Hybrid framework for consensus in directed and asynchronous network of non-holonomic agents

T. Borzone*,†, I.-C. Morărescu*, M. Jungers*, M. Boc† and C. Janneteau†

***Abstract*—The paper presents a hybrid systems strategy for consensus (or formation realization) for a fleet of non-holonomic agents. In the proposed model, each agent has a smooth continuous-time dynamics and a piecewise constant impulsive reference. The jumps on the reference trajectory take place at some updating instants that are decided independently by each agent. The jumps computation is based on the relative distance with respect to some time-varying neighbors at the update instants. Between the updates of its reference each robot will track its own constant reference trajectory. Our results provide consensus (formation realization) as far as a minimum dwell-time condition between consecutive updates is satisfied. A numerical example illustrates the theoretical results.**

***Index Terms*—Agents-based systems; Non-holonomic robots; Network analysis and control**

I. Introduction

MULTI-AGENT systems received an increasing attention during the last decades. The most studied problem within this framework is related to consensus which mathematically formulates the fact that multiple systems with local sensing and actions have to collaborate in order to reach a common goal [1]. Among the consensus applications we are interested here in cooperative control of robotic fleets. Many results already exist on this topic but the robots are often considered as single or double integrators that interact continuously [2], [3], [4], [5], [6]. In reality, most mobile robots have non-holonomic dynamics, which are nontrivial to control, as noticed e.g. by [7], [8] and they interact at some discrete instants of time.

The stabilization and control of unicycle non-holonomic dynamics received a lot of attention during the past decades (see [9]). This is partially due to the fact that Brockett's necessary condition [10] for smooth stabilization is not met for this class of vehicles and therefore, no smooth time invariant state-feedback control law exists to stabilize around a pose this type of dynamics. For this reason both discontinuous control laws [11], [12] and time varying [13], [8] control laws have been studied to stabilize the center of rotation and the orientation of a single robot. The trajectory tracking control problem with smooth references has also been considered for non-holonomic dynamics via linearization of the error model [7], [14] or via dynamic feedback linearization [15]. Global exponential tracking of smooth trajectories is also presented in [16].

In this work we present a decentralized control strategy for fleets of non-holonomic robots which cooperate to obtain the emergent behaviour of realizing a formation. Unlike [17], [18], the proposed algorithm requires sporadic interactions when the robots sense other robots in their neighborhood and based on their relative position with respect to the neighbors they update their references. This is an important constraint that renders the proposed algorithm implementable on real devices. Once the reference is computed, the motion of each robot is completely decoupled from the motion of the other robots in the fleet (see Fig. 1). Although decentralized, this strategy results in a hybrid closed-loop dynamics due to the jumping (non-smooth) references that agents have to track.

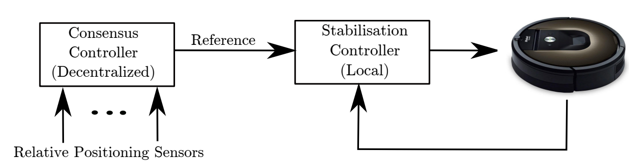

Fig. 1. Control structure

The main contribution of this paper is related to stability analysis of the proposed consensus algorithm that takes into account communication constraints. Since robots evolve continuously but they track a reference which is updated at some discrete instants we end-up with an overall hybrid dynamics. Our results provide a sufficient consensus (formation realization) condition in term of a minimum duration between consecutive updates of the references. It is noteworthy that this work represents an extension of our previous results described in [19]. Unlike [19], the present work considers the more challenging and realistic framework in which the interactions are directed and asynchronous. This generates supplementary difficulties related to the decentralized design of individual time instants at which each agent updates its own trajectory. In other words, the tracking control will be the same, borrowed from [12], but the consensus controller designing the references will be more complex since it has to take into account the asynchronous updates of information.

The rest of the paper is organized as follows. In Section II, we provide some preliminaries related to the network structure and the non-holonomic dynamics under consideration and we introduce the main result for the stability of the overall dynamics. Instrumental results and the proof of the main result

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concerning the stability analysis of the hybrid closed-loop dynamics are proven in Section III. Numerical illustrations are provided in Section IV before providing some concluding remarks.

