Constructive necessary and sufficient condition for the stability of quasi-periodic linear impulsive systems

Mirko Fiacchini and Irinel-Constantin Morărescu.

Abstract

The paper provides a computation-oriented necessary and sufficient condition for the global exponential stability of a class of hybrid systems called linear impulsive systems. Precisely, we focus on continuous-time linear systems whose state undergoes finite jumps referred to as impulsions or resets. We assume that impulsions occur quasi-periodically and the stability analysis is based on a standard tool in set theory that is gauge functional. As a first step, the dynamics is reformulated as a discrete-time parametric uncertain system with the state matrix belonging to a compact but non-polytopic set. Secondly, the necessary and sufficient conditions for the global exponential stability of the linear uncertain systems is provided. These conditions are expressed in term of existence of polyhedral Lyapunov functions. Thirdly, an algorithm is developed for testing the stability of the system and for computing the induced polyhedral Lyapunov function for asymptotically stable systems. Finally, the computational effort of the proposed algorithm is proved to be similar to the standard algorithm related to discrete-time parametric uncertain systems with the state matrix belonging to a convex polytopic set.

Index Terms

Reset systems, stability analysis, set theory, polyhedral Lyapunov functions

M. Fiacchini is with GIPSA-lab, Grenoble Campus, 11 rue des Mathématiques, BP 46, 38402 Saint Martin d'Hères Cedex, France. mirko.fiacchini@gipsa-lab.fr

I.-C. Morărescu is with Université de Lorraine, CRAN, UMR 7039, 2 avenue de la forêt de Haye, Vandœuvre-lès-Nancy Cedex, 54516, France; CNRS, CRAN, UMR 7039, France. constantin.morarescu@univ-lorraine.fr

I. Introduction

Linear control techniques are the most studied and used in theory as well as in applications. Nevertheless, they have many limitations when dealing with implementation on real systems. For this reason, an increasing interest has been oriented toward the analysis and control design of hybrid systems (i.e. systems that are driven both by continuous-time and discrete-time dynamics) [1]. It is noteworthy that real systems are either hybrid by nature or continuous but controlled in a discrete manner [2], [3].

In order to overcome performance limitations of classical controllers, Clegg introduced an integrator with state reset (see [4]). This idea received an increasing attention and generated the reset control systems domain of research. Recent works have been dedicated to stability properties and performances of reset control systems [5], [6]. Among these systems a particular class is defined by the continuous-time linear systems whose state undergoes finite jumps at some discrete-time instants [7], [8]. They are called linear impulsive systems and their study is mainly motivated by sampled-data systems [9], [10]. In this case, the rule defining the reset/impulsion instants is time-dependent (see [11] and the reference therein). In the same framework can be included the periodic triggered stabilization [12], [13].

In the present paper we extend our preliminary results [14] on the stability analysis of linear impulsive systems by means of set theoretic methods. Precisely, we complete the work presented in [14] by giving the proofs of the main results, by proposing an algorithm for testing the stability of the system under consideration and by proving that our method requires similar computational effort as the algorithm related to parametric uncertain systems with polytopic constraints. As in [11], we consider here that two consecutive reset instants are separated by an uncertain time. Instead of searching ellipsoidal Lyapunov functions we are searching polyhedral ones leading to less conservative stability conditions. Our approach starts by rewriting the problem as a Linear Difference Inclusion (LDI). This formalism allows describing complex phenomena as impacts and dry friction [15]–[17] as well as the behavior of power converters and other electronic circuits with switching components [18], [19]. It is noteworthy that our main contribution is that we relax the convexity requirement that is generally imposed in the Difference Inclusion settings. Moreover, we design an algorithm that is able to decide in finite time if the LDI, and thus also the corresponding linear impulsive system, is globally exponentially stable (GES).

Following [11], the period between to reset instants has been considered as the sum of a nominal reset period and a time-varying bounded term. This second term may be interpreted as a jitter due to uncertainties that affect the reset instants. This increases the accuracy of the considered model with respect to practical implementation. It is noteworthy that, stability criterion in [11] is given in the form of a parametric Linear Matrix Inequality (LMI) that presents a certain conservatism as most of the LMI criteria. Moreover, in order to render the problem numerically tractable the authors use a polytopic embedding of a matrix exponential, adding supplementary conservatism. In our development we not only remove the conservatism but also provide computational methods that are able to deal with bounded but arbitrarily large reset time uncertainties. Moreover, the computational complexity is substantially the same as the analogous methods for proving GES for LDI systems, that are based on set theory and invariance. With this meaning we employ in this paper the term tractable.

The main tool used throughout the paper is the set theory. In the context of control and dynamical systems analysis, set theory has been employed since the end of the sixties. The seminal work [20] focuses on the characterization of invariance for discrete time systems. More recently have appeared contributions on invariant set analysis and computation, see [21]–[23] and the monograph [24] on set-theory in control and invariance. The characterization of invariant subspaces, strongly related with the properties of controllability and observability, has been treated in [25]–[27]. The set membership estimation approach for systems with unknown but bounded uncertainty should also be mentioned, see the pioneering works [28]–[30] and the more recent [31], [32]. In the context of robust control and constrained control, the basis of the worst case approach have been posed in [33]–[35] and give raise to modern prediction based control techniques [36]. One of the main benefits of set theoretic methods is the fact that they allow to apply results proper of convex analysis, which often lead to computational affordable solutions of the considered problems. The notable contributions on convex analysis and convex optimization provided in [37]–[41] are usually a solid basis for the application of set theory in control.

Notation. The set of real numbers is denoted by $\mathbb{R}$ and $\mathbb{N}$ stands for the set of positive integer numbers. We denote $\mathbb{N}_n \triangleq \{i \in \mathbb{N} : i \leq n\}$. For any function x defined on $\mathbb{R}$ we denote $x(t^-) \triangleq \lim_{\tau \mapsto t, \tau < t} x(\tau)$ if the limit exists. A C-set is a convex and compact set containing the origin in its interior. For any real λ and any set $S \subseteq \mathbb{R}^n$ we define $\lambda S \triangleq \{\lambda x \in \mathbb{R}^n : \ x \in S\}$. The unitary ball in $\mathbb{R}^n$ with respect to the norm $\|\cdot\|_p$ is $\mathbf{B}_p^n \triangleq \{x \in \mathbb{R}^n : \ \|x\|_p \leq 1\}$, its analogous in the space of matrices is defined in the following.

II. Set-theory for nearly-periodic reset systems

Given the interval $\Delta = [\tau_m, \tau_M]$ with $0 < \tau_m < \tau_M \in \mathbb{R}$ and $t_0 \leq \tau_M$, we define the set of admissible reset sequences as

$$\Theta(\Delta) = \Big\{ \{t_k\}_{k\in\mathbb{N}} : \ t_{k+1} = t_k + \delta_k, \ \delta_k \in \Delta, \ \ \forall k \in \mathbb{N} \Big\}. \tag{1}$$

For any time sequence $\mathscr{T} \in \Theta(\Delta)$ consider the following linear reset system, see [11],

$$\begin{cases} \dot{x}(t) = A_c x(t), & \forall t \in \mathbb{R}^+ - \mathscr{T}, \\ x(t) = A_r x(t^-), & \forall t \in \mathscr{T}, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{2}$$

where $x \in \mathbb{R}^n$ is the system state and $A_c,\ A_r \in \mathbb{R}^{n\times n}$ are matrices defining the continuous and reset dynamics, respectively. Thus, the assumption of strict positivity of τ_m in (1) avoids Zeno behavior (i.e. accumulation of reset instants) and the time interval between two resets is upper-bounded by τ_M, i.e.

$$t_{k+1} - t_k \in [\tau_m,\ \tau_M], \qquad \forall k \in \mathbb{N}.$$

The state at time $t \in [t_k,\ t_{k+1})$, for a given initial state x_0 and a reset sequence $\mathscr{T} \in \Theta(\Delta)$ is given by

$$x(t) = e^{A_c(t-t_k)} A_r x(t_k^-), \qquad \forall t \in [t_k,\ t_{k+1}) \tag{3}$$

then, the dynamics between two successive resets is given by the following discrete dynamics

$$x(t_{k+1}^-) = e^{A_c(t_{k+1}-t_k)} A_r x(t_k^-) = e^{A_c(\delta_k)} A_r x(t_k^-), \tag{4}$$

where $\delta_k = t_{k+1} - t_k \in \Delta$. Thus, denoting $A(\Delta) = \{e^{A_c\delta} A_r : \ \delta \in \Delta\}$, the problem of stability of the linear impulsive system (2) rewrites in terms of stability of the LDI described by the following discrete-time parametric uncertain system

$$x_{k+1} = A(\delta) x_k, \tag{5}$$

where $A(\delta) \in A(\Delta)$ and $k \in \mathbb{N}$. Let us recall the definition of GES for the system (5).

Definition 1: The system (5) is GES if there exist positive scalars $c,\ \lambda \in [0,1)$ such that

$$\|x(k)\| \leq c\lambda^k \|x_0\| \tag{6}$$

for every $x_0 \in \mathbb{R}^n$ and $k \in \mathbb{N}$.

It is noteworthy that system (2) is GES if and only if (5) is so. Notice moreover that the set $A(\Delta)$ is not convex in general but it is compact, while the set in which the parameter δ lies, i.e. the interval Δ, is trivially convex and compact. Then, using the classical result from invariance and set-induced Lyapunov functions for linear (uncertain) discrete-time systems, see for instance [24], [42], [43], a necessary and sufficient condition for GES can be given, as follows.

Theorem 2 ([42], [43]): There exists a Lyapunov function for a linear parametric uncertain system if and only if there exists a polyhedral Lyapunov function for the system.

The theorem above is less conservative than Theorem 1 in [11], since it gives not only sufficient but also necessary condition for GES. It claims that the search of the candidate Lyapunov function can be limited to the family of functions which are induced by polytopes.

Remark 3: It is noteworthy that the functions considered in Theorem 2 are convex, positive definite and homogeneous as in [11] (the fact that they are homogeneous of order one and not of order two does not induce any loss of generality). Nevertheless, polyhedral Lyapunov functions are determined by a finite number of generators (the facets of the polytope they are induced by), then they form a set of functions strictly contained in the one considered in [11]. Therefore, the condition in Theorem 2 is less conservative and leads to necessary and sufficient conditions for stability which are computationally tractable, as shown in the sequel.

We also recall hereafter some definition and property concerning set theory and its application to the problem of stability of linear parametric uncertain systems. In particular we focus on the λ-contractive and invariant sets for systems (5), see [21], [24].

Definition 4: Given $\lambda \in [0,1]$ the C-set $\Omega \subseteq \mathbb{R}^n$ is λ-contractive for the system (5) if $A(\delta)x \in \lambda\Omega$ for all $x \in \Omega$ and $\delta \in \Delta$. If $\lambda = 1$ then Ω is robustly positively invariant.

Given a C-set $\Omega \subseteq \mathbb{R}^n$, consider the following sequence of sets

$$\begin{cases} \Omega_0 &= \Omega, \\ \Omega_{k+1} &= Q_\lambda(\Omega_k, A(\Delta)) \cap \Omega, \end{cases} \tag{7}$$

where

$$Q_\lambda(S, \mathscr{A}) = \{x \in \mathbb{R}^n : \ Ax \in \lambda S, \ \forall A \in \mathscr{A}\} = \bigcap_{A \in \mathscr{A}} A^{-1}(\lambda S), \tag{8}$$

with $S \subseteq \mathbb{R}^n$ and $\mathscr{A} \subseteq \mathbb{R}^{n \times n}$. Some properties of the operator $Q_\lambda(\cdot,\cdot)$, often referred to as one-step operator, are provided in the appendix, see Lemma 35. The one-step operator and the recursion

(7) are related to the maximal robust λ-contractive set, some results on which are recalled in the theorem below.

