Time optimal control for a mobile robot with a communication objective

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J. Lohéac[1] V. S. Varma[1] I. C. Morărescu[1]

Abstract

The paper proposes control design strategies that minimize the time required by a mobile robot to accomplish a certain task (reach a target) while transmitting/receiving a message. The message delivery is done over a wireless network, and we account for path-loss while disregarding any shadowing phenomena, i.e., the transmission rate depends only on the distance to the wireless antenna. We completely analyze the case when the robot dynamics is described by a single integrator. Our minimal-time control design is based on the use of Pontryagin maximum principle. We show how we can use these theoretical results to control more complicated non-holonomic dynamics. Numerical simulations illustrate the effectiveness of the theoretical results.

Key-words: Time optimal control, Pontryagin maximum principle, wireless communication.

1 Introduction

Time optimal control problems under different hard constrains (mainly communication based) are flourishing in the field of mobile robotics (see e.g., [10]). The objective of the robot's mission is often conflicting with the imposed constraints. For instance, minimizing the travel time requires more energy but this resource is limited and in order to increase the robot autonomy we may require to minimize energy. In other context, to minimize the mission time we should follow a straight trajectory but if we want to transmit/receive a message while travelling one may need a higher transmission rate imposing a completely different path. Precisely, when an unmanned aerial vehicle or a ground robot has to collect data from a field of wireless sensors, it typically has to optimize its trajectory to minimize the task time while collecting correctly the data may require longer trajectories (see e.g., [25] and [16]).

To have an easy interpretation of the mathematical results we propose the following problem formulation. A robot has to move from a starting point to a target point within the shortest possible time. Along its travel it must also ensure the transmission of a certain amount of data to a wireless access point. The access point receives the signal with a signal-to-noise ratio (SNR) which primarily depends on the distance between the mobile and the base. Therefore, the mobile has to choose a trajectory which allows the data to be uploaded successfully (which is made possible by having a sufficiently large SNR) and to minimize the time taken for reaching its target point.

While an important number of applications can be formalized in the framework of a mobile robot that has to minimize a cost under some communication constraints, most of the proposed solutions are either numerical or heuristic. For instance, the problem under consideration in this work is solved numerically in [7, 15, 24] and only some analytic insights which provides some

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conditions for the optimal solution are provided. Mainly heuristic solutions are provided for the minimal energy consumption under communication constraints in [2, 3, 4, 13, 17]. This is due to the fact that the SNR is taken as a random variable.

In contrast with the works cited above, the main contribution of this paper is to analytically provide the optimal solution to our problem of time minimization. This is done through the use of the Pontryagin maximum principle, see e.g., [5, 12]. To do that, we consider the robot dynamics described by a single integrator. The results obtained in this context are then used to improve the performances of non-holonomic robots that have to accomplish the same tasks.

The rest of the paper is organized as follows. In Section 2 we state the constrained minimization problem in a rather general case. Then, in Section 3, we give some elementary results on the general situation and provide some numerical simulations in this case. The Section 4, deals with the case where the dynamic of the robot is a simple integrator. In particular, in § 4.2, we apply the Pontryagin maximum principle and compute the optimal control and the corresponding minimal time. The results of Section 4 section are used for the numerical simulations made in § 3.3. The paper ends with some concluding remarks.

2 Problem statement

Dynamic of the robot. We consider a quite general dynamic for the robot. More precisely, we assume that the robot is subject to the following ordinary differential equations.

$$\dot{x} = f(x, p, u), \tag{2.1a}$$
$$\dot{p} = g(x, p, u), \tag{2.1b}$$

where $x \in \mathbb{R}^d$ is the robot position ($d \in \{1, 2, 3\}$), $p \in \mathbb{R}^s$ ($s \in \mathbb{N}$, with the convention $\mathbb{R}^0 = \{0\}$ and $g \equiv 0$, in the case $s = 0$), are some auxiliary variables for the dynamic of the robot, and $u \in \mathbb{R}^m$ ($m \in \mathbb{N}^*$) are the control variables. In order to avoid some technical difficulties, we assume that f and g are of C^1 regularity, and the control u belongs to L^∞.
We also assume that the robot is subject to some constraints. First of all, we assume that the velocity of the robot is bounded and without loss of generality, we assume that it is bounded by 1 (any other bound can be obtained by a trivial time rescaling),

$$|f(x(t), p(t), u(t))| \leqslant 1 \qquad (t \geqslant 0), \tag{2.2a}$$

where $|\cdot|$ is the Euclidean norm of $\mathbb{R}^d$. We also assume some other state and control constraints,

$$(x(t), p(t)) \in \mathcal{C}_s \qquad (t \geqslant 0), \tag{2.2b}$$
$$u(t) \in \mathcal{C}_c \qquad (t \geqslant 0), \tag{2.2c}$$

where $\mathcal{C}_s$ is a closed subset of $\mathbb{R}^d \times \mathbb{R}^s$ and $\mathcal{C}_c$ is a compact subset of $\mathbb{R}^m$.

We assume that the system (2.1)–(2.2) is controllable, that is to say that for every $(x^0, p^0) \in \mathcal{C}_s$ and every $(x^1, p^1) \in \mathcal{C}_s$, there exist $T > 0$ and $u \in L^\infty(0, T)^m$ such that the solution (x, p) of (2.1), with initial condition $x(0) = x^0$ and $p(0) = p^0$, and the control u, satisfies (2.2) together with $x(T) = x^1$ and $p(T) = p^1$.

Let us summarize the assumptions made on the dynamic of the robot.

Assumption 1.

(a) $f \in C^1(\mathbb{R}^d \times \mathbb{R}^s \times \mathbb{R}^m, \mathbb{R}^d)$ *and* $g \in C^1(\mathbb{R}^d \times \mathbb{R}^s \times \mathbb{R}^m, \mathbb{R}^s)$*;*

(b) $\mathcal{C}_s$ *is a closed and nonempty subset of* $\mathbb{R}^d$*, and* $\mathcal{C}_c$ *is a compact and nonempty subset of* $\mathbb{R}^m$*;*

(c) $0 \in \mathcal{C}_c$, $0 \in \mathcal{C}_s$, $f(0,0,0) = 0$ *and* $g(0,0,0) = 0$*;*

(d) the system (2.1) *with constraints* (2.2) *is controllable;*

(e) the set $\left\{ \left(f(x,p,u)^\top, g(x,p,u)^\top \right)^\top,\ u \in \mathcal{C}_c \right\} \in \mathbb{R}^d \times \mathbb{R}^s$ *is convex for almost every* $(x,p) \in \mathcal{C}_s$*;*

(f) there exist $V \in C^1(\mathbb{R}^d \times \mathbb{R}^s)$ *and* $C > 0$ *such that* $V \geqslant 0$, $\lim_{|x|+|p| \to \infty} V(x,p) = \infty$, *and* $\langle \partial_x V(x,p), f(x,p,u) \rangle + \langle \partial_p V(x,p), g(x,p,u) \rangle \leqslant CV(x,p)$ *for every* $(x,p) \in \mathcal{C}_s$*.*

In this set of assumptions, we add some technical ones. Namely, Assumption 1-(c), will be used for the general transmission problem, and the Assumptions 1-(e) and 1-(f) are made in order to ensure the existence of an optimal control.

In this paragraph, we have presented some state variables and control. Let us recall that one task of the robot is to deliver a message over a wireless network. This is represented in the following as the problem of emptying a buffer whose size at time t is denoted by $b(t)$. This last state variable b and its dynamic is presented in the next paragraph.

Communication model. Typically in wireless communication, the communication rate is modeled as a stochastic function which depends on the distance between the transmitting node and the receiving node. We use $R(|x(t)|)$ to denote the communication rate at time t. In practice, communication is performed over certain intervals over which communication packets are transmitted and received with some probability depending on the channel quality, see [23]. The duration of a frame is typically of the order of 10ms (see [19]) in the LTE communication framework. This implies that if a robot moves sufficiently slowly (speeds of 2 or 3 m/s), the rate function can be well approximated by its expectation over channel fast fading as shown in [11]. Therefore, for the rest of this paper, we assume that R satisfies the following assumption:

Assumption 2.

(a) $R : \mathbb{R}_+ \to \mathbb{R}_+$ *is an absolutely continuous, non-increasing function;*

(b) $R(0) > 0$*;*

(c) R *is decreasing on the set* $\{\rho \in \mathbb{R}_+ \mid R(\rho) > 0\}$*.*

In our numerical examples, we will consider a specific rate function similar to the one provided by [22]. With this assumption, it turns out that we end up with a hybrid control problem in which the robot has first to apply a control action to approach the antenna (increase the transmission rate) and second, switch the control to a point stabilization one (reach the destination). Formally, the state of the system will be (x,p,b) where dynamics of (x,p) is given in (2.1) and b is solution of

$$\dot{b} = -R(|x|). \tag{2.3}$$

Remark 1. *The Assumption 2-(b) is used to prove the existence of a time optimal solution. The absolute continuity of* R *is required to apply the Pontryagin maximum principle. The decreasing properties of* R *are motivated by the physical nature of the problem.*

Objective. We are now ready to formalize the time optimal control problem which is studied in this paper. Given some initial buffer $b^0 \geqslant 0$, some initial position x^0 and target position x^1 in $\mathbb{R}^d$, the goal is to move the robot from x^0 to x^1 in minimal time T following a trajectory allowing at emptying the buffer b, i.e., $b(T) \leqslant 0$ (in practice, the data is transmitted as soon as $b(t) > 0$ and the transmission is stopped after the time instant t_0 where $b(t_0) = 0$). In order to tackle the auxiliary variable p, we consider an initial condition $p^0 \in \mathbb{R}^s$ and closed subset $\mathcal{P}^1$ of $\mathbb{R}^s$, and add the constraints $p(0) = p^0$ and $p(T) \in \mathcal{P}^1$. Obviously, in order that such a path is realizable, one needs to make the following assumption.

Assumption 3. *We assume that $(p^0, x^0) \in \mathcal{C}_s$, $\mathcal{P}^1$ is a closed subset of $\mathbb{R}^s$, and there exist $p^1 \in \mathcal{P}^1$ such that $(x^1, p^1) \in \mathcal{C}_s$.*

In other words, given $b^0 \geqslant 0$, x^0, p^0, x^1 and $\mathcal{P}^1$, satisfying Assumption 3 we aim to solve the following constrained time optimal control problem:

$$
\begin{array}{l}
\min \ T \\
\left|\begin{array}{l}
T \geqslant 0, \\
\exists u \in L^\infty(0,T)^m \text{ and } p^1 \in \mathcal{P}^1 \text{ s.t.} \\
(x,p,u) \text{ satisfies (2.2), together with} \\
x(T) = x^1,\ p(T) = p^1 \text{ and } b(T) \leqslant 0, \\
\quad \text{where } (x,p,b) \text{ is solution of (2.1) and (2.3),} \\
\quad \text{with } x(0) = x^0,\ p(0) = p^0 \text{ and } b(0) = b^0.
\end{array}\right.
\end{array}
\tag{2.4}
$$

3 Preliminary results

3.1 Existence of a time optimal control

As detailed in § 3.2, under the Assumptions 1 to 3, one can see that there exist a time $T > 0$, a control $u \in L^\infty(0,T)^m$, and $p^1 \in \mathbb{R}^s$ such that the solution (x, p, b) of (2.1) and (2.3) with $x(0) = x^0$, $p(0) = p^0$ and $b(0) = b^0$ satisfies $x(T) = x^1$, $p(T) = p^1$ and $b(T) \leqslant 0$. This fact, together with the technical assumptions made in Assumptions 1 to 3, ensure, by application of Filippov Theorem (see e.g., [9, Theorem 9.2.i, and its extention in § 9.4]) ensures that the minimal time T given by (2.4) exists.