A. Notation

The following standard notation has been used throughout the paper. The symbols $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$ stand respectively for the set of non-negative integers, the set of real and non-negative real numbers. Given a vector x, we denote by $\|x\|, \|x\|_\infty$ its Euclidean and infinity norms, respectively. The transpose of a matrix A is denoted by $A^\top$. The notation $A > 0$ ($A \geq 0$) is used for a matrix with positive entries; so the symbols $>$ and $<$ are used to represent element-wise inequalities. The $k \times k$ identity matrix is denoted $\mathbf{I}_k$ and $x(t_k^-) = \lim_{t \to t_k, t \leq t_k} x(t)$. The column vector of dimension n with all components equal 1 is denoted $\mathbb{1}_n$. For a vector $x = (x_1, \ldots, x_n)$ we also use $M_x = \max_{1 \geq i \geq n} x_i$, $m_x = \min_{1 \geq i \geq n} x_i$ and $\Delta_x = M_x - m_x$.

II. Problem Formulation

A. Non-holonomic dynamics

In the following we assume that a fleet of n non-holonomic robots have to reach a consensus in the positions without requiring specific final orientation of the agents. To simplify the presentation we remove the argument t when it is not explicitly needed. We denote by $r_i = (r_{x_i}, r_{y_i})$ the 2D reference position for the robot i and we fix $r_{\theta_i} = 0$ its heading reference. The Cartesian coordinates of the center of mass of each vehicle with respect to the fixed inertial frame are denoted using vector $X_i = (x_i, y_i)$. Denoting $e_i = (e_{x_i}, e_{y_i}, e_{\theta_i})^\top$ the dynamics of the i^{th} robot is described by the following differential equations

$$\dot{e}_i = g(e_i)u_i, \quad g(e_i) = \begin{bmatrix} \cos e_{\theta_i} & 0 \\ \sin e_{\theta_i} & 0 \\ 0 & 1 \end{bmatrix}, \quad u_i = \begin{bmatrix} v_i \\ \omega_i \end{bmatrix}. \quad (1)$$

where v_i is the linear velocity and ω_i is the angular velocity of the mobile robot; e_{x_i} and e_{y_i} are the Cartesian coordinates of the center of mass of the vehicle with respect to a frame positioned on the reference position r_i, and e_{θ_i} is the angle between the heading direction and the x-axis of this frame.

The point stabilization control considered in this work is the continuous piecewise smooth control law introduced in [12]. Basically, one considers a map $F : \mathbb{R}^3 \longmapsto \mathbb{R} \times (-\pi, \pi]$ relating $e_i \in \mathbb{R}^3$ to $z_i = (a_i, \alpha_i)^\top \in \mathbb{R} \times (-\pi, \pi]$. Taking $K, \gamma > 0$ the control law $u_i = \kappa(e_i) = (-\gamma b_1(e_i)a, -b_2(e_i)v - Ka)^\top$, where b_1, b_2 are explicitly defined in [12], exponentially stabilizes the origin of the planning reference frame $e_i = 0$.

In the following, we denote $\varepsilon_i = (e_{x_i}, e_{y_i})$ the 2D Cartesian error coordinates *i.e.*

$$\varepsilon_i = X_i - r_i. \quad (2)$$

Following [1], consensus problem is equivalent with the one of translation invariant formation realization. Consequently, we will focus only on the rendez-vous or consensus problem associated with the n non-holonomic robots.

B. Network structure

We assume that the agents interact over a directed and time-varying network topology described by the digraph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$, where the vertex-set $\mathcal{V}$ represents the set of robots and the edge set $\mathcal{E}(t) \subset \mathcal{V} \times \mathcal{V}$ collects the interactions between robots at time t.

Definition 1: A **path of length** p in a digraph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ is a union of directed edges $\bigcup_{k=1}^{p} (i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \ldots, p-1\}$. The node j is **connected** with node i in $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ if there exists at least a path in $\bar{\mathcal{G}}$ from i to j (*i.e.* $i_1 = i$ and $j_p = j$). A **connected digraph** is such that any of its two distinct elements are connected.

We now define the sequence of update instants

$$\mathcal{T} = \left\{ t_k : t_k \in \mathbb{R}^+, t_k < t_{k+1}, \forall k \in \mathbb{N}, \lim_{k \to \infty} t_k = \infty \right\}. \quad (3)$$

Characterizing the vertex i with a generic state $s_i \in \mathbb{R}^{n_s}, n_s \in \mathbb{N}$, a discrete-time linear consensus algorithm is defined by

$$s(t_{k+1}) = (P(t_k) \otimes \mathbf{I}_{n_s}) s(t_k), \quad \text{for } t_k \in \mathcal{T}, \quad (4)$$

where $s(t_k) = (s_1(t_k)^\top, \ldots, s_n(t_k)^\top)^\top$ and $P(t_k) \in \mathbb{R}^{n \times n}$ is a row stochastic matrix associated with the digraph $\mathcal{G}(t_k) = (\mathcal{V}, \mathcal{E}(t_k))$ *i.e.* for all $t_k \in \mathcal{T}$

$$\begin{cases} P_{i,j}(t_k) = 0, \text{ if } (i,j) \notin \mathcal{E}(t_k), \\ P_{i,i}(t_k) > \beta, \ \forall i = \{1, \cdots, N\}, \\ P_{i,j}(t_k) > \alpha, \text{ if } (i,j) \in \mathcal{E}(t_k), \\ \sum_{j=1}^{n} P_{i,j}(t_k) = 1, \ \forall i = \{1, \cdots, n\}. \end{cases} \quad (5)$$

The corresponding individual dynamics of each agent is:

$$s_i(t_{k+1}) = s_i(t_k) + \sum_{j \neq i} P_{i,j}(t_k)(s_j(t_k) - s_i(t_k)), \quad \text{for } t_k \in \mathcal{T}.$$

Throughout the paper we impose the following assumptions.

Assumption 1 (Minimal Influence and Diagonal Dominance): There exist constants $\alpha \in (0,1)$ and $\beta \in \left(\frac{1}{2}, 1\right)$ such that, $\forall t_k \in \mathcal{T}$, $P_{i,i}(t_k) \geq \beta$ and, if $P_{i,j}(t_k) \neq 0$ and thus $(i,j) \in \mathcal{E}(t_k)$ then $P_{i,j}(t_k) > \alpha$.

Assumption 2 (Connectivity): The digraph $\mathcal{G} = \bigcup_{k \geq k_0} \mathcal{G}(t_k)$ is strongly connected for all $k_0 \in \mathbb{N}$.

Assumption 3 (Bounded Intercommunication Interval): If i communicates with j an infinite number of times (that is, if $(i,j) \in \mathcal{E}(t_k)$ infinitely often), then there is some $l \in \mathbb{N}$ such that, for all $t_k \in \mathcal{T}$, $(i,j) \in \mathcal{E}(t_k) \bigcup \mathcal{E}(t_{k+1}) \bigcup \cdots \bigcup \mathcal{E}(t_{k+l-1})$.

Under Assumption 1-3 it is well-known ([20], [21]) that the discrete-time updating rule (4) ensures asymptotic consensus.

C. Main Result

To describe the behavior of the entire fleet it is worth introducing $r = (r_1^\top, \cdots, r_n^\top)^\top$ and $\varepsilon = (\varepsilon_1^\top, \cdots, \varepsilon_n^\top)^\top$, with $r, \varepsilon \in \mathbb{R}^{2n}$ and r_i, ε_i related by (2). The goal of the paper is to show the global asymptotic stability (GAS) of the set $\mathcal{A}$ defined as

$$\mathcal{A} = \left\{ \varepsilon, r \in \mathbb{R}^{2n} : \varepsilon = 0, \ \Delta_{r_x} = 0, \ \Delta_{r_y} = 0 \right\}. \quad (6)$$

The first requirement in (6) is equivalent to state that each robot reaches its own reference and the second and third requirements mean that all the references achieve consensus both over the x and y component.

In the sequel we suppose that each robot implements the following updating rule:

$$r_i(t_k) = X_i(t_k^-) + \sum_{j\neq i} P_{i,j}(t_k)(X_j(t_k^-) - X_i(t_k^-)),\ \forall t_k \in \mathcal{T}_i, \quad (7)$$

where $\mathcal{T}_i$ is the infinite countable subset $\mathcal{T}$ collecting the update instants for the reference r_i. Therefore, mixing (1) and (7), $\forall\ i \in \{1, \ldots, n\}$ we end up with the following hybrid dynamics (see [22] for an exhaustive presentation of hybrid dynamics):

$$\begin{cases} \dot{e}_i(t) = g(e_i(t))\kappa(e_i(t)) \\ \dot{r}_i(t) = 0 \end{cases} \quad \text{for } t \in \mathbb{R}^+ - \mathcal{T}_i, \quad (8)$$

$$\begin{cases} r_i(t_k) = \sum_{j=1}^{n} P_{i,j}(t_k)\varepsilon_j(t_k^-) + \sum_{j=1}^{n} P_{i,j}(t_k) r_j(t_k^-) \\ \varepsilon_i(t_k) = \varepsilon_i(t_k^-) + r_i(t_k^-) - r_i(t_k) \\ e_{\theta_i}(t_k) = e_{\theta_i}(t_k^-) \end{cases} \quad \text{for } t_k \in \mathcal{T}_i. \quad (9)$$

Remark 1: Note that flow dynamics (8) is completely decentralized meaning that each robot tracks its reference and no interaction with other robots is required. Let us also note that the jump map (9) of one robot requires only information from the neighboring (in the interconnection graph) robots.

Theorem 1: Let Assumptions 1-3 hold. If there exist two constants c_ε and λ_ε such that $\forall t \in [t_k, t_{k+1})$

$$\|\varepsilon(t)\|_\infty \leq \sqrt{n} c_\varepsilon \|\varepsilon(t_k)\|_\infty \,\mathrm{e}^{-\lambda_\varepsilon (t-t_k)}, \quad (10)$$

then $\mathcal{A}$ is GAS for the overall dynamics of n systems defined by (8)-(9), if $t_{k+1} - t_k \geq \tau^*,\ \forall k \in \mathbb{N}$ with

$$\tau^* > \frac{1}{\lambda_\varepsilon} \max\left\{ \ln \frac{\sqrt{n}\gamma_{13} c_\varepsilon}{1-\gamma_{11}},\ \ln \frac{\sqrt{n}\gamma_{33} c_\varepsilon}{1-\gamma_{31}-\gamma_{32}} \right\} > 0, \quad (11)$$

for some given γ_{ij} that will be explicitly defined later.

Proof: See the Proof in Section III-C. ■

Note that Theorem 1 does not define the instants of references' updates but it gives a lower bound between these instants.

III. Stability analysis

A. Analysis of the flow dynamics

The flow dynamics is related to the local stabilization of the vehicles with respect to their relative origins given by the reference state *i.e.* $(r_{x_i}(t_k), r_{y_i}(t_k))$. This follows closely the results in [19] and consequently we do not enter into details. We just recall the following instrumental result.

Lemma 1: There exists positive constants c_ε and λ_ε such that $\forall\ k \in \mathbb{N}$ and for $t \in [t_k, t_{k+1}) \subset \mathcal{T}_i \subset \mathcal{T}$

$$\|r_i(t)\| = \|r_i(t_k)\| \quad (12)$$

$$\|\varepsilon_i(t)\| \leq c_\varepsilon \|\varepsilon_i(t_k)\| \,\mathrm{e}^{-\lambda_\varepsilon (t-t_k)} \quad (13)$$

We can now state the following Corollary that is a direct consequence of Lemma 1.

Corollary 1: There exist positive constants c_ε and λ_ε such that $\forall\ k \in \mathbb{N}$ and for $t \in [t_k, t_{k+1})$

$$\begin{aligned} \Delta_{r_x}(t) &= \Delta_{r_x}(t_k), \qquad \Delta_{r_y}(t) = \Delta_{r_y}(t_k) \\ \|\varepsilon(t)\|_\infty &\leq \sqrt{n} c_\varepsilon \|\varepsilon(t_k)\|_\infty \,\mathrm{e}^{-\lambda_\varepsilon (t-t_k)}. \end{aligned} \quad (14)$$

Proof: The first two expressions come trivially from the fact that the reference r is constant for $t \in [t_k, t_{k+1})$ as described in Lemma 1, then the M_{r_x} and m_{r_x} (equivalently M_{r_y} and m_{r_y}) do not change and consequently the diameter Δ_{r_x} (Δ_{r_y}) does not change either.

As for the third statement, starting from inequality (13) in Lemma 1 we extend it to the whole error vector $\|\varepsilon(t)\| \leq c_\varepsilon \|\varepsilon(t_k)\| \,\mathrm{e}^{-\lambda_\varepsilon (t-t_k)}$ for all $t \in [t_k, t_{k+1})$ and thanks to the norms inequality $\|\varepsilon\|_\infty \leq \|\varepsilon\| \leq \sqrt{n} \|\varepsilon\|_\infty$ we obtain the final expression in (1). ■

Corollary 1 basically states that, as far as the reference is fixed, one can design a decentralized controller that exponentially stabilizes system (8) for all $i \in \{1, \ldots, n\}$. Furthermore the constants c_ε and λ_ε correspond to those in Theorem 1 and they are directly related to the control gains and parameters K, γ and b_1, b_2 (see also [19] and [12]).

B. Analysis of the jump map

We start introducing the vectors $\varepsilon_x = (\varepsilon_{x_1}, \cdots, \varepsilon_{x_n})^\top$ and $\varepsilon_y = (\varepsilon_{y_1}, \cdots, \varepsilon_{y_n})^\top$, with $\varepsilon_x, \varepsilon_y \in \mathbb{R}^n$ which collect the x and y components of the errors.

In the previous subsection, we have shown that for $t \in [t_k, t_{k+1})$ the Cartesian positioning error of the vehicles converges toward $\varepsilon = \mathbf{0}$ but nothing can be said about the reference r which is kept constant during the flow. In order to achieve the global asymptotic stability of $\mathcal{A}$ defined in (6), let us investigate the behavior of system (9). First, let us notice that e_{θ_i} does not change during the jumps defined by (9) and therefore we can neglect this variable in the subsequent analysis. Moreover, by rewriting all the dynamics in (9) one obtains for $t_k \in \mathcal{T}$

$$r_i(t_k) = \sum_{j=1}^{n} P_{i,j}(t_k)\varepsilon_j(t_k^-) + \sum_{j=1}^{n} P_{i,j}(t_k) r_j(t_k^-) \quad (15)$$

$$\begin{aligned} \varepsilon_i(t_k) = \varepsilon_i(t_k^-) &- \sum_{j=1}^{n} P_{i,j}(t_k)\varepsilon_j(t_k^-) + r_i(t_k^-) \\ &- \sum_{j=1}^{n} P_{i,j}(t_k) r_j(t_k^-). \end{aligned} \quad (16)$$

We start with an instrumental result concerning equation (15).

Lemma 2: Let Assumptions 1-3 hold. For all $(i,h) \in \mathcal{E}(t_k)$ and for all $t_k \in \mathcal{T}_i$ the following holds

$$\begin{aligned} r_i(t_k) \geq & m_r(t_k^-) + \alpha\left(r_h(t_k^-) - m_r(t_k^-)\right) \\ & + m_\varepsilon(t_k^-) + \alpha\left(\varepsilon_h(t_k^-) - m_\varepsilon(t_k^-)\right) \end{aligned} \quad (17)$$

and

$$\begin{aligned} r_i(t_k) \leq & M_r(t_k^-) - \alpha\left(M_r(t_k^-) - r_h(t_k^-)\right) \\ & + M_\varepsilon(t_k^-) - \alpha\left(M_\varepsilon(t_k^-) - \varepsilon_h(t_k^-)\right), \end{aligned} \quad (18)$$

where $M_r = (M_{r_x}, M_{r_y})^\top$, $M_\varepsilon = (M_{\varepsilon_x}, M_{\varepsilon_y})^\top$, $m_r = (m_{r_x}, m_{r_y})^\top$ and $m_\varepsilon = (m_{\varepsilon_x}, m_{\varepsilon_y})^\top$.

Proof: Since $P(t_k)$ is stochastic one has $\sum_{j=1,j\neq h}^{n} P_{i,j}(t_k) + P_{i,h}(t_k) = 1$, thus from (15) one obtains

$$r_i(t_k) - m_r(t_k^-) - m_\varepsilon(t_k^-) \geq \sum_{j=1,j\neq h}^{n} P_{i,j}(t_k)\left(r_j(t_k^-) - m_r(t_k^-)\right) + P_{i,h}(t_k)\left(r_h(t_k^-) - m_r(t_k^-)\right) + \sum_{j=1,j\neq h}^{n} P_{i,j}(t_k)\left(\varepsilon_j(t_k^-) - m_\varepsilon(t_k^-)\right) + P_{i,h}(t_k)\left(\varepsilon_h(t_k^-) - m_\varepsilon(t_k^-)\right). \quad (19)$$

From Assumption 1 both $\sum_{j=1,j\neq h}^{n} P_{i,j}(t_k)\left(r_j(t_k^-) - m_r(t_k^-)\right)$ and $\sum_{j=1,j\neq h}^{n} P_{i,j}(t_k)\left(\varepsilon_j(t_k^-) - m_\varepsilon(t_k^-)\right)$ are always positive quantities and $P_{i,j}(t_k) > \alpha$ then, (17) holds. Inequality (18) is proved by a symmetrical argument. ∎

We shall now use the previous result in order to show the boundedness of the reference diameters Δ_{r_x} and Δ_{r_y} during the jumps. They will depend on the norm of the Cartesian error ε as pointed out below.

Proposition 1: Under Assumptions 1-3, for all $t_k \in \mathcal{T}$ one has

$$\begin{aligned}\Delta_{r_x}(t_k) \leq& (1-\alpha)\Delta_{r_x}(t_k^-) + 2(1-\alpha)\left\|\varepsilon(t_k^-)\right\|_\infty, \\ \Delta_{r_y}(t_k) \leq& (1-\alpha)\Delta_{r_y}(t_k^-) + 2(1-\alpha)\left\|\varepsilon(t_k^-)\right\|_\infty.\end{aligned} \quad (20)$$

Proof: We first notice that since Lemma 2 is true for all $i \in \mathcal{V}$, then (17) is a valid lower bound also for $m_r(t_k)$ itself and equivalently (18) is an upper bound for $M_r(t_k)$. We can then write

$$\begin{aligned}M_r(t_k) \leq& M_r(t_k^-) - \alpha\left(M_r(t_k^-) - r_h(t_k^-)\right) \\ &+ M_\varepsilon(t_k^-) - \alpha\left(M_\varepsilon(t_k^-) - \varepsilon_h(t_k^-)\right), \\ m_r(t_k) \geq& m_r(t_k^-) + \alpha\left(r_h(t_k^-) - m_r(t_k^-)\right) \\ &+ m_\varepsilon(t_k^-) + \alpha\left(\varepsilon_h(t_k^-) - m_\varepsilon(t_k^-)\right).\end{aligned} \quad (21)$$

Subtracting these two expressions and manipulating the result we then obtain

$$\begin{aligned}M_r(t_k) - m_r(t_k) \leq& (1-\alpha)\left(M_r(t_k^-) - m_r(t_k^-)\right) \\ &+ (1-\alpha)\left(M_\varepsilon(t_k^-) - m_\varepsilon(t_k^-)\right).\end{aligned} \quad (22)$$

We now express (22) only for the x component (for the y component the argument will be equivalent) and thus introduce $\Delta_{r_x} = M_{r_x} - m_{r_x}$

$$\Delta_{r_x}(t_k) \leq (1-\alpha)\Delta_{r_x}(t_k^-) + (1-\alpha)\left(M_{\varepsilon_x}(t_k^-) - m_{\varepsilon_x}(t_k^-)\right). \quad (23)$$

Using the modulus and noticing that $|M_{\varepsilon_x}| = \|\varepsilon_x\|_\infty$ we finally write

$$\begin{aligned}\Delta_{r_x}(t_k) \leq& (1-\alpha)\Delta_{r_x}(t_k^-) + (1-\alpha)\left|M_{\varepsilon_x}(t_k^-) - m_{\varepsilon_x}(t_k^-)\right| \\ \leq& (1-\alpha)\Delta_{r_x}(t_k^-) + (1-\alpha)\left(\left|M_{\varepsilon_x}(t_k^-)\right| + \left|m_{\varepsilon_x}(t_k^-)\right|\right) \\ \leq& (1-\alpha)\Delta_{r_x}(t_k^-) + (1-\alpha)\left(\left|M_{\varepsilon_x}(t_k^-)\right| + \left|M_{\varepsilon_x}(t_k^-)\right|\right) \\ \leq& (1-\alpha)\Delta_{r_x}(t_k^-) + 2(1-\alpha)\left\|\varepsilon(t_k^-)\right\|_\infty.\end{aligned} \quad (24)$$

where we also used the property that $\|\varepsilon_x\|_\infty \leq \|\varepsilon\|_\infty$ since ε_x collects the x components of vector ε (the same holds for the y components). ∎

As it has been done for the reference diameters we analyze the update law (16) to show the boundedness of the error ε during the jump instants with the following result

Proposition 2: Under Assumptions 1-3, for all $t_k \in \mathcal{T}$ one has

$$\|\varepsilon(t_k)\|_\infty \leq (1-\beta)\left(\Delta_{r_x}(t_k^-) + \Delta_{r_y}(t_k^-) + 4\left\|\varepsilon(t_k^-)\right\|_\infty\right). \quad (25)$$

With β chosen as described in Assumption 1.

Proof: We shall decompose the update law (16) in

$$\begin{aligned}\varepsilon_i(t_k) =& (1 - P_{i,i}(t_k))\, r_i(t_k^-) - \sum_{j=1,j\neq i}^{n} P_{i,j}(t_k) r_j(t_k^-) \\ &+ (1 - P_{i,i}(t_k))\, \varepsilon_i(t_k^-) - \sum_{j=1,j\neq i}^{n} P_{i,j}(t_k)\varepsilon_j(t_k^-).\end{aligned} \quad (26)$$

We use the fact that thanks to Assumption 1 the quantity $\sum_{j=1,j\neq i}^{n} P_{i,j}(t_k)$ is positive and $P_{i,i}(t_k) \geq \beta$ together with the stochasticity of the matrix $P(t_k)$ to write

$$\begin{aligned}\varepsilon_i(t_k) \leq& (1 - P_{i,i}(t_k))\, M_r(t_k^-) - \sum_{j=1,j\neq i}^{n} P_{i,j}(t_k) M_r(t_k^-) \\ &+ (1 - P_{i,i}(t_k))\, \varepsilon_i(t_k^-) - \sum_{j=1,j\neq i}^{n} P_{i,j}(t_k) m_\varepsilon(t_k^-) \\ \leq& (1 - P_{i,i}(t_k))\, M_r(t_k^-) - (1 - P_{i,i}(t_k))\, m_r(t_k^-) \\ &+ (1 - P_{i,i}(t_k))\, \varepsilon_i(t_k^-) - (1 - P_{i,i}(t_k))\, m_\varepsilon(t_k^-) \\ \leq& (1-\beta)\left(M_r(t_k^-) - m_r(t_k^-) + \varepsilon_i(t_k^-) - m_\varepsilon(t_k^-)\right).\end{aligned} \quad (27)$$

Inequality (27) can be written with respect to x and y components. Thus, one has

$$\varepsilon_{x_i}(t_k) \leq (1-\beta)\,\Delta_{r_x} + (1-\beta)\left(\varepsilon_{x_i}(t_k^-) - m_{\varepsilon_x}(t_k^-)\right). \quad (28)$$

Since the previous inequality holds $\forall i \in \mathcal{V}$ it holds also for the maximum element M_{ε_x}

$$M_{\varepsilon_x}(t_k) \leq (1-\beta)\,\Delta_{r_x} + (1-\beta)\left(M_{\varepsilon_x}(t_k^-) - m_{\varepsilon_x}(t_k^-)\right). \quad (29)$$

Passing to modulus and recalling the fact that $|M_{\varepsilon_x}| = \|\varepsilon_x\|_\infty$ we can write

$$\begin{aligned}\|\varepsilon_x(t_k)\|_\infty \leq& (1-\beta)\,\Delta_{r_x} + (1-\beta)\left|M_{\varepsilon_x}(t_k^-) - m_{\varepsilon_x}(t_k^-)\right| \\ \leq& (1-\beta)\,\Delta_{r_x} + (1-\beta)\left(\left|M_{\varepsilon_x}(t_k^-)\right| + \left|m_{\varepsilon_x}(t_k^-)\right|\right) \\ \leq& (1-\beta)\,\Delta_{r_x} + (1-\beta)\left(\left|M_{\varepsilon_x}(t_k^-)\right| + \left|M_{\varepsilon_x}(t_k^-)\right|\right) \\ \leq& (1-\beta)\,\Delta_{r_x} + 2(1-\beta)\left\|\varepsilon(t_k^-)\right\|_\infty\end{aligned} \quad (30)$$

Following the same argument we write the bound for the y components too

$$\left\|\varepsilon_y(t_k)\right\|_\infty \leq (1-\beta)\,\Delta_{r_y} + 2(1-\beta)\left\|\varepsilon(t_k^-)\right\|_\infty. \quad (31)$$

Finally we put all together with the following inequality

$$\|\varepsilon(t_k)\|_\infty \leq \|\varepsilon_x(t_k)\|_\infty + \left\|\varepsilon_y(t_k)\right\|_\infty.$$

to get (25). ∎

Under Assumption 1 the following quantities are defined:

$$\begin{aligned}\gamma_{11} = \gamma_{22} = (1-\alpha), \quad \gamma_{31} = \gamma_{32} = (1-\beta), \\ \gamma_{13} = \gamma_{23} = 2(1-\alpha), \quad \gamma_{33} = 4(1-\beta).\end{aligned} \quad (32)$$

We use those constants to condense the results expressed in Proposition 1 and 2 in the following lemma.

Lemma 3: Under Assumption 1, $\forall\, t_k \in \mathcal{T}$, one has that:

$$\begin{aligned} \Delta_{r_x}(t_k) &\le \gamma_{11}\Delta_{r_x}(t_k^-) + \gamma_{13}\left\|\varepsilon(t_k^-)\right\|_\infty, \\ \Delta_{r_y}(t_k) &\le \gamma_{22}\Delta_{r_y}(t_k^-) + \gamma_{23}\left\|\varepsilon(t_k^-)\right\|_\infty, \\ \|\varepsilon(t_k)\|_\infty &\le \gamma_{31}\Delta_{r_x}(t_k^-) + \gamma_{32}\Delta_{r_y}(t_k^-) + \gamma_{33}\left\|\varepsilon(t_k^-)\right\|_\infty. \end{aligned} \tag{33}$$

Proof: The proof is straightforward from inequalities (20) and (25) and from the definition of γ_{ij} constants in (32). ■

C. Overall hybrid system stability analysis

In the sequel we consider the following matrices

$$\Gamma = \begin{pmatrix} \gamma_{13} & 0 & \gamma_{13} \\ 0 & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{pmatrix}, \quad M_\tau = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sqrt{n}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon \tau} \end{pmatrix}.$$

We emphasize that the behavior of $\Delta_{r_x}(t)$, $\Delta_{r_y}(t)$ and $\|\varepsilon(t)\|_\infty$ is characterized within $[t_k,\ t_{k+1})$ by Corollary (1) in term of the matrix $M_{t_{k+1}-t_k}$. On the other hand Lemma 3 gives an upper-bound, in term of Γ, on the jumps that $\|\varepsilon(t)\|_\infty$ and the two reference diameters suffer at time $t_k \in \mathcal{T}$.

Lemma 4: Under Assumption 1, let $\tau^* \ge 0$ such that the positive matrix ΓM_{τ^*} is Schur. Then, for all sequences $\mathcal{T} = (t_k)_{k\ge0}$ of jump times satisfying the dwell-time property $t_{k+1} - t_k \ge \tau^*$, $\forall k \in \mathbb{N}$, $\mathcal{A}$ is GAS with respect to dynamics (8)-(9).

Proof: From Corollary 1 and Lemma 3, it follows that for all $k \in \mathbb{N}$,

$$\begin{pmatrix} \Delta_{r_x}(t_k) \\ \Delta_{r_y}(t_k) \\ \|\varepsilon(t_k)\|_\infty \end{pmatrix} \le \Gamma M_{\tau_{k-1}} \dots \Gamma M_{\tau_0} \begin{pmatrix} \Delta_{r_x}(t_0) \\ \Delta_{r_y}(t_0) \\ \|\varepsilon(t_0)\|_\infty \end{pmatrix},$$

where $\tau_k = t_{k+1} - t_k$, $\forall k \in \mathbb{N}$. Remarking that the coefficients of the positive matrix M_τ are non-increasing with respect to τ, it follows that

$$\begin{pmatrix} \Delta_{r_x}(t_k) \\ \Delta_{r_y}(t_k) \\ \|\varepsilon(t_k)\|_\infty \end{pmatrix} \le (\Gamma M_{\tau^*})^k \begin{pmatrix} \Delta_{r_x}(t_0) \\ \Delta_{r_y}(t_0) \\ \|\varepsilon(t_0)\|_\infty \end{pmatrix},$$

Hence, if the positive matrix ΓM_{τ^*} is Schur, then the sequences $(\Delta_{r_x}(t_k))_{k\ge0}$, $(\Delta_{r_y}(t_k))_{k\ge0}$ and $(\|\varepsilon(t_k)\|_\infty)_{k\ge0}$ go to 0, and the system (8)-(9) converges to some point in $\mathcal{A}$. ■

Hence, the stability of $\mathcal{A}$ with respect to the overall hybrid dynamics of the fleet of robots can be investigated by studying the spectral properties of the positive matrix ΓM_{τ^*}. Let us remark that values τ^* such that ΓM_{τ^*} is Schur provide upper bounds on the minimal dwell-time between two events that ensures $\mathcal{A}$ is GAS. In the following we provide the proof of our main result previously stated as Theorem 1, establishing sufficient conditions for deriving such values τ^*.

At this point, it is interesting to emphasize that we have transformed the problem of stability analysis of the overall hybrid system in a problem of stabilization of a positive system.

Proof of Theorem 1: First of all let us notice that the assumption expressed in inequality (10) is perfectly fulfilled thanks to the exponential stability of the origin $\varepsilon = 0$ showed in Corollary 1. Now let us remark that

$$\Gamma M_{\tau^*} = \begin{pmatrix} \gamma_{11} & 0 & \gamma_{13}\sqrt{n}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} \\ 0 & \gamma_{22} & \gamma_{23}\sqrt{n}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} \\ \gamma_{31} & \gamma_{32} & \gamma_{33}\sqrt{n}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} \end{pmatrix}.$$

Moreover, the positive matrix ΓM_{τ^*} is Schur if and only if there exists $z \in \mathbb{R}^3_+$, such that $(\Gamma M_{\tau^*})z < z$ (see e.g. [23]). