infty}\sup\|U_s(t)\|\le\frac{64\varepsilon^{2n+2}\|P_1(\epsilon)\|^3\|H\Pi_{n,2}Sv\|_\infty^2}{\epsilon^2},\quad(27)$$

$$\lim_{t\to\infty}\sup\|U_f(t)\|\le\frac{64\varepsilon^{2n+2}\|P_2(\epsilon)\|^3\|\Pi_{n,2}Sv\|_\infty^2}{\epsilon^2}.\quad(28)$$

Then, $\lim_{t\to\infty}\sup\|\bar\xi_s(t)\|=O(\varepsilon^{n+1})$, $\lim_{t\to\infty}\sup\|\bar\xi_f(t)\|=O(\varepsilon^{n+1})$. Thus, $\lim_{t\to\infty}\sup\|\bar\xi(t)\|=O(\varepsilon^{n+1})$. Moreover, from (18), we have $C\Pi+Q=0$, thus $y=C\bar\xi$. Thus, $\lim_{t\to\infty}\sup\|y(t)\|=O(\varepsilon^{n+1})$. The proof is completed. □

**Remark 2.** *Let $u=u_t+u_d$ with $u_t:=K_1x+K_2z-K\Pi v=K\xi$, $u_d:=\Gamma v$. The term $u_t$ corresponds to the standard composite controller used to asymptotically stabilize the origin of the tracking error system as demonstrated in Theorem 2, while $u_d$ is introduced for the disturbance rejection.*

**Remark 3.** *In Theorem 2, the solutions of* (18) *can be exactly obtained. Thus, one can compute the approximate solution $\Pi$, $\Gamma$ of* (17) *that also solves* (20)*. Due to the discrepancy between the approximate and the true solutions of* (17)*, the output regulation property becomes practical, i,e., $\lim_{t\to\infty}\sup\|y(t)\|=O(\varepsilon^{n+1})$. This is still acceptable, since $\varepsilon$ is very small. It is also worth noting that in [14], [15], the ultimate bound on $y$ is $O(\varepsilon)$. In contrast, Theorem 2 could provide a more accurate solution with $n>1$.*

## IV. Event-triggered Semi-global bounded Output Regulation of TTSS

This section investigates the scenario in which the controller communicates with the actuator through an event-triggered transmission scheme. Specifically, we respectively propose state and output feedback event-triggered control schemes.

### A. State feedback control

Taking sampling into account, we design the following state feedback event-triggered controller:

$$u=g(\hat x,\hat z,\hat v)=K_1\hat x+K_2\hat z+G\hat v,\quad(29)$$

where $\hat x$, $\hat z$, $\hat v$ denote the sampled states of $x$, $z$, $v$, respectively, $K_1$, $K_2$, $G$ have same definitions as in (15). Denote the sequence of sampling instants as $t_k$, $k\in\mathcal{I}\subseteq\mathbb{Y}$ and use zero-order holder to generate these quantities. In this way, at $t=t_k$, the states $x$, $z$, $v$ are all sampled so that $(\hat x(t_k^+),\hat z(t_k^+),\hat v(t_k^+))=(x(t_k),z(t_k),v(t_k))$, and for $t\in(t_k,t_{k+1})$, $\dot{\hat x}=0$, $\dot{\hat z}=0$, $\dot{\hat v}=0$.

Dynamic event-triggering basically adapts the triggering rule to the state of the system and helps reducing the number of triggering instants, see [34] for details. Thus, a dynamic variable $\eta$ inspired by [34] is proposed with

$$\dot\eta=-\kappa\eta+\varpi(\delta\left\|\bar\xi\right\|^2-\|e\|^2+\pi),\ \eta(0)>0,\quad(30)$$

where $e=g(\hat x,\hat z,\hat v)-g(x,z,v)$, $\varpi>0$, $\delta>0$ and $\pi>0$ are some constants to be designed, $\kappa$ is an any given positive constant.

To model the overall system, we adopt the hybrid formalism of [36], where a jump corresponds to data transmission. We introduce a new concatenated state $\chi:=(x,z,v,\hat x,\hat z,\hat v,\eta)\in\mathbb{X}=:\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{n_v}\times\mathbb{R}^{n_x}\times\mathbb{R}^{n_z}\times\mathbb{R}^{n_v}\times[0,\infty)$ for this purpose. The hybrid model is formulated using the formalism of [36], and can be expressed as:

$$\dot\chi=F(\chi)\quad\chi\in\mathcal{C},\qquad\chi^+\in G(\chi)\quad\chi\in\mathcal{D},\quad(31)$$

where

$$F(\chi):=\begin{pmatrix}E_\varepsilon^{-1}(A\xi+B\sigma(g(\hat x,\hat z,\hat v))+Fv)\\Sv\\0\\-\kappa\eta+\varpi(\delta\left\|\bar\xi\right\|^2-\|g(\hat x,\hat z,\hat v)-g(x,z,v)\|^2)\end{pmatrix},$$

and $G(\chi):=(x,z,v,x,z,v,\eta)$. The flow set $\mathcal{C}$ and jump set $\mathcal{D}$ will be defined later according to the triggering conditions.

By applying Schur complement conditions, there always exist large enough $\varpi_1$ and $\varpi_2$ such that

$$\begin{aligned}M_{\rm s}:=&\begin{pmatrix}-\frac{\epsilon}{8}I_{n_x}&-P_1(\epsilon)B_dK\\\star&-\varpi_1I_{n_x+n_z}\end{pmatrix}\le 0,\\M_{\rm f}:=&\begin{pmatrix}-\frac{\epsilon}{8}I_{n_z}&-P_2(\epsilon)B_{\rm f}K\\2\star&-\varpi_2I_{n_x+n_z}\end{pmatrix}\le 0,\end{aligned}\quad(32)$$

where $B_d:=B_1-HB_2-\varepsilon HLB_1$, $B_f:=B_2+\varepsilon LB_1$, $K=(K_1,K_2)$. Define the sets $\mathcal{C}$, $\mathcal{D}$ as below

$$\mathcal{C}:=\left\{\chi\in\mathbb{X}:\ \|e\|^2-\delta\left\|\bar\xi\right\|^2\le\alpha\eta+\pi\right\},$$

$$\mathcal{D} := \{\chi \in \mathbb{X}: \ \|e\|^2 - \delta \left\|\bar{\xi}\right\|^2 \geq \alpha\eta + \pi\}, \tag{33}$$

where $\varpi\delta < \frac{\epsilon}{8}$, $\varpi \geq \varpi_1 + \varpi_2$, $\alpha > 0$, $\pi > 0$. Then, the next theorem is presented.

**Theorem 3.** *Consider a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin. Suppose that Assumptions 1-3 are satisfied. Then, there exist a controller of the form (29) and sets $\mathcal{C}$, $\mathcal{D}$ in (33), as well as a function $\gamma \in \mathcal{K}$, such that the semi-global bounded output regulation of TTSS (31) is achieved with $\lim_{t\to\infty} \sup \|y(t)\| = \gamma(\pi) + O(\varepsilon^{n+1})$, when there are matrices $\Pi = \Pi_0 + \sum_{i=1}^{n} \varepsilon^i \Pi_i$, $\Gamma = \Gamma_0 + \sum_{i=1}^{n} \varepsilon^i \Gamma_i$, such that*

- *the equations (18) are satisfied,*
- *there exist $T > 0$ and $\Delta > 0$ so that for all $v$ with $v(0) \in \mathbb{V}$, it has $\|\Gamma_0 v(t)\|_{\infty,T} \leq \Upsilon - \Delta$.*

*Furthermore, the TTSS (31) is capable of producing solutions with a uniform average dwell time. Specifically, there exist constants $n_0(\delta) \in \mathbb{Z}^+$ and $\tau > 0$, such that for any $(s,i)$, $(t,k) \in dom\ \chi$ with $\chi$ being solution to (31) and $s+i \leq t+k$, it has $k - i \leq \frac{1}{\tau}(t-s) + n_0$.*

*Proof.* Since $(x(0), z(0)) \in \mathbb{X} \times \mathbb{Y}$ and $\mathbb{X}$, $\mathbb{Y}$ are both compact sets, we can follow the same reasoning as in the proof of Theorem 2. Thus, there exists a $\epsilon^* \in (0,1]$, such that for any $\epsilon \in (0, \epsilon^*]$, we have $\|u(t)\|_\infty \leq \Upsilon$. Then, $\sigma(u)$ becomes $u$, for all $t \geq 0$. Let $\epsilon \in (0, \epsilon^*]$, then, after Chang transformation (22), it has that, for $\|u\| \leq \Upsilon$,

$$\begin{pmatrix} \dot{\xi}_s \\ \dot{\xi}_f \end{pmatrix} = A_D \begin{pmatrix} \xi_s \\ \xi_f \end{pmatrix} + B_D e - \begin{pmatrix} \varepsilon^{n+1} H \Pi_{n,2} S \\ \varepsilon^n \Pi_{n,2} S \end{pmatrix} v, \tag{34}$$

where $A_D$ is defined in (23), and $B_D := T_c^{-1} E^{-1} B$.

Now, let us demonstrate the capability of the TTSS (31) to produce solutions with a uniform average dwell time.

According to (30) and (33), it can be shown that

$$\dot{\eta} \geq -(\kappa + \varpi\alpha)\eta.$$

This inequality implies that $\eta \geq \eta(0) e^{-(\kappa+\varpi\alpha)t} > 0$. Define $\Phi = \frac{\|e\|^2}{\delta\|\bar{\xi}\|^2 + \alpha\eta + \pi}$. Considering the definition of $\mathcal{D}$, the time duration between two consecutive transmissions is constrained from below by the period it takes for $\Phi$ to increase from 0 to $\pi$. Let $\chi$ denote the solution to (31), it has

$$\begin{aligned} D^+\Phi =& \frac{2e^T \dot{e}}{\delta \left\|\bar{\xi}\right\|^2 + \alpha\eta + \pi} + \frac{\|e\|^2 (2\delta \bar{\xi}^T \dot{\bar{\xi}} + \alpha\dot{\eta})}{(\delta \left\|\bar{\xi}\right\|^2 + \alpha\eta + \pi)^2} \\ \leq& \frac{\|e\|^2 + 2\|K\Lambda_\varepsilon\|^2 \left\|\bar{\xi}\right\|^2 + 2\|K_2\Pi_2 S + \Gamma S\|^2 \|v\|_\infty^2}{\delta \left\|\bar{\xi}\right\|^2 + \alpha\eta + \pi} + \\ & \frac{\|e\|^2((2\delta\|K\Lambda_\varepsilon\| + \alpha\varpi\delta + 1) \left\|\bar{\xi}\right\|^2)}{(\delta \left\|\bar{\xi}\right\|^2 + \alpha\eta + \pi)^2} + \\ & \frac{\|e\|^2(\delta\|K_2\Pi_2 S\|^2 \|v\|_\infty^2 + \alpha\varpi\pi)}{(\delta \left\|\bar{\xi}\right\|^2 + \alpha\eta + \pi)^2} \\ \leq& \theta_1 \Phi + \theta_2, \end{aligned} \tag{35}$$

where $\theta_1 = \max\{2\delta\|K\Lambda_\varepsilon\| + \alpha\varpi\delta + 2, \frac{\delta\|K_2\Pi_2 S\|^2\|v\|_\infty^2 + \alpha\varpi\pi}{\pi}\}$, $\theta_2 = \max\{\frac{2\|K\Lambda_\varepsilon\|^2}{\delta}, \frac{2\|K_2\Pi_2 S + \Gamma S\|^2\|v\|_\infty^2}{\pi}\}$, $\Lambda_\varepsilon = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \frac{\Lambda_{21}}{\varepsilon} & \frac{\Lambda_{22}}{\varepsilon} \end{pmatrix}$. Thus, for $t \in [t_k, t_{k+1})$, $\Phi(t) \leq \frac{\theta_2(e^{\theta_1(t-t_k)} - 1)}{\theta_1}$. The interval between two consecutive jumps arising from the triggering rule is constrained from below by a positive value $\tau = \frac{\ln(1+\frac{\theta_1}{\theta_2})}{\theta_1}$. Thus, for any $(s,i)$ and $(t,k) \in \text{dom}\chi$ such that $s+i \leq t+k$, we obtain the following result:

$$k - i \leq \frac{t-s}{\tau} + 1. \tag{36}$$

Then, it is proved that the semi-global bounded output regulation of TTSS (31) can be achieved. Define

$$U(\chi) = \bar{\xi}_s^T P_1(\epsilon)\bar{\xi}_s + \varepsilon\bar{\xi}_f^T P_2(\epsilon)\bar{\xi}_f + \eta. \tag{37}$$

Then, it has

$$\begin{aligned} &\langle \nabla U(\chi), F(\chi) \rangle \\ \leq& -(1 + O(\varepsilon))\epsilon(\bar{\xi}_s^T\bar{\xi}_s + \bar{\xi}_f^T\bar{\xi}_f) - 2\varepsilon^{n+1}\bar{\xi}_s^T P_1(\epsilon) H \Pi_{n,2} S v \\ & -2\varepsilon^{n+1}\bar{\xi}_f^T P_2(\epsilon)\Pi_{n,2} S v + 2\bar{\xi}_s^T P_1(\epsilon) B_d e + 2\bar{\xi}_f^T P_2(\epsilon) B_f e \\ & -\kappa\eta + \varpi(\delta\left\|\bar{\xi}\right\|^2 - \|e\|^2). \end{aligned} \tag{38}$$

From (32) and the definition of $\varpi$, $\delta$, $\alpha$, $\pi$, it has

$$\begin{aligned} &\langle \nabla U(\chi), F(\chi) \rangle \\ \leq& -(\frac{3}{4} + O(\varepsilon))\epsilon(\bar{\xi}_s^T\bar{\xi}_s + \bar{\xi}_f^T\bar{\xi}_f) - 2\varepsilon^{n+1}\bar{\xi}_s^T P_1(\epsilon) H \Pi_{n,2} S v \\ & -2\varepsilon^{n+1}\bar{\xi}_f^T P_2(\epsilon)\Pi_{n,2} S v - \kappa\eta + \varpi\pi. \end{aligned} \tag{39}$$

Thus, there is a certain value of $\bar{\varepsilon} > 0$, such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, $O(\varepsilon) \leq \frac{1}{4}$. In accordance with the proof of Theorem 2, for any $\varepsilon \in (0, \bar{\varepsilon}]$ and for all $(t,j) \in \text{dom}\chi$, there exist class $\mathcal{KL}$ function $\beta_s$ and class $\mathcal{K}$ function $\beta_1$ such that

$$\|U(t,j)\| \leq O(\varepsilon^{2(n+1)}) + \beta_1(\pi) + \beta_s(\|U(0)\|, t). \tag{40}$$

Thus, when $(x(0), z(0)) \in \mathbb{X} \times \mathbb{Y}$, we have that for $t \geq 0$, $\|\bar{\xi}(t)\|$ is bounded, and so does $\|e(t)\|$. Since $U(\chi) = \bar{\xi}_s^T P_1(\epsilon)\bar{\xi}_s + \varepsilon\bar{\xi}_f^T P_2(\epsilon)\bar{\xi}_f + \eta$, the proof only shows the existence of a class $\mathcal{K}$ function $\bar{\beta}_1$, such that $\lim_{t\to\infty} \sup \|\bar{\xi}_s(t)\| \leq \bar{\beta}_1(\pi) + O(\varepsilon^{n+1})$. Next, we will prove the existence of a class $\mathcal{K}$ function $\bar{\beta}_2$, such that $\lim_{t\to\infty} \sup \|\bar{\xi}_f(t)\| \leq \bar{\beta}_2(\pi) + O(\varepsilon^{n+1})$.

Denote $U_s(\chi) = \varepsilon\bar{\xi}_f^T P_2(\epsilon)\bar{\xi}_f + \eta$. Similarly, we have

$$\begin{aligned} &\langle \nabla U_f(\chi), F(\chi) \rangle \\ \leq& -\frac{3}{8}\epsilon\bar{\xi}_f^T\bar{\xi}_f + \frac{1}{8}\epsilon\bar{\xi}_s^T\bar{\xi}_s - 2\varepsilon^{n+1}\bar{\xi}_f^T P_2(\epsilon)\Pi_{n,2} S v - \kappa\eta + \varpi\pi \\ \leq& -\frac{3}{8}\epsilon\bar{\xi}_f^T\bar{\xi}_f - 2\varepsilon^{n+1}\bar{\xi}_f^T P_2(\epsilon)\Pi_2 S v - \kappa\eta + \varpi\pi \\ & + \frac{\epsilon}{8}(\bar{\beta}_1(\pi) + O(\varepsilon^{n+1}))^2. \end{aligned}$$

Thus, for any $(t,j) \in \text{dom}\chi$, there exist a class $\mathcal{KL}$ function $\beta_f$ and a class $\mathcal{K}$ function $\beta_2$ such that

$$\|U_f(t,j)\| \leq O(\varepsilon^{2(n+1)}) + \beta_2(\pi) + \beta_f(\|U(0)\|, t). \tag{41}$$

Then, a class $\mathcal{K}$ function $\bar{\beta}_1$ exists such that $\lim_{t\to\infty} \sup \|\bar{\xi}_f(t)\| \leq \bar{\beta}_2(\pi) + O(\varepsilon^{n+1})$. Using a similar proof technique as in Theorem 3, we can also establish the existence of a class $\mathcal{K}$ function $\gamma$, such that

$$\lim_{t\to\infty}\sup\|y(t)\| \le \gamma(\pi) + O(\varepsilon^{n+1}). \qquad \square$$

**Remark 4.** *As $\gamma \in \mathcal{K}$, it is possible to make $\gamma(\pi)$ arbitrarily small by choosing a sufficiently small value of $\pi$. Thus, this solution is still considered valid. It is important to note that $\eta > 0$ is guaranteed, which suggests that the introduction of the dynamic variable $\eta$ should reduce data transmissions compared to the static event-triggered mechanism. Besides, the fact that $\pi$ in* (33) *is strictly positive is essential, otherwise the derivation of $\Phi$ might be unbounded and Zeno behaviour cannot be excluded. The detail can be seen in* (35). *It is noted that the obtained uniform average dwell time $\tau$ is independent on the initial states of TTSS* (1).

### *B. Observer-based control*

This subsection presents the design of the observer-based control scheme. Accordingly, the next assumption is needed.

**Assumption 4.** *The pairs $(\bar{A}_0, \bar{C}_0)$ and $(A_{22}, C_2)$ are detectable, where $\bar{A}_0 = \bar{A}_{11} - \bar{A}_{12}A_{22}^{-1}\bar{A}_{21}$, $\bar{C}_0 = \bar{C}_1 - C_2A_{22}^{-1}\bar{A}_{21}$, $\bar{A}_{11} = \begin{pmatrix} S & 0 \\ F_1 & A_{11} \end{pmatrix}$, $\bar{A}_{12} = \begin{pmatrix} 0 \\ A_{12} \end{pmatrix}$, $\bar{A}_{21} = (F_2, A_{21})$, $\bar{C}_1 = (Q, C_1)$.*

Assumption 4 is essential and standard for the design of a full-order observer for the system (1)-(2), and it has also been used in [37]. Under Assumption 4, there exist matrices $L_0$ and $L_2$, such that $\bar{A}_0 + L_0\bar{C}_1$ and $A_{22} + L_2C_2$ are both Hurwitz. Next, we proceed to design the observer-based event-triggered controller as follows,

$$u = g(\tilde{x}, \tilde{z}, \tilde{v}) = K_1\tilde{x} + K_2\tilde{z} + G\tilde{v}, \tag{42}$$

$$\begin{pmatrix} \dot{\bar{v}} \\ \dot{\bar{x}} \\ \dot{\bar{z}} \end{pmatrix} = \bar{A}_\varepsilon \begin{pmatrix} \bar{v} \\ \bar{x} \\ \bar{z} \end{pmatrix} + \bar{B}_\varepsilon u + \bar{L}_\varepsilon(\bar{C}\begin{pmatrix} \bar{v} \\ \bar{x} \\ \bar{z} \end{pmatrix} - y), \tag{43}$$

where the variables $\bar{x}$, $\bar{z}$ and $\bar{v}$ are the estimates of $x$, $z$ and $v$, respectively, the variables $\tilde{x}$, $\tilde{z}$, $\tilde{v}$ represent the sampled states of $\bar{x}$, $\bar{z}$ and $\bar{v}$, respectively, $\bar{A}_\varepsilon = \begin{pmatrix} \bar{A}_{11} & \bar{A}_{12} \\ \frac{\bar{A}_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{pmatrix}$, $\bar{B}_\varepsilon = \begin{pmatrix} 0 & B_1^\top & \frac{B_2^\top}{\varepsilon} \end{pmatrix}^\top$, $\bar{L}_\varepsilon = \begin{pmatrix} L_1^\top & \frac{L_2^\top}{\varepsilon} \end{pmatrix}^\top$, $L_1 = \bar{A}_{12}A_{22}^{-1}L_2 + L_0(1 - C_2A_{22}^{-1}L_2)$ and $L_0$, $L_2$ are matrices such that $\bar{A}_0 + L_0\bar{C}_1$ and $A_{22} + L_2C_2$ are Hurwitz, and $K_1$, $K_2$, $G$ have same definitions as in (29). Prior to presenting the main result of this subsection, we provide the following lemma.

**Lemma 2** ( [37]). *Suppose Assumptions 2 and 4 hold. There is a $\bar{\varepsilon} > 0$ such that, $\forall\varepsilon \in (0, \bar{\varepsilon}]$, the full-order observer* (43) *can achieve the observation of the state $v$, $x$, $z$ of TTSS* (1)*, and the estimation error of the observer $e_m := (\bar{v}, \bar{x}, \bar{z}) - (v, x, z)$ would converge to the origin exponentially.*

Regarding concatenated state $(v, x)$ and $z$ as the slow and fast state respectively, the proof of Lemma 2 is similar with the one in [37], thus it is omitted here.

Let us denote the corresponding sequence of sampling instants as $t_k$, $k \in \mathcal{I} \subseteq \mathbb{Y}$ and also use zero-order holder to generate these quantities. In this way, at $t = t_k$, $x$, $z$, $v$ are all sampled so that $(\tilde{x}(t_k^+), \tilde{z}(t_k^+), \tilde{v}(t_k^+)) = (\bar{x}, \bar{z}, \bar{v})$, and for $t \in (t_k, t_{k+1})$, $\dot{\tilde{x}} = 0$, $\dot{\tilde{z}} = 0$, $\dot{\tilde{v}} = 0$. Similarly, an internal dynamic variable $\bar{\eta}$ is designed, which satisfies

$$\dot{\bar{\eta}} = -\kappa\bar{\eta} + \varpi(\delta\left\|\bar{\zeta}\right\|^2 - \|\bar{e}\|^2 + \pi),\ \bar{\eta}(0) > 0, \tag{44}$$

where $\bar{\zeta} := (\bar{\zeta}_1, \bar{\zeta}_2) = \zeta - \Pi\bar{v}$, $\zeta = (\bar{x}, \bar{z})$, $e = g(\tilde{x}, \tilde{z}, \tilde{v}) - g(\bar{x}, \bar{z}, \bar{v})$, $\kappa$ is an any given positive constant, $\varpi$, $\delta$ are some positive constants to be designed. Denote $\bar{\chi} := (x, z, v, \bar{v}, \bar{x}, \bar{z}, \tilde{x}, \tilde{z}, \tilde{v}, \bar{\eta}) \in \bar{\mathbb{X}} =: \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_v} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_v} \times [0, \infty)$. The hybrid model can be expressed as follows,

$$\dot{\bar{\chi}} = \bar{F}(\bar{\chi})\ \ \bar{\chi} \in \bar{\mathcal{C}}, \qquad \bar{\chi}^+ \in \bar{G}(\bar{\chi})\ \ \bar{\chi} \in \bar{\mathcal{D}}, \tag{45}$$

where $\bar{G}(\bar{\chi}) := (x, z, v, \bar{v}, \bar{x}, \bar{z}, \bar{x}, \bar{z}, \bar{v}, \bar{\eta})$,

$$\bar{F}(\bar{\chi}) := \begin{pmatrix} E_\varepsilon^{-1}(A\xi + B\sigma(g(\tilde{x}, \tilde{z}, \tilde{v})) + Fv) \\ Sv \\ \bar{A}_\varepsilon \begin{pmatrix} \bar{v} \\ \bar{x} \\ \bar{z} \end{pmatrix} + \bar{B}_\varepsilon u + \bar{L}_\varepsilon(\bar{C}\begin{pmatrix} \bar{v} \\ \bar{x} \\ \bar{z} \end{pmatrix} - y) \\ 0 \\ -\kappa\bar{\eta} + \varpi(\delta\left\|\bar{\zeta}\right\|^2 - \|g(\hat{x}, \hat{z}, \hat{v}) - g(x, z, v)\|^2) \end{pmatrix},$$

and the sets $\bar{\mathcal{C}}$ and $\bar{\mathcal{D}}$ are defined as follows,

$$\begin{aligned} \bar{\mathcal{C}} &:= \left\{\chi \in \mathbb{X} :\ \|\bar{e}\|^2 - \delta\left\|\bar{\zeta}\right\|^2 \le \alpha\bar{\eta} + \pi\right\}, \\ \bar{\mathcal{D}} &:= \left\{\chi \in \mathbb{X} :\ \|\bar{e}\|^2 - \delta\left\|\bar{\zeta}\right\|^2 \ge \alpha\bar{\eta} + \pi\right\}, \end{aligned} \tag{46}$$

where $\alpha > 0$, $\pi > 0$, $\varpi\delta < \frac{\epsilon}{8}$, $\varpi \ge \varpi_1 + \varpi_2$, and $\varpi_1, \varpi_2$ satisfy (32), Then, the next theorem is obtained.

**Theorem 4.** *Consider a compact set $\mathbb{V} \subset \mathbb{R}^{n_v}$ containing the origin. Suppose Assumptions 4-1 hold. There exist $\gamma \in \mathcal{K}$, a controller of the form* (42)-(43) *and $\bar{\mathcal{C}}$, $\bar{\mathcal{D}}$ in* (46) *solving the semi-global bounded output regulation problem of TTSSs* (45) *with $\lim_{t\to\infty}\sup\|y(t)\| = \gamma(\pi) + O(\varepsilon^{n+1})$, if there exist matrices $\Pi = \Pi_0 + \sum_{i=1}^{n}\varepsilon^i\Pi_i$ and $\Gamma = \Gamma_0 + \sum_{i=1}^{n}\varepsilon^i\Gamma_i$, such that*

- *the equations* (18) *are satisfied,*
- *there exist $T > 0$ and $\Delta > 0$ so that for all $v$ with $v(0) \in \mathbb{V}$, it has $\|\Gamma_0 v(t)\|_{\infty,T} \le \Upsilon - \Delta$.*

*Moreover, the TTSS* (45) *is capable of producing solutions with a uniform average dwell time that is independent of the initial states.*

*Proof.* Similar to the proof in Theorem 4, there is an $\bar{\epsilon}^* \in (0, 1]$, such that, $\forall\epsilon \in (0, \bar{\epsilon}^*]$, $\|u(t)\|_\infty \le \Upsilon$. Let $\epsilon \in (0, \bar{\epsilon}^*]$. Then, $\sigma(u)$ becomes $u$, for all $t \ge 0$. Then, after Chang transformation (22), it has that

$$\begin{pmatrix} \dot{\xi}_s \\ \dot{\xi}_f \end{pmatrix} = A_D\begin{pmatrix} \xi_s \\ \xi_f \end{pmatrix} + \begin{pmatrix} B_d \\ B_f \end{pmatrix}(\bar{e} + e_m) - \begin{pmatrix} \varepsilon^{n+1}H\Pi_{n,2}S \\ \varepsilon^n\Pi_{n,2}S \end{pmatrix}v, \tag{47}$$

where $A_D$ is defined in (23). The proof of TTSS (45) generating solutions with an uniform average dwell time is similar to the one presented in Theorem 3, and is therefore omitted here. Then, it is proved that the semi-global bounded output regulation of TTSS (45) can be achieved.

Similarly, define Lyapunov function candidate

$$\bar{U}(\bar{\chi}) = \bar{\xi}_s^T P_1(\epsilon)\bar{\xi}_s + \varepsilon\bar{\xi}_f^T P_2(\epsilon)\bar{\xi}_f + \bar{\eta}. \tag{48}$$

Utilizing the proof technique similar to that of (39), based on Lemma 2, it can be shown that there exists a threshold value $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$, the estimation error $\|e_m\|$ exponentially converges to zero, and

$$\begin{aligned}&\langle \nabla \bar{U}(\bar{\chi}), F(\bar{\chi})\rangle \\ \le& -\frac{1}{4}\epsilon(\bar{\xi}_s^T\bar{\xi}_s + \bar{\xi}_f^T\bar{\xi}_f) - 2\varepsilon^{n+1}\bar{\xi}_s^T P_1(\epsilon)H\Pi_{n,2}Sv \\ &- 2\varepsilon^{n+1}\bar{\xi}_f^T P_2(\epsilon)\Pi_{n,2}Sv - \kappa\bar{\eta} + \varpi\|e_m\|^2 + \varpi\pi. \quad (49)\end{aligned}$$

Then, a class $\mathcal{K}$ function $\gamma$ can be established using a proof technique similar to that of Theorem 3, demonstrating that $\lim\limits_{t\to\infty}\sup\|y(t)\| \le \gamma(\pi) + O(\varepsilon^{n+1})$. Thus, for all $\epsilon \in (0, \bar{\epsilon}^*]$, TTSS (45) achieves semi-global bounded output regulation with $\lim\limits_{t\to\infty}\sup\|y(t)\| = \gamma(\pi) + O(\varepsilon^{n+1})$. □

## V. Illustrative Example

In this section, two examples are presented to demonstrate the effectiveness of the proposed technique.

***Example 1***: To demonstrate the efficacy of the proposed approach, we present an example involving system (1) with a selected value of $\varepsilon = 0.01$, where

$$A_{11}=\begin{pmatrix} -2.5 & -1 \\ 1 & 2 \end{pmatrix},\ A_{12}=\begin{pmatrix} -2 & -3 \\ 0 & 2 \end{pmatrix},\ B_1=\begin{pmatrix} -1 \\ -2 \end{pmatrix},$$
$$A_{21}=\begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix},\ A_{22}=\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix},\ B_2=\begin{pmatrix} -1 \\ 2 \end{pmatrix},$$
$$F=\begin{pmatrix} -0.2 & 0 & -0.1 & 0 \\ 0.3 & 0.1 & 0 & 0.1 \end{pmatrix}^T,\ S=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$C=\begin{pmatrix} 0 & 0 & 1 & 1 \end{pmatrix}, Q=\begin{pmatrix} -0.5 & 0 \end{pmatrix}.$$

Then, $A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $B_0 = B_1 - A_{12}A_{22}^{-1}B_2 = (-4.5, -1)^T$. Assumptions 1-3 hold.

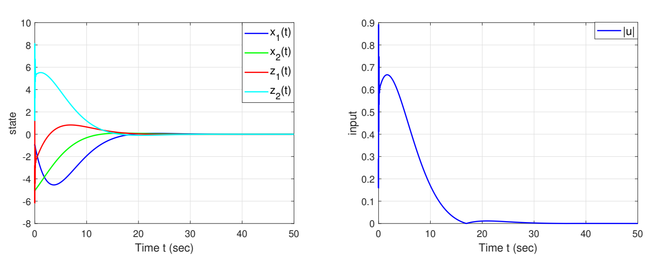
The system is initialized with the following initial conditions: $v(0) = (-1, 0)$ and $(x(0), z(0)) = (-1, -5, 2, 4)$. Let $\Upsilon = 1$. Then, $\|\Gamma v(t)\|_{\infty,0} \le 0.13 \le \Upsilon$. Accordingly, let $\epsilon = 0.01$, we have

$$P_1 = \begin{pmatrix} 0.0145 & 0.0055 \\ 0.0055 & 0.0777 \end{pmatrix}, P_2 = \begin{pmatrix} 0.0431 & 0.0017 \\ 0.0017 & 0.0468 \end{pmatrix}.$$

From (18), it has

$$\Gamma_0 = \begin{pmatrix} -0.0178 & 0.1218 \end{pmatrix},$$
$$\Pi_0 = \begin{pmatrix} -0.4658 & -0.0218 & 0.2042 & 0.2958 \\ 0.2916 & -0.1178 & -0.0329 & 0.0329 \end{pmatrix}^T.$$

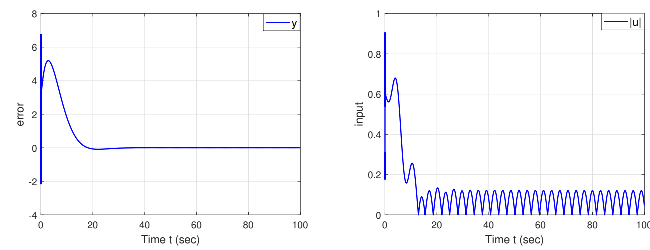
Then controller (15) is obtained with $n = 0$.




(a) Evolution of states $(x, z)$ (b) The controller gain

Fig. 1. The simulation results of Theorem 1 in Example 1.

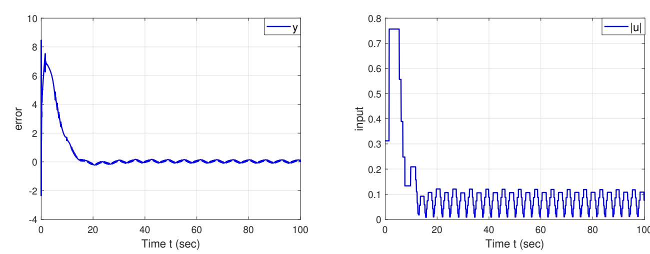
Simulation results of Theorem 1 and Theorem 2 are presented in Fig. 1 and 2. Fig. 1(a) shows that the solutions have asymptotically converged to zero when $v = 0$, which ensures the internal stability and also confirms the effectiveness of Theorem 1. Fig. 2(a) shows that $\lim\limits_{t\to\infty}\sup\|y(t)\| < 0.002$, which confirms the effectiveness of Theorem 2. Moreover, both Fig. 1(b) and Fig. 2(b) show that $\|u(t)\|_{\infty,0} \le \Upsilon$, in other words, the input saturation nonlinearity is avoided.




(a) Evolution of $y$ (b) The controller gain

Fig. 2. The simulation results of Theorem 2 in Example 1.

Then, the controller (15) is implemented via event-triggered scheme. Let $\eta(0) = 1$, $\kappa = 0.1$, $\varpi = 0.1$, $\delta = 0.001$, $\pi = 0.001$. The sets $C$ and $D$ are derived. The simulation results of Theorem 3 are presented in Fig. 3 and Fig. 4. Fig. 3(a) shows that $\lim\limits_{t\to\infty}\sup\|y(t)\| < 0.2$, in agreement with Theorem 3. Similarly, Fig. 3(b) shows that the input saturation nonlinearity is avoided. Fig. 4 depicts the triggering instants and the trajectory of the measurement error.




(a) Evolution of $y$ (b) The controller gain

Fig. 3. The simulation results of Theorem 3 in Example 1.

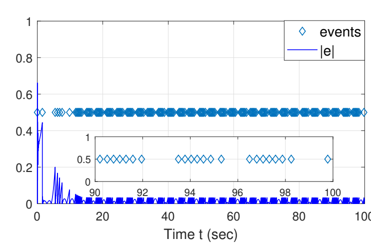
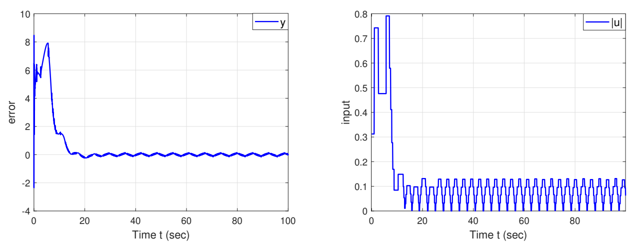



Fig. 4. Event-triggering time and trajectory of $\|e\|$ in Example 1 using Theorem 3.

Finally, the event-triggered controller (42)-(43) designed in Theorem 4 is applied. Since the pairs $(\bar{A}_0, \bar{C}_0)$ and $(A_{22}, C_2)$ are detectable, Assumption 4 is satisfied. Accordingly, we choose

$$L_0 = \begin{pmatrix} -0.3056 & 0.9521 & 1.3258 & 0.5884 \end{pmatrix}^T,$$
$$L_2 = \begin{pmatrix} -0.4690 & -0.8832 \end{pmatrix}^T,$$

so $\bar{A}_0 + L_0\bar{C}_1$ and $A_{22} + L_2C_2$ are both Hurwitz. Then $L_1 = \begin{pmatrix} -0.3689 & 1.1493 & 1.4207 & 0.2412 \end{pmatrix}^T$, and observer (43) is obtained. Let $\bar{\eta}(0) = 1$, $\kappa = 0.1$, $\varpi = 0.1$, $\delta = 0.001$, $\pi = 0.001$. The sets $\bar{C}$ and $\bar{D}$ are derived. The simulation results of Theorem 4 are presented in Fig. 5 and 6. The simulation results show that $\lim_{t\to\infty} \sup \|y(t)\| < 0.2$ with $\|u(t)\|_{\infty,0} \leq \Upsilon$, in agreement with Theorem 4.




(a) Evolution of $y$ (b) The controller gain

Fig. 5. The simulation results of Theorem 4 with $v(0) = (-1, 0)$.

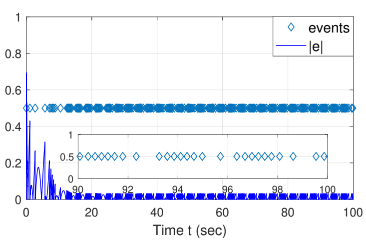



Fig. 6. Event-triggering time and trajectory of $\|e\|$ in Example 1 using Theorem 4.

***Example 2***: Adopting the DC motor model in [38] and further considering the output regulation problems, the dynamics is presented as follows,

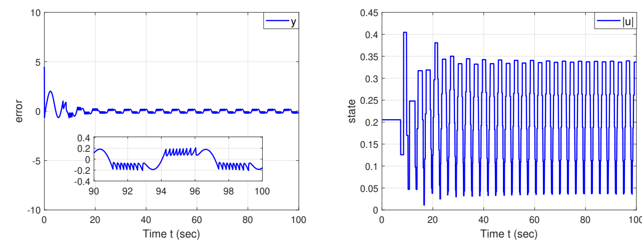
$$\begin{cases} J_m\dot{\omega} = -b\omega + k_mI + F_1v, \\ \bar{L}\dot{I} = -k_b\omega - R_0I + \sigma(u) + F_2v, \end{cases} \tag{50}$$

where $\omega$, $I$ and $u$ are the angular speed, armature current,and control voltage, respectively. $J_m = 0.093$ is the equivalent moment of inertia, $b = 0.008$ is the equivalent viscous friction coefficient, $\bar{L} = 0.006$ is the inductance, $k_b = 0.6$, $k_m = 0.7274$ are respectively the back e.m.f. and torque developed with constant excitation flux, $R_0 = 0.6$ is the resistance, $F_1 = (0.1, -0.2)$, $F_2 = (0.4, -0.5)$, and $v$ is the state of the exosystem (2) with $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The output regulation error is considered with $y = C_1\omega + C_2I + Qv$, where $C_1 = 2$, $C_2 = 1$, and $Q = (-0.5, 0)$.

The system is initialized with the following initial conditions: $v(0) = (-1, 1)$ and $(x(0), z(0)) = (-2, 8)$. Let $\Upsilon = 1$ and $\epsilon = 0.02$. Then, we have $P_1 = 0.0012$ and $P_2 = 0.0162$. From (18), it has

$$\Gamma_0 = \begin{pmatrix} -0.2842 & -0.0063 \end{pmatrix},$$
$$\Pi_0 = \begin{pmatrix} 0.3071 & -0.1562 \\ -0.1141 & 0.3125 \end{pmatrix}^T.$$

Choose $L_0 = \begin{pmatrix} -0.9425 & -0.3342 & -0.6694 \end{pmatrix}^T$ and $L_2 = -0.4681$, then $L_1 = \begin{pmatrix} -1.6778 & -0.5949 & 4.9106 \end{pmatrix}^T$. Let $\bar{\eta}(0) = 2$, $\kappa = 0.2$, $\varpi = 0.1$, $\delta = 0.01$, $\pi = 0.005$. Then, the controller of the form (42)-(43) with $n = 0$ and $\bar{\mathcal{C}}$, $\bar{\mathcal{D}}$ in (46) are derived. The simulation results of Theorem 4 are presented in Fig. 7 and 8. The simulation results show that $\lim_{t\to\infty} \sup \|y(t)\| < 0.2$ with $\|u(t)\|_{\infty,0} \leq \Upsilon$, in agreement with Theorem 4.




(a) Evolution of $y$ (b) The controller gain

Fig. 7. The simulation results of Theorem 4 in Example 2.

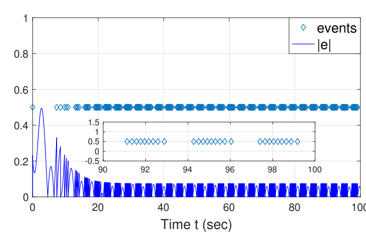



Fig. 8. Event-triggering time and trajectory of $\|e\|$ in Example 2 using Theorem 4.

## VI. Conclusion

The semi-global bounded output regulation problem was investigated for linear TTSSs subject to input saturation, where the impact of the fast subsystem is further considered. The method of asymptotic power series expansion has been applied to provide an approximate solution to the corresponding regulator equation. Accordingly, a time-continuous state feedback control law has been proposed, such that the bounded output regulation is achieved with $\lim_{t\to\infty} \sup \|y(t)\| = O(\varepsilon^{n+1})$. Additionally, a dynamic event-triggered mechanism has been developed to reduce the control updates. In the case of unavailable state information, an observer-based event-triggered control law has also been proposed. Moreover, the effectiveness of the proposed method has been tested on a practical example of a DC motor model. Note that the control design proposed here is vulnerable to structural uncertainty. It would be interesting in future works to consider the robustness issues for the output regulation of TTSSs and its application to microgrids.

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