Theorem 5 ([21]): The maximal robust λ-contractive set contained in the C-set $\Omega \subseteq \mathbb{R}^n$ is given by

$$\Omega_\lambda = \bigcap_{k\in\mathbb{N}} \Omega_k. \tag{9}$$

If Ω_λ, with $\lambda \in [0,1)$, is a C-set then for every $\mu \in (\lambda, 1]$ there exists $k(\mu) \in \mathbb{N}$ such that Ω_k is μ-contractive for all $k \geq k(\mu)$. Moreover, the system (5) is GES with convergence index λ if and only if $0 \in \text{int } (\Omega_\lambda)$.

Then, the linear parametric uncertain system (5) is GES with convergence rate λ if and only if Ω_λ is a C-set, that means that it has non-empty interior. Moreover, a μ-contractive C-set is determined by (7) after a finite number of steps provided that $\mu > \lambda$ and the factor λ is such that $\lambda \geq \lambda^*$, where λ^* denotes the maximal convergence index for which a λ^*-contractive C-set exists.

Remark 6: From Theorem 5, it follows that a contractive set with contraction factor μ can be obtained after a finite number of steps, by applying the recursion (7) with $\lambda \in [0, \mu)$ such that Ω_λ contains the origin in its interior (compactness and convexity of Ω_λ are ensured by linearity of the system and the assumption of Ω C-set). Thus, unless one is interested in a contractive set with contraction factor $\mu \in [0,\ \lambda^*]$ (in which case no smaller contraction factor λ leads to a Ω_λ with non-empty interior), it is always possible to obtain a μ-contractive set by iterating (7) a finite number of steps with $\lambda \in [\lambda^*;\ \mu)$.

Then, the iteration (7) together with an appropriate stop condition, represents one version of the basic algorithm for obtaining a μ-contractive set for the uncertain system in a finite number of steps. Moreover, it terminates after a finite number of steps provided λ is adequately chosen. From this and the fact that every μ-contractive set for a linear parametric uncertain system (5) induces a global exponential Lyapunov function the result below follows, see [24], [43]. Let us first recall that, given a C-set $S \subseteq \mathbb{R}^n$, its gauge (or Minkowski) functional $\Psi_S : \mathbb{R}^n \to \mathbb{R}$ is defined as

$$\Psi_S(x) = \min_{\alpha \geq 0} \{\alpha \in \mathbb{R} :\ x \in \alpha S\}. \tag{10}$$

Proposition 7: The linear parametric uncertain system (5) is GES if and only if there exists

$\lambda \in [0,\ 1)$ such that for all $\mu \in (\lambda, 1)$ and every C-set Ω there is $k = k(\mu, \Omega) \in \mathbb{N}$ such that

$$\Omega_k \subseteq Q_\mu(\Omega_k, A(\Delta)), \tag{11}$$

with Ω_k as in (7). Moreover, $\Psi_{\Omega_k}(x)$ is a global exponential Lyapunov function for the LDI (5).

Proof: The result comes directly from Theorem 5 and the fact that condition (11) is equivalent to μ-contractivity of the set Ω_k. Indeed, (11) is equivalent to the fact that for all $x \in \Omega_k$, x belongs also to $Q_\mu(\Omega_k, A(\Delta))$ which means, by definition (8), that $A(\delta)x \in \mu\Omega_k$ for every $\delta \in \Delta$, definition of μ-contractivity of Ω_k. ■

Alternative, but analogous, formulations of the stop conditions are given in literature, see [24]. Thus, summarizing, classical literature results on invariance and set-induced Lyapunov functions permit to assert that the class of positive definite polyhedral Lyapunov functions, that is, the gauge functions of polytopic C-sets, forms an universal class of Lyapunov functions for assessing GES for parametric uncertain linear systems. Moreover, algorithms exist such that contractive sets (and then also the related set-induced Lyapunov function) can be obtained after a finite number of iterations for exponentially stable parametric uncertain systems whose state matrix belongs to a polytope.

The first question we would like to answer is the following:

Problem 8: Given an exponentially stable uncertain system described by the LDI (5), an initial polytopic C-set Ω and a λ such that a C-set λ-contractive exists, does the recursion (7) with stop condition (11) provide a μ-contractive polytope?

Notice that we are wondering whether the μ-contractive set, that exists from Theorem 5, is a polytope or not. The answer depends on the assumptions on $A(\Delta)$. It has been proven that if $A(\Delta)$ is a polytope, then the algorithm provides μ-contractive polytope, [21], [24]. Such results follow directly from the fact that $Q_\lambda(\cdot, A(\Delta))$ maps polytopes into polytopes provided that $A(\Delta)$ is a polytope in $\mathbb{R}^{n\times n}$. Nevertheless, if $A(\Delta)$ is not a polytopic set, such property is no more ensured in general. The following example illustrates this consideration.

Example 9: Consider the discrete-time linear uncertain system (5) with $A(\delta) = \alpha R(\delta)$ where $R(\delta)$ is the rotation matrix, i.e.

$$R(\delta) = \begin{bmatrix} \cos(\delta) & -\sin(\delta) \\ \sin(\delta) & \cos(\delta) \end{bmatrix}, \tag{12}$$

with $\delta \in \Delta = [0,\ \pi/4]$ and $\alpha \in (0,1)$, which ensures robust asymptotic stability. The set $A(\Delta)$ is not a polytope, neither a convex set, in $\mathbb{R}^{2\times 2}$. Notice, the $A(\delta)$ is related to a contraction and turn dynamics. Given a set Ω, the set of successor and predecessor states of $x \in \Omega$ for the system (5) are

$$A(\Delta)\Omega = \bigcup_{\delta\in\Delta} A(\delta)\Omega = \bigcup_{\delta\in\Delta} \alpha R(\delta)\Omega = \{x \in \mathbb{R}^2 :\ x = \alpha R(\delta) z,\quad z \in \Omega, \forall \delta \in \Delta\},$$

$$\begin{aligned} A(\Delta)^{-1}\Omega &= \bigcap_{\delta\in\Delta} A(\delta)^{-1}\Omega = \bigcap_{\delta\in\Delta} \alpha^{-1} R(-\delta)\Omega = \bigcap_{\delta\in\Delta} \{x \in \mathbb{R}^2 :\ x = \alpha^{-1} R(-\delta) z,\quad z \in \Omega\} \\ &= \bigcap_{\delta\in\Delta} \{x \in \mathbb{R}^2 :\ \alpha R(\delta) x \in \Omega\} = \{x \in \mathbb{R}^2 : \alpha R(\delta) x \in \Omega, \forall \delta \in \Delta\} = Q_1(\Omega, A(\Delta)). \end{aligned}$$

Geometrically it means that, for every $\Omega \subseteq \mathbb{R}^n$, the set $Q_\lambda(\Omega, A(\Delta))$ is given by the intersection of $\alpha^{-1}\Omega$ rotated by $-\delta$, for every $\delta \in [0,\ \pi/4]$. Therefore the set $Q_\lambda(\Omega, A(\Delta))$ is not, in general, a polytope, neither for polytopic Ω. Then there is no insurance that the λ-invariant set potentially provided by the recursion (7) is a polytope. In fact, we have that

$$\Omega_{k+1} = \{x \in \Omega_0 :\ \alpha R(\delta) x \in \lambda\Omega_k,\quad \forall \delta \in \Delta\} = \{x \in \Omega_0 :\ R(\delta) x \in \alpha^{-1}\lambda\Omega_k,\quad \forall \delta \in \Delta\},$$

which is given by the intersection of an infinite number of sets, one for every $\delta \in \Delta$. Consider, for instance the case of $\alpha = 0.9$ and apply the recursion with $\Omega_0 = \mathbf{B}^2_\infty$ and $\lambda = 0.9$. We obtain, at the first step, Ω_1 depicted in Figure 1 (left), non-polytopic.

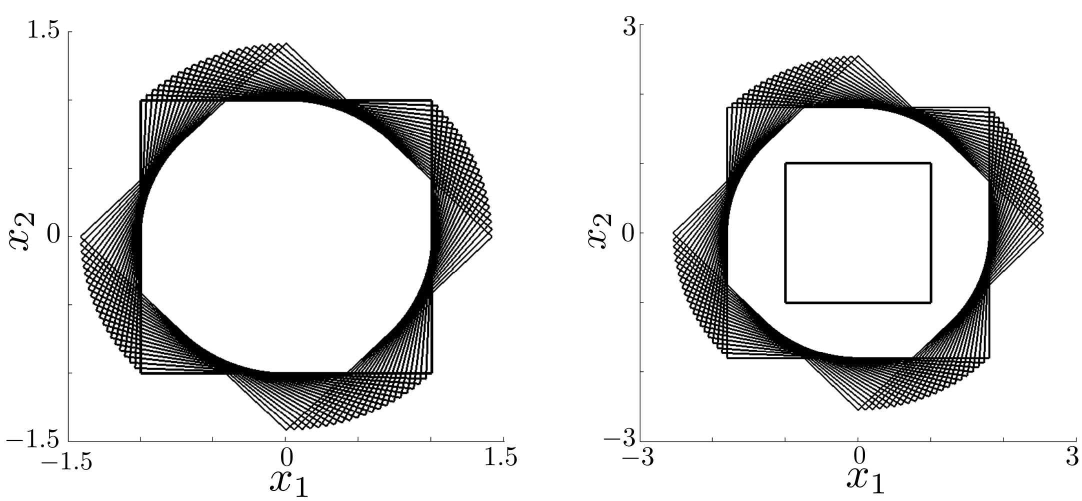

Fig. 1. Ω_0 and $Q_\lambda(\Omega, A(\Delta))$ for $\alpha = 0.9$ (left) and $\alpha = 0.5$ (right).

Nevertheless, if we choose α small enough for the contraction to compensate the rotation due to the uncertainty, then a λ-contractive polytope is obtained at the first step. Figure 1 at right represents Ω_0 and $Q_\lambda(\Omega, A(\Delta))$ for $\alpha = 0.5$. It is obvious that $\Omega_1 = \Omega_0$ and that it is a λ-contractive polytope.

Our main objective is to provide a variation of the classical recursive algorithm for contractive sets computation, such that a polytopic contractive set, and then a polyhedral set-induced Lyapunov function, can be obtained in finite time. Moreover, such algorithm should have a computational complexity similar to the classical one. The algorithm is afterward adapted to the case of study of nearly-periodic reset systems.

III. Gauge functional formalism

In this section we present more details on the gauge functional, which is widely used in the sequel to obtain necessary and sufficient condition for GES of (5).

Definition 10: Given a C-set $\Omega \subseteq \mathbb{R}^n$, define:

- gauge functional of a compact set $S \subseteq \mathbb{R}^{n\times n}$:

$$\Psi_\Omega(S) = \max_{x\in S} \Psi_\Omega(x),$$

- gauge functional of a matrix $A \in \mathbb{R}^{n\times n}$, as induced by the functional for a vector:

$$\Psi_\Omega(A) = \max_{x\in \Omega} \Psi_\Omega(Ax).$$

- gauge functional of compact sets of matrices $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$, as induced by the functional for a matrix:

$$\Psi_\Omega(\mathscr{A}) = \max_{A\in \mathscr{A}} \Psi_\Omega(A).$$

Notice: if Ω is a symmetric C-set, then $\Psi_\Omega(x)$ is a vector norm, see [24], [44].

Definition 11: The (Hausdorff) distance induced by the gauge functional of the C-set $\Gamma \subseteq \mathbb{R}^n$ in the space of matrices $\mathbb{R}^{n\times n}$ is defined

$$d_\Gamma(\mathscr{A}, \mathscr{B}) \triangleq \inf\{\alpha \geq 0 : \mathscr{A} \subseteq \mathscr{B} + \alpha \mathbf{B}_\Gamma^{n\times n};\ \mathscr{B} \subseteq \mathscr{A} + \alpha \mathbf{B}_\Gamma^{n\times n}\}$$

where

$$\mathbf{B}_\Gamma^{n\times n} \triangleq \{A \in \mathbb{R}^{n\times n} : \Psi_\Gamma(A) \leq 1\} = \{A \in \mathbb{R}^{n\times n} : \Psi_\Gamma(Ax) \leq \Psi_\Gamma(x)\}.$$

Some properties of the gauge functional, necessary in the subsequent development, are presented in the following lemmas, whose proofs are provided in the Appendix.

Lemma 12: If Ω is a symmetric C-set, then $\Psi_\Omega(x)$ is a vector norm and $\Psi_\Omega(A)$ is the induced operator norm.

Lemma 13: Given the C-set $\Omega \subseteq \mathbb{R}^n$, one has

$$\Psi_\Omega(\mathscr{A}x) \leq \Psi_\Omega(\mathscr{A})\Psi_\Omega(x) \tag{13}$$

The next lemma follows from convexity of Ω.

Lemma 14: Given the C-set $\Omega \subseteq \mathbb{R}^n$, $\Psi_\Omega(\mathscr{A})$ is such that

$$\Psi_\Omega(\mathscr{A}) = \Psi_\Omega(\mathrm{co}(\mathscr{A})). \tag{14}$$

Remark 15: Given the C-set $\Omega \subseteq \mathbb{R}^n$ and $\mathscr{A}, \mathscr{B} \subseteq \mathbb{R}^{n\times n}$ such that $\mathscr{A} \subseteq \mathrm{co}(\mathscr{B})$ then $\Psi_\Omega(\mathscr{A}) \leq \Psi_\Omega(\mathscr{B})$. The inverse implication is not true in general: consider for instance $\mathscr{A} = \{0\}$ and $\mathscr{B} \neq \{0\}$. Then $\Psi_\Omega(\mathscr{A}) = 0 < \Psi_\Omega(\mathscr{B})$ but $\mathscr{A} \nsubseteq \mathrm{co}(\mathscr{B})$ in general.

IV. Computation-oriented necessary and sufficient condition

First we provide a necessary condition, together with its implication, for a set to be λ-contractive for the linear uncertain system (5).

Proposition 16: If the linear parametric uncertain system (5) is GES with convergence rate $\lambda \in [0,\ 1)$ then for every C-set Ω, every $\mathscr{A} \subseteq \mathrm{co}(A(\Delta))$ and for all $\mu \in (\lambda, 1)$ there exists $p = p(\lambda, \mu) \in \mathbb{N}$ such that condition

$$\Omega_k \subseteq Q_\mu(\Omega_k, \mathscr{A}), \qquad \forall k \geq p, \tag{15}$$

holds, with Ω_k given by

$$\begin{cases} \Omega_0 &= \Omega, \\ \Omega_{i+1} &= Q_\lambda(\Omega_i,\, \mathscr{A}) \cap \Omega, \end{cases} \tag{16}$$

Moreover, if Ω is a polytope in $\mathbb{R}^n$ and $\mathrm{co}(\mathscr{A})$ a polytope in $\mathbb{R}^{n\times n}$ then Ω_k are polytopes and $\Psi_{\Omega_k}(x)$ is a polyhedral global exponential Lyapunov function for the system $x^+ \in \mathscr{A}x$.

Proof: The result follows directly from Theorem 5 and the fact that if $x^+ \in A(\Delta)x$ is GES, also $x^+ \in \mathscr{A}x$ is GES. ∎

The Proposition 16 substantially implies that if one replaces the uncertainty set $A(\Delta)$ with a set which is either polytopic or finite and contained in $\mathrm{co}(A(\Delta))$, then the recursion generates sequences of polytopes and terminates with a polytopic contractive set, if the system is exponentially stable. Notice that this entails a relaxation of the uncertainty bounds and then to an only necessary condition. On the other hand, this leads to a first computationally affordable recursion for obtaining approximation of the polytopic contractive set for the system (5).

Corollary 17: Given $\Omega \subseteq \mathbb{R}^n$ polytope with $0 \in \text{int}\ (\Omega)$ and $\mathscr{A} = \{A_i\}_{i=1}^N \subseteq \text{co}(A(\Delta))$, then there exist $\lambda \in [0,\ 1)$ and $\mu \in (\lambda,\ 1)$ such that the recursion (16) with stop condition (15) terminates in finite steps if the system (5) is GES.

Then, provided the system is GES, every finite selection of matrices in $\text{co}(A(\Delta))$ gives in finite time a polytopic contractive set and a polyhedral Lyapunov functions, for adequate λ and μ. This also means that, if one proves that no contractive set exists for an uncertain system whose matrices forms a subset of $\text{co}(A(\Delta))$, then the system is not exponentially stable.

Corollary 18: Given $\Omega \subseteq \mathbb{R}^n$ polytope with $0 \in \text{int}\ (\Omega)$ and $\mathscr{A} = \{A_i\}_{i=1}^N \subseteq \text{co}(A(\Delta))$, if there are not $\lambda \in [0,\ 1)$ and $\mu \in (\lambda,\ 1)$ such that the stop condition (11) holds for recursion (16), then the system (5) is not GES.

The main practical drawback of the latter result is that, in general, it is not trivial to prove that there are no such pair of λ and μ.

Let us consider an increasing sequence of inner approximations of the set $\text{co}(A(\Delta))$ (for everyone of which a contractive set exists if (5) is GES, from Corollary 17) that converges to $\text{co}(A(\Delta))$. Let us also consider the corresponding sequence of contractive sets obtained by means of (16) and (15). The main idea is to prove that the latter sequence converges to a polytopic contractive set for system (5), if and only (5) is GES.

Remark 19: The metric space of the compact sets of $\mathbb{R}^{n\times n}$ equipped with the Hausdorff distance (determined by the unitary ball with respect to a matricial induced norm) is complete, see [40], [45].

Theorem 20: The linear parametric uncertain system (5) is GES if and only if for every C-set Ω and every increasing sequence of compact convex sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ such that $\mathscr{A}^{(j)} \subseteq \text{co}(A(\Delta))$ and

$$\lim_{j\to\infty} \text{co}(\mathscr{A}^{(j)}) = \text{co}(A(\Delta)), \tag{17}$$

there exist $\lambda \in [0,1)$, $\nu \in (\lambda,1)$, $k = k(\lambda,\nu) \in \mathbb{N}$ and $h = h(\lambda,\nu) \in \mathbb{N}$ such that condition

$$\Omega_k^{(h)} \subseteq Q_\nu(\Omega_k^{(h)}, A(\Delta)) \tag{18}$$

holds, with the sequence of sets $\Omega_k^{(j)}$ given by

$$\begin{cases} \Omega_0^{(j)} = \Omega, \\ \Omega_{i+1}^{(j)} = Q_\lambda(\Omega_i^{(j)}, \mathscr{A}^{(j)}) \cap \Omega, \end{cases} \tag{19}$$

Moreover, if Ω is a polytope in $\mathbb{R}^n$ and $\mathrm{co}(\mathscr{A}^{(j)})$ are polytopes in $\mathbb{R}^{n\times n}$ then $\Omega_k^{(j)}$ are polytopes and $\Psi_{\Omega_k^{(h)}}(x)$ is a polyhedral global exponential Lyapunov function for (5).

Proof: Let us note first that, for Ω convex one has $Q_\lambda(\Omega, \mathscr{A}^{(j)}) = Q_\lambda(\Omega, \mathrm{co}(\mathscr{A}^{(j)}))$ (see Lemma 35 in Appendix for details). So, throughout the proof we consider, without loss of generality, that $\mathscr{A}^{(j)}$ is convex for all $j \in \mathbb{N}$. In other words we suppose that $\mathscr{A}^{(j)} = \mathrm{co}(\mathscr{A}^{(j)})$ for all $j \in \mathbb{N}$.

We need to prove that (18) is necessary and sufficient to guarantee that LDI (5) is GES. We start by necessity, i.e. (5) is GES implies (18). From Theorem 5 and Proposition 16, there exist $\lambda \in [0,1)$, $\mu \in (\lambda, 1)$ and $k \in \mathbb{N}$ such that Ω_k is μ-contractive for dynamics $x^+ \in A(\Delta)x$. Since $\mathscr{A}^{(j)}$ is an increasing sequence, Lemma 35 (see Appendix) yields that for every C-set Ω and every $i \leq j$ one has $Q_\lambda(\Omega, \mathscr{A}^{(j)}) \subseteq Q_\lambda(\Omega, \mathscr{A}^{(i)})$. Thus, by induction, one has

$$\Omega_i \subseteq \Omega_i^{(j+1)} \subseteq \Omega_i^{(j)}, \qquad \forall i \in \mathbb{N},\ \forall j \in \mathbb{N}. \tag{20}$$

In particular, for $i = k$ we have that $\forall j \in \mathbb{N}$ the following inclusions hold $\Omega_k \subseteq \Omega_k^{(j+1)} \subseteq \Omega_k^{(j)}$.

Let us recall that any decreasing sequence of compact convex sets is convergent and it converges to the intersection of all its elements, see Lemma 1.8.1 in [40]. Then, from (17) and (20), it follows that, for all $i \in \mathbb{N}$, the limit of $\left\{\Omega_i^{(j)}\right\}_j \in \mathbb{N}$ exists and contains Ω_i. Throughout the rest of the proof, for all $i \in \mathbb{N}$, we denote $\Omega_i^{(\infty)} = \lim_{j \mapsto \infty} \Omega_i^{(j)}$.

Let us show that $\Omega_i = \Omega_i^{(\infty)}$ for every $i \in \mathbb{N}$. The proof will be done inductively and proceeding by contradiction. We already proved above that, for all $i \in \mathbb{N}$ one has $\Omega_i \subseteq \Omega_i^{(\infty)}$. We start the induction by noting that $\Omega_0 = \Omega_0^{(\infty)} = \Omega$. Suppose that $\exists i \in \mathbb{N}$ such that $\Omega_{i-1} = \Omega_{i-1}^{(\infty)}$ and $\Omega_i \subseteq \Omega_i^{(\infty)}$ but $\Omega_i \neq \Omega_i^{(\infty)}$. Consequently there exists $x \in \Omega_i^{(\infty)}$ such that $x \notin \Omega_i$. From (17), $x \in \Omega_i^{(\infty)}$ and $\Omega_{i-1} = \Omega_{i-1}^{(\infty)}$, it follows that

$$\begin{aligned} x \in Q_\lambda\left(\Omega_{i-1}^{(\infty)}, \lim_{j\to+\infty} \mathscr{A}^{(j)}\right) &= Q_\lambda\left(\Omega_{i-1}, \lim_{j\to+\infty} \mathrm{co}(\mathscr{A}^{(j)})\right) \\ &= Q_\lambda\left(\Omega_{i-1}, \mathrm{co}(A(\Delta))\right) = Q_\lambda\left(\Omega_{i-1}, A(\Delta)\right) = \Omega_i, \end{aligned}$$

that contradicts $x \notin \Omega_i$. We conclude that $\Omega_i = \Omega_i^{(\infty)}$ for all $i \in \mathbb{N}$. Thus, for the particular choice $i = k$ one has $\Omega_k = \Omega_k^{(\infty)}$.

Since $\lim\limits_{j \mapsto \infty} \Omega_k^{(j)} = \Omega_k$, we deduce that: for every $\varepsilon > 0$ there exists $h \in \mathbb{N}$ such that

$$\Omega_k^{(h)} \subseteq \Omega_k + \varepsilon \mathbf{B}^n \subseteq \Omega_k + \varepsilon\rho\Omega_k = (1+\varepsilon\rho)\Omega_k, \tag{21}$$

with $\rho = \Psi_{\Omega_k}(\mathbf{B}^n)$. Therefore, from the μ-contractivity of Ω_k and (20), one has

$$A(\Delta)\Omega_k^{(h)} \subseteq (1+\varepsilon\rho)A(\Delta)\Omega_k \subseteq (1+\varepsilon\rho)\mu\Omega_k \subseteq (1+\varepsilon\rho)\mu\Omega_k^{(h)}, \tag{22}$$

for every $\varepsilon > 0$. By choosing ε such that $1+\varepsilon\rho < \mu^{-1}$, that exists since $\mu < 1$ and ρ is bounded, and defining $\nu = (1+\varepsilon\rho)\mu$ then condition (18) holds and $\Omega_k^{(h)}$ is ν-contractive for (5) with $\nu < 1$.

If Ω and $\mathrm{co}(\mathscr{A}^{(j)})$ are polytopes, then $Q_\lambda^j(\Omega, \mathrm{co}(\mathscr{A}^{(j)}))$ are polytopes too, see Lemma 35 in appendix, and then also $\Omega_k^{(h)}$ is so. Thus, $\Psi_{\Omega_k^{(h)}}(x)$ is a polyhedral global exponential Lyapunov function for (5).

To prove sufficiency, consider $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ such that $\mathscr{A}^j \subseteq \mathrm{co}(A(\Delta))$ and (17) holds. Suppose that there exist appropriate $\lambda \in [0,1)$, $\nu \in \nu(\lambda,1)$ and $k = k(\lambda,\nu)$ such that (18) is satisfied. Thus $\Omega_k^{(j)}$ is a ν-contractive, by definition, which is a sufficent condition for the parametric uncertain system (5) to be globally exponentially stable with $\Psi_{\Omega_k^{(j)}}(x)$ Lyapunov function, polyhedral if Ω and $\mathscr{A}^{(j)}$ are polytopes. ■

Remark 21: Notice that λ and k in the proof of Theorem 20 do not necessarily depend on $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$, whereas ν and h do. Moreover, from the practical point of view, it is worth noting that the value of μ, ρ and h don't have to be computed. Theorem 20 claims that, by choosing appropriate $\lambda \in [0,1)$ and $\nu \in (\lambda,1)$, the sets $\Omega_k^{(j)}$ are ν-contractive for all j big enough. Thus, the computational complexity is analogous to that of classical algorithm for contractive sets computation. The main difference is that the sequence of sets could be more complex, but this is related to the higher complexity of the problem itself.

Thus, any sequence of compact sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ whose convex hull converges from the interior to the convex hull of $A(\Delta)$ generates a sequence of C-sets $\Omega_k^{(j)}$ that converges to a contractive set for (5). Remarkably, if the sets $\mathscr{A}^{(j)}$ are polytopes or finite sets (and Ω is a polytope), the sets $\Omega_k^{(j)}$ are also polytopes.

Corollary 22: Let the linear parametric uncertain system (5) be GES and consider $\lambda \in [0,1)$, $\mu \in (\lambda,1)$, $k = k(\lambda,\mu) \in \mathbb{N}$ such that Ω_k is μ-contractive. Then, for every $\nu \in (\mu,1)$ and every increasing sequence of compact convex sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ such that $\mathscr{A}^{(j)} \subseteq \mathrm{co}(A(\Delta)$ with (17) there exists $h = h(\lambda, \nu)$ such that $\Omega_k^{(j)}$ given by (19) is ν-contractive for (5) for all $j \geq h$.

Proof: As in the proof of Theorem 20, for every $\varepsilon > 0$ there is $h \in \mathbb{N}$ such that $A(\Delta)\Omega_k^{(h)} \subseteq (1+\varepsilon\rho)\mu\Omega_k^{(h)}$, see (21) and (22). Then, for every $\nu \in (\mu, 1)$, with $\nu = (1+\varepsilon\rho)\mu$ for appropriate ε, there exists h such that $\Omega_k^{(h)}$ is ν-contractive for (5). Moreover since $\Omega_k^{(j)}$ is decreasing with respect to j, then

$$\Omega_k^{(j)} \subseteq \Omega_k^{(h)} \subseteq \Omega_k + \varepsilon\mathbf{B}^n \subseteq (1+\varepsilon\rho)\Omega_k, \qquad \forall j \geq h,$$

and then, from μ-contractivity of Ω_k,

$$A(\Delta)\Omega_k^{(j)} \subseteq (1+\varepsilon\rho)\mu\Omega_k \subseteq (1+\varepsilon\rho)\mu\Omega_k^{(j)}, \quad \forall j \geq h.$$

This means that all the sets $\Omega_k^{(j)}$ are ν-contractive for (5) for all $j \geq h$. ■

A. Computation of contractive polytopes and polyhedral Lyapunov functions

Resuming the previous sections, the basic idea for certifying if a nearly periodic reset system is GES is to generate appropriate inner approximations of the set $A(\Delta)$ and use it to compute a contractive C-set. As far as the LDI (5) is GES, every sequence $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ whose convex hull converges to the one of $A(\Delta)$ leads to a contractive C-set for (5) . Thus, we can restrict our attention to finite sets $\mathscr{A}^{(j)}$. This, together with polytopic Ω would lead to sequences of polytopic $\Omega_k^{(j)}$, thus numerically analogous to similar methods for linear systems.

Remark 23: An important computational implication of considering inner approximations of $\mathrm{co}(A(\Delta))$ rather than outer ones, as for instance in [11], is that they are obtained much easier. In fact, every finite set $\mathscr{A}$ contained in $\mathrm{co}(A(\Delta))$ is an inner approximation. Moreover, adding a matrix $A \notin \mathscr{A}$ such that $A \in \mathrm{co}(A(\Delta))$ to $\mathscr{A}$ leads to a tighter approximation of $\mathrm{co}(A(\Delta))$. Then, the sequences of $\mathscr{A}^{(j)}$ can be easily generated by adequately selecting sequences of points in $\partial(\mathrm{co}(A(\Delta)))$. Therefore, no relevant computational effort is required to generate the sequence $\mathscr{A}^{(j)}$.

In other words, generating an appropriate sequence $\left\{\mathscr{A}^{(j)}\right\}_{j\in\mathbb{N}}$ such that (17) is satisfied is a tractable problem in general, even for non-polytopic and nonconvex sets $A(\Delta)$. Thus, the main computational issue for the practical application of the result in Theorem 20 is checking whether the condition (18) is satisfied, i.e. if

$$Ax \in \nu\Omega_k^{(j)}, \qquad \forall A \in A(\Delta),\ \forall x \in \Omega_k^{(j)},$$
$$\Leftrightarrow \quad A\Omega_k^{(j)} \subseteq \nu\Omega_k^{(j)}, \qquad \forall A \in A(\Delta).$$

If $A(\Delta)$ is polytopic, it is sufficient to check the previous inclusion only for a finite number of matrices $A \in A(\Delta)$, i.e. the vertices of $A(\Delta)$. When $A(\Delta)$ is not polytopic, convex or not, condition (18) concerns an uncountable number of matrices in $A(\Delta)$. A possible approach could consist in evaluating the condition for an outer polytopic set $\mathscr{A}$, i.e. for $A \in \mathscr{A}$ with $\mathscr{A}$ polytopic and $\mathrm{co}(A(\Delta)) \subseteq \mathscr{A}$. Nevertheless, this would introduce a certain conservatism. Moreover, the computation of outer approximations of $\mathrm{co}(A(\Delta))$ could be numerically inefficient.

The following considerations are aimed at providing tractable conditions to check whether (18) is satisfied.

Given the two generic sets $\Lambda \subseteq \mathbb{R}^{p\times n}$ and $\mathscr{A} \subseteq \mathbb{R}^{n\times m}$ define

$$\Lambda\mathscr{A} = \bigcup_{\Gamma\in\Lambda} \Gamma\mathscr{A} = \bigcup_{\Gamma\in\Lambda}\bigcup_{\Sigma\in\mathscr{A}} \Gamma\Sigma.$$

Proposition 24: Suppose that $\mathscr{A} \subseteq \mathbb{R}^{n\times n}$ compact is such that for every C-set Ω there exists $\lambda \in [0,\ 1)$ such that for all $\mu \in (\lambda, 1)$ there is $k = k(\lambda,\mu) \in \mathbb{N}$ such that condition (15) holds, with Ω_k given by the sequence of sets given by (16). If $\Lambda \subseteq \mathbb{R}^{n\times n}$ is such that

$$A(\Delta) \subseteq \Lambda\,\mathrm{co}(\mathscr{A}), \tag{23}$$

with

$$\Psi_{\Omega_k}(\Lambda) < \mu^{-1}, \tag{24}$$

then the linear parametric uncertain system (5) is GES and $\Psi_{\Omega_k}(x)$ is a global exponential Lyapunov function for (5).

Proof: From the stop condition (15), it follows that

$$\Omega_k \subseteq Q_\mu(\Omega_k,\mathscr{A}) = \bigcap_{A\in\mathscr{A}} A^{-1}\mu\Omega_k = \{x\in\mathbb{R}^n:\ Ax \in \mu\Omega_k,\ \forall A\in\mathscr{A}\}$$
$$\Leftrightarrow\ Ax \in \mu\Omega_k,\ \forall A \in \mathscr{A},\ \forall x \in \Omega_k, \quad \Leftrightarrow \quad \Psi_{\Omega_k}(\mathscr{A}x) \le \mu\Psi_{\Omega_k}(x).$$

Let us notice that (13), (14) and (23) imply

$$\Psi_{\Omega_k}(A(\Delta)x) \le \Psi_{\Omega_k}(\Lambda\mathrm{co}(\mathscr{A})x) = \max_{A\in\mathrm{co}(\mathscr{A})} \Psi_{\Omega_k}(\Lambda Ax) \le \max_{A\in\mathrm{co}(\mathscr{A})} \Psi_{\Omega_k}(\Lambda)\Psi_{\Omega_k}(Ax)$$
$$< \mu^{-1}\max_{A\in\mathrm{co}(\mathscr{A})}\Psi_{\Omega_k}(Ax) = \mu^{-1}\Psi_{\Omega_k}(\mathrm{co}(\mathscr{A})x) = \mu^{-1}\Psi_{\Omega_k}(\mathscr{A}x) \le \mu^{-1}\mu\Psi_{\Omega_k}(x) = \Psi_{\Omega_k}(x),$$

where the equality is due to (14). Then there exists $\varepsilon \in [0,1)$ such that Ω_k is ε-contractive for the system (5) and $\Psi_{\Omega_k}(x)$ is a global exponential Lyapunov function. ■

The results above leads to the following theorem, that implicitly provides the method for checking whether system (5) is GES.

Theorem 25: The system (5) is GES if and only if for every two sequences of compact sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$ increasing, and $\{\Lambda^{(j)}\}_{j\in\mathbb{N}}$ such that

$$\mathscr{A}^{(j)} \subseteq \mathrm{co}(A(\Delta)) \subseteq \Lambda^{(j)}\mathrm{co}(\mathscr{A}^{(j)}), \tag{25}$$

and

$$\lim_{j\to+\infty} \Lambda^{(j)} = I, \tag{26}$$

there exist $\lambda \in [0,1)$, $\nu \in (\lambda,1)$, $k \in \mathbb{N}$ and $h \in \mathbb{N}$ such that $\Omega_k^{(h)}$, given by (19) is ν-contractive for $x^+ \in \mathscr{A}^{(h)}x$ and

$$\Psi_{\Omega_k^{(h)}}(\Lambda^{(j)}) < \nu^{-1}. \tag{27}$$

Proof: The sufficiency part, that is (27) implies the system (5) is GES, follows directly from Proposition 24.

