Reset strategy for consensus in networks of clusters ⋆

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Abstract

This paper addresses the problem of consensus in networks structured in several clusters. The clusters are represented by fixed, directed and strongly connected graphs. They are composed by a number of agents which are able to interact only with other agents belonging to the same cluster. To every agent we associate a scalar real value representing its state. The states continuously evolve following a linear consensus protocol and approach local agreements specific to each cluster. In order to enforce a global agreement over the whole network, we consider that each cluster contains an agent that can be exogenously controlled. The state of this agent, called leader, will be quasi-periodically reseted by a local master controller that receives information from some neighboring leaders. In order to control the consensus value we have to firstly characterize it. Precisely we show that it depends only on the initial condition and the interaction topologies. Secondly, we provide sufficient Linear Matrix Inequality (LMI) conditions for the global uniform exponential stability of the consensus in presence of a quasi-periodic reset rule. The study of the network behavior is completed by a decay rate analysis. Finally we design the interaction network of the leaders which allows to reach a prescribed consensus value. Numerical implementation strategy is provided before illustrating the results by some simulations.

Key words: Multiagent systems; consensus; reset systems, LMI, opinion dynamics.

1 Introduction

Networks appear naturally in diverse areas of science and engineering as biology (Reynolds 2001, Ratmann et al. 2009, Pavlopoulos et al. 2011), physics (Gfeller & Rios 2008) and sociology (Hegselmann & Krause 2002, Lorenz 2005) as well as robotics (Bullo et al. 2009) and communication (Pastor-Satorras & Vespignani 2004). Studies concerning real networks revealed that the topology of interactions in communication, social or biological systems presents a cluster/community structure (Pastor-Satorras & Vespignani 2004, Boccaletti et al. 2006, Hanski 1998). In order to detect these communities, different algorithms are available in the literature (Newman & Girvan 2004, Lambiotte et al. 2009, Morărescu & Girard 2011). A consequence of the presence of decoupled clusters in the network is that consensus/synchronization cannot be reached and different local agreements are obtained (Morărescu & Girard 2011, Touri & Nedic 2012). To overcome this problem, we propose a quasi-periodic discrete controller meant to force the consensus in this type of clustered networks.

As in many works in the literature, in this paper we call agents the constitutive elements of the network and their number will define the network dimension. The consensus of the agents attracted a lot of interest in the last decade and it was studied in different frameworks: directed or undirected interactions, fixed or time-varying interaction graph, delayed or un-delayed, synchronized or desynchronized interactions, linear or nonlinear, continuous or discrete agent dynamics (Jadbabaie et al. 2003, Olfati-Saber & Murray 2004, Ren & Beard 2005, Moreau 2005, Morărescu et al. 2012). The agreement speed in various frameworks has also been quantified (see for instance (Xiao & Boyd 2004, Olshevsky & Tsitsiklis 2009)). In order to guarantee the global coordination in networks with dynamic topologies, some works proposed controller designs that are able to maintain the network connectivity (Zavlanos & Pappas 2008, Fiacchini & Morărescu 2014).

A major concern in the last decade has been the control over networks with communication constraints (Anta &

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Tabuada 2010, Postoyan et al. 2011, Heemels et al. 2012). The focus was on the control of one system over a communication network by limiting the interactions between the controller and the plant. In this paper, only the exogenous control actions are constrained. We assume the network topology is known and the network is partitioned in several clusters. The agents continuously evolve by interacting with some other agents belonging to the same cluster. The global behavior of one cluster can be exogenously controlled by acting on the state of only one agent called leader. Due to communication constraint we assume that the exogenous actions take place at specific isolated instants that, for practical reasons, will be defined in the next section as a nearly/quasi periodic sequence. This strategy can be applied to control fleets of robots that are spatially clustered (Mahacek et al. 2011). The robots belonging to one cluster continuously interact but due to energy and communication constraint long distance interactions occurs discretely.
As mentioned in (Bragagnolo et al. 2014), this model can be interpreted in terms of opinion dynamics. Each agent has an opinion that continuously evolves towards a local agreement representing the opinion of the community in which it lies. At specific instants, the leaders of the communities interact and they reset their opinion taking into account the ones of other leaders. The new opinions of the leaders will reset the values of the local agreements in each community. Iterating this process all the opinions will tend to a common value that depends only on the initial conditions and the network topology.
All the clusters are represented by fixed, strongly connected directed graphs. In order to enforce consensus the discrete control action of the leaders will be designed in a decentralized manner by taking into account only informations provided by some other leaders. In other words, we address the problem of consensus for agents subject to both continuous and discrete dynamics.
The aim of the paper is to control the consensus behavior in the network. A first contribution is related to the characterization of the consensus value in the framework under consideration. This is an important step that has to be done before imposing the consensus value. We note that the consensus value depends only on the initial conditions and the topologies of the involved networks (i.e the networks associated with the clusters and that associated with the leaders). It is noteworthy that the consensus value does not depend on the reset sequence used for the leaders' state. In order to study the stability of consensus we propose a LMI based condition that can be adapted for further goals of the paper encompassing the design of resets that allows reaching some network performances. The analysis of the network behavior finishes with the characterization of the convergence speed.
Another contribution of the paper is related to the design of the reset strategy of the leaders' state. In this part we design the interaction topology between leaders allowing to reach an a priori specified consensus value. In this part the network topology continue to be considered fixed and known for each cluster. The objective is to modify the consensus value of the whole network by changing the weights in the network of leaders. The set of consensus values that can be reached is contained in the interval defined by the minimum and maximum initial local agreements.
The paper is organized as follows. In Section 2 we formulate the problem under consideration. The agreement behavior and the possible consensus value are studied in Section 3. Sufficient conditions for the global uniform exponential stability of the consensus are provided in Section 4. These conditions are given in the form of a parametric LMI. Complementary results concerning the design of the network of interactions between the leaders allowing to reach a prescribed consensus value and the convergence speed are presented in Section 5. The problem of numerical implementation of the proposed developments is considered in Section 6. Precisely we show how the parametric LMI can be replaced by a finite number of LMIs. Section 7 is dedicated to numerical simulations which illustrate the results. Some conclusions and perspectives are presented at the end of the paper.
Notation. The following standard notation will be used throughout the paper. The sets of nonnegative integers, real and nonnegative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For a vector x we denote by $\|x\|$ its Euclidian norm. The transpose of a matrix A is denoted by $A^\top$. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, notation $A > 0$ ($A \geq 0$) means that A is positive (semi-)definite. By I_k we denote the $k \times k$ identity matrix. $\mathbb{1}_k$ and $\mathbf{0}_k$ are the column vectors of size k having all the components equal 1 and 0, respectively. We also use $x(t_k^-) = \lim_{t\to t_k, t\leq t_k} x(t)$. Throughout the paper we say that the LMI: $A > 0$ is satisfied on the subspace $\mathcal{K}$ if and only if $x^\top Ax > 0$ for all $x \in \mathcal{K}$.

2 Problem formulation

We consider a network of n agents described by the digraph (i.e. directed graph) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the vertex set $\mathcal{V}$ represents the set of agents and the edge set $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ represents the interactions.

Definition 1 *A* **path** *in a given digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a union of directed edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \dots, p-1\}$.*
Two nodes i, j are **connected** *in a digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if there exists at least a path in $\mathcal{G}$ joining i and j (i.e. $i_1 = i$ and $j_p = j$).*
A **strongly connected digraph** *is such that any two distinct nodes are connected. A* **strongly connected component** *of a digraph is a maximal subset of the vertex set such that any of its two distinct elements are connected.*

In the sequel, we consider that the agent set $\mathcal{V}$ is partitioned in m strongly connected clusters/communities $\mathcal{C}_1, \dots, \mathcal{C}_m$ and no link between agents belonging to different communities exists. Each community possesses one particular agent called leader and denoted in the following by $l_i \in \mathcal{C}_i$, $\forall i \in \{1, \dots, m\}$. The set of leaders will be referred to as $\mathcal{L} = \{l_1, \dots, l_m\}$. At specific time instants t_k, $k \geq 1$, called reset times, the leaders interact between them following a predefined interaction map $\mathcal{E}_l \subset \mathcal{L} \times \mathcal{L}$. We also suppose that $\mathcal{G}_l = (\mathcal{L}, \mathcal{E}_l)$ is strongly connected. The rest of the agents will be called followers and denoted by f_j. For the sake of clarity we consider that the leader is the first element of its community:

$$\mathcal{C}_i = \{l_i, f_{m_{i-1}+2}, \dots, f_{m_i}\},\ \forall i \in \{1, \dots, m\} \quad (1)$$

where $m_0 = 0,\ m_m = n$ and the cardinality of $\mathcal{C}_i$ is given by

$$|\mathcal{C}_i| \triangleq n_i = m_i - m_{i-1}, \forall i \geq 1.$$

Example 1 *To illustrate the notation* (1) *we consider a simple network of 6 agents partitioned in 2 clusters having 3 elements. Then* $\mathcal{C}_1 = \{l_1, f_2, f_3\}$ *and* $\mathcal{C}_2 = \{l_2, f_5, f_6\}$.

In order to keep the presentation simple and making an abuse of notation, each agent will have a scalar state denoted also by l_i for the leader l_i and f_j for the follower f_j. We also introduce the vectors $x = (l_1, f_2, \dots, f_{m_1}, \dots, l_m, \dots, f_{m_m} = f_n)^\top \in \mathbb{R}^n$ and $x_l = (l_1, l_2, \dots, l_m)^\top \in \mathbb{R}^m$ collecting all the states of the agents and all the leaders' states, respectively.
We are ready now to introduce the linear reset system describing the overall network dynamics:

$$\begin{cases} \dot{x}(t) = -Lx(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\ x_l(t_k) = P_l x_l(t_k^-) & \forall t_k \in \mathcal{T} \\ x(0) = x_0 \end{cases} \quad (2)$$

where $\mathcal{T} = \{t_k \in \mathbb{R}_+ \mid t_k < t_{k+1},\ \forall k \in \mathbb{N}, t_k \text{ reset time}\}$, $L \in \mathbb{R}^{n \times n}$ is a generalized Laplacian matrix associated to the graph $\mathcal{G}$ and $P_l \in \mathbb{R}^{m \times m}$ is a row stochastic (Perron) matrix associated to the graph $\mathcal{G}_l = (\mathcal{L}, \mathcal{E}_l)$. Precisely, the entries of L and P_l satisfies the following relations:

$$\begin{cases} L_{(i,j)} = 0, \text{ if } (i,j) \notin \mathcal{E} \\ L_{(i,j)} < 0, \text{ if } (i,j) \in \mathcal{E},\ i \neq j \\ L_{(i,i)} = -\sum_{j \neq i} L_{i,j}, \forall i = 1, \dots, n \end{cases}, \quad (3)$$

$$\begin{cases} P_{l(i,j)} = 0, \text{ if } (i,j) \notin \mathcal{E}_l \\ P_{l(i,j)} > 0, \text{ if } (i,j) \in \mathcal{E}_l,\ i \neq j \\ \sum_{j=1}^{m} P_{l(i,j)} = 1, \forall i = 1, \dots, m \end{cases}. \quad (4)$$

The values $L_{(i,j)}$ and $P_{l(i,j)}$ represent the weight of the state of the agent j in the updating process of the state of agent i when using the continuous and discrete dynamics, respectively.

In particular, L has the following block diagonal structure

$$L = \begin{pmatrix} L_1 & & \\ & \ddots & \\ & & L_m \end{pmatrix},\ L_i \in \mathbb{R}^{n_i} \quad (5)$$

with $L_i \mathbb{1}_{n_i} = \mathbf{0}_{n_i}$ and $P_l \mathbb{1}_m = \mathbb{1}_m$. Due to strong connectivity of $\mathcal{C}_i,\ i = 1, \dots, m$ and $\mathcal{G}_l$, 0 is simple eigenvalue of each L_i and 1 is simple eigenvalue of P_l.

3 Agreement behavior

In this section we assume that system (2) achieves consensus and we characterize its possible values. Firstly, we show that each agent tracks a local agreement function which is piecewise constant. In the second subsection we prove that the vector of local agreements lies in a subspace defined by the system's dynamics and initial condition. Therefore, if the consensus is achieved and the corresponding consensus value is x^* then $x^* \mathbb{1}_m$ belongs to the same subspace. Moreover, this value is determined only by the initial condition of the network and by the interconnection structure.
As we have noticed $\mathbb{1}_{n_i}$ is the right eigenvector of L_i associated with the eigenvalue 0 and $\mathbb{1}_m$ is the right eigenvector of P_l associated with the eigenvalue 1. In the sequel, we denote by w_i the left eigenvector of L_i associated with the eigenvalue 0 such that $w_i^\top \mathbb{1}_{n_i} = 1$. Similarly, let $v = (v_1, \dots, v_m)^\top$ be the left eigenvector of P_l associated with the eigenvalue 1 such that $v^\top \mathbb{1}_m = 1$. Due to the structure (1) of the communities, we emphasize that each vector w_i can be decomposed in its first component $w_{i,l}$ and the rest of its components grouped in the vector $w_{i,f}$.

3.1 Local agreements

Let us first recall a well known result concerning the consensus in networks of agents with continuous time dynamics (see (Olfati-Saber & Murray 2004) for instance).

Lemma 2 *Let* $\mathcal{G}$ *be a strongly connected digraph and* L *the corresponding Laplacian matrix. Consider a network of agents whose collective dynamics is described by* $\dot{x}(t) = -Lx(t)$. *Let us also consider* $L\mathbb{1} = \mathbf{0}$, $\omega^\top L = \mathbf{0}$ *and* $\omega^\top \mathbb{1} = 1$. *Then, the agents asymptotically reach a consensus and the consensus value is given by* $x^* = \omega^\top x(0)$. *Moreover, the vector* ω *defines an invariant subspace for the collective dynamics:* $\omega^\top x(t) = \omega^\top x(0), \forall t \geq 0$

Remark 1 *When dynamics* (2) *is considered, Lemma 2 implies that between two consecutive reset instants* t_k *and* t_{k+1}*, the agents belonging to the same community try to approach a local agreement defined by* $x_i^*(k) = w_i^\top x_{\mathcal{C}_i}(t_k)$ *where* $x_{\mathcal{C}_i}(\cdot)$ *is the vector collecting the states of the agents belonging to the cluster* $\mathcal{C}_i$*. Nevertheless, at the reset times* *the value of the local agreement can change. Thus,*

$$w_i^\top x_{\mathcal{C}_i}(t) = w_i^\top x_{\mathcal{C}_i}(t_k),\ \forall t \in (t_k, t_{k+1}) \text{ and possibly}$$
$$w_i^\top x_{\mathcal{C}_i}(t) \neq w_i^\top x_{\mathcal{C}_i}(t_k),\ \text{ for } t \notin (t_k, t_{k+1})$$

Therefore, the agents whose collective dynamics is described by the hybrid system (2), may reach a consensus only if the local agreements converge one to each other.

3.2 Consensus value

Before presenting our next result, let us introduce the following vectors:

$$\begin{aligned} x^*(t) &= (x_1^*(t), x_2^*(t), ..., x_m^*(t))^\top \in \mathbb{R}^m \\ u &= (v_1/w_{1,l}, v_2/w_{2,l}, ..., v_m/w_{m,l})^\top \in \mathbb{R}^m \end{aligned} \tag{6}$$

where $x_i^*(\cdot)$ represents the local agreement of the cluster $\mathcal{C}_i$ and $v \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^{n_i}$ are defined at the beginning of the section as left eigenvectors associated with the matrices describing the reset dynamics of the leaders and the continuous dynamics of each cluster, respectively. Let us also introduce the matrix of the left eigenvectors of the communities:

$$W = \begin{bmatrix} w_1^\top & 0 & \cdots & 0 \\ 0 & w_2^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m^\top \end{bmatrix} \in \mathbb{R}^{m \times n}. \tag{7}$$

It is noteworthy that $x^*(t)$ is time-varying but piecewise constant: $x^*(t) = x^*(k)\ \forall t \in (t_k, t_{k+1})$.

Proposition 3 *Consider the system* (2) *with* L *and* P_l *defined by* (3) *and* (4)*, respectively. Then,*

$$u^\top x^*(t) = u^\top x^*(0),\ \forall t \in \mathbb{R}_+. \tag{8}$$

PROOF. The following relation holds:

$$x^*(t) = Wx(t) \qquad \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \tag{9}$$

Since $w_i = (w_{il}, w_{if})$, we define a permutation matrix T such that $WT^\top = U = (U_1, U_2)$. The matrix U_1 is a diagonal matrix corresponding to the leaders' components $w_{i,l}$, while U_2 is a block diagonal matrix corresponding to the followers' components $w_{i,f}$. In other terms

$$U_1 = \begin{bmatrix} w_{1,l} & 0 & \cdots & 0 \\ 0 & w_{2,l} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{m,l} \end{bmatrix} \in \mathbb{R}^{m \times m} \tag{10}$$

$$U_2 = \begin{bmatrix} w_{1,f} & 0 & \cdots & 0 \\ 0 & w_{2,f} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{m,f} \end{bmatrix} \in \mathbb{R}^{m \times (n-m)}. \tag{11}$$

Finally, we can rewrite equation (9) as:

$$x^*(t) = WT^\top x(t) = U \cdot (x_l(t), x_f(t)). \tag{12}$$

Note that at the reset time t_k one has $x_f(t_k) = x_f(t_k^-)$. This yields

$$\begin{aligned} &x^*(t_k) - x^*(t_k^-) = U \cdot (x_l(t_k) - x_l(t_k^-), x_f(t_k) - x_f(t_k^-)) \\ &= U \cdot (x_l(t_k) - x_l(t_k^-), 0) = U_1 \cdot (x_l(t_k) - x_l(t_k^-)) + U_2 \cdot 0 \\ &= U_1(x_l(t_k) - x_l(t_k^-)). \end{aligned}$$

Thus,

$$x^*(t_k) = x^*(t_k^-) + U_1 \cdot (P_l - I_m) x_l(t_k^-). \tag{13}$$

Multiplying equation (13) by $u^\top$ and using $u^\top U_1 = v^\top$ as well as $v^\top P_l = v^\top$ one obtains

$$\begin{aligned} u^\top x^*(t_k) &= u^\top x^*(t_k^-) + u^\top U_1 (P_l - I_m) x_l(t_k^-) \\ &= u^\top x^*(t_k^-) + v^\top (P_l - I_m) x_l(t_k^-) \\ &= u^\top x^*(t_k^-) + v^\top x_l(t_k^-) - v^\top x_l(t_k^-) = u^\top x^*(t_k^-) \end{aligned} \tag{14}$$

According to Remark 1, $x^*(t)$ remains constant for all $t \in (t_k, t_{k+1})$ leading to

$$u^\top x^*(t) = u^\top x^*(0) \qquad \forall t \in \mathbb{R}_+. \tag{15}$$

Corollary 1 *Consider the system* (2) *with* L *and* P_l *defined by* (3) *and* (4)*, respectively. Assuming the agents of this system reach a consensus, the consensus value is*

$$x^* = \frac{u^\top W x(0)}{\sum_{i=1}^m u_i}. \tag{16}$$

PROOF. Let x^* be the consensus value reached by the system (2). It means that $x(t) \to x^* \mathbb{1}_n$. Thus, when t goes to ∞ in (15) one obtains

$$u^\top x^* \mathbb{1}_n = u^\top x^*(0) = u^\top W x(0)$$

leading to (16).

In order to simplify the presentation and without loss of generality, in what follows, we consider that $\sum_{i=1}^m u_i = 1$.

Remark 2 *It is important to note that the consensus value depends only on the system matrices* L*,* P_l *and does not depend on the reset sequence* $\mathcal{T}$*.*

A trivial result which may be seen as a consequence of Corollary 1 is the following.

Corollary 2 *If the matrices L, P_l are symmetric (i.e. i^{th} agent takes into account the state of j^{th} agent as far as j^{th} takes into account the i^{th} one and they give the same importance one to another) the consensus value is the average of the initial states.*

PROOF. In this case $w_i = \frac{1}{n_i}\mathbb{1}_{n_i}$ and $v = \frac{1}{m}\mathbb{1}_m$ which leads to $u = (\frac{n_1}{m}, \frac{n_2}{m}, \dots, \frac{n_m}{m})$. The result follows from (16).

4 Stability analysis

In this section, the stability analysis of the equilibrium point x^* will be given by means of some LMI conditions. Basically we are searching a quadratic Lyapunov function that ensures stability. Even if the method introduces a certain conservatism it can be adapted for the computation and control of the convergence speed as well as for the design of the reset matrix that guarantees the convergence towards a prescribed consensus value.

It is important to note that the consensus problem for the dynamics (2) can be rewritten in term of consensus of discrete dynamics with switching topology studied for instance in (Ren & Beard 2005). Although sufficient conditions for stability exist in the literature, the existing tools cannot provide the consensus value and they cannot be used for the design of the network allowing to reach prescribed consensus value with prescribed convergence speed. Thus, it is important to introduce our tool for the simpler problem of stability before going further and complexify it for network design purposes. It is noteworthy that, the convergence towards a prescribed consensus value can be interpreted as the control of the network through the design of few interconnection weights defined by the matrix P_l.

4.1 Prerequisites

Since the consensus value is computed in the previous section we can first define the disagreement vector $y = x - x^*\mathbb{1}_n$. We also introduce an extended stochastic matrix P_{ex} as follows:

$$P_{ex} = T^\top \begin{bmatrix} P_l & 0 \\ 0 & I_{n-m} \end{bmatrix} T \tag{17}$$

where T is the permutation matrix used in the proof of Proposition 3. It is noteworthy that $L\mathbb{1}_n = \mathbf{0}_n$ and $P_{ex}\mathbb{1}_n = \mathbb{1}_n$. Thus, the disagreement dynamics is exactly the same as the system one:

$$\begin{cases} \dot{y}(t) = -Ly(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\ y(t_k) = P_{ex} y(t_k^-) & \forall t_k \in \mathcal{T} \\ y(0) = y_0 \end{cases} . \tag{18}$$

Due to uncertainties that affect the reset instant in practice, instead of considering a periodic reset sequence, we consider a nearly periodic one defined by $t_{k+1} - t_k = \delta + \delta'$ where $\delta \in \mathbb{R}_+$ is the fixed period and $\delta' \in \Delta$ is a jitter belonging to the compact set $\Delta \subset \mathbb{R}_+$. Thus the set of reset times $\mathcal{T}$ belongs to the set of all admissible reset sequences associated with Δ:

$$\Phi(\Delta) \triangleq \left\{ \{t_k\}_{k\in\mathbb{N}}, t_{k+1} - t_k = \delta + \delta'_k, \delta'_k \in \Delta, \forall k \in \mathbb{N} \right\} \tag{19}$$

where we always consider $t_0 = 0$.

Remark 3 • *We note that in practice periodic events are difficult to ensure while nearly periodic is simple. In the case of social network periodic meetings can be impossible while quasi periodic ones are more realistic. Thus, quasi-periodic reset sequences increase the accuracy of the model with respect to practical applications.*
• *The case $\delta'_k = 0, \forall k \in \mathbb{N}$ recovers the purely periodic reset strategy. In this situation system* (2) *rewrites as a discrete dynamics $x(t_{k+1}) = P_{ex}e^{-L\delta}x(t_k)$. The stability issue in this case can be solved without using the LMI based criterium presented below. Indeed, in order to guarantee the consensus, we can use the strong connectivity of the clusters and of the graph of leaders in order to prove that $P_{ex}e^{-L\delta}$ is not only stochastic but also primitive (i.e. irreducible and aperiodic). This is a necessary and sufficient condition to reach consensus starting from any initial condition.*

We recall that for any $\mathcal{T} \in \Phi(\Delta)$ and any initial condition x_0 the system (2) has a unique solution denoted by $\varphi(t, x_0)$.

Definition 4 *We say that the equilibrium $y^* = \mathbf{0}_n$ of the system* (18) *is Globally Uniformly Exponentially Stable (GUES) with respect to the set of reset sequences $\Phi(\Delta)$ if there exist positive scalars c, λ such that for any $\mathcal{T} \in \Phi(\Delta)$, any $y_0 \in \mathbb{R}^n$, and any $t \geq 0$*

$$\|\varphi(t, y_0)\| \leq ce^{-\lambda t}\|y_0\| \tag{20}$$

The following theorem is instrumental:

Theorem 5 (Theorem 1 in (Hetel et al. 2013)) *Consider the system* (18) *with the set of reset times $\mathcal{T} \in \Phi(\Delta)$. The equilibrium $y^* = \mathbf{0}_n$ is GUES if and only if there exists $S_{[\cdot]} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, $S_{[y]} = S_{[y]}^\top = S_{[ay]} > 0$, $\forall x \neq 0, a \in \mathbb{R}$, $a \neq 0$ defining a positive function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ strictly convex,*

$$V(y) = y^\top S_{[y]} y,$$

homogeneous (of second order), $V(0) = 0$, such that $V(y(t_k)) > V(y(t_{k+1}))$ for all $y(t_k) \neq 0$, $k \in \mathbb{N}$ and any of the possible reset sequences $\mathcal{T} \in \Phi(\Delta)$.

4.2 Periodic reset case

In order to fix the ideas and provide a simpler version of our stability result we start by briefly discussing the case

$\Delta = \{0\}$ i.e. the resets take place periodically with the period δ.

Proposition 6 *The consensus $x^*\mathbb{1}_n$ is GUES for* (2) *if and only if there exists a positive definite matrix S such that the LMI*

$$\begin{aligned}&\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S\Big(I_n - \mathbb{1}_n u^\top W\Big) - \\ &\Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)^\top S\Big(Y(\delta) - \mathbb{1}_n u^\top W\Big) > 0, \\ &Y(\delta) \triangleq P_{ex} e^{-L(\delta)}\end{aligned} \tag{21}$$

is satisfied on $span\{\mathbb{1}_n\}^\perp$. Moreover, the stability is characterized by the quadratic Lyapunov function $V(t) = (x(t) - x^*\mathbb{1}_n)^\top S(x(t) - x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.

PROOF. The stability of $x^*\mathbb{1}_n$ is equivalent to the existence of a quadratic Lyapunov function $V(t) = (x(t)-x^*\mathbb{1}_n)^\top S(x(t)-x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$, $\forall x(t_k) \notin span\{\mathbb{1}_n\}$. Therefore, $x^*\mathbb{1}_n$ is GUES for (2) if and only if

$$\begin{aligned}&\Big(x(t_k) - \mathbb{1}_n x^*\Big)^\top S\Big(x(t_k) - \mathbb{1}_n x^*\Big) > \\ &\Big(x(t_{k+1}) - \mathbb{1}_n x^*\Big)^\top S\Big(x(t_{k+1}) - \mathbb{1}_n x^*\Big), \\ &\qquad \forall\ x(t_k) \notin span\{\mathbb{1}_n\}\end{aligned}$$

Consequently, using $u^\top W x(t_k) = x^*$ and $x(t_{k+1}) = Y(\delta)x(t_k)$ one gets that $x^*\mathbb{1}_n$ is GUES for (2) if and only if

$$\begin{aligned}&x(t_k)^\top\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S\Big(I_n - \mathbb{1}_n u^\top W\Big)x(t_k) > \\ &x(t_k)^\top\Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)^\top S\Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)x(t_k), \\ &\qquad \forall\ x(t_k) \notin span\{\mathbb{1}_n\}\end{aligned} \tag{22}$$

Let us note that any $x(t_k) \in \mathbb{R}^n$ can be decomposed as $x(t_k) = \bar{x}(t_k) + \tilde{x}(t_k)$ with $\bar{x}(t_k) \in span\{\mathbb{1}_n\}^\perp$ and $\tilde{x}(t_k) \in span\{\mathbb{1}_n\}$. Moreover,

$$\Big(I_n - \mathbb{1}_n u^\top W\Big)\tilde{x}(t_k) = 0,\ \Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)\tilde{x}(t_k) = 0$$

hence

$$\begin{aligned}&\Big(I_n - \mathbb{1}_n u^\top W\Big)x(t_k) = \Big(I_n - \mathbb{1}_n u^\top W\Big)\bar{x}(t_k), \\ &\Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)x(t_k) = \Big(Y(\delta) - \mathbb{1}_n u^\top W\Big)\bar{x}(t_k)\end{aligned}$$

Therefore (22) means that LMI-simple is satisfied on $span\{\mathbb{1}_n\}^\perp$.

Remark 4 *From theoretical point of view Proposition* (6) *is equivalent to the stability of the consensus value x^*. In practice, solving* (21) *may not a simple task in the case of very large networks. Indeed, existing LMI solvers may fail to solve LMI problems involving a large number of variables. Nevertheless, we can say that our method provides numerically tractable necessary and sufficient conditions for stability of x^*.*

Throughout the rest of the paper the admissible reset sequences are quasi-periodic, i.e. defined by (19). As explained in Remark 3 this increases the accuracy of the model but also its complexity. Since the matrix exponential is not monotone we cannot consider that the worst case scenario occurs for the maximum $t_{k+1} - t_k$ allowed by (19). Therefore, we cannot reduce the stability of system (2) to stability analysis of the discrete time linear time invariant system defined by $P_{ex}e^{-L(\delta+\delta_{max})}$. Consequently, the results will be formulated in terms of parametric LMIs. Nevertheless, we propose to use convex embedding techniques to transform these infinite dimensional inequalities into more tractable LMIs at the expense of introducing some conservatism.

4.3 Parametric LMI condition

In the sequel, we define a quasi-quadratic Lyapunov function satisfying Theorem 5 by means of some LMI. Therefore, the following result gives sufficient conditions for the stability of the equilibrium point $y^* = \mathbf{0}_n$ for the system (18) or equivalently of $x^*\mathbb{1}_n$ for the system (2). Even if other sufficient condition for GUES can be given, we present the following result since it will be useful in the next section.

Theorem 7 *Consider the system* (2) *with $\mathcal{T}$ in the admissible reset sequences $\Phi(\Delta)$. If there exist matrices $S(\delta')$, $S(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $S(\delta') = S^\top(\delta') > 0,\ \delta' \in \Delta$ such that the LMI*

$$\begin{aligned}&\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S(\delta_a)\Big(I_n - \mathbb{1}_n u^\top W\Big) - \\ &\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top S(\delta_b)\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) > 0, \\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)}\end{aligned} \tag{23}$$

is satisfied on $span\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (2). *Moreover, the stability is characterized by the quasi-quadratic Lyapunov function $V(t) = V(x(t)) \triangleq \max_{\delta'\in\Delta}(x(t) - x^*\mathbb{1}_n)^\top S(\delta')(x(t) - x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.*

PROOF. Using the disagreement vector $y(t) = x(t) - x^*\mathbb{1}_n$ and supposing that there exist matrices $S(\delta')$ satisfying (23) for all $\delta_a, \delta_b \in \Delta$ we define the Lyapunov matrix

$$S_{[y]} = S(\delta^*(y)) \text{ with } \delta^*(y) = \arg\max_{\delta'\in\Delta} y^\top S(\delta')y \tag{24}$$

Following (Hetel et al. 2013) the Lyapunov function

$$V(y) = y^\top S_{[y]} y = \max_{\delta' \in \Delta} y^\top S(\delta') y,$$

is convex and homogeneous of the second order.
Let us show now that $S(\cdot)$ solution of (23) ensures that $V(\cdot)$ defined above satisfies Theorem 5.
Following the proof of Proposition 6, the LMI (23) yields

$$\begin{aligned} &\Big(x(t_k) - \mathbb{1}_n x^*\Big)^\top S(\delta_a)\Big(x(t_k) - \mathbb{1}_n x^*\Big) > \\ &\Big(Y(\delta_a)x(t_k) - \mathbb{1}_n x^*\Big)^\top S(\delta_b)\Big(Y(\delta_a)x(t_k) - \mathbb{1}_n x^*\Big), \\ &\quad \forall \delta_a, \delta_b \in \Delta,\ x(t_k) \in span\{\mathbb{1}_n\}^\perp \end{aligned} \tag{25}$$

For any $\{t_k\}_{k\in\mathbb{N}} \in \Phi(\Delta)$ we have $x(t_{k+1}) = Y(\delta'_k)x(t_k)$ with some $\delta'_k \in \Delta$. Thus, for $\delta_a = \delta'_k$, (25) rewrites as:

$$\begin{aligned} &\Big(x(t_k) - x^*\mathbb{1}_n\Big)^\top S(\delta'_k)\Big(x(t_k) - x^*\mathbb{1}_n\Big) > \\ &\Big(x(t_{k+1}) - x^*\mathbb{1}_n\Big)^\top S(\delta_b)\Big(x(t_{k+1}) - x^*\mathbb{1}_n\Big) \\ &\quad \forall \delta'_k,\ \delta_b \in \Delta,\ x(t_k) \in span\{\mathbb{1}_n\}^\perp \end{aligned}$$

or equivalently

$$y(t_k)^\top S(\delta'_k) y(t_k) > y(t_{k+1})^\top S(\delta_b) y(t_{k+1}) \quad \forall \delta'_k, \delta_b \in \Delta$$

Taking $\delta_b = \delta^*(y(t_{k+1}))$, defined by (24) one obtains

$$V(y(t_k)) > y(t_k)^\top S(\delta'_k) y(t_k) > V(y(t_{k+1}))$$

for all $y(t_k)$, which ends the proof.

4.4 *Decay rate analysis*

Once the global uniform exponential stability of x^* is ensured by Theorem 7 we can compute the convergence speed of the state of system (2). In other words, we are searching to evaluate λ in (20). Straightforward computation shows that $\lambda = \dfrac{\ln \lambda_d}{\delta + \delta_{max}}$ where $\delta_{max} = \max_{\delta' \in \Delta} \delta'$ and λ_d defined as the decay rate of the linear difference inclusion (LDI)

$$x(t_{k+1}) \in \mathcal{F}(x(t_k)), \quad k \in \mathbb{N} \tag{26}$$

where

$$\mathcal{F}(x) = \left\{ P_{ex} e^{-L(\delta+\delta')},\ \delta' \in \Delta \right\}.$$

Precisely, for the LDI (26) there exist $M > 0$ and $\xi \in [0, 1]$ such that

$$\|x(t_k) - x^*\mathbb{1}_n\| \le M\xi^k \|x(0) - x^*\mathbb{1}_n\|, \quad \forall k \in \mathbb{N} \tag{27}$$

and λ_d is defined as the smallest ξ satisfying (27).
Thus, in order to quantify the convergence speed of system (2), we only have to evaluate λ_d. Let us denote again $y = x - x^*\mathbb{1}_n$ and note that $V(y)$ defined by Theorem 7 is a norm. That implies there exist $\alpha, \beta > 0$ such that

$$\alpha\|y\|^2 \le V(y) \le \beta\|y\|^2.$$

Consequently, one obtains that the decay rate λ_d coincides with the decay rate of V. Thus, the following result can be derived directly from Theorem 7.

Proposition 8 *Assume there exist $\alpha > 0,\ \beta > 0,\ \xi \in (0, 1]$ and the matrices $S(\delta') = S^\top(\delta') > 0,\ \delta' \in \Delta$ defined by $S(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', fulfilling the following constraints*

$$\begin{aligned} &\alpha I_n \le S(\delta') \le \beta I_n, \quad \forall \delta' \in \Delta \\ &\xi^2\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S(\delta_a)\Big(I_n - \mathbb{1}_n u^\top W\Big) - \\ &\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top S(\delta_b)\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) > 0, \\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)}. \end{aligned} \tag{28}$$

on $span\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$. Then, the decay rate is defined as

$$\lambda_d = \min_{\xi \text{ satisfies } (28)} \xi$$

and

$$\|x(t_k) - x^*\mathbb{1}_n\| \le \frac{\beta}{\alpha}(\lambda_d)^k \|x(0) - x^*\mathbb{1}_n\|, \quad \forall k \in \mathbb{N}.$$

Remark 5 *It is noteworthy that $0 < \lambda_d \le 1$ and for a priori fixed values of $\alpha,\ \beta$ we can use the bisection algorithm to approach as close as we want the value of λ_d.*

Remark 6 *To complete the decay rate analysis, we can consider that $P_{ex},\ L$ and λ_d are fixed and perform a line search to find the nominal reset period δ that ensures the convergence speed constraint. In other words, we check if (28) has solutions for $\xi = \lambda_d$ and δ heuristically sweeping the positive real axis. Moreover, we can progressively decrease λ_d and re-iterate the line search in order to find the smaller reachable decay rate.*

5 Convergence toward a prescribed value

In what follows we assume that the value x^* is a priori fixed and at least a vector u satisfying (16) exists. Under this assumption we are wondering if there exists a matrix P_l that allows system (2) to reach the consensus value x^*. It is worth noting that the network topology is considered fixed and known for each cluster. Under these assumptions, a consensus value is imposed by a certain choice of v such that $v^\top \mathbb{1}_m = 1$ and v left eigenvector of P_l associated with the eigenvalue 1. In other words we arrive to a joint design of P_l and the Lyapunov function V guaranteeing the trajectory of (2) ends up on x^*.

Theorem 9 *Let us consider the system* (2) *with $\mathcal{T}$ in the admissible reset sequences $\Phi(\Delta)$ and let x^* be a priori fixed by a certain choice of v. If there exist matrices $R(\delta')$, $R(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $R(\delta') = R^\top(\delta') > 0,\ \delta' \in \Delta$ and P_l stochastic such that the LMI*

$$\begin{bmatrix} Z(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & R(\delta_b) \end{bmatrix} > 0,$$
$$Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \tag{29}$$
$$Z(\delta_a) \triangleq \Big(I_n - \mathbb{1}_n u^\top W\Big)^\top + \Big(I_n - \mathbb{1}_n u^\top W\Big) - R(\delta_a)$$

with the constraint

$$v^\top P_l = v^\top$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (2)*. Moreover, the stability is characterized by the quasi-quadratic Lyapunov function $V(t) = V(x(t)) = \max_{\delta' \in \Delta} (x(t) - x^*\mathbb{1}_n)^\top R(\delta')^{-1}(x(t) - x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.*

Remark 7 *To obtain the results in the periodic resets framework, one has to replace $\delta_a = \delta_b = 0$ in* (29)*. Doing so one obtain an LMI instead of a parametric LMI. The same remark holds for Theorem 10.*

PROOF of Theorem 9. First notice that

$$\Big((I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a) - I_n\Big) S(\delta_a)^{-1} \times$$
$$\Big(S(\delta_a)(I_n - \mathbb{1}_n u^\top W) - I_n\Big) \geq 0$$

leads to

$$(I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a)(I_n - \mathbb{1}_n u^\top W) \geq$$
$$(I_n - \mathbb{1}_n u^\top W)^\top + (I_n - \mathbb{1}_n u^\top W) - S(\delta_a)^{-1}$$

Thus, once the solution to the LMI problem (29) is obtained we can define $S(\delta_a) = R(\delta_a)^{-1}$ and $S(\delta_b) = R(\delta_b)^{-1}$. Then:

$$\begin{bmatrix} Z(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & S(\delta_b)^{-1} \end{bmatrix} > 0$$

where

$$Z(\delta_a) = (I_n - \mathbb{1}_n u^\top W)^\top + (I_n - \mathbb{1}_n u^\top W) - S(\delta_a)^{-1}$$

and hence

$$\begin{bmatrix} \bar{Z}(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & S(\delta_b)^{-1} \end{bmatrix} > 0$$

where

$$\bar{Z}(\delta_a) = (I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a)(I_n - \mathbb{1}_n u^\top W).$$

By Schur complement, the last LMI is nothing than (23) in Theorem 7. Moreover, the constraints $v^\top P_l = v^\top$, $P_l \mathbb{1}_m = \mathbb{1}_m$ and the coefficients of P_l positive ensure the matrix P_l is stochastic and the consensus value is exactly x^*.

5.1 Convergence toward a prescribed value with a prescribed decay rate

Combining the results of Proposition 8 and Theorem 9 we can design the matrix P_l that allows to reach an a priori given consensus value x^* with a decay rate inferior to an a priori fixed value. Precisely, the following result holds.

Theorem 10 *Let us consider the system* (2) *with $\mathcal{T}$ in the admissible reset sequences $\Phi(\Delta)$ and let x^* be a priori fixed by a certain choice of v. Let us also consider $\bar{\lambda} \in (0, 1)$ a priori fixed. If there exist matrices $R(\delta')$, $R(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $R(\delta') = R^\top(\delta') > 0,\ \delta' \in \Delta$ and P_l row stochastic such that the LMI*

$$\begin{bmatrix} Z(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & \bar{\lambda}^2 R(\delta_b) \end{bmatrix} > 0,$$
$$Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \tag{30}$$
$$Z(\delta_a) \triangleq \Big(I_n - \mathbb{1}_n u^\top W\Big)^\top + \Big(I_n - \mathbb{1}_n u^\top W\Big) - R(\delta_a)$$

with the constraint

$$v^\top P_l = v^\top$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (2) *and* (27) *is satisfied for $\xi = \bar{\lambda}$ and $M = \beta/\alpha$ where β and α are the minimum and the maximum eigenvalue of $R(\delta'),\ \delta' \in \Delta$, respectively.*

PROOF. If (30) holds, following the proof of Theorem 9 one obtains that x^* is GUES for (2) and (28) is satisfied for $\xi = \bar{\lambda}$ and $S(\delta') = R(\delta')^{-1},\ \delta' \in \Delta$. Thus from Proposition 8 one concludes that $\lambda_d \leq \bar{\lambda}$.

Remark 8 *It is noteworthy that LMI* (30) *implies LMI* (28) *but they are not equivalent. Therefore, the decay rate λ_d is smaller than $\bar{\lambda}$.*

6 Numerical implementation

In order to render this paper self-contained, in this section we consider the problem of approximation of the parametric LMI (23) by a finite number of conditions using polytopic embeddings. Obviously this approximation introduces a supplementary conservatism that can be reduced increasing the number of vertices of the polytope. On the other hand, as we will see in the following, increasing the number of vertices increases the computational load.

As in (Hetel et al. 2013) the matrix exponential $e^{-L\delta_a}$ is approximated by its h^{th}- order Taylor expansion $\sum_{i=0}^{h} \frac{(-L)^i}{i!}\delta_a^i$. Thus the set $\{X \in \mathbb{R}^{n\times n} \mid X = e^{-L\delta_a},\ \delta_a \in \Delta\}$ can be embedded into the polytopic set defined by the vertices $Z_1, \dots, Z_{h+1}$ where

$$Z_1 = I_n$$
$$Z_i = \sum_{l=0}^{i-1} \frac{(-L)^l}{l!}\delta_{max}^l,\ \forall i \in \{2, \dots, h+1\}$$

with $\delta_{max} = \max_{\delta' \in \Delta} \delta'$, $(-L)^0 = I_n$ and $0! = 1$. Then, Theorem 7 can be replaced by the following more conservative but numerically tractable result.

Theorem 11 *Consider the system* (2) *with $\mathcal{T}$ in the admissible reset sequences* $\Phi(\Delta)$. *If there exist symmetric positive definite matrices $S_i,\ 1 \le i \le h+1$ such that the LMI*

$$\begin{aligned}&\left(I_n - \mathbb{1}_n u^\top W\right)^\top S_i \left(I_n - \mathbb{1}_n u^\top W\right) - \\ &\left(Y(\delta)Z_i - \mathbb{1}_n u^\top W\right)^\top S_j \left(Y(\delta)Z_i - \mathbb{1}_n u^\top W\right) > 0, \\ &Y(\delta) \triangleq P_{ex} e^{-L(\delta)}\end{aligned} \tag{31}$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ *for all* $i, j \in \{1, \dots, h+1\}$, *then* x^* *is GUES for* (2).

PROOF. Assume that the set of LMIs (31) is satisfied for a set of matrices $S_i,\ 1 \le i \le h+1$. Thus,

$$\begin{aligned}&\left(I_n - \mathbb{1}_n u^\top W\right)^\top \left(\sum_{i=1}^{h+1} \mu_i S_i\right) \left(I_n - \mathbb{1}_n u^\top W\right) - \\ &\left(Y(\delta)\sum_{i=1}^{h+1} \mu_i Z_i - \mathbb{1}_n u^\top W\right)^\top \left(\sum_{j=1}^{h+1} \mu_j S_j\right) \times \\ &\left(Y(\delta)\sum_{i=1}^{h+1} \mu_i Z_i - \mathbb{1}_n u^\top W\right) > 0,\end{aligned}$$

is satisfied for all $\mu_i, \mu_j \in [0,1],\ i, j \in \{1, \dots, h+1\}$ such that $\sum_{i=1}^{h+1} \mu_i = \sum_{j=1}^{h+1} \mu_j = 1$. It is noteworthy that the polytopic embedding provided above implies that for all $\delta_a \in [0, \delta_{max}]$ there exists the set of scalars $\mu_i \in [0,1]$ such that $e^{-L\delta_a} = \sum_{i=1}^{h+1} \mu_i Z_i$ and $\sum_{i=1}^{h+1} \mu_i = 1$. In other words, Theorem 7 holds with $S(\delta') = \sum_{i=1}^{h+1} \mu_i(\delta') S_i$.

7 Illustrative examples

Throughout this section the parametric LMIs in Theorems 7, 9 and 10 are replaced by a finite number of LMIs following the reasoning in Section 6. In order to limit the number of LMIs to solve, we have chosen $h = 5$ and embed the set $\{X \in \mathbb{R}^{n\times n} \mid X = e^{-L\delta_a},\ \delta_a \in \Delta\}$ into the polytopic set defined by the vertices $Z_1, \dots, Z_{h+1}$.

7.1 Small network analysis

An academic example consisting in a network of 5 agents partitioned in 2 clusters ($n_1 = 3, n_2 = 2$) is used in the sequel to illustrate the theoretical results. We consider the dynamics (2) with

$$L = \begin{bmatrix} 4 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},\ P_l = \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix} \tag{32}$$

and the reset sequence given by $\delta = 0.5$ and δ'_k randomly chosen in $\Delta = [0, 0.2]$. The initial condition of the system is $x(0) = (8, 7, 9, 2, 3)$ and the corresponding consensus value computed by (16) is $x^* = 4.6757$. The convergence of the 5 agents towards x^* is illustrated in Figure 1 emphasizing that the leaders trajectories are non-smooth while the followers trajectories are. The table below collects the first 10 time

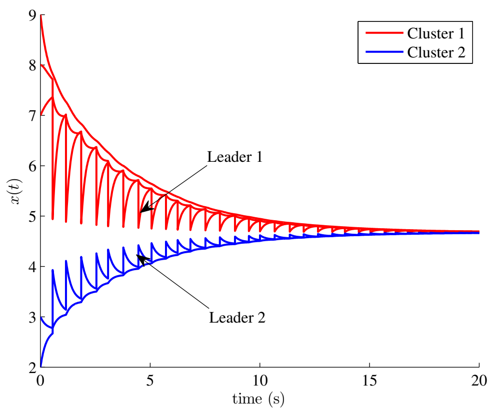

Fig. 1. The state-trajectories of the agents converging to the calculated consensus value.

intervals between consecutive reset instants. As expected, these time intervals have random lengths within $[0.5, 0.7]$ and no monotony occurs. The jumps and decreasing of the

$t_1 - t_0$	0.6189 s	$t_6 - t_5$	0.5979 s
$t_2 - t_1$	0.6131 s	$t_7 - t_6$	0.5372 s
$t_3 - t_2$	0.6433 s	$t_8 - t_7$	0.6401 s
$t_4 - t_3$	0.6023 s	$t_9 - t_8$	0.6965 s
$t_5 - t_4$	0.6553 s	$t_{10} - t_9$	0.6613 s

Table 1
The length of the first 10 time intervals between consecutive reset instants.

Lyapunov function defined by Theorem 7 are pointed out in Figure 2. We emphasize that the matrices $S(\delta')$ used to define V are obtained as in the proof of Theorem 11 after solving (31) for $h = 5$. In order to illustrate the independence

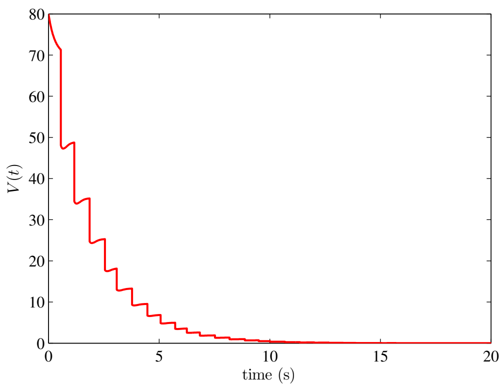

Fig. 2. The behavior of the Lyapunov function given by Theorem 7.

of the consensus value on the reset sequence (see Remark 2), we also considered $\delta = 5$. In this case, as can be seen in Figure 3, the local agreements are reached before each reset and we better emphasize their piece-wise constant behavior (see Remark 1). As expected the consensus value remains $x^* = 4.6757$. Coming back to $\delta = 0.5$, to find the decay rate λ_d we use the bisection algorithm as stated in Remark 5. In Figure 4 we demonstrate that the initial conditions of Lyapunov function may vary due to the conditioning of the matrix $S(\delta)$, but the decay rate remains the same. The value of λ_d obtained was $\lambda_d = 0.855$ and the number of iterations of the bisection algorithm is $k = 30$. Analyzing equation (15) we obtain that the consensus value is always a convex combination of the initial agreement values of the clusters. In the present case, one has two clusters and the two initial agreements are 2.75 and 7.5. Thus, we can try to reach only consensus values belonging to $[2.75, 7.5]$. In Figure 5 the consensus value is fixed at $x^* = 6.5$ and the associated P_l matrix is

$$P_l = \begin{bmatrix} 0.6870 & 0.3130 \\ 0.7825 & 0.2175 \end{bmatrix}.$$

The decay rate associated with this P_l is $\lambda_d = 0.782$ and

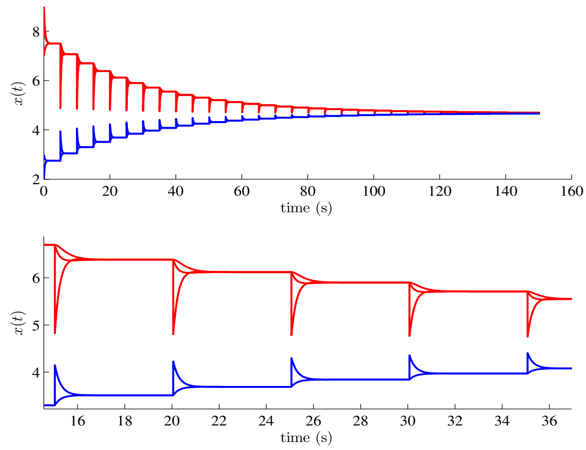

Fig. 3. Top: State trajectories of the agents converging to the piece–wise constant local agreements. The local agreements approach one of each other at the reset times. Bottom: Zoom emphasizing the state behavior. Jumps are present only in the leaders trajectories.

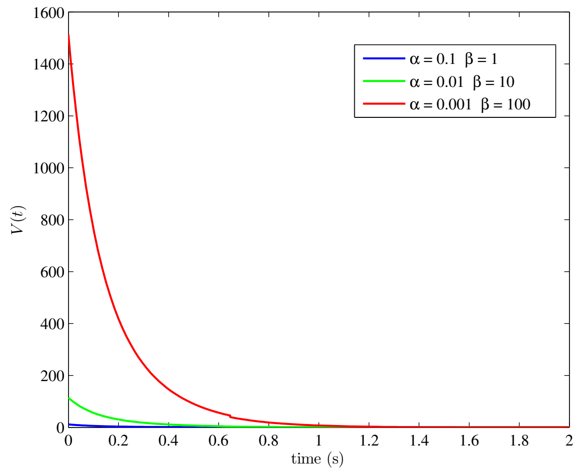

Fig. 4. Lyapunov function for different conditionings

can be improved by using Theorem 10. Imposing $\bar{\lambda} = 0.82$ in (30) one gets

$$P_l = \begin{bmatrix} 0.6425 & 0.3575 \\ 0.8937 & 0.1063 \end{bmatrix}. \tag{33}$$

and for this P_l, the corresponding decay rate is $\lambda_d = 0.756 < \bar{\lambda}$ as noticed in Remark 8. Similar analysis has been done for $x^* = 6$. When P_l is designed without decay rate constraint one gets $\lambda_d = 0.799$ and it is improved to $\lambda_d = 0.747$ designing P_l based on Theorem 10.

In Figure 6 the consensus value is fixed as $x^* = 3.5$ and based on Theorem 9 and the polytopic embedding described

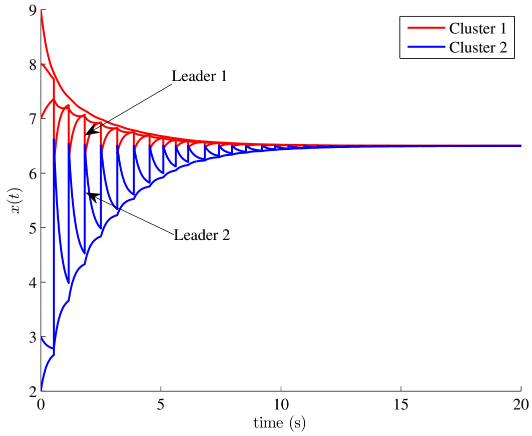

Fig. 5. The states of a system ($x^* = 6.5$).

in the previous Section we get P_l matrix is

$$P_l = \begin{bmatrix} 0.3010 & 0.6990 \\ 0.0874 & 0.9126 \end{bmatrix}. \quad (34)$$

Finally, in Figure 7 the associated Lyapunov function is

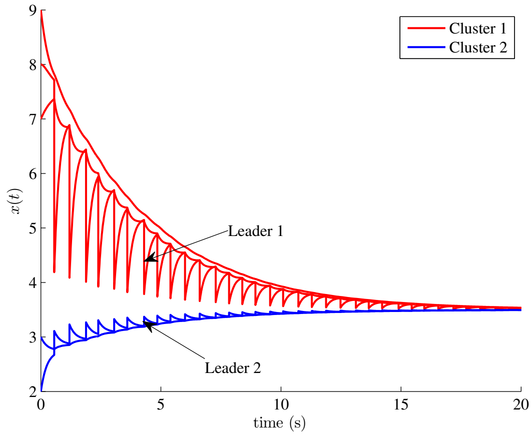

Fig. 6. The states of a system ($x^* = 3.5$).

plotted for both consensus values. Numerical simulations have confirmed the intuition that P_l tends to $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ when x^* approaches the initial local agreement of the second cluster 2.75 while P_l tends to $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ when x^* approaches 7.5 the initial local agreement of the first cluster .

7.2 Larger network analysis

In order to prove that the algorithms are implementable in real networks we consider in the following a larger system.

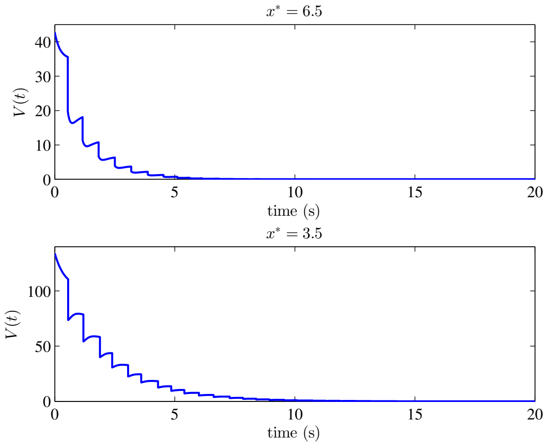

Fig. 7. The Lyapunov function of the system

Precisely, we present an example consisting of a network of 100 agents partitioned in 3 clusters. The size of the clusters as well as the connections between agents are randomized, resulting in non-symmetric matrices L and P_l. The random initialization leads at $n_1 = 59, n_2 = 20, n_3 = 21$ and

$$P_l = \begin{bmatrix} 0.1538 & 0.8080 & 0.0382 \\ 0.4886 & 0.3876 & 0.1238 \\ 0.1266 & 0.2805 & 0.5929 \end{bmatrix}. \quad (35)$$

The initial condition is also randomized but, in order to guarantee a relatively large interval for the possible consensus value, for the first cluster the initial states of the agents are randomly chosen within $[0, 3]$, for the second one within $[3, 7]$ and for the third one within $[7, 10]$. The corresponding initial local agreement values are 1.2970, 5.2578 and 8.7556, respectively. We illustrate the theoretical results by using the dynamics (2) with the reset sequence given by $\delta = 0.5$. The corresponding consensus value computed by (16) is $x^* = 4.2562$. The convergence of the 100 agents towards x^* is shown in Figure 8.

To find the decay rate λ_d we use the bisection algorithm as stated in Remark 5. In Figure 9 we demonstrate that the initial conditions of Lyapunov function may vary due to the conditioning of the matrix $S(\delta)$, but the decay rate remains the same. The value of λ_d obtained was $\lambda_d = 0.9712$ and the number of iterations of the bisection algorithm is $k = 10$.

As noticed before, from (15) we deduce that the consensus value is always a convex combination of the initial agreement values of the clusters. Therefore, for the initialization above, any consensus value can be imposed between 1.2970 and 8.7556. In Figure 10 the consensus value was fixed at

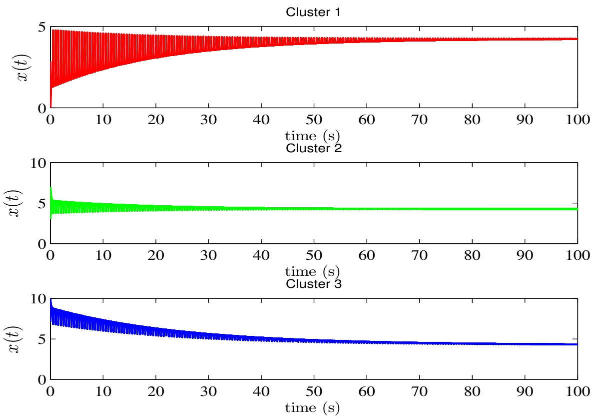

Fig. 8. The state-trajectories of the agents converging to the calculated consensus value.

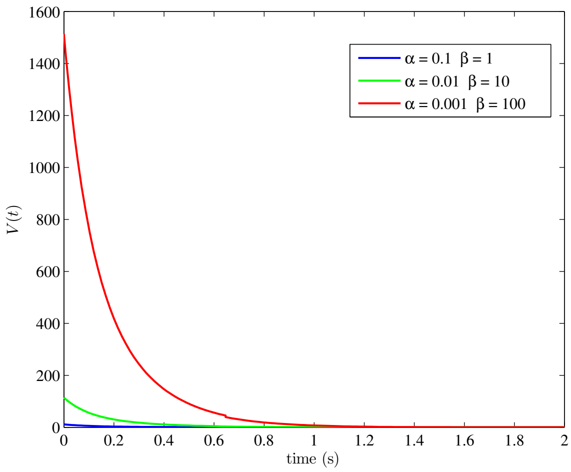

Fig. 9. Lyapunov function for different conditionings

$x^* = 6.5$ and one obtained P_l matrix is

$$P_l = \begin{bmatrix} 0.0643 & 0.3720 & 0.5637 \\ 0.3064 & 0.0358 & 0.6578 \\ 0.0360 & 0.1917 & 0.7723 \end{bmatrix}.$$

8 Conclusions and perspectives

In this work we have considered networks of linear agents partitioned in several clusters disconnected one of each other. The consensus is forced by designing a decentralized reset strategy that exogenously control the state of one agent in each cluster. On one hand we have characterized the consensus value for this type of networks and we have analyzed its stability as well as the convergence speed. On the other hand, we have designed the interconnection network between the

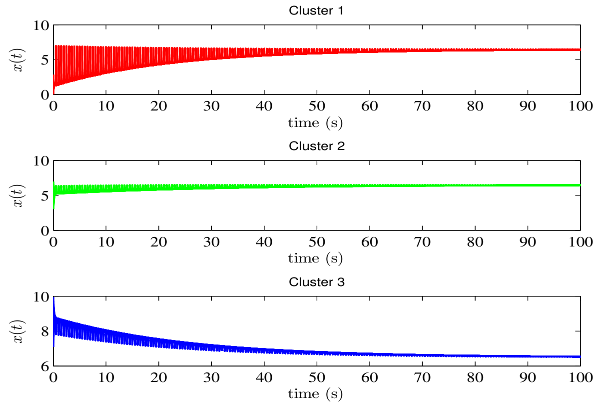

Fig. 10. The states of a system ($x^* = 6.5$).

leaders allowing to reach a prescribed consensus value. Our results are computationally oriented since they are given in LMI form. Two academic examples illustrate the entire theoretical developments.

Future investigations may consider the influence of the leaders centrality on the convergence speed. Other interesting issue would be related to the influence of the nominal reset period δ on the decay rate λ_d. Finally, we consider that networks with impulsive leaders having event-based reset rules may be of particular interest.

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