Analysis of opinion dynamics under binary exogenous and endogenous signals

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Abstract

We propose and analyze a stochastic model for opinion dynamics over social networks. In the scenario considered, each agent has an opinion level which belongs to a discrete set. At any given time, the agent takes an action 0 or 1 depending on the opinion, and this action can be seen as a binary signal that can influence the other agents in the network. The opinion updates based on the signal from a random neighbor or from an external entity who attempts to manipulate or control the network. In the absence of the external signal or a constant signal, this model is shown to asymptotically produce consensus with a finite number of connected agents. Additionally, the consensus is determined by the signal. On the other hand, when the number of agents is large, the time to achieve consensus can become exponentially large and the dynamics exhibit population equilibrium points that are ”metastable”. These equilibria can be observed with a finite (but large) number of agents through numerical simulations and are shown to persist for a long duration.

Keywords: Opinion dynamics, Markov chains, agent based models, stability analysis

1. Introduction

Recent social science experiments [1, 2] have shown that information propagated via social media impact not only the opinions and decisions of the users who directly observe this information but also their friends. The understanding of different features related to opinion dynamics in social networks gained more and more relevance in politics and economics. One such feature is observational learning which refers to the fact that individuals extract information from others' actions. This type of learning has been studied in economics [3, 4] in contexts where agents make decisions – for example, buy product A or product B – sequentially, so that agent 2 observes the action of agent 1, agent 3 observes the actions of agents 1 and 2 and so on. Assuming that product A is better than product B but that each agent observes an imperfect private signal about the relative qualities of these products, rational agents can reach a consensus on the wrong decision, i.e. there is a positive probability that all agents buy product B. Such an outcome is called an information cascade: if enough[1] agents at the beginning of the sequence have a bad private signal indicating that B is better than A, this information is aggregated in a way that makes all subsequent agents ignore their private information and follow the decisions of previous agents by choosing product B. Interestingly, information cascades can also occur in networks where agents engage in observational learning [5, 6]. This motivated several researchers from various disciplines such as sociology, mathematics, physics, computer science, and engineering to study the dynamics of opinions in social networks. The aim of these studies is to better understand the propagation of ideas and fads in social networks. A major issue in this framework is developing realistic models which capture the behavior of large-scale social networks [7]. This problem is complex since it includes different hard challenges as: the dynamics of the interaction network (who talks with whom at a given instant), the dynamics during the interaction (what is the update rule when an individual interacts) and the heterogeneity of the network (each individual has a certain influence and sensibility to other opinions in the network).

In early studies of the opinion dynamics, the heterogeneity is neglected and the dynamics of the network, as well as the update rule, are simplified. For instance, DeGroot [8] considered a fixed network in which individuals

[1] The number of agents necessary to start an information cascade depends on the quality of their private signals.

repeatedly update their opinion by averaging the opinions of their neighbors. Under some mild assumption on the network structure, this dynamics always leads to consensus. After two decades this model was modified in different manners. On one hand, Friedkin-Johnsen [9] proposed a model in which the initial condition (representing initial beliefs, social class membership, etc) plays an important role since it weights at each step the individual opinion evolution. On the other hand, a bounded confidence dynamics on the network topology was added to the simple updating rule proposed by DeGroot. This results in the Deffuant [10] and the Hegselmann-Krause [11] models which consider that each individual can be influenced only by those having opinions close their own. The main difference between these models is that the former supposes that each agent updates its opinion by using only the opinion of a random neighbor while the latter assumes that the update is done by averaging the opinions of all the neighbors. These dynamics often lead to local agreements that correspond to a partitioning in the network. Hegselmann-Krause model was adapted in [12] to a decaying confidence model that can be used both for negotiation processes or cluster detection in a network.

While in the models above the opinion can take any real number, there exist models, in which the opinions are restricted to a discrete set of values. These models generally come from statistical physics, and the most employed are the Ising [13], voter [14], and Sznajd [15] models. They are used to model processes in which individuals have to choose one among a certain number of decisions. It is noteworthy that these models led to the so-called threshold-based rules [16] that basically say that an individual adopts a certain opinion whenever a certain number of its neighbors have this opinion. Moreover, they allowed the consideration of heterogeneity of the network by defining an agent-dependent threshold [17] or the presence of stubborn agents [18].

It is worth noting that all the models presented above assume that each individual has access to the exact value of the neighbor's state. To model a more realistic behavior, a mix of discrete actions, but continuous opinion was proposed in [19]. This model, a type of social learning, considers the opinion as a measure of confidence in the binary choices or actions, and only these actions are accessible to the neighbors. This work was inspired by [19], but in contrast, we provide a theoretical analysis in order to support our numerical results. In [20], a deterministic version of [19] was studied and it was shown that this deterministic model leads to a variety of asymptotic behaviors including consensus, oscillation of opinions or clustering according to local agreements.

In this paper, we present a theoretical analysis of the stochastic model with discrete opinion and binary actions proposed in [19]. In order to simplify the developments, we assume that opinions can be approximated by a given number of levels within (0, 1), while the actions can be only 0 or 1. As an example, if the opinions levels define which car is preferred, from company A or B, an action would be buying the car, or sharing advertisements for A or B in social media. This action is what influences other agents to shift their opinion. As in [20] we consider that extremal opinions (close to 0 or 1) present more inertia. This corresponds to the fact that people who have higher confidence in their action are less likely to change their opinion. Considering stochastic behavior instead of deterministic results in a more realistic model, and discretizing the opinion space is not very restrictive since quantifying the opinion with a precise real number is almost impossible. While many studies in the social sciences literature focus on the emergence of a consensus in social networks [21, 22, 23], our goal here is to analyze the emergence of persistent agreements as well as the preservation and propagation of the opinions in the network. Nevertheless, one main result of the paper states that as long as the network is connected, in the absence of an external signal the opinions will asymptotically reach consensus. Moreover, the consensus value is close to one of the extreme opinions 0 or 1. However, when the number of agents is large, for certain network structures, we observe transient behavior that persists for a long duration of time and is characterized as metastable equilibria. In the presence of a persistent constant external signal, the opinions converge to the action corresponding to the signal asymptotically. However, when a sufficiently large number of agents start with actions opposite to the signal, we show that a metastable equilibrium is reached with majority action being in opposition to the signal.

Opinion dynamics is very complex and each of the existing models emphasizes some of its particular features. We do not claim that the model proposed in this work is more realistic than existing ones but like the others it highlights particular characteristics such as the fact that actions are visible to neighbors while opinions are often not, the interactions between individuals are random and people with strong opinions are more difficult to influence. In this paper, we consider a discretized version of the model from [19] which can be mathematically treated as an interactive Markov chain. Similar approaches have been considered in the literature (see for instance [24, 25]) for other opinion dynamics models like the ones by Hegselmann-Krause or Deffuant. In addition to classical opinion dynamics papers, we also study the impact of an external influence or signal. In contrast with other Markov based approaches, the novelty of our paper comes from the fact that instead of agents accessing the opinion of the neighbors, we suppose that only the action is accessible, which leads to very different dynamics. The main contributions of this work are summarized as follows:

- Firstly, we propose and analyze the asymptotic behavior of a **stochastic model for social learning, where agents update their opinions based on an external signal or the action of a random neighbor**. This is in contrast to other works on opinion dynamics which assume that the opinions of neighboring agents are perfectly observable, which may not be true in many practical situations.
- Secondly, in other OD models like [8], if a stubborn agent can influence all the agents within the network, the consensus value is determined by the opinion of the stubborn agent which is reached asymptotically. While this behavior holds even in our model, we have shown that agents may stay in a metastable equilibrium for an arbitrarily long duration. From a social point of view, this transient metastable phase is the one that matters because the lifecycle of products and duration of campaigns (either electoral or selling) are finite.
- When the number of agents is assumed to be arbitrarily large in communities with certain properties, we show **preservation and propagation** of actions inside communities or clusters of agents within the social network through the characterization of the metastable equilibria.

Some preliminary results provided in this paper were also reported without proofs in [26]. The current work goes further by completing the analysis, considering external signals and providing more consistent illustrations. The rest of the paper is organized as follows. Section 2 introduces the main notation and concepts necessary for the model description detailed in Section 3. The asymptotic behaviour of the dynamics under consideration is described in Section 4. These results are valid for general networks represented by connected graphs. Taking into account the small world features of social networks, in Sections 5 and 6 we analyze large-scale networks with uniformly random gossiping and clustered topology, respectively. Some numerical studies and illustrative simulations are provided in Section 7 before presenting some concluding remarks.

2. Preliminaries

We consider a set of agents $\mathcal{V} = \{1, 2, \dots, K\}$ who are connected through a social network and influence the opinions of each other. The opinions of any agent belong to a discrete set $\Theta = \{\theta_1, \theta_2, \dots, \theta_N\}$. This set is structured such that $\theta_n \in (0, 1)\backslash\{0.5\}$ and $\theta_n < \theta_{n+1}$ for all $n \in \{1, 2, \dots, N\}$. We also take *N to be positive and even*, i.e. $N \in \{2, 4, \dots\}$. We define Θ such that $\theta_{\frac{N}{2}} < 0.5$ and $\theta_{\frac{N}{2}+1} > 0.5$. Note that this definition *strictly prohibits* an opinion level of 0.5. To have the opinion levels symmteric around 0.5, we also take $\theta_n = \theta_{N-n+1}$.

Next, we introduce some graph notions defining the interaction over the social network under consideration.

Definition 1 (Directed graph). *A directed graph G is an ordered triplet $(\mathcal{V}, \mathcal{E}, \pi)$ with $\mathcal{V}$ being a finite set denoting the vertices, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ being the set of directed edges, and $\pi_{i,j}$ for any $i, j \in \mathcal{V}$ represents the trust that agent j has on the actions of agent i. Agent j is a neighbor of agent i if and only if $\pi_{i,j} > 0$.*

We say that agent i is *connected* with agent j, if there exists at least one sequence $(i_1, i_2, \dots, i_{p+1})$ such that $\pi_{i_k, i_{k+1}} > 0$, $\forall k \in \{1, 2, \dots, p\}$.

Definition 2 (Strongly connected). *If any two distinct agents $i, j \in \mathcal{V}$ are connected, the graph G is said to be strongly connected.*

As our opinion dynamics has stochastic elements, we also introduce $\Pr(\cdot)$ to denote the probability and $E(\cdot)$ for the expectation of a random variable. We also use *almost surely* to describe events that happen with probability 1 although the set of possible exceptions to the event occurring is non-empty. In the sequel the discrete-time will be denoted by the variable $t \in \mathbb{Z}_{\geq 0}$ representing the curent index of sampling time. In order to characterize the asymptotic and transient behavior of the system, we also introduce some relevant stability notions.

Definition 3 (Stability notions). *For any autonomous discrete-time dynamics of a stochastic variable $p(t)$, a steady state or an* **equilibrium** *p_e is*

- *an absorbing state if $p(t^*) = p_e$ implies $p(t) = p_e$ for all $t > t^*$ surely,*
- **almost surely stable** *if there exists $\epsilon^* > 0$ such that if $\|p(t^*) - p_e\| < \epsilon$ at some t^*, then $\|p(t) - p_e\| < \epsilon$ almost surely for any finite $t > t^*$ and any $\epsilon \leq \epsilon^*$.*
- **almost surely exponentially locally stable** *if there exists $R, r > 0$ and $\epsilon \in (0, 1)$ such that if $\|p(t^*) - p_e\| < r$ at some t^*, then $\|p(t) - p_e\| < R\epsilon^{(t-t^*)}$ almost surely for any finite $t > t^*$.*
- **metastable** *if p_e is almost surely stable but does not correspond to an absorbing state.*

The notion of metastability comes from physics where systems can have local minima for their potential energy function which is not a global minimum. Such states are called metastable as the system persists around such states in the absence of any perturbation. A large enough perturbation can push the system out of the locally stable state. In the context of stochastic systems and Markov chains, metastable states (or region) correspond to a state (or set of states) of the Markov chain which have an infinitely large exit rate, i.e. the probability of reaching a state outside the meta-stable state approaches zero [27].

3. Model

We denote the opinion of any agent $i \in \mathcal{V}$ as $x_i(t) \in \Theta$ at time $t \in \{1, 2, \dots\}$. We use $X(t) = (x_1(t), x_2(t), \dots, x_K(t))$ to denote the state of the network at any time t. Let $\lfloor \cdot \rceil$ denote the nearest integer function. The action taken by agent i is then given by

$$q_i(t) = \lfloor x_i(t) \rceil.$$

Therefore, if an agent has an opinion more than 0.5, it will take the action 1 and 0 otherwise. Since we suppose that $0.5 \notin \Theta$, agents always pick action 1 or 0. This kind of opinion quantization is suitable for many practical applications such as in politics where the agents must make binary decisions despite believing in their action strongly or weakly.

Let the total population share of users with opinion θ_n be denoted by $p_n(t)$ such that $\sum_n p_n(t) = 1$, calculated as

$$p_n(t) = \frac{\sum_{i=1}^{K} \delta_{x_i(t),\theta_n}}{K} \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta function (i.e., $\delta_{a,b} = 0$ if $a \neq b$ and $\delta_{a,b} = 1$ when $a = b$). Define $p_-(t)$ as the population share of agents with *action* 0, and $p_+(t)$ as the population share with the *action* 1. We have

$$p_-(t) = \frac{\sum_{i=1}^{K} \delta_{q_i(t),0}}{K} = \sum_{n=1}^{N/2} p_n(t) \tag{2}$$

and

$$p_+(t) = \frac{\sum_{i=1}^{K} \delta_{q_i(t),1}}{K} = \sum_{n=N/2+1}^{N} p_n(t) \tag{3}$$

More generally, if we consider a set $C \subset \mathcal{V}$, we can define the population shares of agents in C as

$$p_n^C(t) = \frac{\sum_{i \in C} \delta_{x_i(t),\theta_n}}{|C|}. \tag{4}$$

and similarly define p_+^a and p_-^a.

3.1. Exogenous signal

In addition to the interaction of agents in the graph, we also account for the presence of an external exogenous signal which sometimes influence the opinion of agents. In order to smoothly integrate the presence of these signals, we add two agent labeled E_0 and E_1 to the graph which results in an extended set of agents $\mathcal{V}_E := \{E_0, E_1, 1, 2, \dots, K\}$. We define the influence on these two agents to be only be themselves, i.e., $\pi_{i,E_0} = \pi_{i,E_1} = 0$ for all $i \in \mathcal{V}$ and

$\pi_{E_0,E_0} = \pi_{E_1,E_1} = 1$. We also define $X_{E_0}(1) = \theta_1$ and $X_{E_1}(1) = \theta_N$. These conditions implies than these agents never change their opinion and that $q_{E_0}(t) = 0$ and $q_{E_1}(t) = 1$ for all t.

We have identified four cases of interest

1. $\pi_{E_0,i} = \pi_{E_1,i} = 0$ for all $i \in \mathcal{V}$.
2. $\pi_{E_1,i} = 0$ for all $i \in \mathcal{V}$ and $\exists j \in \mathcal{V}$ such that $\pi_{E_0,j} > 0$.
3. $\pi_{E_0,i} = 0$ for all $i \in \mathcal{V}$ and $\exists j \in \mathcal{V}$ such that $\pi_{E_1,j} > 0$.
4. $\exists i, j \in \mathcal{V}$ such that $\pi_{E_0,i} > 0$ and $\pi_{E_1,j} > 0$.

The exogenous signals considered here represent external players who defend a fixed position in favor of one of the alternatives 0 or 1, and who influence each agent in the network with some given probability at each period. For example, these external players can be firms competing for customers or opinion leaders competing for the support of the public on a political or social matter. Although the behavior of these players is not modeled explicitly and is taken as an exogenous shock to the network, it is interesting to study the effect of a variation of the relative strength of their influence on the resulting opinion dynamics.

3.2. Opinion dynamics

Now, we describe the model of opinion evolution by random gossiping. This implies that agents $i \in \mathcal{V}$ at any time instant $t \in \mathbb{Z}_{\geq 0}$ is influenced by the external signal or the action of one random neighboring agent. This influential agent is denoted as $J_i(t) \in \mathcal{V}_E$ which includes the external influence E_0 or E_1. The probability of having an agent j influencing agent i is given by $\pi_{j,i}$, which implies

$$\forall i, j \in \mathcal{V}_E, \qquad \Pr(J_i(t) = j) = \pi_{j,i}. \tag{5}$$

Agent i will shift his opinion towards the action of $J_i(t)$ with a certain probability f_n when $x_i(t) = \theta_n$, the *inertia factor*. That is, the opinion also measures the confidence of agents in their actions and an agent with opinion close to 0.5 is more likely to shift as he is less confident in his action. Whereas an agent with a strong opinion (close to 1 or 0) is less likely to shift his opinion as he is more convinced by his action choice. The opinion of agent i may shift towards the action of agent i based on the *inertia factor* $f_n \in (0, 1)$ for any $n \in \{1, 2, \ldots, N\}$. For example, $f_n = 4\theta_n(1 - \theta_n)$, which is inspired by the model used in [19]. This factor is symmetric w.r.t. $n = N/2$, i.e., $f_n = f_{N+1-n}$ and is typically increasing from $n = 1$ to $n = N/2$ and decreasing later [2] Therefore, we can write the stochastic opinion dynamics of any agent $i \in \mathcal{V}$ as

$$\begin{array}{lll}
\forall n \in \{1, 2, \ldots, N-1\}, & \Pr(x_i(t+1) = \theta_{n+1} | x_i(t) = \theta_n, q_{J_i(t)} = 1) & = f_n, \\
\forall n \in \{2, 3, \ldots, N\}, & \Pr(x_i(t+1) = \theta_{n-1} | x_i(t) = \theta_n, q_{J_i(t)} = 0) & = f_n, \\
\forall n \in \{2, 3, \ldots, N-1\}, & \Pr(x_i(t+1) = \theta_n | x_i(t) = \theta_n) & = 1 - f_n, \\
\forall n \in \{1, N\}, & \Pr(x_i(t+1) = \theta_n | x_i(t) = \theta_n, q_{J_i(t)} = \lfloor \theta_n \rceil) & = 1, \\
 & \Pr(x_i(t+1) = \theta_n | x_i(t) = \theta_n, q_{J_i(t)} \neq \lfloor \theta_n \rceil) & = 1 - f_n.
\end{array} \tag{6}$$

for all $t \in \mathbb{Z}_{\geq 0}$.

Before proceeding to the analysis of this Markov chain, we provide a list of notations and their meaning in Table 1 for ease of exposition.

4. Asymptotic behavior of opinions in finite networks

We denote the combined states of the overall network by α_m where $\alpha_m \in \Theta^K$ for all $m \in \{1, 2, \ldots, N^K\}$. In other words, for any $m \in \{1, 2, \ldots, N^K\}$ one defines $\alpha_m := (\theta_{m_1}, \theta_{m_2}, \ldots, \theta_{m_K}) \in \Theta^K$ as the vector collecting the opinions of the K agents in the network. Moreover, we consider that $\alpha_1 = (\theta_1, \theta_1, \ldots, \theta_1)$ and $\alpha_2 = (\theta_N, \theta_N, \ldots, \theta_N)$.

[2] Note that $\frac{1}{f_n}$ would be the actual 'inertia' as a higher f_n implies a higher willingness to change its opinion.

Symbol	Meaning
Θ	Set of opinion values, $\lvert\Theta\rvert = N$, N even.
$\mathcal{V}$	Set of agents, $\lvert\mathcal{V}\rvert = K$
f_n	Willingness to shift opinion with current opinion $\theta_n \in \Theta$
$x_i(t)$	Opinion of agent i at time t, $x_i(t) \in \Theta$
$q_i(t)$	Action of agent i at time t $q_i(t) = \lfloor x_i(t) \rceil \in \{0, 1\}$
$\pi_{j,i}$	Probability of agent i observing action of j
$p_n(t)$	Expected fraction of population with opinion θ_n, $p_n = \sum_{i \in \mathcal{K}} \frac{E(x_i(t)=\theta_n)}{K}$
$p_+(t)$, $p_-(t)$	Expected fraction of population with action 1, $p_+(t) = \sum_{i \in \mathcal{K}} \frac{E(q_i(t)=1)}{K}$ or action 0, $p_-(t) = \sum_{i \in \mathcal{K}} \frac{E(q_i(t)=0)}{K}$
$p_n^C(t)$	Expected population share within set $C \subseteq \mathcal{K}$ with opinion θ_n
$p_+^C(t)$, $p_-^C(t)$	Expected population share within set $C \subseteq \mathcal{K}$ with action 1 $p_+^C = \sum_{i \in C} \frac{E(q_i(t)=1)}{\lvert C \rvert}$

Table 1: Notations used

Now define a sequence of random variables $X(t)$, where the realizations of $X(t)$ belong to Θ^K. From the opinion dynamics model (6), we have:

$$\Pr(X(t+1) = \alpha_m | X(t) = \alpha_{m'}, X(t-1) = \alpha_{m''}, \dots) = \Pr(X(t+1) = \alpha_m | X(t) = \alpha_{m'}) \tag{7}$$

which by definition, is a Markov process and represents the opinion dynamics of all the K agents. Additionally, the transition of each agent is independent of the other transitions yielding

$$\Pr(X(t+1) = \alpha_m | X(t) = \alpha_{m'}) = \Pi_{k=1}^{K} \Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_{m'}) \tag{8}$$

Here, the term $\Pr(x_k(t+1) = \theta_{n_k} | X(t) = \alpha_m)$ can be calculated based on (5) and (6) as

$$\Pr(x_k(t+1) = \theta_{n_k} | X(t) = \alpha_m) = \begin{cases} 0 & \text{if } |m_k - n_k| > 1 \\ f_{m_k} Q_k^-(\alpha_m) & \text{if } n_k = m_k - 1 \\ f_{m_k} Q_k^+(\alpha_m) & \text{if } n_k = m_k + 1 \\ 1 - f_{m_k} & \text{if } m_k = n_k \ \&\ 1 < n_k < N \\ 1 - f_{m_k} Q_k^+(\alpha_m) & \text{if } m_k = n_k = 1 \\ 1 - f_{m_k} Q_k^-(\alpha_m) & \text{if } m_k = n_k = N \end{cases} \tag{9}$$

where $Q_k^-(\alpha_m) = \sum_{j \in \mathcal{V} \setminus \{k\}} \pi_{j,k} (1 - \lfloor \theta_{m_j} \rceil)$ and $Q_k^+(\alpha_m) = \sum_{j \in \mathcal{V} \setminus \{k\}} \pi_{j,k} \lfloor \theta_{m_j} \rceil$. We use $\xi(t)$ to denote the probability distribution over the states $(\alpha_1, \alpha_2, \dots)$ at time t, i.e., for all $m \in \{1, \dots, N^K\}$, $\xi_m(t) = \Pr(X(t) = \alpha_m)$. The Markov process in matrix form can be written as

$$\xi(t+1) = M\xi(t) \tag{10}$$

where M, the *transition matrix* has elements

$$M_{m,n} = \Pr(X(t+1) = \alpha_m | X(t) = \alpha_n)$$

which are obtained from (8) and (9).

Definition 4 (Absorbing state). *A state α_m of the Markov process* (8) *is called an absorbing state if and only if*

$$\Pr(X(t+1) = \alpha_m | X(t) = \alpha_m) = 1.$$

This implies that once the Markov process reaches an absorbing state, then it stays in that state and will never move to another state in the future. Consider the following assumption.

Assumption 1. *The graph defined by the probability of interactions* $(\mathcal{V}, \mathcal{E}, \pi)$ *is strongly connected.*

In this context, we are able to get the following result.

Proposition 1. *We define* $\alpha_1 := (\theta_1, \theta_1, \ldots, \theta_1)$ *and* $\alpha_2 := (\theta_N, \theta_N, \ldots, \theta_N)$. *Under Assumption 1, the Markov process* $X(t)$ *defined in (8) has the following characteristic for each case defined in Section 3.1:*

1. $\pi_{E_0,i} = \pi_{E_1,i} = 0$ *for all* $i \in \mathcal{V} \Rightarrow$ *both* α_1 *and* α_2 *are absorbing states,*
2. $\pi_{E_1,i} = 0$ *for all* $i \in \mathcal{V}$, $\exists j \in \mathcal{V}$ *such that* $\pi_{E_0,j} > 0 \Rightarrow \alpha_1$ *is the only absorbing state,*
3. $\pi_{E_0,i} = 0$ *for all* $i \in \mathcal{V}$, $\exists j \in \mathcal{V}$ *such that* $\pi_{E_1,j} > 0 \Rightarrow \alpha_2$ *is the only absorbing state,*
4. $\exists i, j \in \mathcal{V}$ *such that* $\pi_{E_0,i} > 0, \pi_{E_1,j} > 0 \Rightarrow$ *there are no absorbing states.*

Proof See Appendix A. ■

The probability of reaching absorbing states α_1 or α_2 can be computed based on the results in [28]. Since α_1 and α_2 are absorbing states, we can rewrite (10) as

$$\xi(t+1) = \begin{bmatrix} I_2 & 0 \\ R_1 & R_2 \end{bmatrix} \xi(t) \tag{11}$$

where I_2 is the 2×2 identity matrix, and R_1 and R_2 are the remaining elements of M (M takes this form).

Corollary 1 (From [28]). *The Markov process* $X(t)$ *defined in (8), under Assumption 1, will asymptotically* almost surely *reach* α_1 *or* α_2 *irrespective of the starting state. The probability to reach a specific absorbing state can be calculated as* $B = (I' - R_2)^{-1} R_1$, *where* I' *is the* $(N^K - 2) \times (N^K - 2)$ *identity matrix,* $B_{m,n}, m \in \{1, 2\}, n \in \{1, 2, \ldots, N^K - 2\}$ *is the probability to reach the absorbing state* α_m *from a non-absorbing state* α_n.

Also note that the absorbing states α_1 and α_2 are almost surely locally asymptotically stable when $x(t) \in S_1$ or $x(t) \in S_2$ respectively, where $S_i = \{y \in \Theta^N \mid \lfloor y_n \rceil = i - 1 \;\; \forall n \in \mathcal{V}\}$ for $i = 1, 2$. However, note that the number of states of this Markov system is given by $|N^K|$, i.e if there are 4 opinion levels and 20 agents, there will be 4^{20} states corresponding to a $4^{20} \times 4^{20}$ sized transition matrix. This makes an analysis of the Markov chain in terms of rate of convergence to the absorbing states or probability to converge to the absorbing states etc. computationally heavy and almost impossible when K is large. In many social networks, K is indeed large and of the order of thousands if not millions. This motivates us to study the large-scale limit in the following section.

As pointed out in literature [29, 30], social networks are characterized by a small world structure. In other words, any two nodes in the network are connected through a small path. To account for this feature, in the sequel we are considering two particular cases: uniform random interactions among all agents and clustered communications. In the first case any agent can interact with any other agent with equal probability while in the second case we have the generic modeling of small world networks in which agents have a large probability to interact with agents in a certain group and small probability to interact outside the group. By focusing on these particular network typologies we are able to obtain analytic characterizations of the opinion dynamics.

5. Uniformly random gossiping in large-scale networks

In this section we consider the case of an arbitararily large number of agents, i.e., $K \to \infty$. In practice, this case is common as real social networks contain large number of individuals/agents. Indeed the convergence in such networks to the absorbing states may be very slow, and the transient behavior may be of interest, see [24, 25] for instance. The goal of this section is to provide an approximation of the transient behavior of large-scale networks. Beside the absorbing states emphasized before, in large networks, we prove that the system may also reach other *population equilibria*, even if the individual opinion of agents vary in time. Define the population share vector for the entire set of agents as $\underline{p}(t) = (p_1(t), \ldots, p_N(t))^{\mathrm{T}}$ (where T is the transpose) and $\underline{p}^C(t) = (p_1^C(t), \ldots, p_N^C(t))^{\mathrm{T}}$ as the population share with a set $C \subset \mathcal{V}$ for convenience.

Definition 5 (Equilibrium in population). *A population vector p_e denotes an equilibrium in population if $\underline{p}(t) = p_e$ implies $\underline{p}(t+1) = p_e$.*

First, we assume that $\pi_{E_0,i} = \pi_{E_0,j} := \pi^-$ and $\pi_{E_1,i} = \pi_{E_1,j} := \pi^+$ for any $i, j \in \mathcal{V}$. Define by $\lambda = 1 - \pi^- - \pi^+$, the probability of any agent to interact with other agents in the network at a given time. We consider a uniformly random selection of agents for gossiping, i.e., we assume $\pi_{i,j} = \frac{\lambda}{K-1}$ for all $i, j \in \mathcal{V}$. As every agent has an identical probability of interacting with the rest, we can do a large number or mean field approximation (using the central limit theorem) to study the dynamics of the population shares.

For finite K, the number of agents that shift their opinion from θ_m at time t to opinion θ_n at time $t+1$ is a stochastic variable and can be expressed as the sum of $Kp_m(t)$ independent and identically distributed random variables (recall that $p_m(t)$ is the population share of agents with opinion θ_m). In the limit of $K \to \infty$, the total number of agents migrating converges to a deterministic value (the expectation) if we apply the central limit theorem as shown in the following proposition.

Proposition 2. *When $X(t)$ has the stochastic dynamics* (9) *with $K \to \infty$ and $\pi_{i,j} = \frac{\lambda}{K-1}$, we have*

$$
\begin{aligned}
&p_1(t+1) = p_1(t) - f_1 p_1(t)\left(\pi_{E_1,1} + \lambda \textstyle\sum_{n=N/2+1}^{N} p_n(t)\right) + p_2(t) f_2 \left(\pi_{E_0,1} + \lambda \textstyle\sum_{n=1}^{N/2} p_n(t)\right), \\
&p_m(t+1) = (1 - f_m) p_m(t) + f_{m-1} p_{m-1}(t)\left(\pi_{E_1,1} + \lambda \textstyle\sum_{n=N/2+1}^{N} p_n(t)\right) + f_{m+1} p_{m+1}(t)\left(\pi_{E_0,1} + \lambda \textstyle\sum_{n=1}^{N/2} p_n(t)\right), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall m \in \{2, \ldots, N-1\} \\
&p_N(t+1) = p_N(t) - f_N p_N(t)\left(\pi_{E_1,0} + \lambda \textstyle\sum_{n=1}^{N/2} p_n(t)\right) + p_{N-1}(t) f_{N-1} \left(\pi_{E_1,1} + \lambda \textstyle\sum_{n=N/2+1}^{N} p_n(t)\right).
\end{aligned} \tag{12}
$$

which holds almost surely.

Proof We transform the agent dynamics given in (9), into a deterministic population dynamics which occurs almost surely using the central limit theorem as follows. For this, we evaluate

$$
\begin{aligned}
\Pr(x_i(t+1) = \theta_n) = (1 - f_n) \Pr(x_i(t) = \theta_n) + f_{n-1} \Pr(x_i(t) = \theta_{n-1}) \Pr(q_{J_i(t)}(t) = 1) + \\
f_{n+1} \Pr(x_i(t) = \theta_{n+1}) \Pr(q_{J_i(t)}(t) = 0)
\end{aligned} \tag{13}
$$

Since we have $K \to \infty$ and $\pi_{i,j} = \frac{\lambda}{K-1}$ for all $i, j \in \mathcal{V}$, we have $\Pr(q_{J_i(t)}(t) = 1) = \lambda p_+(t) + \pi_{E_1,1}$ and $\Pr(q_{J_i(t)}(t) = 0) = \lambda p_-(t) + \pi_{E_0,1}$ for any $i \in \mathcal{V}$. $p_n(t+1)$ is in fact the sum of K (with $K \to \infty$) Bernoulli variables taking the value 1 with a probability given by (13). This allows us to evaluate

$$
E(p_n(t+1)) = f_{n-1} p_{n-1}(t)(\lambda p_+(t) + \pi_{E_1,1}) + (1 - f_n) p_n(t) + f_{n+1} p_{n+1}(t)(\lambda p_-(t) + \pi_{E_0,1}) \tag{14}
$$

for all $n \in \{2, 3, \ldots, N-1\}$. However, since the variances of each of the K Bernoulli variables are bounded by $1/4$ (all of them are between 0 and 1), we can use the Central limit theorem (Lyapunov version), to conclude that the variance of $p_n(t+1)$ will be arbitararily small as $K \to \infty$, or alternately, $p_n(t+1) = E(p_n(t+1))$ almost surely. Applying similar arguments to $p_1(t+1)$ and $p_N(t+1)$, we get the result (12). ∎

With Proposition 2, we have transformed the behavior of the stochastic dynamic system (6) with an infinitely large number of agents into an almost sure deterministic quadratic dynamical system (12) which characterizes the evolution of the population shares of agents with a given opinion. When the number of agents $K \to \infty$, (12) holds almost surely and can therefore be used to study the behavior of the distribution of agents with each opinion.

5.1. Without the influence of any exogenous signal

Note that $\pi_{E_0,i} = \pi_{E_1,i} = 0$ for all $i \in \mathcal{V}$ for this subsection and we have $\lambda = 1$. Consider the population vectors $p_{e_1} = (1, 0, \ldots, 0)$, $p_{e_2} = (0, \ldots, 0, 1)$ and $p_{e_3} = (\frac{\kappa}{f_1}, \frac{\kappa}{f_2}, \ldots, \frac{\kappa}{f_N})$, where $\kappa = \frac{1}{\sum_{n=1}^{N} \frac{1}{f_n}}$.

Theorem 1. *When $N \in \{4, 6, 8, \dots\}$, $\pi_{i,j} = \frac{1}{K-1}$ $\forall i, j \in \mathcal{V}$ and $K \to \infty$, the system* (12) *allows for exactly three equilibria p_{e_1}, p_{e_2} and p_{e_3}. p_{e_1} and p_{e_2} characterize the absorbing states α_1 and α_2 and are almost surely locally exponentially stable equilibria, and the system also allows for one unstable equilibrium point p_{e_3}.*

Proof See Appendix B. ■

Theorem 1 shows that the absorbing states are almost surely exponentially locally stable while the only other equilibrium is unstable when $\pi_{i,k} = \frac{1}{K-1} \forall i, j \in \mathcal{V}$. When the number of agents is large, several works on Markov chains like [31] have the absorbing states that are not stable, leading to metastable or quasi-stationary states that persist for an arbitrarily long duration. However, in our model, when all agents have identical probabilities of interacting, the absorbing states are almost surely locally exponentially stable. This suggests that the convergence time to one of these absorbing states will be quite small, and this can be observed from our numerical results in Section 7.

Remark 1. *The assumptions made in the preliminaries on θ_n being symmetric around* 0.5 *are only required to prove the properties of p_{e_3}. All the other results stated in Theorem 1 only require that there are at least* 2 *opinion levels above and below* 0.5*, i.e. there exists $\theta_n, \theta_m < 0.5$ and another $\theta_n, \theta_m > 0.5$ belonging to* Θ.

Proposition 3. *When the number of opinion levels N is exactly* 2*, then every population distribution is a stable (but not asymptotically stable) equilibrium. Additionally, for finite N, the randomness of the opinion dynamics results in perturbations that disturb the equilibrium for any distribution other than the two corresponding to the absorbing states.*

Proof We observe that when $N = 2$, i.e., when there are exactly two opinion levels, equation (12) can be simplified into

$$\begin{aligned} p_1(t+1) &= p_1(t) - f_1 p_1(t) p_2(t) + p_2(t) f_2 p_1(t) \\ p_2(t+1) &= p_2(t) - f_2 p_2(t) p_1(t) + p_1(t) f_1 p_2(t) \end{aligned} \tag{15}$$

As $f_n = f_{N+1-n}$ is symmetric by definition, we get $p_1(t+1) = p_1(t)$ and $p_2(t+1) = p_2(t)$ for any $p_1(t)$ or $p_2(t)$. Therefore, when there are a very large number of agents and exactly two opinion levels, any population distribution becomes an equilibrium. These equilibrium points are neutral as they are neither stable nor unstable and any perturbation just brings it to another equilibrium. ■

The case of $N = 2$ corresponds to a voter model and the resulting population dynamics is that of a random walk. Such systems have already been well studied in literature [32] and we therefore focus our attention on the case of $N > 2$, i.e $N \in \{4, 6, \dots\}$.

5.2. With the influence of a exogenous signals

Let us consider that x is the probability for an agent to be influenced by action 1 (the influence can come from another agent or from an exogenous signal). Then, the following function: $\phi : [0,1] \to [0,1]$

$$\phi(x) = \begin{cases} \dfrac{\sum_{n=N/2+1}^{N} \frac{1}{f_n} (\frac{x}{1-x})^{n-1}}{\sum_{n=1}^{N} \frac{1}{f_n} (\frac{x}{1-x})^{n-1}} & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases}, \tag{16}$$

computes the equilibrium population share with action 1 provided that x does not change in time. Moreover, the component m of the function $\Phi : [0,1] \to [0,1]^N$ defined by

$$\Phi_m(x) = \begin{cases} \dfrac{\frac{1}{f_m} (\frac{x}{1-x})^{m-1}}{\sum_{n=1}^{N} \frac{1}{f_n} (\frac{x}{1-x})^{n-1}} & \text{if } x < 1 \\ 1_{\{N\}}(m) & \text{if } x = 1 \end{cases} \tag{17}$$

represents the equilibrium population share with opinion θ_m provided that x does not change in time. We note that $1_S(s)$ is the indicator function which takes the value 1 when $s \in S$ and 0 otherwise.

Let us recall that λ is the probability for an agent to be influenced by agents belonging to the network (excluding the external signals). Then, one can define the set

$$\Lambda = \{\lambda \in (0.5, 1] \,|\, \exists y \in (0.5, 1] \text{ s.t. } \phi(\lambda y) = y\} \tag{18}$$

collecting the values of λ that allow at least a population share y to preserve the action 1.

Theorem 2. *When $N \in \{4, 6, 8, \dots\}$, $\pi_{i,j} = \frac{\lambda}{K-1}$ $\forall i, j \in \mathcal{V}$, $K \to \infty$ and $\lambda < 1$, the equilibria of the system* (12) *is given by $\Phi(p_+^*)$ where p_+^* satisfies $\phi(\lambda p_+^* + \pi^+) = p_+^*$. Furthermore, if $\lambda \in \Lambda$, the system* (12) *has at least two equilibria, one with $p_+^* > 0.5$ and another with $p_+^* < 0.5$, i.e., $\lambda \in \Lambda$ is a sufficient condition for the preservation of majority opinion. This allows us to characterize the equilibrium points of* (12).

Proof Consider that $x := \lambda p_+ + \pi_+$. We have $1 - x = \lambda p_- + \pi_-$. The equilibrium points p_n^* of (12) can be found by setting $p_n^*(t+1) - p_n^*(t) = 0$ for all n resulting in

$$p_n^* := \frac{\frac{1}{f_n}(\frac{x}{1-x})^{n-1}}{\sum_{m=1}^{N} \frac{1}{f_m}(\frac{x}{1-x})^{m-1}} \tag{19}$$

when $x < 1$ and $p_N^{C*} = 1$ when $x = 1$. Since we have $p_+^C(t) = \sum_{n=N/2+1}^{N} p_n^C(t)$, we can write

$$p_+^* = \phi(x)$$

Since $x = \lambda p_+ + \pi_+$, we have that at equilibrium

$$p_+^* = \phi(\lambda p_+^* + \pi_+),$$

and

$$p_-^* = \phi(\lambda p_-^* + \pi_-).$$

Therefore $p_n^* = \Phi(p_+^*)$ is an equilibrium if it satisfies the above condition. Finally, if $\lambda \in \Lambda$ and $\pi_+ = 0$, then $p_+ > 0.5$ satisfying $p_+ = \phi(\lambda p_+)$ exists by definition of Λ. We have ϕ continuous and increasing. We can use the mean-value theorem on $\phi(\lambda p_+ + \pi_+) - p_+$, which is greater than 0 at $p_+ = \phi(\lambda p_+)$ and less than or equal to 0 at $p_+ = 1$ (because $\lambda + \pi_+ \leq 1$). Therefore there exists at least one point in between satisfying the equilibrium condition. ■

If multiple equilibria $\Phi(p_+^*)$ exist, then the initial population distribution determines the equilibrium that is reached almost surely. Theorem 2 states that even in the presence of a single external signal, an equilibrium point with the majority of agents holding the opposite action to the signal can exist if λ is sufficiently high. This behavior is surprising as in the finite number of agents regime, in the presence of a single external signal, there is only one absorbing state for the Markov system and in this absorbing state, all agents have the same action as the external agent. In contrast, when the number of agents is large, if λ is sufficiently large and the initial population share of agents with an action opposite to the external agent is sufficiently large, this majority action is preserved (almost surely) due to frequent interactions within the network.

When the opinion of agents is concerning social or economic issues, the external agents can be interpreted as political parties or firms which can, at a cost (e.g. ads, meetings, debates), increase its own probability of influencing voters or consumers. This can be seen as a non-cooperative game when E_1 and E_2 represent competing firms. As the probabilities change, the equilibrium points are modified according to Theorem 2, and therefore, this result can be of use to design advertising strategies and provide thresholds below which the agents continue to preserve their original opinion (and the impact of the external signal is marginal). The exact determination of the outcomes of the games described above needs to be addressed in future work.

6. Behavior of a generic cluster based graph

In this section, we introduce the notion of clusters of agents and analyze a generic social network model. The notion of clusters is inspired by communities in social networks, which are groups of agents that strongly interact with each other and interact less with agents outside the group. This is a generic way to model small world networks including social networks [33]. Basically, we consider that they are the union of a number of clusters (for instance, see [12] for a cluster detection algorithm) which are weakly influencing each other.

Definition 6 (Cluster). *A set of agents C is said to be a cluster with coefficient $\lambda_C \in [0.5, 1]$ if the following property is satisfied*

$$\pi_{i,j}|C| \geq \lambda_C, \qquad \forall i, j \in C \tag{20}$$

The case of $\lambda_C = 1$ corresponds to a cluster which has no influence from agents outside the cluster, i.e. the graph is not strongly connected or $C = \mathcal{V}$. However, when the number of agents is arbitrarily large, and $|C|, K \to \infty$, then $\lambda_C \to 1$ is possible even when the graph is strongly connected. This is because C and $\mathcal{V} \setminus C$ can have a finite number of agents with $\pi_{i,j} > 0$. With this definition, we are in a position to study the dynamics of opinion within a cluster in terms of its population.

Proposition 4 (Population dynamics in C). *Let C be a cluster with a coefficient λ_C and $|C| \to \infty$, the population shares of agents in C evolve* almost surely *with the following dynamics*

$$\begin{aligned} p_1^C(t+1) = &\quad p_1^C(t) + f_2 p_2^C(\lambda_C p_-^C + (1-\lambda_C)(1-\sigma_2)) - f_1 p_1^C(\lambda_C p_+^C + (1-\lambda_C)\sigma_1) \\ p_n^C(t+1) = &\quad f_{n-1} p_{n-1}^C(\lambda_C p_+^C + (1-\lambda_C)(1-\sigma_{n-1})) + f_{n+1} p_{n+1}^C(\lambda_C p_-^C + (1-\lambda_C)\sigma_{n+1}) + f_n p_n^C(t) \\ p_N^C(t+1) = &\quad f_{N-1} p_{N-1}^C(\lambda_C p_+^C + (1-\lambda_C)(1-\sigma_{N-1})) + p_N^C(t) - f_N p_N^C(\lambda_C p_-^C + (1-\lambda_C)\sigma_N) \end{aligned} \tag{21}$$

for all $n \in \{2, 3, \ldots, N-1\}$, and with some parameters $\sigma_1, \sigma_2, \ldots, \sigma_N \in [0, 1]$.

Proof For any agent $i \in C$, we can derive the following inequality using (20),

$$\sum_{k \in C} q_k(t)\pi_{k,i} \geq \sum_{k \in C} q_k(t)\frac{\lambda_C}{|C|} = \lambda_C p_+^C(t) \tag{22}$$

and similarly $\sum_{k=1}^K \pi_{k,i}(1 - q_k(t)) \geq \lambda_C p_-^C(t)$ which results in $\sum_{k=1}^K \pi_{k,i} q_k(t) \leq 1 - \lambda_C p_-^C(t)$ or

$$k = 1^K \pi_{k,i} q_k(t) \leq \lambda_C p_+^C(t) + (1 - \lambda_C) \tag{23}$$

From (6), we can always write the expectation of i having opinion θ_1 at $t+1$ as

$$E(x_i(t+1) = \theta_1) = \Pr(x_i(t) = \theta_1) - \Pr(x_i(t) = \theta_1) f_1 \left(\sum_{k=1}^K \pi_{k,i} q_k(t)\right) + \Pr(x_i(t) = \theta_2) f_2 \left(\sum_{k=1}^K \pi_{k,i}(1 - q_k(t))\right) \tag{24}$$

We can rewrite this expression as

$$E(x_i(t+1) = \theta_1) = \Pr(x_i(t) = \theta_1) - \Pr(x_i(t) = \theta_1) f_1(\lambda_C p_-^C(t) + (1-\lambda_C)e_i) + \Pr(x_i(t) = \theta_2) f_2(\lambda_C p_+^C(t) + (1-\lambda_C)(1-e_i)) \tag{25}$$

where $e_i \in [0, 1]$. This is possible because of the two inequalities (22) and (23). Now, we can look at $p_1^C(t+1)$, which is equivalent to $\frac{1}{|C|}\sum_{i \in C} E(x_i(t+1) = \theta_1)$ in the limit of $|C| \to \infty$. Therefore, we have

$$\begin{aligned} E(p_1(t+1)) = &\sum_{i \in C} \tfrac{1}{|C|} \Pr(x_i(t) = \theta_1) + \lambda_C \sum_{i \in C} \tfrac{1}{|C|} f_1 \Pr(x_i(t) = \theta_2) p_-^C(t) - \lambda_C \sum_{i \in C} \tfrac{1}{|C|} \Pr(x_i(t) = \theta_1) f_1 p_+^C(t) \\ &+ (1-\lambda_C) \sum_{i \in C} \tfrac{1}{|C|} f_2 \Pr(x_i(t) = \theta_2)(1 - e_i) - (1-\lambda_C) \sum_{i \in C} \tfrac{1}{|C|} \Pr(x_i(t) = \theta_1) f_1 e_i \end{aligned} \tag{26}$$

However, note that $\sum_{i \in C} \Pr(x_i(t) = \theta_n)$ is just p_n^C in the limit of $|C| \to \infty$. Additionally, as $e_i \in [0, 1]$ for all i, we can write $\sum_{i \in C} \frac{1}{|C|} \Pr(x_i(t) = \theta_n) e_i = p_1^C \sigma_n$, where $\sigma_n \in [0, 1]$. This results in

$$E(p_1^C(t+1)) = p_1^C(t) + f_2 p_2^C(\lambda_C p_-^C + (1-\lambda_C)(1-\sigma_2)) - f_1 p_1^C(\lambda_C p_+^C + (1-\lambda_C)\sigma_1) \tag{27}$$

However, due to the central limit theorem, we have that

$$\lim_{|C| \to \infty} \Pr(p_1^C(t+1) - E(p_1^C(t+1)) > \epsilon) = 0 \tag{28}$$

for any $\epsilon > 0$. This allows us to write that $\lim_{C\to\infty} p_1^C(t+1) = E(p_1^C(t+1))$ *almost surely*. Performing a similar calculation for other p_n^C, we can get

$$p_n^C(t+1) = (1-f_n)p_n^C(t) + f_{n-1}p_{n-1}^C(\lambda_C p_+^C + (1-\lambda_C)(1-\sigma_{n-1})) + f_{n+1}p_{n+1}^C(\lambda_C p_-^C + (1-\lambda_C)\sigma_{n+1}) \tag{29}$$

■

This allows us to study the behavior of agents in a well-connected cluster.

6.1. Preservation of opinion

In particular, when $\lambda_C \to 1$, we can show the following.

Proposition 5 (Preservation with $\lambda_C \to 1$). *If C is a cluster as per Definition 6, with an interconnection coefficient $\lambda_C \to 1$ and $|C| \to \infty$, the population shares of agents in C allows for three distinct equilibrium points irrespective of the actions of agents in $\mathcal{V} \setminus C$. Two of these are metastable equilibrium points, and are given by $\underline{p}^C = p_{e_1}$ and $\underline{p}^C = p_{e_2}$ and the third is an unstable equilibrium.*

Proof For any configuration of actions for agents in $\mathcal{V} \setminus C$, we can write that almost surely,

$$\begin{aligned}
p_1^C(t+1) &= p_1^C(t) + f_2 p_2^C(\lambda_C p_-^C + (1-\lambda)(1-\sigma_2)) - f_1 p_1^C(\lambda_C p_+^C + (1-\lambda_C)\sigma_1)\\
p_n^C(t+1) &= (1-f_n)p_n^C(t) + f_{n-1}p_{n-1}^C(\lambda_C p_+^C + (1-\lambda_C)(1-\sigma_{n-1})) + f_{n+1}p_{n+1}^C(\lambda_C p_-^C + (1-\lambda_C)\sigma_{n+1})\\
p_N^C(t+1) &= p_N^C(t) - f_N p_N^C(\lambda_C p_-^C + (1-\lambda_C)\sigma_N) + f_{N-1}p_{N-1}^C(\lambda_C p_+^C + (1-\lambda_C)(1-\sigma_{N-1}))
\end{aligned} \tag{30}$$

where $\sigma_1, \dots, \sigma_N \in [0,1]$ from Proposition 4. In the limit of $\lambda_C \to 1$, we have

$$\begin{aligned}
\lim_{\lambda_C\to 1} p_1^C(t+1) &= p_1^C(t) + f_2 p_2^C p_-^C - f_1 p_1^C p_+^C\\
\lim_{\lambda_C\to 1} p_n^C(t+1) &= (1-f_n)p_n^C(t) + f_{n-1}p_{n-1}^C p_+^C + f_{n+1}p_{n+1}^C p_-^C\\
\lim_{\lambda_C\to 1} p_N^C(t+1) &= p_N^C(t) - f_N p_N^C p_-^C + f_{N-1}p_{N-1}^C p_+^C
\end{aligned} \tag{31}$$

which are mathematically identical to the dynamics of the all to all network described in (12). From theorem 1, we know that this dynamics allows for exactly two stable equilibria corresponding to the points $p_1^C = 1$ and $p_N^C = 1$. However, since (30) is true only almost surely, the equilibrium is metastable. The third equilibrium is unstable in both cases. ■

This result implies that regardless of external opinion, agents inside C will have their own equilibria, and if all agents in C start with action 1, they will all continue to have action 1 regardless of the opinions of external agents, i.e., the agents inside the cluster *preserve* their opinion. The main property is that this equilibrium is local to the cluster and is independent of the rest of the agents, i.e., $\underline{p}^C = p_{e_1}$ does not imply that $\underline{p} = p_{e_1}$, i.e., these equilibria may not correspond to the absorbing states and are then metastable. If the graph comprises several clusters, each cluster may preserve a local consensus at p_{e_1} or p_{e_2} for an arbitrarily long duration. This kind of phenomenon is often seen in epidemic dynamics [31]. The metastable state (in population) persists for any finite duration almost surely with an infinitely large number of agents in the cluster. The study of metastable states is highly relevant as this can approximate the transient behavior well for a large (but finite) network.

Practically, $\lambda_C \to 1$ implies that the number of connections from $\mathcal{V} \setminus C$ to C is finite, while $|C|$ and K are arbitrarily large and all agents inside C may connect to each other with a large probability. For $|C|$ finite, the only absorbing state of the finite Markov chain is with $\phi_N = 1$ (all agents in $\mathcal{V}$ have an identical opinion, not just agents in C). The equilibrium point with $p_1^C = 1$ in the limit of $|C| \to \infty$ and $\lambda_C \to 1$ is referred to as a *metastable equilibrium*.

Definition 7 (Action preservation). *If a set C has $p_+^C(t_0) = 1$ (or 0) for some t_0, then $p_+^C(t) > 0.5$ (or < 0.5) must be satisfied for any finite $t > t_0$.*

When $\lambda_C \to 1$, we have seen than the cluster can indeed preserve its action as $p_+^C = 1$ or $p_+^C = 0$ are metastable equilibria. The next step in our analysis is to find what values of λ allow action preservation for a cluster. That is, equilibrium points for the dynamics of $\underline{p}^C$, that are not the absorbing states of the whole network.

Theorem 3. *A necessary condition for action preservation in a cluster C with coefficient λ_C, is that $\lambda_C \in \Lambda$. The resulting metastable equilibria will satisfy $p_+^C \geq \alpha$ for an initial condition with $p_+^C = 1$ (preserving majority action 1) or the vice-verse for action 0.*

Proof in Appendix C.
Theorem 3 gives a lower-bound on the parameter λ_C characterizing the ratio between the internal influence in the cluster and external influence from outside the cluster. The action preservation inside a cluster can be achieved only if λ_C is sufficiently large.

6.2. Propagation of opinion

To study the propagation of opinion from a set of agents to others, we define the set of followers.

Definition 8 (Followers). *An agent $j \subset \mathcal{V}$ is a follower of cluster C with a degree $\gamma > 0$ if*

$$\sum_{i \in C} \pi_{i,j} \geq \gamma \tag{32}$$

With this definition, we can now constraint the expected probability of any agent j to pick action 1.

Proposition 6 (Propagation). *If agent $j \in \mathcal{V} \setminus C$ is following C with a trust degree γ, then*

$$\phi(\gamma p_+^C) \leq Pr(q_j(t) = 1) \leq 1 - \phi(\gamma p_-^C) \tag{33}$$

when t is sufficiently large so that the system is in a metastable state.

Proof The transition probabilities of the opinion of any agent j can always be written as follows

$$\Pr(x_j(t+1) = \theta_{n+1} | x_j(t) = \theta_n) = f_n(\gamma p_+^C + (1-\gamma) d_j)$$

for all $n < N$ and

$$\Pr(x_j(t+1) = \theta_{n-1} | x_j(t) = \theta_n) = f_n(\gamma p_-^C + (1-\gamma)(1 - d_j))$$

for all $n > 1$ where $d_j \in [0, 1]$, by exploiting definition 8.

Therefore, we can study the steady state distribution of the opinion of agent j, by formulating the above process as a Markov chain. This steady state can be found as

$$\Pr(x_j(t+1) \geq \theta_{N/2+1}) = \phi(\gamma p_+^C + (1-\gamma) d_j)$$

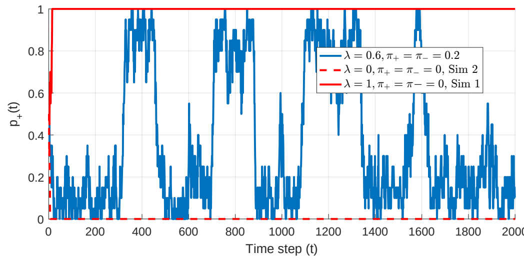
due to mathematical similarity with (21).

The above result implies that if γ is large enough, a tight constraint on the probability of agent j picking action 1 can be provided. This result is especially significant when the cluster C has a sufficiently large λ to preserve its opinion resulting in $p_+^{C*} \geq \alpha > 0.5$ satisfying $\alpha = \phi(\lambda_C \alpha)$.

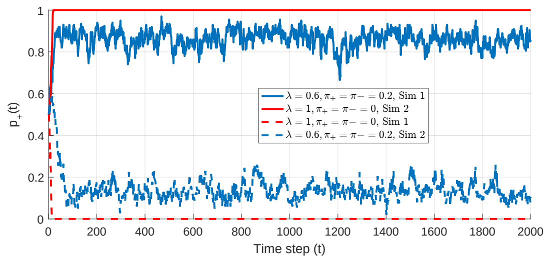
7. Numerical results

For all our simulations, we use $N = 4$, i.e., four levels of opinions with $\Theta = \{0.2, 0.4, 0.6, 0.8\}$ are considered. We also take the inertia factor to be $f_n = 4\theta_n(1 - \theta_n)$. We demonstrate simulation results that validate our results from Section 5.

In Fig. 1, we consider a network where all agents are connected to each other with the same degree of influence, i.e., $\pi_{i,j} = \frac{\lambda}{K} \forall i, j \in \mathcal{V}$, $\pi_{i,E_0} = \pi_-$ and $\pi_{i,E_E} = \pi_+$ for all $i \in \mathcal{V}$. We initialize the opinion levels of agents with opinion 0.4 for half of the agents and 0.6 for the rest. Fig. 1a illustrates Proposition 1 for $K = 20$ and we see that there are two absorbing states when $\lambda = 1$ and no absorbing states when $\pi_+, \pi_- > 0$. Once the Markov process reaches an absorbing state it stays there for the remaining time. On the other hand, if we use the large K approximation as in Section 5, we

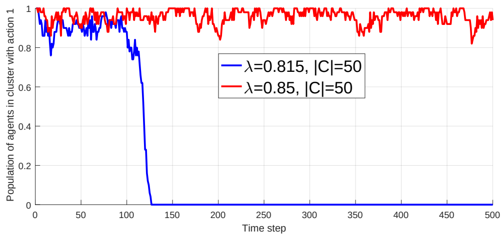


(a) Symmetric starting conditions and $K = 20$. When $\lambda = 1$, there are two absorbing states which are reached in two different simulations. When $\pi_+ = \pi_- = 0.2$, there are no absorbing states.

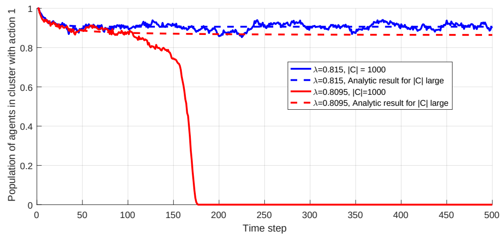


(b) Symmetric starting conditions and $K = 200$. When $\pi_+ = \pi_- = 0.2$, the system in two simulations reach one of the two metastable equilibrium.

Figure 1: Population share with the action 1, i.e., $p_+(t)$ plotted against t for a graph with all to all identical connections with $\pi_{i,j} = \frac{\lambda}{K} \forall i, j \in \mathcal{V}$.



(a) Number of agents $K = 50$. Due to the small number of agents, there is a large perturbation due to the stochastic nature, but with a large enough λ, (of 0.85) the network can preserve its opinion despite the external signal of 0 before collapsing to the absorbing state α_1.



(b) Large number of agents, i.e., $K = 1000$ and this makes the dynamics approximated by (12) less perturbed and it allows for preservation with $\lambda = 0.815$ (for a long time).

Figure 2: Population share of agents in C with the action 1, i.e., $p_+^C(t)$ plotted against t. All agents start with opinion level 0.8 and the external signal is always 0 with $\pi_- = 1 - \lambda$.

find p_+^* satisfying the condition in Theorem 2 when $\pi_+ = \pi_- = 0.2$. We see in Fig. 1b that when $K = 200$, the system to converges to one of the two metastable equilibrium and stays there for a long duration.

For the next simulation (Fig. 2), we look at the case where we have $\lambda < 1$, $\pi_+ = 0$ and $\pi_- = 1 - \lambda$. This corresponds to the case where the finite Markov process has exactly one absorbing state α_1. Here, we do simulations with $K = 50$ and $K = 1000$ with all agents in the network starting with opinion 0.8. Fig. 2a shows that for small K, i.e. 50 agents in the network, even the external signal of $\pi_- = 0.185$ can influence the agents to shift their opinion to 0 in finite time. Fig. 2b shows that when a large number of agents are taken, i.e., $K = 1000$, we see that the agents in the network *preserve* their majority opinion even for $\lambda = 0.815$. This illustrates Theorem 2, which states that even in the presence of a persistent external signal, an equilibrium point with the majority of agents holding the opposite action to the signal can exist if λ is sufficiently high. A larger number of agents reduces the stochastic perturbation to the dynamics approximated by the central limit theorem (12). At the absorbing state the system has no more perturbations but the metastable points are perturbed.

In Fig. 3, we plot the final expected fraction of agents with action 1, $p_+(T)$ against the initial fraction of agents with action 1, $p_+(0)$. The expectation is calculated by running the simulation 1000 times for each initial configuration. We take $K = 200$, $T = 200$ (the final time considered) and by setting $p_1(0) = p_-(0)$ and $p_N(0) = p_+(0)$, i.e all agents start with one of the extreme opinions. Fig. 3 illustrates that in absence of exogenous influence ($\pi^+ = \pi^- = 0$) the probability to reach α_2 increases with $p_+(0)$. Moreover, it is almost sure that the system converges to α_1 when $p_+(0) \leq 0.4$ and to α_2 when $p_+(0) \geq 0.6$. When the initial population is symmetrically distributed (half at θ_1 and half at θ_N) the system converges towards α_1 or α_2 with the same probability 0.5. On the other hand when exogenous influence is present ($\pi^+ > 0$), α_2 is the only absorbing state and this is almost always reached for a larger set of initial conditions. However, for sufficiently small π^+ and $p_+(0)$, a meta-stable equilibrium with majority action 0 is reached. When $\pi^+ \geq 0.2$, we observe that α_2 is reached for all initial conditions within the considered finite final time $T = 200$. This further illustrates Theorem 2 and characterizes the equilibrium reached for a given initial condition.

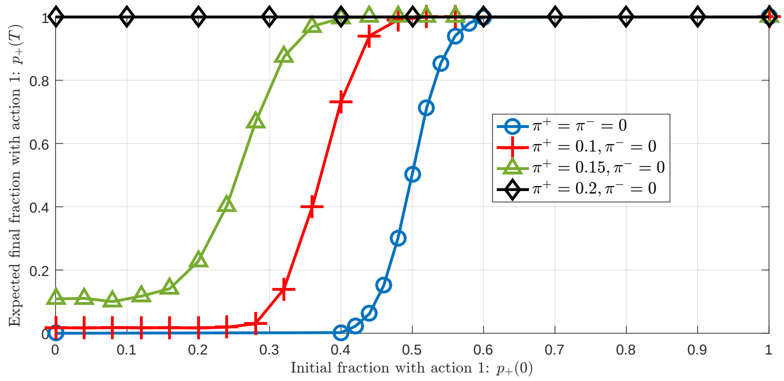
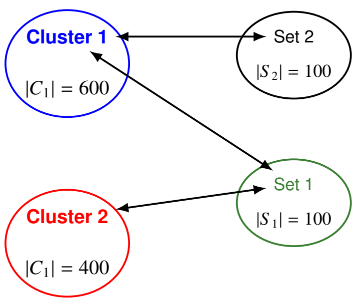


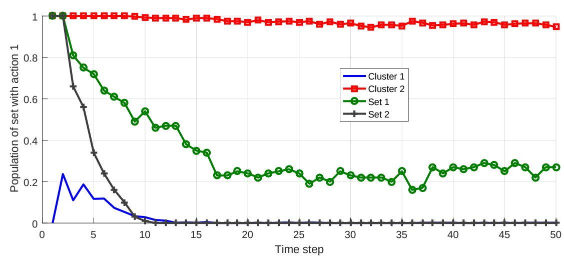
Figure 3: Expected final population fraction of agents with action 1 for a given initial fraction of agents with action 1 with $K = 100$. As the strength of the external signal to 1 increases, the number of initial conditions that lead to a meta-stable equilibrium with majority action 0 decreases. For $\pi^+ \geq 0.2$, almost all initial configuration converges to the absorbing state with all agents at action 1.

Remark: Note that in Fig. 2, a global consensus is not achieved when λ is sufficiently large as the external signal influences a small fraction to take action 0. For $\lambda = 0.815$, on average a bit more than 90% of the agents have action 1 but the remaining 10% or so are with action 0.

Finally, we study the opinion dynamics when the graph has a very specific structure, i.e. the agents are partitioned into two clusters C_1 and C_2 and two other follower sets S_1 and S_2 with $|S_1| = |S_2| = 100$ and $|C_1| = 600$ and $|C_2| = 400$, with $\mathcal{V} = C_1 \cup C_2 \cup S_1 \cup S_2$. All agents in C_1 are connected identically and in an undirected manner to all agents in C_1, S_1 and S_2, while all agents in C_2 are connected identically to all agents in C_2 and S_1 as shown in Fig. 4a. Therefore, C_1 and C_2 define clusters while S_1 is a follower set for both C_1 and C_2, and S_2 just follows C_1. The population share of agents in each set with action 1 is plotted in Fig. 4b, when agents in C_1 start with opinion level 0.2 and all the other agents start with opinion level 0.8. This figure demonstrates both the properties of preservation and propagation developed in Theorem 3 and Proposition 6.



(a) Graph structure: all connections between agents are of identical strength when they exist and undirected.



(b) This figure shows how C_1 and C_2 preserve their opinion while propagating their opinion to the follower sets.

Figure 4: Opinion preservation and propagation illustrated in a structured interaction graph. S_1 following both C_1 and C_2 results in its average action fluctuating between the two contrasting opinions, but with more influence from C_1. S_2 following only C_1 results in its action shifting from a majority with 1 to 0.

8. Conclusion

We have proposed and studied a model of stochastic opinion dynamics models. This model features multi-leveled opinions of each agent, which represents the confidence of an agent in choosing an action 0 or 1. These opinions are influenced by an external signal (which can be 0 or 1) or the internal signal (or action) of a random neighbor. We show that for a finite number of connected agents, a global consensus is asymptotically achieved at either all agents with the lowest opinion θ_1, when the external signal is only 0 or no signal; or at all agents with the highest opinion θ_N when the external signal is only 1 or no signal. On the other hand, when the number of agents is infinitely large, the time to achieve consensus may become exponentially high as illustrated in both our analytic results and numerical simulations with a large number of agents. Of particular interest is the case where a cluster structure exists in the network graph (agents in the cluster are strongly interconnected and weakly connected to external agents). For a cluster, we analytically (for an arbitrarily large number of agents) and through simulation (for a large number of agents) demonstrate the property of preservation, i.e., when agents in a cluster start with a common opinion (or action), a majority of agents in the cluster can preserve this action regardless of the opinion of external agents or signals. Theoretically, this can be identified as a meta-stable equilibrium point, and this equilibrium is different from the global consensus.

Future works will study how external signals can control the system to achieve a certain population distribution in the network. When each of the external signals/actions correspond to an economic or political entity, we can formulate a non-cooperative game with the utility of each player (each of the external entities in this case) depending on the actions of the agents in the network and the cost for sending the signal.

Appendix A. Proof of Proposition 1

Proof Case 1: When $\pi_{E_0,i} = \pi_{E_1,i} = 0$ for all $i \in \mathcal{V}$. Here, we can verify that α_1 and α_2 are absorbing states by evaluating $\Pr(x_k(t+1) = \theta_{m_k} | X(t) = \alpha_n)$ through equation (9). We know $Q_k^-(\alpha_1) = 1$, $Q_k^+(\alpha_1) = 0$, $Q_k^-(\alpha_2) = 0$ and $Q_k^-(\alpha_2) = 1$ for any k and for any graph G. This results in

$$\Pr(x_k(t+1) = \theta_1 | X(t) = \alpha_1) = 1, \quad \forall k \in \mathcal{V}.$$

Similarly

$$\Pr(x_k(t+1) = \theta_N | X(t) = \alpha_2) = 1, \quad \forall k \in \mathcal{V}.$$

Consequently, $\alpha_1,\ \alpha_2$ are absorbing states. Now take any state $\beta \neq \alpha_1, \alpha_2$, then;
Case 1-A: β is such that there exists at least one k for which $x_k(t) \neq \theta_1, \theta_N$. This agent by (9) has a non zero probability to shift its opinion (towards either θ_1 or θ_N) as the graph is connected resulting in either

$$0 < Q_k^+(\beta) < 1 \ \text{ or } \ 0 < Q_k^-(\beta) < 1.$$

As agent k also has a non-zero probability to stay, β is therefore a non-absorbing state.
Case 1-B: Some of the agents have opinion θ_1 while the other agents have the opinion θ_N (any agent with a non-extreme opinion will result in Case 1-A). Now, since G is a strongly connected graph, there exists at least one agent with opinion θ_1 influenced by another agent with opinion θ_N resulting in a non-absorbing state.

Therefore there are exactly two absorbing states for the Markov process defined in (8) which are α_1 and α_2.

Case 2: When $\pi_{E_1,i} = 0$ for all $i \in \mathcal{V}$, $\exists j \in \mathcal{V}$ such that $\pi_{E_0,j} > 0$.

We can use the same arguments as in Case 1-A and 1-B to prove that any state other than α_1 and α_2 can not be absorbing states. Additionally, since $\exists j \in \mathcal{V}$ such that $\pi_{E_0,j} > 0$, α_2 can not be an absorbing state as agent j will have a non zero probability to shift its opinion to θ_{N-1}. α_1 remains an absorbing state.

Case 3: Due to symmetry, we can prove the proposition result by following the arguments for Case 2 with α_2 being the absorbing state.

Case 4: Here, neither α_1 nor α_2 remain absorbing states as there always exists at least one agent which has a probability to shift its opinion (similar arguments as in Case 2).

Appendix B. Proof of Theorem 1

Proof From (12), if $p_+(t) = 1$, it can be verified that

$$p_N(t+1) = p_N(t) + p_{N-1}(t) f_{N-1},$$

which is an increasing function. Similarly, as $p_+(t) = 1$ one obtains that that $p_n(t) = 0,\ \forall n < N/2$. Moreover $p_n(t),\ \forall n < N/2$ are decreasing functions as

$$p_n(t+1) = (1 - f_n) p_n(t).$$

We can also see that $p_{N/2+1}(t')$ becomes a decreasing function when $p_+(t) = 1$ as

$$p_{N/2+1}(t+1) = (1 - f_m) p_{N/2+1}(t) + f_{m-1} p_{N/2}(t) \tag{B.1}$$

and $p_{N/2}(t) = 0$. Therefore, as $t \to \infty$, and $\underline{p}(t)$ will asymptotically converge to $(0, 0, \ldots, 1)$. Similarly, we can show that when $p_+(t) = 0$, the system maintains $\underline{p_+}(t') = 0,\ \forall t' > t$ and $\underline{p}(t)$ asymptotically converges to $(0, 0, \ldots, 1)$. These points correspond to the same absorbing states identified in Section 4 for the original Markov system.

Next, assume that $p_+(t) = 0.5$ and $p_n f_n = p_m f_m),\ \forall m, n \in \{1, 2, \ldots, N\}$. Substituting these values into (12) we obtain that $p_n(t+1) = p_n(t),\ \forall n$, resulting in another equilibrium. The stability of these equilibria are studied with a linear analysis of the dynamics around these points.

First, we evaluate the Jacobian of the discrete dynamical system (12). Denote by $g_i(\mathbf{p})$ the dynamics of the population with opinion θ_i, i.e. $p_i(t+1) = g_i(\mathbf{p})$. If we denote the Jacobian elements by $J_{i,j}$, where $J_{i,j} = \frac{\partial g_i}{\partial p_j}$, then for all $1 < i \leq \frac{N}{2}$, and for all $\frac{N}{2} < j < N$, we have:

$$J_{11} = 1 + f_2 p_2 - f_1 \left(\textstyle\sum_{n=N/2+1}^{N} p_n(t) \right), \quad J_{i,i} = 1 - f_i + f_{i+1} p_{i+1}$$
$$J_{j,j} = 1 - f_j + f_j p_{j-1}, \quad J_{NN} = 1 + f_{N-1} p_{N-1} - f_n \left(\textstyle\sum_{n=1}^{N/2} p_n(t) \right)$$

We also have ,

$$\forall\, 2 < i \leq N/2, \quad J_{1i} = -f_1 p_1$$

$$\forall\, N/2 < i \leq N-2, \quad J_{1i} = f_2 p_2$$

and

$$\forall\, 2 < i \leq N/2, \quad J_{Ni} = f_{N-1} p_{N-1}$$

$$\forall\, N/2 < i \leq N-2, \quad J_{Ni} = -f_n p_N.$$

For all $i, j \in \{2, 3, \ldots, N-1\}$ such that $|i - j| > 1$ we have

$$J_{ij} = f_{i+1} p_{i+1}$$

when $j \leq N/2$ and

$$J_{ij} = f_{i-1} p_{i-1}$$

when $j > N/2$. Finally, we have

$$J_{12} = f_2 \left(p_2 + \sum_{n=1}^{N/2} p_n(t) \right)$$

$$J_{N-1,N} = f_{N-1} \left(p_{N-1} + \sum_{n=N/2+1}^{N} p_n(t) \right)$$

and for all $i, j \in \{2, 3, \ldots, N-1\}$ such that $|i - j| = 1$ we have

$$J_{i,i+1} = f_k p_k + f_{i+1}\left(\sum_{n=1}^{N/2} p_n(t)\right)$$

where $k = i + 1$ if $i + 1 \leq N/2$ and $k = i - 1$ otherwise; and

$$J_{i,i-1} = f_k p_k + f_{i-1}\left(\sum_{n=N/2+1}^{N} p_n(t)\right)$$

where $k = i + 1$ if $i - 1 \leq N/2$ and $k = i - 1$ otherwise.

The stability of an equilibrium point is checked by studying the eigenvalues of the Jacobian matrix (evaluated at the equilibrium points). We evaluate the Jacobian matrix at $p_1 = 1$, $p_{n>1} = 0$ as

$$\begin{bmatrix} 1 & f_2 & 0 & \ldots \\ 0 & 1 - f_2 & f_3 & \ldots \\ 0 & 0 & 1 - f_3 & \ldots \\ \vdots & & & \end{bmatrix}.$$

Note that this is a triangular matrix and therefore has the eigenvalues $1, 1 - f_n$ for all $n \in \{2, \ldots, N\}$. This can be easily verified by evaluating the determinant of the $\mathbf{J}(1, 0, \ldots) - \lambda\mathbf{I}$. Therefore this equilibrium is stable. Since the system is symmetric around 0.5, we can also show the same for the equilibrium with $p_N = 1$. Note that the eigenvalue 1 corresponds to the equilibrium point itself. Since the population space is inside a simplex and not $\mathbb{R}^N$, we have local asymptotic stability around $(1, 0, \ldots)$ and $(0, \ldots, 0, 1)$ and as (12) holds almost surely, the stability property is also almost sure.

Now, we evaluate the Jacobian at the other equilibrium, which has $p_n f_n = \kappa$, where $\kappa > 0$ for all n. The first column of the Jacobian at this point is given by

$$(1 + \kappa - \frac{f_1}{2}, \kappa + \frac{f_1}{2}, \kappa, \ldots, \kappa, -\kappa)^T$$

The columns j for $2 \leq j \leq N/2$ has the following form

$$(\kappa, \ldots, 1 + \kappa + \frac{f_j}{2}, \kappa - f_j, \kappa + \frac{f_j}{2}, \ldots, \kappa, -\kappa)^T$$

where $1 + \kappa + \frac{f_j}{2}$ is the diagonal term of the Jacobian. The j-th column where $N/2 < j \leq N - 1$ has the following form

$$(-\kappa, \kappa, \ldots, 1 + \kappa + \frac{f_j}{2}, \kappa - f_j, \kappa + \frac{f_j}{2}, \ldots, \kappa)^T$$

and the N-th column is given by

$$(-\kappa, \kappa, \ldots, \kappa, \kappa + \frac{f_1}{2}, 1 + \kappa - \frac{f_1}{2})^T.$$

We observe that the Jacobian at each column has exactly one element which is $-\kappa$. This is located either at the first row (after column index is more than $N/2$) or at the final row. Subtracting this matrix by $1 + \kappa(N-2)$ times the identity matrix, we have a new matrix. For the new matrix, the sum of each column is 0 and therefore has determinant zero. This shows that one of the eigenvalues of the matrix above is $1 + \kappa(N - 2)$. Therefore, when $N > 2$, since $\kappa > 0$, this point is unstable. ■

Appendix C. Proof of Theorem 3

We first prove a preliminary result before proving Theorem 3.

Lemma 1. *The function $\phi(x)$ has values $\phi(x) = x$ only at $x = 0, 0.5, 1$. Additionally, $\phi(x)$ is increasing, $\phi(x) < x$ when $x \in (0, 0.5)$ and $\phi(x) > x$ when $x \in (0.5, 1)$.*

Proof Note that $\phi(0) = 0, \phi(1) = 1$ and $\phi(0.5) = 0.5$. Additionally, it can be verified that $\phi(x)$ is continuous in x. We will now show that $\phi(x) = x$ is never satisfied at any point other than at $0, 0.5, 1$. Setting $\phi(x) = x$ and $1 - \phi(x) = 1 - x$ in (17) and dividing, we get

$$\sum_{n=N/2+1}^{N} (1-x)\frac{1}{f_n}(\frac{x}{1-x})^{n-1} = \sum_{n=1}^{N/2} x\frac{1}{f_n}(\frac{x}{1-x})^{n-1} \tag{C.1}$$

since $f_n = f_{N+1-n}$ by definition, we have

$$0 = \sum_{n=1}^{N/2}(2x-1)\frac{1}{f_n}(\frac{x}{1-x})^{n-1}(1-(\frac{x}{1-x})^{N-n}) \tag{C.2}$$

which can never be satisfied unless $x = 0$ or $x = 0.5$. Additionally, we can easily verify that $\phi(x) < x$ when $x \to 0$. Since $\phi(x)$ is a continuous function, $\phi(x) > x$ is not possible in the interval $(0, 0.5)$. Similarly, $\phi(x) < x$ is not possible in the interval $(0.5, 1)$. ■

Now we provide the proof of Theorem 3.

Proof Consider the situation in which all agents outside C have action 0. This results in σ_n in (21) being 0 for all n. If we denote by $x := \lambda p_+^C$, (21) can be rewritten as

$$\begin{aligned} p_1^C(t+1) &= p_1^C(t) + f_2 p_2^C(1-x) - f_1 p_1^C x \\ p_n^C(t+1) &= (1-f_n)p_n^C(t) + f_{n-1}p_{n-1}^C x + f_{n+1}p_{n+1}^C(1-x) \\ p_N^C(t+1) &= p_N^C(t) - f_n p_N^C(1-x) + f_{N-1}p_{N-1}^C x \end{aligned} \tag{C.3}$$

Now the equilibrium points p_n^{C*} of (21) can be found by setting $p_n^C(t+1) - p_n^C(t) = 0$ resulting in

$$p_n^{C*} := \frac{\frac{1}{f_n}(\frac{x}{1-x})^{n-1}}{\sum_{m=1}^{N}\frac{1}{f_m}(\frac{x}{1-x})^{m-1}} \tag{C.4}$$

when $x < 1$ and $p_N^{C*} = 1$ when $x = 1$. Since we have $p_+^C(t) = \sum_{n=N/2+1}^{N} p_n^C(t)$, we can write

$$p_+^{C*} = \phi(x)$$

However, at equilibrium, $x = \lambda_C p_+^{C*}$, which means that p_+^{C*} at equilibrium must satisfy $p_+^{C*} = \phi(\lambda_C p_+^{C*})$. One trivial solution to this equation is when $p_+^{C*} = 0$ and this corresponds to the absorbing state as all agents outside C also have action 0, which will result in all agents having opinion θ_1 asymptotically. However, if some other $p_+^{C*} > 0$ exists satisfying this equation, this corresponds to a potential metastable equilibrium.

We have thus shown that if all agents outside C have action 0, a metastable equilibrium exists only if λ is such that $p_+^{C*} = \phi(\lambda_C p_+^{C*})$ is satisfied for $p_+^{C*} > 0$. Denote this p_+^{C*} by α, i.e. $\alpha = \phi(\lambda_C\alpha)$. Next, we will prove that an equilibrium point with $p_+^{C*} \geq \alpha$ exists for any action profile of $\mathcal{V} \setminus C$.

Denote by $p_n^{C:\alpha}$ the equilibrium points when $\sigma_n = 0$. Then, for any $\sigma_n \in [0, 1]$, we have

$$\frac{p_{n+1}^{C*}}{p_n^{C*}} = \frac{f_N}{f_{n+1}}\frac{x + (1-\lambda_C)\sigma_{n+1}}{1 - x - (1-\lambda_C)\sigma_n} \tag{C.5}$$

which can be proven by induction as follows. For $n = 1$, we have

$$\frac{p_2^{C*}}{p_1^{C*}} = \frac{f_1}{f_2}\frac{x + (1-\lambda)\sigma_1}{1 - x - (1-\lambda_C)\sigma_2} \tag{C.6}$$

For any $n > 1$, we have

$$f_{n+1} p_{n+1}^{C*}(1 - x - (1-\lambda_C)\sigma_{n+1}) = f_n p_n^{C*} - f_{n-1} p_{n-1}^{C*}(x + (1-\lambda_C)\sigma_{n-1}) \tag{C.7}$$

if we assume that (C.5) is true for $n-1$, then we get

$$f_{n+1} p_{n+1}^{C*}(1 - x - (1-\lambda_C)\sigma_{n+1}) = f_n p_n^{C*} - f_N p_n^{C*}(1 - x - (1-\lambda_C)\sigma_{n1}) \tag{C.8}$$

which satisfies (C.5) for n. Since $\sigma_n \geq 0$ and $0.5 < \lambda \leq 1$, we have that

$$\frac{p_{n+1}^{C*}}{p_n^{C*}} \geq \frac{f_N}{f_{n+1}} \frac{x}{1-x} = \frac{p_{n+1}^{C:\alpha}}{p_n^{C:\alpha}} \tag{C.9}$$

This implies that $\sum_{N/2+1}^{N} p_n^{C*} \geq \alpha$ as we have $\sum_1^N p_n^{C*} = 1$, with the $\sum_{N/2+1}^{N} p_n^{C*} = \alpha$ when (C.9) are a set of equalities.

This implies that regardless of external opinions, the cluster C allows an equilibrium p_n^{C*} with $p_+^{C*} \geq \alpha$ if λ_C is such that $\exists \alpha \in (0,1]$ satisfying $\alpha = \phi(\lambda_C \alpha)$. By symmetry, we can show the same result for preservation of action 0.

Finally, we will prove that an $\alpha > 0$ satisfying $\alpha = \phi(\lambda_C \alpha)$ must also be larger than 0.5. Using Lemma 1, since $\phi(x)$ is a continuous function, $\phi(x) > x$ is not possible in the interval $(0, 0.5)$. Similarly, $\phi(x) < x$ is not possible in the interval $(0.5, 1)$. This implies that any α satisfying $\alpha = \phi(\lambda_C \alpha)$ must have $\alpha > 0.5$, which concludes our proof. ■

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