In the following we treat the necessity part. Thus, we suppose that (5) is GES and we prove the existence of $\lambda \in [0,1)$, $k,h \in \mathbb{N}$ and $\nu \in (\lambda,1)$ such that $\Omega_k^{(h)}$ is ν-contractive for $x^+ \in \mathscr{A}^{(h)}x$ and (27) holds, for all $\mathscr{A}^{(j)}$, $\Lambda^{(j)}$ such that (25) and (26) are satisfied. First notice that $\Psi_\Gamma(\Lambda^{(j)}) \geq 1$ for every symmetric C-set Γ, otherwise, i.e. if $\Psi_\Gamma(\Lambda^{(j)}) < 1$, one has

$$\Psi_\Gamma(\mathscr{A}^{(j)}) = \Psi_\Gamma(\mathrm{co}(\mathscr{A}^{(j)})) \leq \Psi_\Gamma(\Lambda^{(j)}\mathrm{co}(\mathscr{A}^{(j)})) \leq \Psi_\Gamma(\Lambda^{(j)})\Psi_\Gamma(\mathrm{co}(\mathscr{A}^{(j)})) < \Psi_\Gamma(\mathrm{co}(\mathscr{A}^{(j)})), \tag{28}$$

that is absurd. Moreover $\Lambda^{(j)} \to I$ with respect to the Hausdorff distance stated in Definition 11, for a given symmetric C-set Γ. This means that, for every $\varepsilon > 0$ there exists $j_\varepsilon \in \mathbb{N}$ such that

$$\begin{cases} \Lambda^{(j)} \subseteq I + \varepsilon \mathbf{B}_\Gamma^{n\times n}, \\ I \subseteq \Lambda^{(j)} + \varepsilon \mathbf{B}_\Gamma^{n\times n}, \end{cases}$$

for all $j \geq j_\varepsilon$, which yields

$$\Psi_\Gamma(\Lambda^{(j)}) \leq \Psi_\Gamma(I + \varepsilon \mathbf{B}_\Gamma^{n\times n}) \leq \Psi_\Gamma(I) + \Psi_\Gamma(\varepsilon \mathbf{B}_\Gamma^{n\times n}) = 1 + \varepsilon.$$

Thus, for every $\nu \in (\lambda, 1)$ and every symmetric C-set Γ, there exists $j_\nu \in \mathbb{N}$ such that

$$\Psi_\Gamma(\Lambda^{(j)}) < \nu^{-1}, \qquad \forall j \geq j_\nu.$$

From Proposition 16, for every $\mathscr{A}^{(j)}$ and every $\nu \in (\lambda, 1)$, there exists $p = p(\lambda, \nu)$ such that $\Omega_k^{(j)}$ is ν-contractive $x^+ \in \mathscr{A}^{(h)}x$, for all $k \geq p$. Therefore, for $h \in \mathbb{N}$ and $k \in \mathbb{N}$ big enough, (27) holds for $\Gamma = \Omega_k^{(h)}$ and $\Omega_k^{(h)}$ is ν-contractive for $x^+ \in \mathscr{A}^{(h)}x$. ■

V. Finitely determined polytopic Lyapunov functions

In the previous sections we proved that, in general, the classical iterative procedure for obtaining a λ-contractive polytopic C-set entails notable computational concerns when parametric uncertain systems are considered. In what follows the two main computational issues are addressed to provide an affordable algorithm for obtaining polyhedral exponential Lyapunov functions, and thus for checking if the nearly-periodic reset system is GES. Both issues are solved by applying the theoretical results presented above.

The first issue is the fact that the one-step backward operator, basis of such algorithms, does not necessarily generates polytopic sets unless the set of matrices and the initial set are both polytopic. Unfortunately, in the case of nearly-periodic reset systems, the set of possible matrices $A(\Delta)$ that relate the states between two resets is not polytopic in general. The second main issue, related to the computation of polytopic contractive sets (and hence of polyhedral robust exponential Lyapunov functions) concerns the stop condition of the procedure. In fact, the iteration illustrated in Theorem 20 reaches an end in finite time if the condition (18) holds. But such condition involves a potentially uncountable number of constraints to be checked, one for every element of $A(\Delta)$. This leads to computational problems in particular if $A(\Delta)$ is not polytopic.

A. *First issue: sequence of polytopes*

The proposed solution to the first issue is to generate a sequence of polytopic sets $\{\mathscr{A}^{(j)}\}_{j\in\mathbb{N}}$, in the space of matrices, that converges to the set $A(\Delta)$. It has been proved, in fact, that if the original system is GES, then the contractive set for the sets of the sequence is contractive also for the original system, for j big enough, see Theorem 20.

Thus, the first step concerns a possible method to generate the sequence of sets in the space of matrices $\left\{\Lambda^{(j)}\right\}_{j\in\mathbb{N}}$ and $\left\{\mathscr{A}^{(j)}\right\}_{j\in\mathbb{N}}$ such that conditions (25) and (26) hold. Although this particular class of converging sequence of approximations of $A(\Delta)$ might seem rather arbitrary, it fits very well within the framework considered in this paper. In fact, consider the set of matrices $A(\Delta)$ defined in Section II that is:

$$A(\Delta) = \{e^{A_c\delta}A_r : \ \delta \in [\tau_m, \tau_M]\} = \{e^{A_c\delta}e^{A_c\tau_m}A_r : \ \delta \in [0, \tau_m - \tau_M]\} = \bigcup_{\delta\in\Delta_m} e^{A_c\delta}A_m = \Lambda_m A_m,$$

with

$$A_m = e^{A_c\tau_m}A_r, \qquad \Delta_m = [0, \tau_M - \tau_m], \qquad \Lambda_m = \bigcup_{\delta\in\Delta_m} e^{A_c\delta}. \tag{29}$$

The sequences $\left\{\Lambda^{(j)}\right\}_{j\in\mathbb{N}}$ and $\left\{\mathscr{A}^{(j)}\right\}_{j\in\mathbb{N}}$ satisfying (25) and (26) can be constructed as stated below.

Fact 26: Given the nearly-periodic reset system (2), the sequences of sets $\Lambda^{(j)} \subseteq \mathbb{R}^{n\times n}$ and $\mathscr{A}^{(j)} \subseteq \mathbb{R}^{n\times n}$, with $j \in \mathbb{N}$ defined as

$$\begin{aligned} \tau^{(j)} &= (\tau_M - \tau_m)j^{-1}, \qquad & \mathscr{A}^{(j)} &= \bigcup_{i=0}^{j} e^{A_c\tau^{(j)}i}A_m, \\ \Delta^{(j)} &= [0, \tau^{(j)}], & \Lambda^{(j)} &= \bigcup_{\delta\in\Delta^{(j)}} e^{A_c\delta}, \end{aligned} \tag{30}$$

satisfy (25) and (26) and $\{\mathscr{A}^{(j)}\}$ is increasing.

B. Second issue: evaluating the stop condition

The second computational issue to address concerns the stop condition. To solve this problem, we employed Proposition 24, that provides an alternative method to check the stop condition, method that involves a measure of the uncertainty effect. Using such a result, it has been proved in Theorem 25 that if the sequence of sets converging to $A(\Delta)$ is appropriately generated then the stop condition is satisfied in finite time, provided the system is GES. Moreover if the set $\Omega_k^{(h)}$ satisfies condition (27), with $\Lambda^{(j)}$ defined in (30), then the stop condition holds. This means that, if one is able to check efficiently condition (27), then the stop condition is computationally affordable as shown in Theorem 25.

Such a problem consists in practice in finding an affordable method to evaluate function $\Psi_\Omega(e^{A_c\delta})$ with $\delta \in \Delta$, or at least its maximum, for appropriate polytope Ω and $\Delta = [0, \tau]$. For this aim, we provide hereafter a computationally affordable upper bound of the gauge functional of a set $\Lambda^{(j)}$ that converges to 1 as (26) holds.

We recall hereafter some results that are the basis of the set-theory and invariance for control, see [23], [24].

Definition 27: The C-set Ω is β-contractive for the system $\dot{x} = f(x)$ if and only if for each point on the boundary $x \in \partial\Omega$ the following condition holds

$$D^+\Psi_\Omega(x, f(x)) \leq -\beta, \tag{31}$$

with $\beta > 0$.

The link between β-contractivity and exponential stability follows.

Theorem 28: If the C-set Ω is β-contractive for the system $\dot{x} = A_c x$ with $\beta > 0$ then $\Psi_\Omega(x)$ is a global Lyapunov function and

$$\Psi_\Omega(x(t)) \leq e^{-\beta t}\Psi_\Omega(x(0)), \tag{32}$$

for all $t \geq 0$ and $x(0) \in \mathbb{R}^n$.

In case of polyhedral sets Ω, a necessary and sufficient algebraic condition for contractivity can be given.

Theorem 29: The polyhedral C-set $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, where $H \in \mathbb{R}^{h\times n}$, is β-contractive for the system $\dot{x} = A_c x$ with $\beta > 0$ if and only if there exists $T \in \mathbb{R}^{h\times h}$ such that

$$\begin{cases} HA_c = TH, \\ T\mathbf{1} \leq -\beta\mathbf{1}, \\ T_{i,j} \geq 0, \qquad \forall i \neq j. \end{cases} \tag{33}$$

Summarizing, the existence of a β-contractive C-set Ω is necessary and sufficient for a linear (and also a linear parametric uncertain) system to be GES. Moreover, the gauge function induced by Ω is a global Lyapunov function. Finally, a polytope Ω is a β-contractive set, and then condition (32) holds, if and only if the matrix T exists such that (33) are satisfied.

Notice that, although the requirement of $\beta > 0$ is always imposed for achieving exponential stability, the fact that (33) implies the bound (32) does not depend on the sign of β. Indeed, such a relation relies essentially on the comparison lemma, that holds for every β. Thus, the following Lemma holds.

Lemma 30: Given the polyhedral C-set $\Omega = \{x \in \mathbb{R}^n : Hx \leq 1\}$, where $H \in \mathbb{R}^{h \times n}$, the system $\dot{x} = A_c x$ and $\beta \in \mathbb{R}$, if there exists $T \in \mathbb{R}^{h \times h}$ such that (33) holds then (32) is satisfied for all $t \geq 0$.

Proof: The proof, based on the comparison lemma, is substantially the same as Theorem 29, see [23], [24], without positivity constraint on β. ■

Thus, an upper bound on $\Psi_\Omega(\Lambda)$ converging to 1 as $\tau^{(j)}$ goes to 0 can be obtained.

Proposition 31: Given the polytope $\Omega = \{x \in \mathbb{R}^n : \|Hx\|_\infty \leq 1\}$, with $H \in \mathbb{R}^{h \times n}$, and the set $\Delta = [0,\, \tau]$, then

$$\Psi_\Omega(\Lambda) \leq \max\left\{1,\ e^{-\beta^* \tau}\right\}, \tag{34}$$

with $\Lambda = \bigcup_{\delta \in \Delta} e^{A_c \delta}$ and $\beta^* = \sup_{T,\beta}\{\beta \in \mathbb{R} :\ \text{s.t. } (33)\}$.

Proof: From Lemma 30 and the fact that $x(t) = e^{A_c t} x$ with $x(0) = x$, then

$$\Psi_\Omega(\Lambda) = \max_{\delta \in \Delta}\ \Psi_\Omega(e^{A_c \delta}) = \max_{\delta \in \Delta}\ \max_{x \in \Omega}\ \Psi_\Omega(e^{A_c \delta} x) \leq \max_{\delta \in \Delta}\ \max_{x \in \Omega}\ e^{-\beta^* \delta} \Psi_\Omega(x) = \max_{\delta \in \Delta}\ e^{-\beta^* \delta},$$

since $e^{-\beta^* \delta} > 0$ for all β^* and δ. Since $e^{-\beta^* \delta}$ is a monotonic function of δ, its maximum is attain at the boundary of the interval Δ and then (34) follows. ■

The Proposition 31 provides an easily computable (i.e. through an LMI problem in T and β) upper bound of the function $\Psi_\Omega(\Lambda)$ for Λ that is a nonconvex set in $\mathbb{R}^{n \times n}$. Most importantly, such an upper bound converges to one as τ tends to zero. This means that $\Psi_\Omega(\Lambda^{(j)})$ converges to 1 as $j \to \infty$. This implies that, the condition (27) holds if and only there exists a $r \in \mathbb{N}$ for which also the upper bound is smaller than ν^{-1}, and then such a bound can be employed to formulate a computationally affordable stop condition. It is noteworthy that, given the matrix H, the upper bound in Proposition 31 requires to solve a convex optimization problem and the explicit dependence on the nonconvex set $\Lambda^{(j)}$ is removed.

C. Algorithm

A sketch of the algorithm for checking whether the system (5), and then also system (2), is GES follows.

Algorithm 1 Checking GES of system (5) with $A(\delta) \in A(\Delta)$

Input: Values $\tau_m, \tau_M \in \mathbb{R}$, $\lambda \in [0,1)$ and $\nu \in [\lambda, 1)$; C-set Ω.

1: **for** $j \in \mathbb{N}_N$ **do**
2:     Get $\tau^{(j)}, \mathscr{A}^{(j)}, \Delta^{(j)}$ and $\Lambda^{(j)}$ from (30).
3:     From (19), get $\Omega_k^{(j)}$ s.t. $\Omega_k^{(j)} \subseteq Q_\nu(\Omega_k^{(j)}, \mathscr{A}^{(j)})$.
4:     **if** $\Omega_k^{(j)} \neq \{0\}$ **then**
5:         From Proposition 31 get $b^{(j)}$ s.t. $\Psi_{\Omega_k^{(j)}}(\Lambda^{(j)}) \leq b^{(j)}$.
6:         **if** $b^{(j)} < \nu^{-1}$ **then**
7:             The system (5) is GES. **return**
8:         **end if**
9:     **end if**
10: **end for**

Notice that, if condition in point 4 does not hold, one cannot asses that the system is not GES. In this case, it may be necessary to re-initialize the algorithm with a different (higher) value of λ.

Remark 32: The procedure above should be more appropriately defined a semi-algorithm, since it provides an answer in finite time only if the systems is GES. Indeed, in the other case it may run indefinitely (for this reason we impose a finite maximal number of iteration N). Nevertheless, notice that the non-finite termination is related to the problem of assuring that there does not exist an exponential Lyapunov function, which is an undecidable problem in general.

VI. Numerical examples

In this section we present two illustrative examples that provide a comparison with results from the literature.

Example 33: Consider the system (2) with

$$A_c = \begin{bmatrix} 0 & -3 & 1 \\ 1.4 & -2.6 & 0.6 \\ 8.4 & -18.6 & 4.6 \end{bmatrix}, \quad A_r = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon \end{bmatrix}, \tag{35}$$

where $\varepsilon = 10^{-10}$. Such a system has been taken from [11] in order to compare our results with those reported there. Unlike [11] in the matrix A_r we use $\varepsilon \approx 0$ instead of 0. This is done to avoid some numerical complications in our approach. Nevertheless, we note that a non-zero value of ε has a negative effect (the eigenvalues of A_r are $1,1$ and ε instead of $1,1$ and 0). Choosing $\varepsilon \approx 0$ we can reasonably assert that the results obtained hold also for the system in [11].

In the cited paper, the authors fix $\tau_m = 0.1$ and search for an estimation of the maximal value of τ_M for which the reset system is globally asymptotically stable. The method consists in practice in computing a polytopic outer bound of the set $A(\Delta)$. This is done by embedding $e^{A_c\Delta}$ into a polytope whose vertices are defined using Taylor expansion. Then an LMI constraint for every vertex of the polytope is imposed and the resulting stability condition is checked.

In the particular case, the time interval $[\tau_m, \tau_M]$ is partitioned in 5 subintervals and an 8th order Taylor expansion is used to obtain the polytopic outer bound of $A(\Delta)$. This choice seems to be imposed by the complexity related to outer bound computation and to the LMI condition for testing asymptotic stability. The result obtained is that the reset system given by matrices (35) is guaranteed to by globally asymptotically stable for $\tau_M \leq 0.3$.

We applied the method presented in this paper, see Algorithm 1, to obtain an estimation of the maximal value of τ_M for which global asymptotic stability is assured. After several choices of λ, we found out that, using $\lambda = 0.945$ and with $j = 52$ in (30) and $k = 17$, the set $\Omega_{17}^{(52)}$ is λ-contractive for the system with $\mathscr{A}^{(52)}$ and $\Lambda^{(52)}$ such that

$$\Psi_{\Omega_{17}^{(52)}}(\Lambda^{(52)}) \leq 1.0575 < 1.0582 = \frac{1}{\lambda},$$

for $\tau_M = 0.475$, where the first inequality results from the application of Proposition 31. The geometric meaning is that the effect of the implicit approximation process, i.e. considering $x^+ \in \mathscr{A}^{(52)}x$ in spite of $x^+ \in A(\Delta)x$, is an expansion of the set $\Omega_{17}^{(52)}$ bounded by the value of 1.0575. Then, the contraction of $\Omega_{17}^{(52)}$ compensates the expansion due to the approximation error and the set results to be contractive (with contraction rate smaller than λ) also for the system (5). This means that the necessary and sufficient condition for the system to be GES is satisfied. In particular, the contraction ensured for the set $\Omega_{17}^{(52)}$ is given by

$$\lambda \cdot \Psi_{\Omega_{17}^{(52)}}(\Lambda^{(52)}) \leq 0.945 \cdot 1.0575 = 0,9993375 < 1.$$

The fact that the guaranteed contraction is so close to 1 means that the value of τ_M might be close to the maximal one, i.e. the critical one over which the exponential stability is lost. On the other hand, is has been necessary to use a really complex approximation of $A(\Delta)$, by partitioning the set Δ in 52 subintervals.

The obtained result is illustrated in Figure 2 where the sets $\Omega_{17}^{(52)}$ and $e^{A_c\tau^{(52)}}\lambda\Omega_{17}^{(52)}$ are depicted. For every state $x \in \Omega_{17}^{(52)}$ we have that $\mathscr{A}^{(52)}x \subseteq \lambda\Omega_{17}^{(52)}$, from the λ-contractivity of the set, and then $e^{A_c\tau^{(52)}}\mathscr{A}^{(52)}x \subseteq \text{int }(\Omega_{17}^{(52)})$ for all $x \in \Omega_{17}^{(52)}$. Figure 2 shows also that the guaranteed contraction is very close to one, since the boundaries of the the two sets are very close. Remarkably, the estimation of maximal time interval $[\tau_m, \tau_M]$ for which GES is preserved is almost twice the value obtained in [11], i.e. 0.375 in spite of 0.2. We think that the improvement is due to the higher complexity of the approximation of $A(\delta)$. Notice that in our approach the time interval $[\tau_m, \tau_M]$ has been implicitly partitioned in 52 subintervals instead of 5 as done in [11].

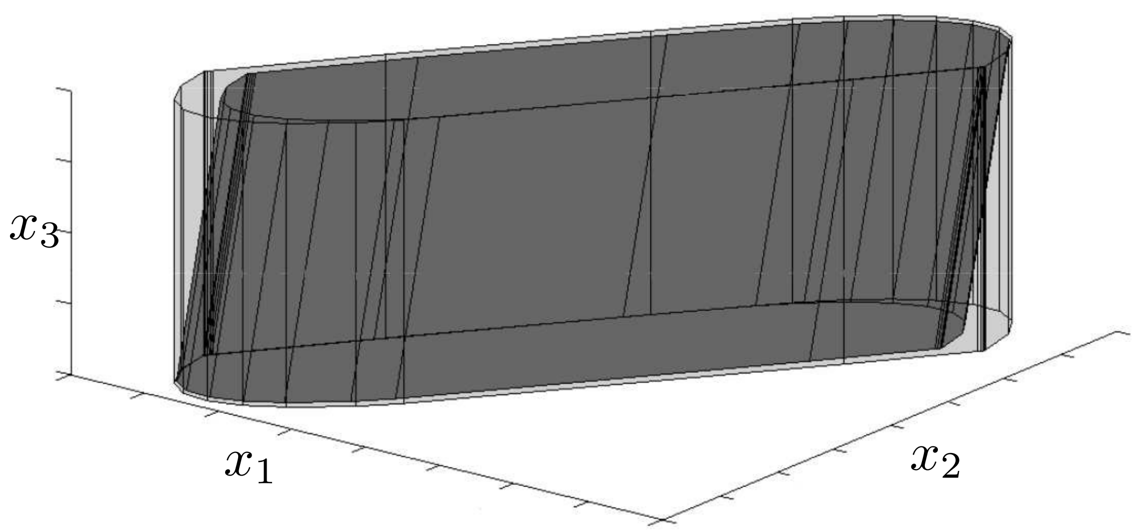

Fig. 2. $\Omega_{17}^{(52)}$ in light gray and the set $e^{A_c\tau^{(52)}}\lambda\Omega_{17}^{(52)}$ in dark gray.

Example 34: This example has been chosen to provide a comparison in terms of conservatism, with the recent interesting results presented in [10]. The method proposed in the cited work for checking whether an impulsive system, substantially equal to (5), is GES, consists in finding a matrix $P > 0$ such that the parameterized LMI condition

$$A_r^T e^{A_c^T\delta} P e^{A_c\delta} A_r < P, \tag{36}$$

holds for all $\delta \in \Delta$. This is equivalent to look for a quadratic Lyapunov function for the uncertain system (5) which might be very conservative when dealing with uncertain linear systems. Recall that, as noticed in Section II, the conservatism is completely removed by considering homogeneous polyhedral functions in spite of quadratic ones. Consider the system (2) with

$$\begin{array}{l} A_c = \begin{bmatrix} -0.01 & 0 \\ 0 & -0.1 \end{bmatrix} R(0.4\pi) = \begin{bmatrix} -0.031 & 0.0095 \\ -0.0951 & -0.0309 \end{bmatrix}, \\ A_r = \begin{bmatrix} 0.9589 & 0.2230 \\ -0.0687 & 1.0561 \end{bmatrix}, \end{array} \tag{37}$$

with $\tau_m = 2$ and $\tau_M = 5$. A necessary condition for (36) to hold for all $\delta \in \Delta$ is clearly that is must be satisfied in particular for $\delta = \tau_m$ and $\delta = \tau_M$. It can be analytically proved that a $P > 0$ such that (36) is satisfied does not exists. To have geometrical hints on this fact, consider the discrete-time system (5) supposing that the reset occurs always at τ_m and τ_M. Their the dynamics are given by

$$A(\tau_m) = e^{A_c \tau_m} A_r = \begin{bmatrix} 0.9500 & 0.2406 \\ -0.2406 & 0.9500 \end{bmatrix}, \qquad A(\tau_M) = e^{A_c \tau_M} A_r = \begin{bmatrix} 0.9589 & 0.2230 \\ -0.0687 & 1.0561 \end{bmatrix}, \tag{38}$$

whose eigenvalues are complex and the spectral radius are 0.98 and 0.9313, respectively. In Figure 3, the trajectories of the systems $x^+ = A(\tau_m)x$ and $x^+ = A(\tau_M)x$, related to resets occurring always at τ_m and τ_M respectively, are drawn together with the level sets of the related Lyapunov functions. One can notice that the first system admits Lyapunov function whose level sets must not be dissimilar from circle whereas for the second one they must be ellipsoids with different axis. Then, there could not be an ellipsoid inducing a common Lyapunov function and the method [10] cannot provide a Lyapunov function for the system (5).

On the other hand, a polytopic set that is λ-contractive for both systems $x^+ = A(\tau_m)x$ and $x^+ = A(\tau_M)x$ exists. Moreover, a polytope that is contractive with $\lambda = 0.995$ for the system $x^+ \in \mathscr{A}^{(16)}x$ is obtained with $k = 12$ and it is such that

$$\Psi_{\Omega_{12}^{(16)}}(\Lambda^{(16)}) \leq 1.0047 < 1.0050 = \frac{1}{\lambda},$$

which implies that $\Omega_{12}^{(16)}$ is contractive also for $x^+ \in A(\Delta)x$, with contraction $0.995 \cdot 1.0047 = 0,9996765$. Thus, the system (5) is GES, although the guaranteed contraction of the robust contractive set $\Omega_{12}^{(16)}$ is very close to one, see Figure 4.

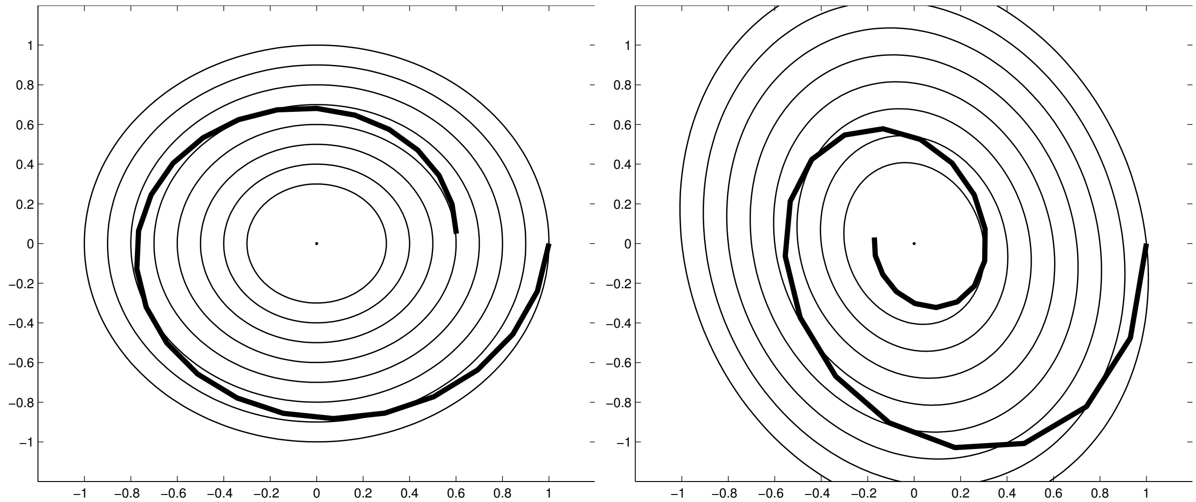

Fig. 3. Trajectories of the systems $x^+ = A(\tau_m)x$ (left) and $x^+ = A(\tau_M)x$ (right), with Lyapunov functions level sets.

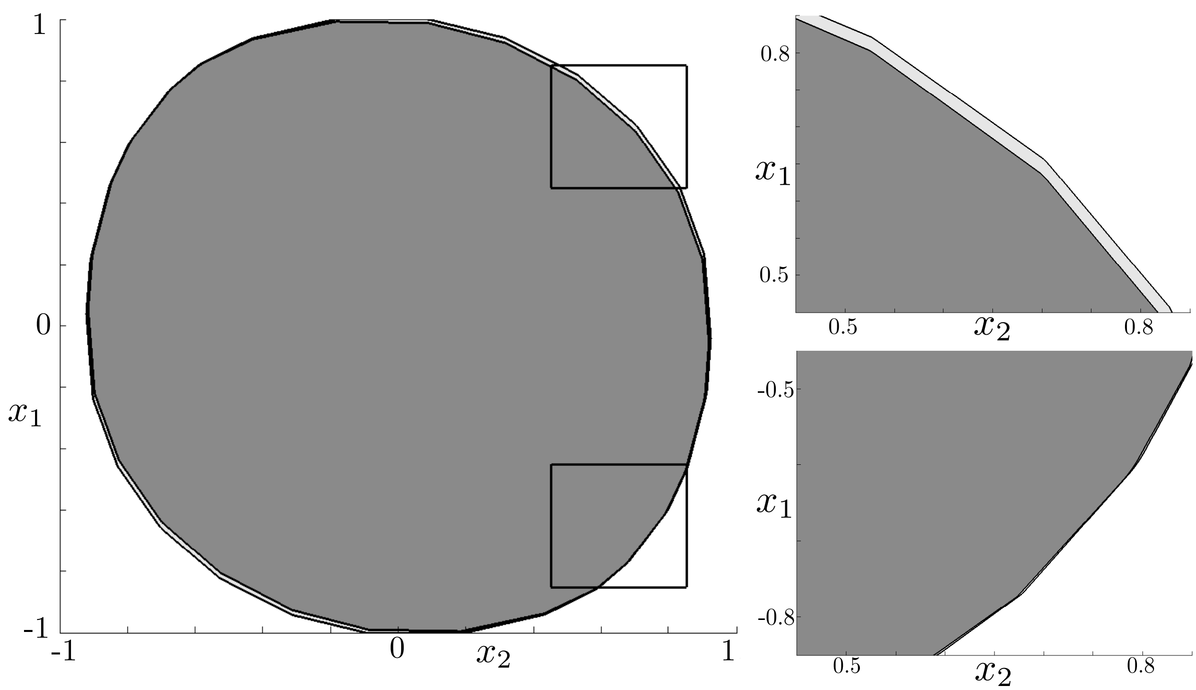

Fig. 4. $\Omega_{12}^{(16)}$ in light gray and the set $e^{A_c\tau^{(16)}}\lambda\Omega_{12}^{(16)}$ in dark gray. Zoom images on the right.

VII. Conclusions

In this paper we employed set theory to provide a computation-oriented method for testing whether an impulsive linear system is globally exponentially stable. The rule defining the occurrence of impulses considered in this paper is assumed to be nearly-periodic. Firstly, we rewrote the system as an LDI associated with linear parametric uncertain non-polytopic system. Next, we provided a method for obtaining a polyhedral Lyapunov function, whose existence is necessary and sufficient for the system to be GES. The approach is particularly suitable since the computational burden is analogous to that required for linear uncertain polytopic systems. The computational aspects related to the implementation of the method have been considered. Finally, the results have been applied to illustrative examples that provide comparisons with analogous existing methods from the literature.

Appendix

A. Instrumental lemma

Lemma 35: Given $\lambda \in \mathbb{R}^n$, $\Omega, \Gamma \subseteq \mathbb{R}^n$ and $\mathscr{A}, \mathscr{B} \subseteq \mathbb{R}^{n\times n}$ then

a) $\mathscr{A} \subseteq \mathscr{B} \quad \Rightarrow \quad Q_\lambda(\Omega, \mathscr{A}) \supseteq Q_\lambda(\Omega, \mathscr{B})$.

b) $\Omega \subseteq \Gamma \quad \Rightarrow \quad Q_\lambda(\Omega, \mathscr{A}) \subseteq Q_\lambda(\Gamma, \mathscr{A})$.

c) If Ω is convex, then $Q_\lambda(\Omega, \mathscr{A}) = Q_\lambda(\Omega, \mathrm{co}(\mathscr{A}))$.

d) If the Ω and $\mathscr{A}$ are polytopes, i.e.

$$\begin{aligned} &\Omega = \{x \in \mathbb{R}^n : Hx \leq b\}, \\ &\mathscr{A} = \{A \in \mathbb{R}^{n\times n} : \; A = \sum_{i\in\mathbb{N}_a} \alpha_i A_i,\, \alpha_i \geq 0,\, \sum_{i\in\mathbb{N}_a} \alpha_i = 1\}, \end{aligned} \tag{39}$$

then $Q_\lambda(\Omega, \mathscr{A})$ is a polytope too.

Proof: a) If $\mathscr{A} \subseteq \mathscr{B}$ it is straightforward that $\bigcap_{A\in\mathscr{A}} A^{-1}(\lambda\Omega) \supseteq \bigcap_{A\in\mathscr{B}} A^{-1}(\lambda\Omega)$ since the second set is obtained by intersecting the first one with some supplementary pre-images of $\lambda\Omega$.

b) If $\Omega \subseteq \Gamma$ let $x \in Q_\lambda(\Omega, \mathscr{A})$ then

$$Ax \in \lambda\Omega,\ \forall A \in \mathscr{A}$$

yielding

$$Ax \in \lambda\Gamma,\ \forall A \in \mathscr{A} \Rightarrow x \in Q_\lambda(\Gamma, \mathscr{A}).$$

Therefore

$$\Omega \subseteq \Gamma \Rightarrow Q_\lambda(\Omega, \mathscr{A}) \subseteq Q_\lambda(\Gamma, \mathscr{A}).$$

c) Since $\mathscr{A} \subseteq \mathrm{co}(\mathscr{A})$ from a) one has $Q_\lambda(\Omega, \mathrm{co}(\mathscr{A})) \subseteq Q_\lambda(\Omega, \mathscr{A})$. Let us show now that $Q_\lambda(\Omega, \mathscr{A}) \subseteq Q_\lambda(\Omega, \mathrm{co}(\mathscr{A}))$. Let $x \in Q_\lambda(\Omega, \mathscr{A})$ and $A \in \mathrm{co}(\mathscr{A})$ i.e. there exist $n \in \mathbb{N}_a$ and

$A_i \in \mathscr{A}, \alpha_i \geq 0$, for all $i \in \mathbb{N}_n$ such that $\sum_{i\in\mathbb{N}_a} \alpha_i = 1$ and $\sum_{i\in\mathbb{N}_a} \alpha_i A_i = A$. Thus, $A_i x = \lambda \omega_i$ with $\omega_i \in \Omega$ and

$$Ax = \sum_{i\in\mathbb{N}_a} \alpha_i A_i x = \lambda \sum_{i\in\mathbb{N}_a} \alpha_i \omega_i \in \lambda\Omega$$

since Ω is convex. Therefore, $\forall A \in \mathrm{co}(\mathscr{A})$ one has $x \in A^{-1}(\lambda\Omega)$, meaning that $x \in Q_\lambda(\Omega, \mathrm{co}(\mathscr{A}))$.
d) It is sufficient to prove that $Q_\lambda(\Omega,\mathscr{A}) = Q_\lambda(\Omega,\{A_i\}_{i\in\mathbb{N}_a})$. Indeed, the latter is the intersection of a finite number of polytopes, hence a polytope itself. From a) one has $Q_\lambda(\Omega,\mathscr{A}) \subseteq Q_\lambda(\Omega,\{A_i\}_{i\in\mathbb{N}_a})$. It remains to prove that one also has $Q_\lambda(\Omega,\mathscr{A}) \supseteq Q_\lambda(\Omega,\{A_i\}_{i\in\mathbb{N}_a})$. Suppose that $x \in Q_\lambda(\Omega,\{A_i\}_{i\in\mathbb{N}_a})$, i.e. $HA_i x \leq b$ for all $i \in \mathbb{N}_a$. Then for every $A \in \mathscr{A}$ it follows that

$$HAx = H\sum_{i\in\mathbb{N}_a} \alpha_i A_i x = \sum_{i\in\mathbb{N}_a} \alpha_i HA_i x \leq \sum_{i\in\mathbb{N}_a} \alpha_i b = b,$$

since $\alpha_i \geq 0$ and $\sum_{i\in\mathbb{N}_a} \alpha_i = 1$. Then $x \in Q_\lambda(\Omega,\mathscr{A})$, which implies that $Q_\lambda(\Omega,\{A_i\}_{i\in\mathbb{N}_a}) \subseteq Q_\lambda(\Omega,\mathscr{A})$. ■

B. Proof of Lemma 12:

By definition $\Psi_\Omega(x) \geq 0$, for all $x \in \mathbb{R}^n$. Moreover, it can be easily proved that $\Psi_\Omega(\cdot)$ defines a norm on $\mathbb{R}^n$ as follows.

1) For all $x \in \mathbb{R}^n$ one has

$$\Psi_\Omega(x) = 0 \quad\Leftrightarrow\quad \min\{\alpha \geq 0 : x \in \alpha\Omega\} = 0 \quad\Leftrightarrow\quad x \in \{0\} \quad\Leftrightarrow\quad x = 0$$

2) Since Ω is symmetric, for all $\lambda \in \mathbb{R}$, $\alpha \geq 0$ and $x \in \mathbb{R}^n$ the following is true $\lambda x \in \alpha\Omega \Leftrightarrow -\lambda x \in \alpha\Omega$ yielding $\Psi_\Omega(\lambda x) = \Psi_\Omega(|\lambda| x)$. For $\lambda \neq 0$, one straightforwardly gets:

$$\Psi_\Omega(|\lambda|x) = \min\{\alpha \geq 0 : |\lambda|x \in \alpha\Omega\} = |\lambda| \min\left\{\frac{\alpha}{|\lambda|} \geq 0 : x \in \frac{\alpha}{|\lambda|}\Omega\right\} = |\lambda|\Psi_\Omega(x)$$

where the last equality is obtained by rescaling α as $\dfrac{\alpha}{|\lambda|}$. For $\lambda = 0$:

$$\Psi_\Omega(|\lambda|x) = \Psi_\Omega(0) = 0 = |\lambda|\Psi_\Omega(x).$$

3) For $x, y \in \mathbb{R}^n$ let us consider $\Psi_\Omega(x) = \alpha_x \in \mathbb{R}_+$ and $\Psi_\Omega(y) = \alpha_y \in \mathbb{R}_+$. Therefore one has

$$\alpha_x = \min\{\alpha \geq 0 : x \in \alpha\Omega\}, \quad \alpha_y = \min\{\alpha \geq 0 : y \in \alpha\Omega\},$$

Thus $\alpha_y x \in \alpha_x\alpha_y\Omega$ and $\alpha_x y \in \alpha_x\alpha_y\Omega$. Since Ω is convex one has

$$\frac{\alpha_x}{\alpha_x+\alpha_y}\alpha_y x + \frac{\alpha_y}{\alpha_x+\alpha_y}\alpha_x y \in \alpha_x\alpha_y\Omega \quad \Leftrightarrow$$
$$\frac{\alpha_x\alpha_y}{\alpha_x+\alpha_y}(x+y) \in \alpha_x\alpha_y\Omega \quad \Leftrightarrow \quad x+y \in (\alpha_x+\alpha_y)\Omega \Leftrightarrow$$
$$\Psi_\Omega(x+y) \leq \alpha_x+\alpha_y = \Psi_\Omega(x)+\Psi_\Omega(y)$$

Let us now consider the operator norm induced by Ψ_Ω:

$$\Psi_\Omega(A) = \sup_{x\neq 0}\frac{\Psi_\Omega(Ax)}{\Psi_\Omega(x)} = \sup_{\Psi_\Omega(x)\leq 1}\Psi_\Omega(Ax) = \sup_{x\in\Omega}\Psi_\Omega(Ax)$$

Since Ω is compact $\sup_{x\in\Omega}\Psi_\Omega(Ax) = \max_{x\in\Omega}\Psi_\Omega(Ax)$ and we recover the definition of $\Psi_\Omega(A)$. From the definition of the operator norm induced by Ψ_Ω one gets

$$\Psi_\Omega(Ax) \leq \Psi_\Omega(A)\Psi_\Omega(x).$$

■

C. *Proof of Lemma 13:*

Using (3) one has

$$\Psi_\Omega(\mathscr{A}x) = \max_{A\in\mathscr{A}}\Psi_\Omega(Ax) \leq \max_{A\in\mathscr{A}}\Psi_\Omega(A)\Psi_\Omega(x) = \Psi_\Omega(\mathscr{A})\Psi_\Omega(x).$$

■

D. *Proof of Lemma 14:*

Since $\mathscr{A} \subseteq \mathrm{co}(\mathscr{A})$ one has $\Psi_\Omega(\mathscr{A}) \leq \Psi_\Omega(\mathrm{co}(\mathscr{A}))$.

Reversely, let us consider

$$\Psi_\Omega(A_{max}) = \max_{A\in\mathrm{co}(\mathscr{A})}\Psi_\Omega(A) = \Psi_\Omega(\mathrm{co}(\mathscr{A})).$$

From the definition of the convex hull $A_{max} \in \mathrm{co}(\mathscr{A})$ implies there exist $A_i \in \mathscr{A}, \alpha_i \geq 0$, for all $i \in \mathbb{N}_n$ such that $\sum_{i\in\mathbb{N}_a}\alpha_i = 1$ and $\sum_{i\in\mathbb{N}_a}\alpha_i A_i = A_{max}$. Therefore,

$$\Psi_\Omega(\mathrm{co}(\mathscr{A})) = \Psi_\Omega(A_{max}) = \Psi_\Omega\left(\sum_{i\in\mathbb{N}_a}\alpha_i A_i\right)$$

and using Lemma 12 one has

$$\Psi_\Omega(\mathrm{co}(\mathscr{A})) \leq \sum_{i\in\mathbb{N}_a}\alpha_i\Psi_\Omega(A_i).$$

We recall that $A_i \in \mathscr{A}$, for all $i \in \mathbb{N}_{N_a}$. Consequently, $\Psi_\Omega(A_i) \leq \Psi_\Omega(\mathscr{A})$ and using $\sum_{i\in\mathbb{N}_a} \alpha_i = 1$ one concludes that

$$\Psi_\Omega(\text{co}(\mathscr{A})) \leq \Psi_\Omega(\mathscr{A})$$

which finishes the proof. ■

References

[1] C. Prieur, R. Goebel, and T. A.R., "Hybrid feedback control and robust stabilization of nonlinear systems.," *IEEE Transactions on Automatic Control*, vol. 52, no. 11, pp. 2103–2117, 2007.

[2] I.-C. Morărescu and B. Brogliato, "Passivity-based tracking control of multiconstraint complementarity lagrangian systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1300–1310, 2010.

[3] P. Hauroigné, P. Riedinger, and C. Iung, "Switched affine systems using sampled-data controllers: robust and guaranteed stabilization," *IEEE Transactions on Automatic Control*, 2011 (To appear).

[4] J. Clegg, "A nonlinear integrator for servomechanisms," *Transactions of the American Institute of Electrical Engineering*, vol. 77(Part II), pp. 41–42, 1958.

[5] O. Beker, C. Hollot, Y. Chait, and H. Han, "Fundamental prperties of reset control systems," *Automatica*, vol. 40, pp. 905–915, 2004.

[6] D. Nešić, L. Zaccarian, and A. R. Teel, "Stability properties of reset systems," *Automatica*, vol. 44, pp. 2019–2026, 2008.

[7] O. Beker, C. Hollot, and Y. Chait, "Plant with an integrator: an example of reset control overcoming limitations of linear feedback," *IEEE Transactions on Automatic Control*, vol. 46, no. 11, pp. 1797–1799, 2001.

[8] W. Aangenent, G. Witvoet, W. Heemels, M. van de Moloengraft, and M. Steinbuch, "Performance analysis of reset control systems," *International Journal of Robust and Nonlinear Control*, vol. 20, no. 11, pp. 1213–1233, 2009.

[9] J. Hespanha, D. Liberzon, and A. Teel, "Lyapunov conditions for input-to-state stability of impulsive systems," *IEEE Transactions on Automatic Control*, vol. 44, no. 11, pp. 2735–2744, 2008.

[10] C. Briat, "Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints," *Automatica*, vol. 49, no. 11, pp. 3349–3457, 2013.

[11] L. Hetel, J. Daafouz, S. Tarbouriech, and C. Prieur, "Stabilization of linear impulsive systems through a nearly-periodic reset," *Nonlinear Analysis: Hybrid Systems*, vol. 7, no. 1, pp. 4–15, 2013.

[12] W. Heemels, M. Donkers, and A. Teel, "Periodic event-triggered control for linear systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 847–861, 2013.

[13] R. Postoyan, P. Tabuada, D. Nesic, and A. Anta, "Event-triggered and self-triggered stabilization of distributed networked control systems," in *Joint IEEE Conference on Decision and Control and European Control Conference*, pp. 2565–2570, 2011.

[14] M. Fiacchini and I.-C. Morărescu, "Set theory conditions for stability of linear impulsive systems," in *Proceedings of the 40th IEEE Conference on Decision and Control*, (Los Angeles, CA, USA), p. To appear, 2014.

[15] F. Facchinei and J. Pang, *Finite-dimensional Variational Inequalities and Complementarity ProblemsResearch, New-York, (2003)*. Springer Series in Operations Research, New-York, 2003.

[16] A. Filippov, *Differential equations with discontinuous righthand sides*, vol. 18. Mathematics and its Applications (Soviet Series), 1988.

[17] J. J. Moreau, *Unilateral contact and dry friction in finite freedom dynamics*, vol. 302. Springer-Verlag, 1998.

[18] F. Vasca, L. Iannelli, M. Camlibel, and R. Frasca, "A new perspective for modeling power electronics converters: complementarity framework," *IEEE Trans. on Power Electronics*, vol. 24, no. 2, pp. 456–468, 2009.

[19] J. Schumacher, "Complementarity systems in optimisation," *Mathematical Programming, Serie B*, vol. 101, pp. 263–295, 2004.

[20] D. P. Bertsekas, "Infinite-time reachability of state-space regions by using feedback control," *IEEE Transactions on Automatic Control*, vol. 17, pp. 604–613, 1972.

[21] F. Blanchini, "Ultimate boundedness control for discrete-time uncertain systems via set-induced Lyapunov functions," *IEEE Transactions on Automatic Control*, vol. 39, pp. 428–433, 1994.

[22] I. Kolmanovsky and E. G. Gilbert, "Theory and computation of disturbance invariant sets for discrete-time linear systems," *Mathematical Problems in Engineering*, vol. 4, pp. 317–367, 1998.

[23] F. Blanchini, "Set invariance in control," *Automatica*, vol. 35, pp. 1747–1767, 1999.

[24] F. Blanchini and S. Miani, *Set-Theoretic Methods in Control*. Birkhäuser, 2008.

[25] G. Basile and G. Marro, "Controlled and conditioned invariant subspaces in linear system theory," *Journal of Optimization Theory and Applications*, vol. 3, pp. 306–315, 1969.

[26] G. Basile and G. Marro, "On the robust controlled invariant," *Systems and Control Letters*, vol. 9, pp. 191–195, 1987.

[27] G. Basile and G. Marro, *Controlled and Conditioned Invariants in Linear System Theory*. Prentice Hall, 1992.

[28] F. C. Schweppe, "Recursive state estimation: Unknown but bounded errors and system inputs," *IEEE Trans. on Automatic Control*, vol. 13, pp. 22–28, 1968.

[29] H. S. Witsenhausen, "Sets of possible states of linear systems given perturbed observations," *IEEE Trans. on Automatic Control*, vol. 13, pp. 556–558, 1968.

[30] D. P. Bertsekas and I. B. Rhodes, "Recursive state estimation for a set-membership description of uncertainty," *IEEE Trans. on Automatic Control*, vol. 16, pp. 117–128, 1971.

[31] D. Maksarov and J. P. Norton, "State bounding with minimal volume ellipsoids," in *Proceedings of UKACC International Conference on Control '96*, 1996.

[32] F. L. Chernousko, "Optimal ellipsoidal estimation of dynamic systems subject to uncertain disturbances," *Cybernetics and systems analysis*, vol. 38, pp. 221–229, 2002.

[33] H. S. Witsenhausen, "A minimax control problem for sampled linear systems," *IEEE Trans. on Automatic Control*, vol. 1, pp. 5–21, 1968.

[34] D. P. Bertsekas and I. B. Rhodes, "On the minimax reachability of target sets and target tubes," *Automatica*, vol. 7, pp. 233–247, 1971.

[35] J. D. Glover and F. C. Schweppe, "Control of linear dynamic systems with set constrained disturbances," *IEEE Transactions on Automatic Control*, vol. 16, no. 5, pp. 411–423, 1971.

[36] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert, "Constrained model predictive control: Stability and optimality," *Automatica*, vol. 36, pp. 789–814, 2000.

[37] S. Boyd, L. El Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in system and control theory*. SIAM, 1994.

[38] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[39] R. T. Rockafellar, *Convex Analysis*. Princeton University Press, USA, 1970.

[40] R. Schneider, *Convex bodies: The Brunn-Minkowski theory*. Cambridge University Press, Cambridge, England, 1993.

[41] A. Ben-Tal and A. Nemirovski, *Lectures on modern convex optimization*. MPS–SIAM, 2001.

[42] A. P. Molchanov and Y. S. Pyatnitskiy, “Criteria of asymptotic stability of differential and difference inclusions encounterd in control theory,” *Systems & Control Letters*, vol. 13, pp. 59–64, 1989.

[43] F. Blanchini, “Nonquadratic Lyapunov functions for robust control,” *Automatica*, vol. 31, pp. 451–461, 1995.

[44] D. G. Luenberger, *Optimization by vector space methods*. John Wiley & Sons, Inc, 1969.

[45] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*. Springer, 1998.