3.2 Bounds on the minimal time

Assumptions 1-(c) and 1-(d), ensures the existence of a time T_0, a control $u_0 \in L^\infty(0,T_0)^m$, such that the solution (x,p) of (2.1) with initial conditions x^0 and p^0, satisfies $x(T_0) = 0$ and $p(T_0) = 0$. We also have that b, solution of (2.3) with initial condition b^0 satisfies $b(T_0) = b^0 - \int_0^{T_0} R(|x(t)|)\,\mathrm{d}t \leqslant b_0$. Let us now set $T_1 = b^0/R(0)$ (recall that due to Assumption 2, we have $R(0) > 0$), and consider during the times $(T_0, T_0 + T_1)$ we chose the null control, and we have that the position of the robot is fixed to 0 during this time interval (see Assumption 1-(c)), and $b(T_0 + T_1) \leqslant 0$. Finally, Assumptions 1-(c) and 1-(d), ensures the existence of a time T_2, a control $u_2 \in L^\infty(T_0 + T_1, T_0 + T_1 + T_2)^m$, such that the solution (x,p) of (2.1) with initial conditions $x(T_0 + T_1) = 0$ and $p(T_0 + T_1) = 0$, satisfies $x(T_0 + T_1 + T_2) = x^1$ and $p(T_0 + T_1 + T_2) = p^1$, for some $p^1 \in \mathcal{P}^1$ such that $(x^1, p^1) \in \mathcal{C}_s$. Obviously, we also have that $b(T_0 + T_1 + T_2) \leqslant 0$. It is then clear that the minimal time T given by (2.4) is non-greater that $T_0 + T_1 + T_2$.

It is also possible to provide lower bounds on the minimal time T. Trivially, one has $T \geqslant \max\{|x^1 - x^0|, b^0/R(0)\}$. Indeed, $|x^1 - x^0|$ is a lower bound on the minimal time required by the robot to reached x^1 from x^0, and $b^0/R(0)$ is a lower bound on the minimal time to empty the buffer of the robot (recall that due to Assumption 2-(a), we have $R(0) = \max\limits_{\rho \geqslant 0} R(\rho)$).
In addition to these trivial lower bounds, one can be easily convinced that the minimum time T

given by (2.4) is non-lower than the minimal time $\underline{T}$ given by the following minimization problem,

$$\begin{array}{l}\min\ \underline{T} \\ \left|\begin{array}{l}\underline{T} \geqslant 0, \\ \exists \underline{u} \in L^\infty(0,T)^d,\ \text{s.t. } \|\underline{u}\|_{L^\infty(0,T)^d} \leqslant 1 \text{ and} \\ (\underline{x}, \underline{b}) \text{ satisfies } \underline{x}(T) = x^1 \text{ and } \underline{b}(T) \leqslant 0, \\ \quad \text{where } \underline{x} \text{ is solution of (3.2a) with } \underline{x}(0) = x^0, \\ \quad \text{and } \underline{b} \text{ is solution of (3.2b) with } \underline{b}(0) = b^0,\end{array}\right.\end{array} \tag{3.1}$$

where we have set,

$$\dot{\underline{x}} = \underline{u}, \tag{3.2a}$$

$$\dot{\underline{b}} = -R(|\underline{x}|). \tag{3.2b}$$

As we will see later in Section 4, we will be able to provide a description of the time optimal controls for the optimal control problem (3.1). In addition, as we will see in § 3.3.1, the optimal solution of (3.1) will give a nice initialization for the minimization problem (2.4).

3.3 Numerical experiments

3.3.1 Discretization of the problem

In order to numerically compute a time optimal control and a time optimal path, we use the total discretization strategy, as presented for instance in [20, § 9.2.1] (see also [6, 21]). This method will be compbined with explicit Euler method. To this end, let us set $n_t \geqslant 2$, the number of time discretization points. We also set $x^{(i)}$ (respectively $p^{(i)}$, $b^{(i)}$ and $u^{(i)}$) the approximation of x (respectively p, b and u) at time $t = iT/n_t$. So that the discretized version of the time optimal control problem (2.4) becomes

$$\text{Minimize } T, \tag{3.3a}$$

subject to $T \geqslant 0$ and for every $i \in \{0, \dots, n_t\}$, there exist $x^{(i)} \in \mathbb{R}^d$, $p^{(i)} \in \mathbb{R}^s$, $b^{(i)} \in \mathbb{R}$ and $u^{(i)} \in \mathbb{R}^m$, such that for all $i \in \{0, \dots, n_t - 1\}$ we have,

$$x^{(i+1)} - x^{(i)} = \tfrac{T}{n_t} f(x^{(i)}, p^{(i)}, u^{(i)}), \tag{3.3b}$$

$$p^{(i+1)} - p^{(i)} = \tfrac{T}{n_t} g(x^{(i)}, p^{(i)}, u^{(i)}), \tag{3.3c}$$

$$b^{(i+1)} - b^{(i)} = -\tfrac{T}{n_t} R(|x^{(i)}|), \tag{3.3d}$$

and for every $i \in \{0, \dots, n_t\}$ we have,

$$u^{(i)} \in \mathcal{C}_c,\ (x^{(i)}, p^{(i)}) \in \mathcal{C}_s \text{ and } |f(x^{(i)}, p^{(i)}, u^{(i)})| \leqslant 1, \tag{3.3e}$$

together with

$$x^{(0)} = x^0,\ p^{(0)} = p^0, b^{(0)} = b^0, \quad \text{and} \quad x^{(n_t)} = x^1,\ p^{(n_t)} \in \mathcal{P}^1,\ b^{(n_t)} \leqslant 0. \tag{3.3f}$$

In order to numerically solve this problem, we use the sequential quadratic programming routine of the `matlab` `fmincon` function. This iterative routine can be initialized using different ways. We proposed the following ones:

(i) use the optimal solution of (3.1);

(ii) initialize all the variables to 0;

(iii) initialize with the robot path formed by the segment $[x^0, x^1]$;

(iv) initialize with the robot path formed by the two segments $[x^0, 0]$ and $[0, x^1]$.

Let us briefly explain how the initialization (i) of the minimization problem (3.3) is made. We explain this in continuous time, the real initialization will be obtained after time discretization, which is not a key point.
We set $\underline{T}$, $\underline{x}$, $\underline{b}$ and $\underline{u}$ the optimal solution of the minimization problem (3.1). We then initialize with $T = \underline{T}$, $x = \underline{x}$, $b = \underline{b}$ and p and q are adjusted so that $\underline{u} = f(\underline{x}, p, q)$ and $\dot{p} = g(\underline{x}, p, q)$. Of course, a solution (p, q) of this system might not exist. In this case, one can think to minimize some norm of $t \in [0, T] \mapsto (\underline{u} - f(\underline{x}, p, q), \dot{p} - g(\underline{x}, p, q))$, but this is case dependent. Similarly, it can happen that multiple solutions exist, in this case, we could choose the one minimizing some distance between $(p(0), p(T))$ and $\{p^0\} \times \mathcal{P}^1$.

As title of illustration, we consider the following dynamic for the robot,

$$\dot{x} = v\,(\cos\gamma,\, \sin\gamma)^\top, \tag{3.4a}$$

$$\dot{v} = a, \tag{3.4b}$$

$$\dot{\gamma} = \omega. \tag{3.4c}$$

In addition, we will assume that the velocity of the robot v is nonnegative and bounded by 1, the acceleration a of the robot is bounded by some constant $\overline{a}$, and the angular velocity ω is also bounded by some constant $\overline{\omega}$. This leads to the state constraint

$$0 \leqslant v(t) \leqslant 1 \qquad (t \geqslant 0), \tag{3.4d}$$

and the control constraints,

$$|a(t)| \leqslant \overline{a} \quad \text{and} \quad |\omega(t)| \leqslant \overline{\omega} \qquad (t \geqslant 0). \tag{3.4e}$$

The problem considered in this paragraph is given $b^0 \in \mathbb{R}_+$, $x^0, x^1 \in \mathbb{R}^2$, $v_0, v_1 \in [0, 1]$ and $\gamma^0 \in \mathbb{R}$, find the minimal time $T \geqslant 0$ such that there exist a and ω in $L^\infty(0, T)$ satisfying (3.4e) such that the solution $t \in [0, T] \mapsto (x(t), v(t), \gamma(t))$ with initial conditions

$$x(0) = x^0, \quad v(0) = v^0 \quad \text{and} \quad \gamma(0) = \gamma^0, \tag{3.4f}$$

satisfies (3.4d) together with the final constraints,

$$x(T) = x^1, \quad v(T) = v^1 \quad \text{and} \quad b(T) \leqslant 0, \tag{3.4g}$$

where b is solution of (2.3) with initial condition b^0, i.e.,

$$\dot{b} = -R(|x|), \qquad b(0) = b^0. \tag{3.4h}$$

Note that when $\overline{a} = \overline{\omega} = +\infty$, this is exactly the minimization problem considered in § 4.3.

Note that this system coincide with the one given by (2.1) and (2.2), with $d = s = m = 2$, $p = (v, \gamma)$, $u = (a, \omega)$, $f(x, p, u) = v(\cos\gamma, \sin\gamma)^\top$, $g(x, p, u) = u$, $\mathcal{C}_s = \mathbb{R}^2 \times \mathbb{R}^2$, $\mathcal{C}_c = [-\overline{a}, \overline{a}] \times [-\overline{\omega}, \overline{\omega}]$, and $\mathcal{P}^1 = \{v^0\} \times \mathbb{R}$. Note also that Assumptions 1 and 3 are clearly satisfied.

Remark 2. *Let us discuss the initialization (i) in the particular situation of* (3.4)*. In addition to the trivial initialization of x and b, we initialize the system with $v = |\underline{u}|$, $\gamma = \arg(\underline{u}_1 + i\underline{u}_2)$, $a = \dot{v}$ and $\omega = \dot{\gamma}$. Note also that for this particular case, γ is defined modulo 2π, we thus look for a continuous realization of $t \in [0, T] \mapsto \arg(\underline{u}_1(t) + i\underline{u}_2(t))$ such that $|\gamma^0 - \arg(\underline{u}_1(0) + i\underline{u}_2(0))|$ is minimal.*

We also consider the transmission rate R, given by

$$R(\rho) = p_s R_0 \log_2 \left(1 + \frac{P}{(\delta+\rho)^a}\right) \qquad (\rho \geqslant 0), \tag{3.5}$$

with $R_0 > 0$, $P > 0$, $\delta > 0$, $a \geqslant 2$ and $0 \leqslant p_s \leqslant 1$. In practice, we chose $p_s = e^{-c}$ with $c \geqslant 0$ (see the next subsection § 3.3.2 for more details). This particular choice represent the expected data rate based on information theory results, and is often used in wireless literature as an upper bound on the achievable rate [22]. Here, p_s denotes the probability of packet success caused by wireless channel uncertainty and will play a role in § 3.3.2. It is also obvious that the function given in (3.5) satisfies all the requirements of Assumption 2.

In the numerical simulations below, we chose,

$$R_0 = 1,\ \delta = 1/10,\ a = 3,\ P = 1 \text{ and } c = 1/10, \tag{3.6}$$

leading to $p_s = e^{-c} \simeq 0.9048$. The function R defined by (3.5), with the parameters given in (3.6), is displayed on Figure 1.

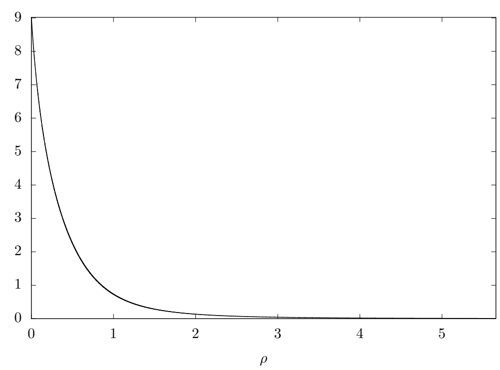


Figure 1: Illustration of the expected transmission rate function R given by (3.5), with parameters given in (3.6).

The other parameters, used in the numerical simulation, are

$$x^0 = (4\sqrt{2},\ 0)^\top, \quad x^1 = \sqrt{2}(1,\ 1)^\top, \tag{3.7a}$$

$$v^0 = v^1 = 0, \quad \gamma^0 = -\pi/2, \tag{3.7b}$$

$$b^0 \simeq 3.448784 \tag{3.7c}$$

and the chosen control bounds are

$$\overline{a} = 2 \quad \text{and} \quad \overline{\omega} = 3. \tag{3.8}$$

As we can see on Table 1, the choice of initialization is rather important for the computational time and the convergence of the algorithm. This is expected since the minimization problem (3.3) is a priori non-convex. In addition, on Figure 2, we plot the paths obtained with initializations (i) and (iv) (the results obtained with initializations (iii) and (iv) are almost the same).

On Table 1, one can see that the best results (in term of minimal value obtained and computational time) are obtained with the initialization strategy (i). In any cases, as it is expected, the minimal time obtained is always greater that the minimal time obtained for the simple integrator robot (see Figure 2).

n_t	Initialization	Number of iterations	CPU time (sec.)	Minimal time obtained
10	(i)	32	0.43	7.877915
	(ii)	450	7.65	13.624892
	(iii)	61	0.97	8.736611
	(iv)	306	3.43	8.736611
20	(i)	61	0.99	7.719635
	(ii)	218	6.57	8.923896
	(iii)	147	4.44	8.662189
	(iv)	226	4.15	8.662189
50	(i)	140	11.22	7.692913
	(ii)	5	25.91	did not converge
	(iii)	202	32.73	8.667430
	(iv)	423	28.92	8.667430
75	(i)	208	22.75	7.687747
	(ii)	6	130.77	did not converge
	(iii)	266	129.87	10.058427
	(iv)	330	61.79	8.667643
100	(i)	261	52.41	7.688806
	(ii)	8	538.80	did not converge
	(iii)	356	349.71	8.666217
	(iv)	317	89.24	8.666217
150	(i)	358	190.15	7.686823
	(ii)	5	1584.41	did not converge
	(iii)	573	2113.11	8.665094
	(iv)	421	293.25	8.665094
200	(i)	450	563.15	7.685786
	(ii)	7	5439.79	did not converge
	(iii)	690	4330.84	8.664531
	(iv)	509	860.40	8.664531

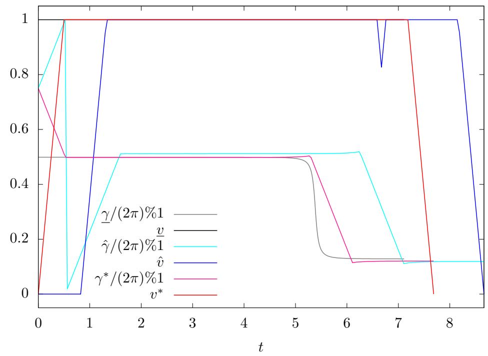
Table 1: Results obtained for the minimization problem (3.3) with different parameters n_t and initializations proposed in (i)–(iv).

3.3.2 Case of a noisy transmission rate

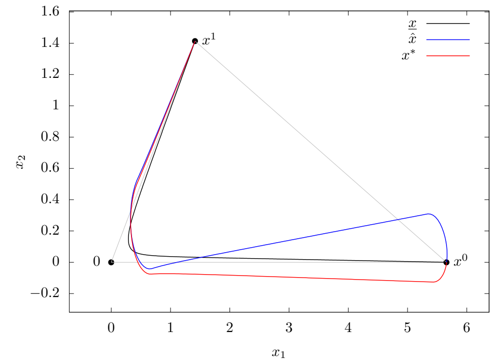
In this paragraph, we will only perform some numerical experimentation. To this end, we consider the robot dynamic introduced given in (3.4). In this paragraph, we also assume that the wireless transmission is subject to noise. More precisely, we consider that the signal-to-noise ratio (SNR) at distance ρ from the base station is inversely proportional to $(\delta+\rho)^a$ where $a \geqslant 2$ is the path loss exponent (determined by the environment type) and $\delta > 0$. When transmitting with a bandwidth of R_0 and a spectral rate of $\overline{R}$, we assume that the packet success event is a Bernoulli process, i.e., it is received with a probability $p_s \in [0,1]$. Therefore, when a packet is received $R_0\overline{R}$ amount of information is communicated, while no bits are communicated when the packet is lost. The expected communication rate is thus given by $R = p_s R_0 \overline{R}$.

Exploiting the results in [18], we know that the probability of successful reception for Rayleigh slow-fading channels can be well-approximated by

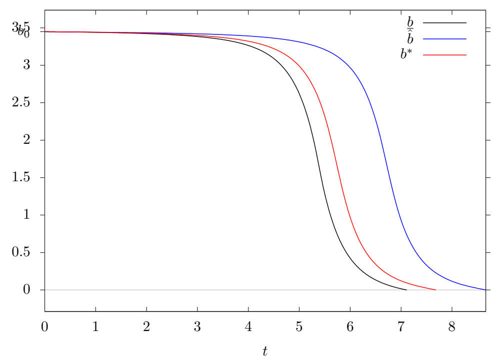
$$p_s(\overline{R}, \rho) = \exp\left(-c(2^{\overline{R}} - 1)\frac{(\delta+\rho)^a}{P} \right),$$



(a) Time optimal values computed for v and γ.



(b) Time optimal path computed for the robot position.



(c) Time optimal buffer discharge computed.

Figure 2: Time optimal state trajectories computed for initializations (i) (named γ^*, v^*, x^* and b^*) and (iv) (named $\hat{\gamma}$, $\hat{v}$, $\hat{x}$ and $\hat{b}$). The state trajectory used for initialization (i) is also displayed (named $\underline{\gamma}$, $\underline{v}$, $\underline{x}$ and $\underline{b}$) (The displayed result has been obtained with $n_t = 200$).

where $c > 0$ is a constant associated to the wireless channel. In the numerical simulations below, we set the spectral rate as

$$\overline{R} = \overline{R}(\rho) = \log_2\left(1 + \frac{P}{(\delta+\rho)^a}\right) \qquad (\rho \geqslant 0). \tag{3.9}$$

resulting in $p_s = e^{-c}$ and an expected rate R given by (3.5).

In order that the robot performed the required task with this noisy transmission rate, we update the control at some given times. Roughly speaking, for a given update time $\tau > 0$, the strategy is the following,

(a) we compute the control minimizing (3.3);

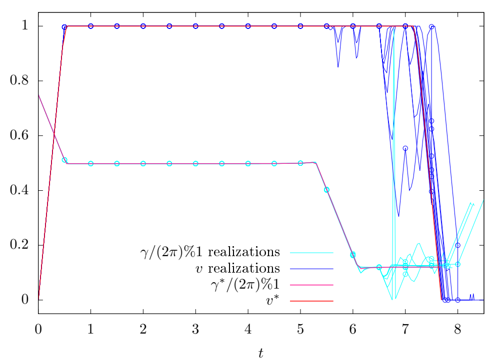
(b) we apply this control during time $T_u = \min\{\tau, T\}$ (where T is the minimal time computed in (3.3)). In particular $b(t)$ is given by

$$\dot{b}(t) = -\mathfrak{T}(t)\overline{R}(|x(t)|), \qquad b(0) = b^0, \tag{3.10}$$

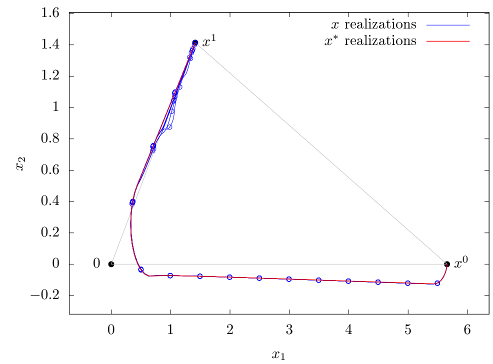
where $\overline{R}$ is given by (3.9), and $\mathfrak{T} \in \{0,1\}^{\mathbb{R}_+}$ describes the chance of success in the transmission and is a realization of the Bernoulli process. Recall that the probability of success is $p_s = e^{-c}$;

(c) at time T_u, we update the current value of the buffer, and we go back to step (a) with new initial condition.
We repeat this until the buffer is not empty and the robot's position target is not reached.

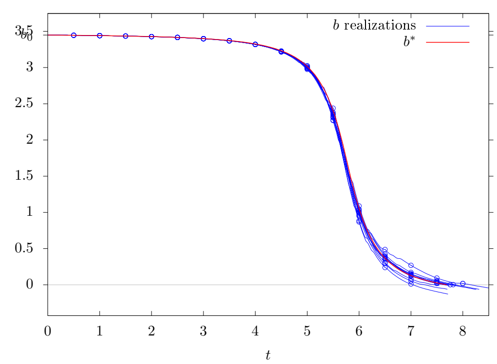
On Figure 3, we display some realizations of this algorithm.



(a) Values computed for v and γ.



(b) Path computed for the robot position.



(c) Buffer discharge computed.

Figure 3: Realization of the process described in § 3.3.2, items (a)–(c), with $\tau = 1/2$. Update points are marked with circles and the state path $(\gamma^*, v^*, x^*, b^*)$ is the optimal solution of (3.3). (Parameters used for this simulation are given in (3.6), (3.7) and (3.8).)

4 Study of the simple integrator case

Although we develop, in this section, results for a simple dynamics, the reader has to keep in mind that we have shown in Section 3 how these results can be used to control more complex non-holonomic ones. In addition, the choice of a simple integrator dynamics is largely motivated in the literature (see e.g., [8]). Besides the relevance of this choice for practical applications, we will also see that mathematical analysis of this simple dynamics is not trivial. However, keeping the dynamics simple facilitates the mathematical presentation of the results. Throughout this section, we consider the time optimal control problem (3.1).

Before entering the core of this section, let us give a brief summary of the obtained results. See also Figure 8, at the end of this paper, for an example of time optimal controlled path.

- When b^0 is small, the optimal time is given by Proposition 1.
- When b^0 is large, the optimal time is given by Proposition 2.
- When b^0 takes intermediate values and x^0, x^1 and 0 are aligned, the optimal time is given by Proposition 3.
- When b^0 takes intermediate values and x^0, x^1 and 0 are not aligned, the optimal time is given by Theorem 1.

This section is organized as follows. Based on some geometrical facts, in § 4.1, we reduce the above minimization problem to a simpler one. Precisely, we show that it is sufficient to consider that the robot evolves in the two-dimensional plane. Moreover, if the size of the message to transmit is small or large enough the solution can be easily found. In § 4.2, we apply the Pontryagin maximum principle and compute the optimal control and the corresponding minimal time. Finally, in § 3.3, we illustrate the result with some numerical simulations.

4.1 Preliminary observations

As for the discussion made in § 3.1, Filippov theorem and its extension can also be applied in the context of the minimal time control problem (3.1) (under Assumption 2-(b)). Using some simple geometric facts we reduce the general d-dimensional problem (3.1) to the same problem but with $d = 2$. Furthermore, we will show that there exist an optimal path x such that $x(t)$ belongs to the convex hull of $\{x^0, 0, x^1\}$ for every time $t \in [0, T]$. Finally, we will give the closed form of the control signal and of the minimal time in some particular situations. Precisely, the analytic solution is provided when b^0 is small or large enough, as well as when the initial position (x^0), the position of the antenna (0) and the final position (x^1) are aligned.

4.1.1 Reduction to a planar motion

Let us first emphasize an invariance property with respect to the change of the basis used to express the vectors $\underline{x} \in \mathbb{R}^d$.

Remark 3. *It is straightforward to show that for every orthogonal matrix $Q \in \mathbb{R}^{d\times d}$, if $(\underline{T}, \underline{u})$ is an optimal solution of the constrained minimization problem* (3.1)*, then $(\underline{T}, Q\underline{u})$ is also an optimal solution of* (3.1) *with x^0 replaced by Qx^0 and x^1 replaced by Qx^1.*

Secondly, we can show that there exist a time-optimal solution for which the motion of the robot is performed in a 2D space, namely $\mathrm{Span}\{x^0, x^1\}$.

Lemma 1. *Given $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$, then there exists a solution $(\underline{T}, \underline{u})$ of* (3.1) *such that the trajectory $\underline{x}$ of* (3.2a) *associated with $\underline{u}$, satisfies $\underline{x}(t) \in \mathrm{Span}\left\{x^0, x^1\right\}$ for every $t \in [0, \underline{T}]$.*

Proof. Let us define $P \in \mathbb{R}^{d\times d}$ the orthonormal projector from $\mathbb{R}^d$ to $\mathrm{Span}\left\{x^0, x^1\right\} \subset \mathbb{R}^d$. Let $(\underline{T}, \underline{u})$ be an optimal solution, and set $(\underline{x}, \underline{b})$ the corresponding time-optimal trajectory. We set $\tilde{\underline{u}} = P\underline{u}$ and $\tilde{\underline{x}} = P\underline{x}$, and we obtain $\dot{\tilde{\underline{x}}} = \tilde{\underline{u}}$, $\tilde{\underline{x}}(0) = x^0$, $\tilde{\underline{x}}(T) = x^1$ and $|\tilde{\underline{u}}(t)| \leqslant |\underline{u}(t)| \leqslant 1$. Let us also define $\tilde{\underline{b}}(t) = b^0 - \int_0^t R(|\tilde{\underline{x}}(\tau)|)\,\mathrm{d}\tau$. Since $|\tilde{\underline{x}}(t)| \leqslant |\underline{x}(t)|$ and since R is a decreasing function, we have $\tilde{\underline{b}}(t) \leqslant \underline{b}(t)$ for every $t \in [0, \underline{T}]$ and in particular, $\tilde{\underline{b}}(T) \leqslant 0$.

In conclusion we have found a control in time $\underline{T}$ which is admissible (i.e., is of L^∞-norm lower than 1 for which we have $\underline{x}(T) = x^1$ and $\underline{b}(T) \leqslant 0$) such that the trajectory of the robot belongs to Span $\{x^0, x^1\}$. □

From Remark 3 and Lemma 1, we can assume without loss of generality that $d = 2$.

The next result shows that, there always exists a time-optimal solution such that the trajectory $\underline{x}$ belongs to the following convex and bounded set co $\{0, x^0, x^1\}$, where co A denotes the convex hull of the set A.

Lemma 2. *Given $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$, there exist a solution $(\underline{T}, \underline{u})$ of (2.4) such that the trajectory $\underline{x}$ of (3.2a) associated with $\underline{u}$ satisfies,*

$$\underline{x}(t) \in \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\} \qquad (t \in (t_0, t_1)), \tag{4.1}$$

for every t_0, t_1 such that $0 \leqslant t_0 \leqslant t_1 \leqslant \underline{T}$.

Proof. Assume that $(\underline{T}, \underline{u})$ is optimal and let $\underline{x}$ be the corresponding path. Since $\underline{x}$ is continuous, if this property is not satisfied, there exist two times t_0 and t_1 such that $0 \leqslant t_0 < t_1 \leqslant \underline{T}$ and $\underline{x}(t) \notin \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\}$ for every $t \in (t_0, t_1)$.
Then, for every $t \in (t_0, t_1)$, we define $\tilde{\underline{x}}(t) \in \mathrm{co}\{0, \underline{x}(t_0), \underline{x}(t_1)\}$ such that $\tilde{\underline{x}}(t)$ minimizes $y \mapsto |\underline{x}(t) - y|$ under the constraint $y \in \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\}$, and for $t \in [0, \underline{T}] \setminus (t_0, t_1)$, we simply set $\tilde{\underline{x}}(t) = \underline{x}(t)$. It is easy to see that $\tilde{\underline{x}}$ is almost everywhere differentiable on (t_0, t_1) and $|\dot{\tilde{\underline{x}}}| \leqslant 1$. We thus have build an admissible path $\tilde{\underline{x}}$ satisfying $|\underline{x}(t)| \geqslant |\tilde{\underline{x}}(t)|$ for every $t \in [0, \underline{T}]$, and since R is non-increasing, we have $\int_{t_0}^{t_1} R(|\underline{x}(t)|)\,\mathrm{d}t \leqslant \int_{t_0}^{t_1} R(|\tilde{\underline{x}}(t)|)\,\mathrm{d}t$. □

Remark 4. *The result of Lemma 2 ensures that there always exists a time optimal control $\underline{u}$ such that the solution $\underline{x}$ of (3.2a) satisfies (4.1). However, it can be possible that some other time optimal trajectories do not satisfy the property (4.1). This is, in particular, the case when R is constant on a ball centered on 0. However, when b^0 is large enough and when the Assumption 2-(c) is used, we will see in Lemma 4 that the time optimal trajectory of the robot necessarily satisfies (4.1).*

4.1.2 Minimal time for small or large enough initial buffer

For every $x^0, x^1 \in \mathbb{R}^d$, let us define

$$B(x^0, x^1) = |x^1 - x^0| \int_0^1 R\left(|x^0 + s(x^1 - x^0)|\right)\,\mathrm{d}s, \tag{4.2}$$

representing the quantity of buffer transmitted when going on a straight line from x^0 to x^1 with velocity one.

In the next proposition, we give the optimal time and the optimal control when b^0 is small.

Proposition 1. *Given $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$. If $B(x^0, x^1) \geqslant b^0$, then the minimal time is $|x^1 - x^0|$ and the optimal control is $\underline{u}(t) = (x^1 - x^0)/|x^1 - x^0|$.*

In other words, the optimal path of the robot is to go straight to the target.

Proof. Indeed, when the buffer to transmit is smaller than the quantity of information that can be transmitted while going from x^0 to x^1 with the maximum speed, it is optimal to apply a control that achieves this straight line motion. It is clear that, with this control, we get $\underline{b}(T) \leqslant 0$. □

The next result considers the other extreme case (b^0 large) in which going straight to antenna (where the transmission rate is maximal) and then going to the target provides a time which is not sufficient to empty the buffer. Therefore, we basically show that the optimal strategy is to go to the antenna, stay there for a certain period, and then go straight to the target.

Proposition 2. *Given $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$. If $B(x^0,0) + B(0,x^1) \leqslant b^0$, then the optimal time is*

$$\underline{T} = |x^0| + |x^1| + \left(b^0 - B(x^0,0) - B(0,x^1)\right)/R(0) \tag{4.3a}$$

and an optimal control is

$$\underline{u}(t) = \begin{cases} -x^0/|x^0| & \text{if } 0 < t < |x^0|, \\ 0 & \text{if } |x^0| < t < \underline{T} - |x^1|, \\ x^1/|x^1| & \text{if } \underline{T} - |x^1| < t < \underline{T}. \end{cases} \tag{4.3b}$$

Proof. It is easy to see that the maximal amount of buffer that can be transmitted during the time interval $[0, |x^0|]$ is $B(x^0, 0)$, and the maximal amount of buffer that can be transmitted during the time interval $[\underline{T} - |x^1|, \underline{T}]$ is $B(0, |x^1|)$, and finally, the maximal amount of buffer that can be transmitted during the time interval $[|x^0|, \underline{T} - |x^1|]$ is $R(0)\left(\underline{T} - |x^1| - |x^0|\right)$. Consequently, the minimal time cannot be lower than $\underline{T}$ given by (4.3a). We conclude the proof by noticing that the control $\underline{u}$ given by (4.3b) allows to reach the target in this time $\underline{T}$, hence is optimal. □

4.1.3 Minimal time when x^0, x^1 and 0 are aligned

From the convexity result Lemma 2, we know that an optimal trajectory belongs to the triangle formed by x^0, x^1 and 0. Furthermore, if $b^0 \leqslant B(x^0, x^1)$ or $b^0 \geqslant B(x^0, 0) + B(0, x^1)$, an optimal trajectory has been obtained in Propositions 1 and 2 respectively. Note that these cases include the case where 0 is included in the segment $[x^0, x^1]$. Let us state the result for the other cases.

Proposition 3. *Let $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$ and assume that x^0, x^1 and 0 are aligned and that $B(x^0, x^1) < b^0 < B(x^0, 0) + B(0, x^1)$, then there exist $\lambda \in (0, 1)$ such that*

$$\lambda|x^0| < |x^1| \quad \text{and} \quad b^0 = B(x^0, \lambda x^0) + B(\lambda x^0, x^1).$$

Furthermore, the minimal time is given by

$$\underline{T} = |x^0| + |x^1| - 2\lambda|x^0|$$

and an optimal control is

$$\underline{u}(t) = \begin{cases} -x^0/|x^0| & \text{if } 0 < t < (1-\lambda)|x^0|, \\ x^0/|x^0| & \text{if } (1-\lambda)|x^0| < t < \underline{T}. \end{cases}$$

The proof of this result is a direct application of Lemma 2 and is not detailed here.

4.1.4 A priori conditions when b^0 takes intermediate values

To conclude this paragraph, we consider the last case, i.e., the case where x^0, x^1 and b^0 satisfy the following assumptions:

Assumption 4.

(a) $\dim \mathrm{Span}\{x^0, x^1\} = 2$ *(i.e., x^0, x^1 and 0 are not aligned);*

(b) $B(x^0, x^1) < b^0 < B(x^0, 0) + B(0, x^1)$.

In the following lemma, we give some preliminary observations on the optimal solution when Assumption 4 is satisfied.

Lemma 3. *Let $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$ satisfying the Assumption 4, let $(\underline{T}, \underline{u})$ be a minimizer of* (3.1) *and let $(\underline{x}, \underline{b})$ be the corresponding optimal state trajectory. Then we have:*

(i) $|x^1 - x^0| < \underline{T} < |x^0| + |x^1|$*;*

(ii) $\min_{t\in[0,\underline{T}]} |\underline{x}(t)| > 0$*;*

(iii) $\underline{b}(\underline{T}) = 0$.

Proof. The first item is trivial. In fact, the first inequality says that the optimal time $\underline{T}$ is larger than the time required to go from x^0 to x^1 on a straight line, and the second inequality says that the optimal time $\underline{T}$ is smaller than the time required to go from x^0 to 0 and then to x^1 following two straight lines.

The second item can be proven by contradiction. If there exist a time $t \in [0, T]$ such that $\underline{x}(t) = 0$, we necessarily have $\underline{T} \geqslant |x^0| + |x^1|$, which contradicts the first item.

For the last item, assume by contradiction that $\underline{b}(\underline{T}) < 0$. For every $\tau \in [0, \underline{T}]$, let us define the path $\underline{x}_\tau$ by

$$\underline{x}_\tau(t) = \begin{cases} \underline{x}(t) & \text{if } 0 \leqslant t \leqslant \tau, \\ \underline{x}(\tau) + \dfrac{t-\tau}{\underline{T}_\tau - \tau}(x^1 - \underline{x}(\tau)) & \text{if } \tau < t \leqslant \underline{T}_\tau, \end{cases}$$

with $\underline{T}_\tau = \tau + |x^1 - \underline{x}(\tau)|$. Note that we have by construction $\underline{x}_\tau(0) = x^0$, $\underline{x}_\tau(\underline{T}_\tau) = x^1$ and $|\dot{\underline{x}}_\tau(t)| \leqslant 1$ for almost every $t \in [0, \underline{T}_\tau]$. Note also that $\underline{T}_\tau \leqslant \underline{T}$ for every $\tau \in [0, \underline{T}]$. Let us also define $\underline{b}_\tau$ the buffer size associated with the path $\underline{x}_\tau$. We then have $\underline{b}_\tau(T_\tau) = \underline{b}(\tau) - B(\underline{x}(\tau), x^1)$. Note that $\tau \mapsto \underline{b}_\tau(\underline{T}_\tau)$ is continuous, $\underline{b}_0(\underline{T}_0) = \underline{b}^0 - B(x^0, x^1) > 0$ and $\underline{b}_{\underline{T}}(\underline{T}_{\underline{T}}) = \underline{b}(\underline{T}) < 0$. Consequently, there exist $\tau^* \in (0, \underline{T})$ such that $\underline{b}_{\tau^*}(\underline{T}_{\tau^*}) = 0$. Note now that we have $\underline{T}_{\tau^*} < \underline{T}$. In fact if $\underline{T}_{\tau^*} = \underline{T}$ then we have $\underline{x} = \underline{x}_{\tau^*}$ on $[0, \underline{T}]$, and hence $\underline{b}_{\tau^*} = \underline{b}$, which is impossible (because $\underline{b}(\underline{T}) < 0$). This leads to a contradiction with the optimality of $\underline{T}$. □

Remark 5. *Note that the item (iii) is valid with the only assumption being $b^0 > B(x^0, x^1)$.*

This last result, together with a more careful use of Assumption 2, leads to a refined version of Lemma 2.

Lemma 4. *Let $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$ satisfying the Assumption 4, let $(\underline{T}, \underline{u})$ be a minimizer of* (3.1) *and let $(\underline{x}, \underline{b})$ be the corresponding optimal state trajectory. Then for every $t_0, t_1 \in [0, \underline{T}]$, with $t_0 \leqslant t_1$, the convex property* (4.1) *is fulfilled.*

Proof. We reproduce here the proof of Lemma 2. If this property is not satisfied, there exist two time t_0 and t_1 such that $0 \leqslant t_0 \leqslant t_1 \leqslant \underline{T}$ and $\underline{x}(t) \notin \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\}$ for every $t \in (t_0, t_1)$.
For every $t \in [t_0, t_1]$, we then define $\tilde{\underline{x}}(t) \in \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\}$ such that $\tilde{\underline{x}}(t)$ minimizes $y \mapsto |\underline{x}(t) - y|$ under the constraint $y \in \mathrm{co}\,\{0, \underline{x}(t_0), \underline{x}(t_1)\}$. Thus, we have built an admissible path $\tilde{\underline{x}}$ on $[t_0, t_1]$ satisfying $|\underline{x}(t)| \geqslant |\tilde{\underline{x}}(t)|$ for every $t \in [t_0, t_1]$, and since R is non-increasing, we have $\int_{t_0}^{t_1} R(|\underline{x}(t)|)\,\mathrm{d}t \leqslant \int_{t_0}^{t_1} R(|\tilde{\underline{x}}(t)|)\,\mathrm{d}t$. Two situations can happen: either we have $R(|\tilde{\underline{x}}(t)|) = 0$ for every $t \in [t_0, t_1]$, or there exist a time $t \in [t_0, t_1]$ such that $R(|\tilde{\underline{x}}(t)|) > 0$.

1. In the first case ($R(|\tilde{\underline{x}}|) = 0$ on $[t_0, t_1]$) no buffer is transmitted and obviously the optimal path to steer $\underline{x}(t_0)$ to $\underline{x}(t_1)$ is a straight line. This contradicts the fact that $\underline{x}$ was optimal and $\underline{x}(t) \notin \mathrm{co}\{0, \underline{x}(t_0), \underline{x}(t_1)\}$ for every $t \in (t_0, t_1)$.
2. In the second case ($\exists\, t \in [t_0, t_1] \mid R(|\tilde{\underline{x}}(t)|) > 0$), the continuity of R and $\underline{x}$, ensure the existence of a time $\bar{t} \in (t_0, t_1)$ such that $R(|\tilde{\underline{x}}(\bar{t})|) > 0$. Since, by assumption, we have $\underline{x}(\bar{t}) \notin \mathrm{co}\{0, x^0, x^1\}$, we can conclude that $|\tilde{\underline{x}}(\bar{t})| < |\underline{x}(\bar{t})|$. This means (using the strict monotonicity of R, see Assumption 2), that $R(|\underline{x}(\bar{t})|) < R(|\tilde{\underline{x}}(\bar{t})|)$, and hence, from the continuity of $\underline{x}$ and R, we conclude

that $\int_{t_0}^{t_1} R(|\underline{x}(t)|)\,\mathrm{d}t < \int_{t_0}^{t_1} R(|\tilde{\underline{x}}(t)|)\,\mathrm{d}t$. In conclusion, we have built an admissible path, for which the corresponding buffer size satisfies $\tilde{\underline{b}}(T) < 0$. This leads to a contradiction with the item (iii) of Lemma 3.

□

In the next paragraph, we give some more precise results when Assumption 4 is satisfied. These result will be derived from the Pontryagin maximum principle.

4.2 Pontryagin maximum principle

In this section, we are going to apply the well-known Pontryagin maximum principle to solve the problem (3.1). We first write the Pontryagin maximum principle in the general case $d \in \mathbb{N}^*$. Next, based on the results in § 4.1, we reduce the analysis without loss of generality to the particular case $d = 2$. Finally, the results obtained are summarized in Theorem 1. In addition, since the optimal controls and times have already been obtained for x^0, x^1 and 0 aligned and for b^0 small or large (see Propositions 1 to 3), we will assume in this section that x^0, x^1 and b^0 satisfy the Assumption 4.

4.2.1 General case $d \in \mathbb{N}^*$

Let us recall that we assume that R is an absolutely continuous function.
The Hamiltonian associated to the optimal control problem (3.1) is defined by

$$H(\underline{x}, \underline{b}, \underline{u}, \xi, \beta, s_0) = -s_0 + \langle \xi, \underline{u} \rangle - \beta R(|\underline{x}|),$$

for $(\underline{x}, \underline{b}, \underline{u}, \xi, \beta, s_0) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}_+$. The Pontryagin maximum principle (see e.g., [1, Chapter 12] or [14, Chapter 11]) ensures that if $(\underline{T}, \underline{x}, \underline{b}, \underline{u})$ is an optimal solution, then, for almost every $t \in [0, \underline{T}]$, we have,

$$0 = \max_{v \in D} H(\underline{x}(t), \underline{b}(t), v, \xi(t), \beta(t), s_0) = H(\underline{x}(t), \underline{b}(t), \underline{u}(t), \xi(t), \beta(t), s_0), \tag{4.4}$$

where D is the closed unit ball of $\mathbb{R}^d$.
We note that in (4.4), ξ and β (the adjoint states) are solutions of

$$\dot{\xi} = -\frac{\partial H(\underline{x}, \underline{b}, \underline{u}, \xi, \beta, s_0)}{\partial \underline{x}} = \beta R'(|\underline{x}|)\frac{\underline{x}}{|\underline{x}|}, \tag{4.5a}$$

$$\dot{\beta} = -\frac{\partial H(\underline{x}, \underline{b}, \underline{u}, \xi, \beta, s_0)}{\partial \underline{b}} = 0, \tag{4.5b}$$

with $R' \in L^\infty_{loc}(\mathbb{R}_+)$ is the derivative of R. Recall that if x^0, x^1 and b^0 satisfy the Assumption 4 then the optimal path of the robot do not pass through 0 (see item (i) of Lemma 3). This ensures the validity of the relation (4.5a). The relation (4.5b) trivially ensures that β is constant.

Note also that the relation (4.4), ensures that

$$\underline{u}(t) = \frac{\xi(t)}{|\xi(t)|} \qquad (t \in [0, \underline{T}] \text{ s.t. } \xi(t) \neq 0). \tag{4.6}$$

Using (4.4) together with the expression of $\underline{u}$, we deduce that

$$s_0 + \beta R(|\underline{x}|) = |\xi| \qquad (t \in [0, \underline{T}]). \tag{4.7}$$

The next proposition summarize the above discussion.

Proposition 4. *Let $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$ satisfying the Assumption 4, let $(\underline{T}, \underline{u})$ be a minimizer of* (3.1) *and let $(\underline{x}, \underline{b})$ be the corresponding optimal state trajectory. Then there exist $s_0 \geqslant 0$, $\beta \in \mathbb{R}$ and an absolutely continuous function $\xi : [0, T] \to \mathbb{R}^d$, such that (s_0, β, ξ) is not trivial and satisfies* (4.5a) *together with* (4.7).
Furthermore, $\underline{u}$ is given by (4.6) *for every $t \in [0, \underline{T}]$ such that $\xi(t) \neq 0$.*

In addition to this initial result, we can give some more properties on s_0 and β.

Lemma 5. *With the notations and assumptions introduced in Proposition 4, we have in addition $\beta < 0$ and $s_0 > 0$.*

Proof. The fact that $\beta \leqslant 0$ follows from transversality conditions (see e.g., [5]). Consequently, we only have to prove that $\beta \neq 0$ and $s_0 \neq 0$.

Let us assume by contradiction that $\beta = 0$, which yields, from (4.5a), ξ is constant. Since (s_0, β, ξ) shall not be trivial, we necessarily have (using (4.7)) $\xi \neq 0$. Consequently, using (4.6), $\underline{u}$ is a constant vector of the unit sphere of $\mathbb{R}^d$. In order to reach the target x^1, we necessarily have $\underline{u} = (x^1 - x^0)/|x^1 - x^0|$ and $\underline{T} = |x^1 - x^0|$. But with this path, the transmitted information will be $B(x^0, x^1)$ which is strictly smaller than b^0. Consequently, one has $\underline{b}(T) > 0$ which is a contradiction with item (iii) of Lemma 3. This proves that $\beta < 0$.

Since $b^0 > 0$, there exist a time $\tau \in [0, \underline{T}]$ such that $R(|\underline{x}(\tau)|) > 0$. Consequently, from (4.7) (together with $\beta < 0$), we deduce that $s_0 \geqslant |\beta| R(|\underline{x}(\tau)|) > 0$. □

As a consequence of this result, we assume without loss of generality that $\beta = -1$ (recall that s_0, ξ and β are defined up to a multiplicative constant).

Let us now show that the adjoint state ξ vanishes at most one time.

Lemma 6. *With the notations and assumptions of Proposition 4, the adjoint state ξ vanishes at most one time.*

Proof. Assume there exist two times t_0 and t_1 such that $0 \leqslant t_0 < t_1 \leqslant \underline{T}$ and $\xi(t_0) = \xi(t_1) = 0$. Using (4.7) this yields that $R(|\underline{x}(t_0)|) = R(|\underline{x}(t_1)|) = s_0$. By Assumption 2-(c), we have that R is injective on $R^{-1}(\mathbb{R}_+^*)$. In addition, since $s_0 > 0$ (see Lemma 5), we conclude that $R^{-1}(\{s_0\})$ is a single point and $|\underline{x}(t_0)| = |\underline{x}(t_1)|$. Using (4.7), once again, we have for every $t \in [t_0, t_1]$, $R(|\underline{x}(t)|) = s_0 - |\xi(t)| \leqslant s_0 = R(|\underline{x}(t_0)|) = R(|\underline{x}(t_1)|)$. By Assumption 2 ($R$ is nonincreasing), we deduce that $0 \leqslant |\underline{x}(t_0)| = |\underline{x}(t_1)| \leqslant |\underline{x}(t)|$ for every $t \in [t_0, t_1]$. Obviously, this situation is impossible due to the convexity result of Lemma 4 and the fact that $|\underline{x}(t)| > 0$ for every $t \in [0, T]$ (see item (iii) of Lemma 3). □

Remark 6. *This last result ensures that any time optimal control u is given by* (4.6) *for almost every time $t \in [0, \underline{T}]$.*

4.2.2 Case $d = 2$

Let us now particularise the consequences of the Pontryagin maximum principle to the particular case $d = 2$. Recall that the study of the case $d = 2$ is not a restriction, see Remark 3 and Lemma 1. In order to integrate the Pontryagin maximum principle, we identify $\mathbb{R}^2$ with $\mathbb{C}$. Consequently, we set $\underline{x}(t) = \rho(t)e^{i\theta(t)}$ and $\xi(t) = \sigma(t)e^{i\gamma(t)}$, with ρ and σ non-negative. Recall also that due to the item (ii) of Lemma 3, if $\underline{x}$ is a time optimal path, then $\rho(t)$ is positive for every time t. From (4.5), we deduce that σ and γ satisfy (recall that we have chosen, without loss of generality, $\beta = -1$):

$$\dot{\sigma} = -R'(\rho)\cos(\theta - \gamma), \tag{4.8a}$$
$$\sigma\dot{\gamma} = -R'(\rho)\sin(\theta - \gamma) \tag{4.8b}$$

and we have from (4.7),

$$\sigma = s_0 - R(\rho). \tag{4.9}$$

According to Remark 6, the optimal control $\underline{u}$ is given by $\underline{u}(t) = e^{i\gamma(t)}$ for almost every $t \in [0, \underline{T}]$. Thus, from (3.2), we deduce that ρ, θ and b satisfy:

$$\dot{\rho} = \cos(\theta - \gamma), \tag{4.10a}$$

$$\rho\dot{\theta} = -\sin(\theta - \gamma), \tag{4.10b}$$

$$\dot{b} = -R(\rho). \tag{4.10c}$$

In addition, from Remark 3 (with Assumption 4-(a)), we can assume without loss of generality that the initial and final state constraints are

$$\rho(0) = \rho^0 > 0, \quad \rho(\underline{T}) = \rho^1 > 0 \quad \text{and} \quad \theta(\underline{T}) = -\theta(0) = \Theta \in (0, \pi/2).$$

By eventually performing the change of variables $t \mapsto \underline{T} - t$ and using again Remark 3, it can also be assumed that $\rho^0 \geqslant \rho^1$. Hence, in the rest of this paragraph, we assume that

$$\rho^0 \geqslant \rho^1 > 0 \quad \text{and} \quad \Theta \in (0, \pi/2). \tag{4.11}$$

Let us finally define $\alpha = \gamma - \theta$. We are now ready to state the following lemma.

Lemma 7. *Let $x^0 = \rho^0 e^{-i\Theta}$ and $x^1 = \rho^1 e^{i\Theta}$, with ρ^0, ρ^1 and Θ satisfying the assumptions* (4.11), *and let $b^0 \in \mathbb{R}_+$ satisfying the Assumption 4-(b). Given any minimizer $(\underline{T}, \underline{u})$ of* (3.1), *we set $\underline{x} = \rho e^{i\theta}$ and $\underline{b}$ the corresponding optimal state trajectory. Then there exist three constants $\overline{\rho} \in [0, \rho^1]$ and $s_0 > R(\overline{\rho})$, and an absolutely continuous function $\xi = \sigma e^{i\gamma}$ such that ρ, θ, b, σ, γ and s_0 satisfy* (4.8), (4.9) *and* (4.10), *and in addition,*

$$\sigma(t) \neq 0 \qquad (t \in [0, \underline{T}]) \tag{4.12}$$

and $\alpha = \gamma - \theta$ satisfies:

$$\sin \alpha(t) = \frac{\overline{\rho}\,(s_0 - R(\overline{\rho}))}{\rho(t)\,(s_0 - R(\rho(t)))} \qquad (t \in [0, \underline{T}]), \tag{4.13}$$

together with $\alpha(0) \in (\pi/2, \pi]$ (modulo 2π), and α is non-increasing. Finally, the time optimal control is given by $\underline{u} = e^{i\gamma}$ everywhere on $[0, \underline{T}]$.

Proof. Most of the results of this lemma are direct consequences of the previous results introduced in this paper. In fact, it remains to prove (4.12), the existence of $\overline{\rho}$, that $s_0 > R(\overline{\rho})$ and the claimed properties on α.

Using the notation $\alpha = \gamma - \theta$, (4.8) and (4.10), become:

$$\dot{\rho} = \cos\alpha, \qquad \rho\dot{\theta} = \sin\alpha,$$

$$\dot{\sigma} = -R'(\rho)\cos\alpha, \qquad \sigma\dot{\gamma} = R'(\rho)\sin\alpha.$$

and, for every $t \in [0, \underline{T}]$ such that $\sigma(t) \neq 0$, we have

$$\dot{\alpha} = \left(\frac{-1}{\rho} + \frac{R'(\rho)}{\sigma}\right)\sin\alpha. \tag{4.14}$$

Let us denote by $\mathcal{T}$ a connected component of $[0, \underline{T}] \setminus \sigma^{-1}(\{0\})$ (recall that according to Lemma 6, $\sigma^{-1}(\{0\})$ is either the empty set or a singleton). Note that, if $\alpha(\bar{t})$ is given for some $\bar{t} \in \mathcal{T}$, then α solution of (4.14), is uniquely determined in $\mathcal{T}$. Consequently, we have either $\alpha(t) = 0$, or $\alpha(t) \neq 0$ (modulo π) for every $t \in \mathcal{T}$.

1. In the first situation ($\alpha = 0$ (modulo π) on $\mathcal{T}$), by continuity of α on $\mathcal{T}$, we conclude that α is constant equal to 0 (modulo π) on $\mathcal{T}$.

2. In the second situation ($\alpha(t) \neq 0$ (modulo π) for every $t \in \mathcal{T}$), we have,

$$\frac{\cos\alpha}{\sin\alpha}\dot{\alpha} = -\frac{\dot{\rho}}{\rho} - \frac{\dot{\sigma}}{\sigma} \qquad (\text{on } \mathcal{T}).$$

   From which we obtain,

$$\sin\alpha = \frac{c}{\rho\sigma} = \frac{c}{\rho\,(s_0 - R(\rho))} \qquad (\text{on } \mathcal{T}), \tag{4.15}$$

   with c a constant depending only on of the connected component $\mathcal{T}$ of $[0, \underline{T}] \setminus \sigma^{-1}(\{0\})$. Note that if $\sigma^{-1}(\{0\}) = \{\bar{t}\}$ is not empty, we have

$$\lim_{t\to\bar{t}} \rho(t)\,(s_0 - R(\rho(t))) = 0.$$

   Hence, this ensures that if $\sigma^{-1}(\{0\})$ is not empty, we have $c = 0$.

In both cases, if $\sigma^{-1}(\{0\})$ is not empty or if there exist $\bar{t} \in [0, \underline{T}] \setminus \sigma^{-1}(\{0\})$ such that $\alpha(\bar{t}) = 0$ (modulo π), then α is constant equal to 0 (modulo π) on each connected component of $[0, \underline{T}] \setminus \sigma^{-1}(\{0\})$. Thus, using the expression of $\dot{\theta}$ and $\dot{\gamma}$, we deduce that θ and γ are constant on each component of $[0, \underline{T}] \setminus \sigma^{-1}(\{0\})$. Note that θ is also continuous on each connected component of $[0, \underline{T}] \setminus \sigma^{-1}(\{0\})$, and the only point of discontinuity of θ can be when $\rho = 0$. But, since $\theta(\underline{T}) = -\theta(0) = \Theta \in (0, \pi/2)$, θ necessarily have a discontinuity point, meaning that there exist a time $\bar{t}$ such that $\rho(\bar{t}) = 0$. This leads to a contradiction with the item (ii) of Lemma 3. In conclusion, we have $\sigma(t) \neq 0$ for every $t \in [0, \underline{T}]$ and there exist a constant $c \neq 0$ such that (4.15) holds on $[0, \underline{T}]$. In particular, this ensures that ρ, σ, θ, γ and α are continuous on $[0, \underline{T}]$. Note that this also ensure that $u = e^{i\gamma}$ everywhere on $[0, \underline{T}]$.

Due to the assumption $\rho^0 \geqslant \rho^1$ and due to the convexity result (Lemma 4), one has $\gamma(0) \in (-\Theta + \pi/2, -\Theta + \pi]$ (modulo 2π), and hence $\alpha(0) \in (\pi/2, \pi]$ (modulo 2π) (see Figure 4). This, in particular, ensures that $c > 0$, and hence $\alpha(t) \in (0, \pi)$ (modulo 2π) for every $t \in [0, T]$. From the above result, and using (4.14), we can now state that α is a non-increasing function.

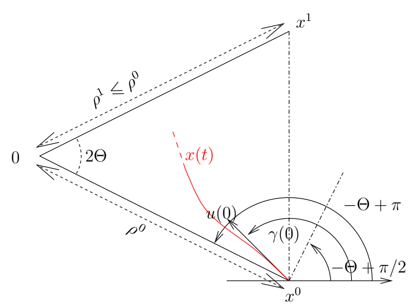


Figure 4: Graphical illustration of the fact that $\gamma(0) \in (-\Theta + \pi/2, -\Theta + \pi]$ (modulo 2π), and $\alpha(0) = \gamma(0) - \theta(0) = \gamma(0) + \Theta \in (\pi/2, \pi]$ (modulo 2π).

Let us finally prove the existence of $\overline{\rho}$ such that $c = R(\overline{\rho})$. Note that $\rho \mapsto \rho(s_0 - R(\rho))$ is increasing on $[\rho_m, \infty)$ with $\rho_m = 0$ if $s_0 \geqslant R(0)$, and $\rho_m \in \mathbb{R}_+$ is such that $s_0 = R(\rho_m)$ otherwise. In any cases, we have $0 = \rho_m(s_0 - R(\rho_m)) \leqslant c \leqslant \rho^1(s_0 - R(\rho^1))$ (the second inequality follows from the fact that (4.15) holds on the full interval $[0, \underline{T}]$). Consequently, there exist $\overline{\rho} \in (\rho_m, \rho^1]$ such that $c = \overline{\rho}(s_0 - R(\overline{\rho}))$ yielding that $s_0 > R(\overline{\rho})$. $\square$

Remark 7. *Let us mention that once s_0 and $\overline{\rho}$ are found, the control law is purely a feed-back control law. More precisely, the optimal control is given by $\underline{u} = e^{i\gamma}$ with γ given by the ordinary differential equation:*

$$\dot{\gamma} = \frac{\overline{\rho} R'(\rho)(s_0 - R(\overline{\rho}))}{\rho(s_0 - R(\rho))^2}, \tag{4.16a}$$

with the initial condition:

$$\gamma(0) = -\Theta + \pi - \arcsin \frac{\overline{\rho}(s_0 - R(\overline{\rho}))}{\rho^0(s_0 - R(\rho^0))}. \tag{4.16b}$$

In what follows we distinguish two possible situations $\alpha(\underline{T}) < \pi/2$ and $\alpha(\underline{T}) \geqslant \pi/2$ (modulo 2π). Let us first set

$$f_{s_0,\overline{\rho}}(\rho) = \frac{\overline{\rho}(s_0 - R(\overline{\rho}))}{\rho(s_0 - R(\rho))} \qquad (\rho^1 \geqslant \overline{\rho} \geqslant 0,\ s_0 > R(\overline{\rho}),\ \rho \geqslant \overline{\rho}). \tag{4.17}$$

- Case $\alpha(\underline{T}) \leqslant \pi/2$ (modulo 2π):

In this case, there exist $\overline{t} \in [0, \underline{T}]$ such that $\sin \alpha(\overline{t}) = 1$, we then have $\overline{\rho} = \rho(\overline{t})$. Note that according to (4.10a), we have $\overline{\rho} = \min_{[0,\underline{T}]} \rho$. Using the monotonicity properties of ρ, we can see that $t \in [0, \overline{t}] \mapsto \rho(t) \in [\overline{\rho}, \rho^0]$ and $t \in [\overline{t}, \underline{T}] \mapsto \rho(t) \in [\overline{\rho}, \rho^1]$ are two diffeomorphisms. Thus, using (4.10a)–(4.10c) and the expression (4.13), we deduce that s_0 and $\overline{\rho}$ shall, in addition to $s_0 > R(\overline{\rho})$, satisfy

$$2\Theta = \int_{\overline{\rho}}^{\rho^0} \frac{f_{s_0,\overline{\rho}}(\rho)}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho + \int_{\overline{\rho}}^{\rho^1} \frac{f_{s_0,\overline{\rho}}(\rho)}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho \tag{4.18a}$$

and

$$b^0 = \int_{\overline{\rho}}^{\rho^0} \frac{R(\rho)}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho + \int_{\overline{\rho}}^{\rho^1} \frac{R(\rho)}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho. \tag{4.18b}$$

The corresponding minimal time is given by

$$\underline{T} = \int_{\overline{\rho}}^{\rho^0} \frac{\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}} + \int_{\overline{\rho}}^{\rho^1} \frac{\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}. \tag{4.18c}$$

Note that the above integrals are well-defined as soon as $s_0 > R(\overline{\rho})$.

- Case $\alpha(T) > \pi/2$ (modulo 2π):

In this case, ρ is strictly decreasing and $t \in [0, \underline{T}] \mapsto \rho(t) \in [\rho^1, \rho^0]$ is a diffeomorphism. Using (4.10a)–(4.10c) and the expression (4.13), we deduce that s_0 and $\overline{\rho}$ shall, in addition to $s_0 > R(\overline{\rho})$, satisfy

$$2\Theta = \int_{\rho^1}^{\rho^0} \frac{f_{s_0,\overline{\rho}}(\rho)}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho \tag{4.19a}$$

and

$$b^0 = \int_{\rho^1}^{\rho^0} \frac{R(\rho)}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\, \mathrm{d}\rho. \tag{4.19b}$$

The corresponding minimal time is given by

$$\underline{T} = \int_{\rho^1}^{\rho^0} \frac{\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}. \tag{4.19c}$$

Remark 8.

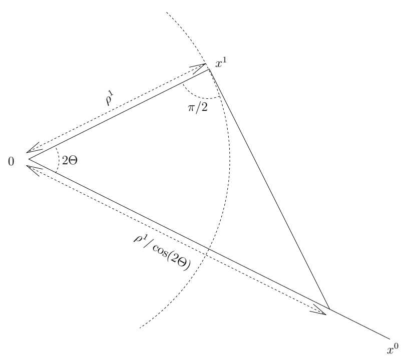
1. *We can have* $\alpha(\underline{T}) > \pi/2$ *only if* $\Theta < \pi/4$ *and* $\rho^0 > \rho^1/\cos(2\Theta)$.
   If $\alpha(\underline{T}) > \pi/2$, *then we have* $\min_{[0,\underline{T}]} \rho = \rho^1$ *and* $\rho'(\underline{T}) \neq 0$. *This together with the convexity result Lemma 4 leads to the claim of the remark. To clarify the reasoning, we refer to Figure 5.*

2. *If* $\alpha(\underline{T}) \geqslant \pi/2$, *then* $\overline{\rho} > \rho_m$, *with*

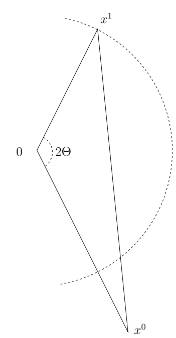
$$\rho_m = \min_{\lambda \in \mathbb{R}} |x^0 + \lambda(x^1 - x^0)| = \sqrt{|x^0|^2 - \left\langle \frac{x^1 - x^0}{|x^1 - x^0|}, x^0 \right\rangle^2}.$$

   If $\overline{\rho} = \rho^1$, *the fact that* $\overline{\rho} > \rho_m$ *is obvious. If* $\overline{\rho} < \rho^1$, *then for every* $\tau > 0$, *the optimal trajectory can be continued on* $[\underline{T}, \underline{T} + \tau]$, *to create a new optimal path for some other initial buffer* $b^0_\tau > b^0$. *Knowing that at time* $\underline{T}$, $\rho(\underline{T}) = \rho^1$, *this result can be proved using the convexity result Lemma 4 for* $t_1 < \underline{T}$ *and* $t_2 > \underline{T}$. *This is illustrated on Figure 6.*

3. *In any cases, if* $(s_0, \overline{\rho})$ *is a solution to the equation* (4.18b) *or* (4.19b), *with* $b^0 > 0$, *we necessarily have* $\overline{\rho} > 0$.



(a) Case $\Theta < \pi/4$.



(b) Case $\Theta \geqslant \pi/4$.

Figure 5: Illustration of the 1st claim of Remark 8.

Remark 9. *In both situations, the time* t, *the angle* θ *and the buffer size* b *can be recovered in term of* ρ, *for instance, in the situation* $\alpha(\underline{T}) < \pi/2$, *we set,*

$$\bar{t} = \int_{\overline{\rho}}^{\rho^0} \frac{\mathrm{d}r}{\sqrt{1 - f_{s_0,\overline{\rho}}(r)^2}}$$

and for $t \leqslant \bar{t}$, *we have,*

$$t = t(\rho) = \int_{\rho}^{\rho^0} \frac{\mathrm{d}r}{\sqrt{1 - f_{s_0,\overline{\rho}}(r)^2}}$$

and for $t \geqslant \bar{t}$, *we have,*

$$t = t(\rho) = \bar{t} + \int_{\overline{\rho}}^{\rho} \frac{\mathrm{d}r}{\sqrt{1 - f_{s_0,\overline{\rho}}(r)^2}}.$$

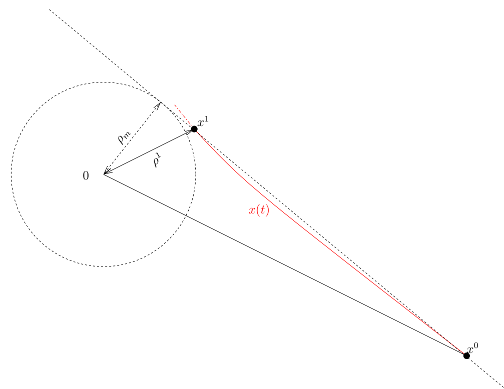


Figure 6: Illustration of the 2^{nd} claim of Remark 8.

Let us also define for $\epsilon = \pm 1$ the maps

$$J_\epsilon^T(s_0, \overline{\rho}) = \int_{\overline{\rho}}^{\rho^0} \frac{\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}} + \epsilon \int_{\overline{\rho}}^{\rho^1} \frac{\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}, \tag{4.20a}$$

$$C_\epsilon^\Theta(s_0, \overline{\rho}) = \int_{\overline{\rho}}^{\rho^0} \frac{f_{s_0,\overline{\rho}}(\rho)\,\mathrm{d}\rho}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}} + \epsilon \int_{\overline{\rho}}^{\rho^1} \frac{f_{s_0,\overline{\rho}}(\rho)\,\mathrm{d}\rho}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}} \tag{4.20b}$$

and

$$C_\epsilon^b(s_0, \overline{\rho}) = \int_{\overline{\rho}}^{\rho^0} \frac{R(\rho)\,\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}} + \epsilon \int_{\overline{\rho}}^{\rho^1} \frac{R(\rho)\,\mathrm{d}\rho}{\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}, \tag{4.20c}$$

and the set

$$E_\epsilon = \left\{(s_0, \overline{\rho}) \in \mathbb{R}_+ \times [0, \rho^1] \;\middle|\; s_0 > R(\overline{\rho}),\ C_\epsilon^\Theta(s_0, \overline{\rho}) = 2\Theta \text{ and } C_\epsilon^b(s_0, \overline{\rho}) = b^0\right\}. \tag{4.21}$$

We are now in a position to give the main result.

Theorem 1. *Let $x^0, x^1 \in \mathbb{R}^d$ and $b^0 \in \mathbb{R}_+$ satisfying the Assumption 4, and let $(\underline{T}, \underline{u})$ be a minimizer of* (3.1)*. Let us define $\rho^0 = \min\left\{|x^0|, |x^1|\right\}$ and $\rho^1 = \max\left\{|x^0|, |x^1|\right\}$. Then $\underline{T} = \min\{T_{-1}, T_{+1}\}$, where for $\epsilon = \pm 1$, T_ϵ is the minimum of J_ϵ^T (defined by* (4.20a)*) on the set E_ϵ (defined by* (4.21)*) (by convention, we have set $T_\epsilon = \infty$ if $E_\epsilon = \emptyset$).*

In addition, once $\epsilon \in \{-1, 1\}$ and the parameters s_0 and $\overline{\rho}$ minimizing J_ϵ^T on E_ϵ are found, the control $\underline{u} = e^{i\gamma}$ can be recovered from (4.16)*, and the state trajectories can be recovered by using the process described in Remark 9.*

Recall that according to Remark 8, we already know that $E_{-1} = \emptyset$ for $\Theta \geqslant \pi/4$, or for $\Theta < \pi/4$ and $\rho^0 \leqslant \rho^1/\cos(2\Theta)$. The problem is now to minimize J_ϵ^T on the set E_ϵ. We conjecture that the set E_ϵ is the empty set or a singleton. Proving such a fact does not seem to be an easy task. However, we can make the next remark ensuring that s_0 is uniquely determined by $\overline{\rho}$.

Remark 10. *Let $\overline{\rho} \in (0, \rho^1]$, then when* (4.11) *holds, for every $\epsilon \in \{-1, 1\}$, there exists at most one $s_0 = s_0(\overline{\rho})$ such that $(s_0, \overline{\rho}) \in E_\epsilon$.*

Let us first recall that according to the 3^{rd} claim of Remark 8, we necessarily have $R(\overline{\rho}) > 0$. To prove the claim of the remark, we first define,

$$I_\Theta^{\hat{\rho}}(s_0, \overline{\rho}) = \int_{\overline{\rho}}^{\hat{\rho}} \frac{f_{s_0,\overline{\rho}}(\rho)}{\rho\sqrt{1 - f_{s_0,\overline{\rho}}(\rho)^2}}\,\mathrm{d}\rho \qquad (\hat{\rho} \in \mathbb{R}_+^*,\ (s_0, \overline{\rho}) \in D(\hat{\rho})),$$

where $D(\hat{\rho}) = \{(s_0, \overline{\rho}) \in \mathbb{R}_+ \times (0, \hat{\rho}) \ \mid \ s_0 > R(\overline{\rho}) > 0\}$, and where $f_{s_0,\overline{\rho}}$ is given by (4.17). Note that we have:

$$C_\epsilon^\Theta(s_0, \overline{\rho}) = I_\Theta^{\rho^0}(s_0, \overline{\rho}) + \epsilon I_\Theta^{\rho^1}(s_0, \overline{\rho}).$$

Note also that the above functions are continuously differentiable on $D(\hat{\rho})$ for every $\hat{\rho} > 0$, and, after some computations, we obtain,

$$\partial_{s_0} I_\Theta^{\hat{\rho}}(s_0, \overline{\rho}) = \int_{\overline{\rho}}^{\hat{\rho}} \frac{\partial_{s_0} f_{s_0,\overline{\rho}}(\rho)}{\rho\,(1 - f_{s_0,\overline{\rho}}(\rho)^2)^{3/2}}\,\mathrm{d}\rho,$$

with,

$$\partial_{s_0} f_{s_0,\overline{\rho}}(\rho) = \frac{\overline{\rho}\,(R(\overline{\rho}) - R(\rho))}{\rho\,(s_0 - R(\rho))^2}.$$

Using the monotonicity of R, we obtain (recall that $R(\overline{\rho}) > 0$) that

$$\partial_{s_0}\left(I_\Theta^{\rho^0}(s_0, \overline{\rho}) + \epsilon I_\Theta^{\rho^1}(s_0, \overline{\rho})\right) > 0.$$

This in particular ensures that given $\overline{\rho}$, there exist at most one $s_0 = s_0(\overline{\rho})$ such that

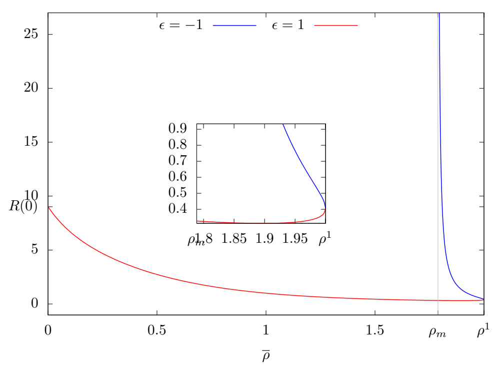
$$2\Theta = I_\Theta^{\rho^0}(s_0(\overline{\rho}), \overline{\rho}) + \epsilon I_\Theta^{\rho^1}(s_0(\overline{\rho}), \overline{\rho}).$$

Furthermore, if such an s_0 exist, then, using the implicit function Theorem, $\overline{\rho} \mapsto s_0(\overline{\rho})$ is absolutely continuous.

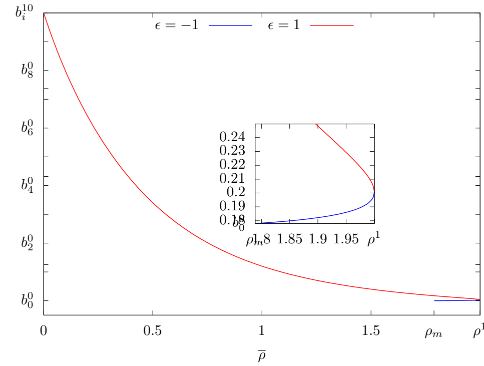
4.3 Numerical illustration

We illustrate the results obtained Section 4, with the transmission rate given by (3.5), with parameters given in (3.6). We place ourselves in the case $d = 2$, and we consider the initial and final condition x^0 and x^1 given by (3.7a). Up to a rotation, this leads, with the notation of § 4.2, to $\Theta = \pi/8$, $\rho^1 = 2$ and $\rho^0 = 2\rho^1/\cos(2\Theta)$. This choice has been made in order to allow the existence of a minimizer of J_{-1}^T on the set E_{-1} (see Remark 8). Let us define $b_0^0 = B(x^0, x^1)$ and $b_{10}^0 = B(x^0, 0) + B(0, x^1)$ (note that the initial buffer chosen in § 3.3 is $(b_0^0 + b_{10}^0)/2$). Numerically we obtain $b_0^0 \simeq$ `0.1780` and $b_{10}^0 \simeq$ `6.7196`, for the initial condition on $\underline{b}$, we will take $b^0 = b_k^0$, with $b_k^0 = b_0^0 + k(b_{10}^0 - b_0^0)/10$, for $k = 0, \ldots, 10$. In order to show the optimal trajectory in the situations given by Propositions 1 and 2, we will also consider initial conditions $b^0 < b_0^0$ and $b^0 > b_{10}^0$. In order to obtain the optimal solution for the given initial condition b^0, we chose to use the property claimed in Remark 10, that is to say that for every $\overline{\rho} \in [0, \rho^1]$, and every $\epsilon \in \{-1, +1\}$ we try to compute an $s_0 \geqslant R(\overline{\rho})$ such that $C_\epsilon^\Theta(s_0, \overline{\rho}) = 2\Theta$.

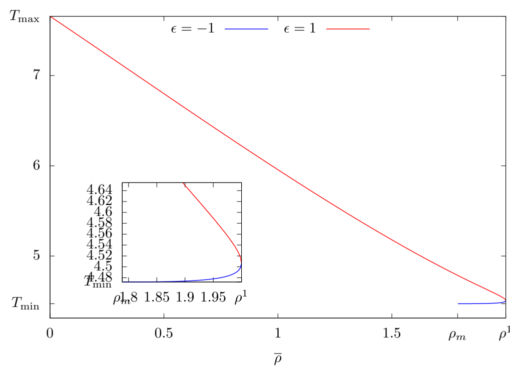
The results of these computations are given on Figure 7a, and the computation has been done with the `fsolve` function of `matlab`. Once s_0 is known in term of ϵ and $\overline{\rho}$, we draw the corresponding buffer and time, see Figures 7b and 7c.
Let us also mention that in order to numerically compute the integrals given in (4.18) and (4.19), we use the midpoint rule, which is known to be convergent for improper integrals. For the computation presented here, we have used 10^5 discretization points.
On Figure 7b, we observe that given $b^0 \in (b_0^0, b_{10}^0)$, there exist one and only one corresponding value for $\overline{\rho}$ and ϵ, leading to the time optimal solution. Finally, on Figure 8, we display the optimal state trajectories associated to $b_0^0, \ldots, b_{10}^0$.



(a) Value of $s_0 = s_0(\overline{\rho};\epsilon)$. ρ_m was introduced in the 3rd claim of Remark 8.



(b) Value of $C^b_\epsilon(s_0,\overline{\rho})$, with $s_0 = s_0(\overline{\rho};\epsilon)$ given in Figure 7a.

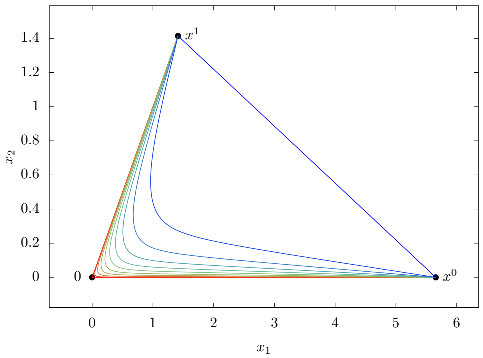


(c) Value of $J^T_\epsilon(s_0,\overline{\rho})$, with $s_0 = s_0(\overline{\rho};\epsilon)$ given in Figure 7a. $T_{\min} = |x^1 - x^0|$ and $T_{\max} = |x^0| + |x^1|$.

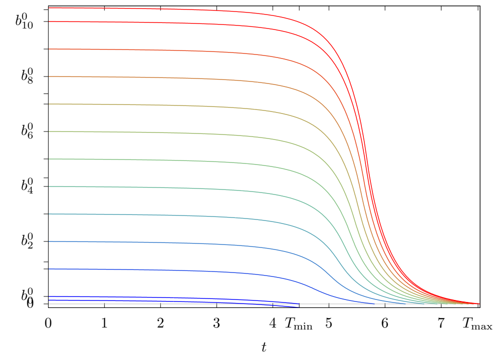
Figure 7: Computation of s_0 such that $C^\Theta_\epsilon(s_0,\overline{\rho}) = 2\Theta$ and associated values of C^b_ϵ and J^T_ϵ. (Explicit values of ρ^0, ρ^1 and Θ are given in the introduction of § 3.3.)

5 Conclusions

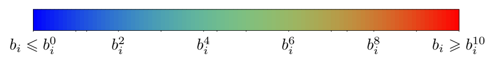
In this paper, we design a time-optimal control strategy allowing a mobile robot to reach a target while transmitting a message in minimum time. Analytical results have been obtained in one of the simplest situations in which there is only one antenna, there are no shadowing effects (the transmission rate only depend on the distance to the antenna) and the dynamics of the robot is described by a single integrator. In this framework, we give both a theoretical description of the optimal control and a strategy for its numerical implementation. We also show how this theoretical result could be useful for the numerical computation of optimal trajectories with robots having non-holonomic dynamics and in the case where the transmission debit is subject to noise. Further works should consider the presence of multiple antennas and noises.



(a) Time optimal path.



(b) Time optimal buffer discharge ($T_{\min} = |x^1 - x^0|$ and $T_{\max} = |x^0| + |x^1|$).



(c) Color-map for the different values of b^0.

Figure 8: Time optimal state trajectories for different values of b^0. (Explicit values of x^0 and x^1 are given in (3.7a).)

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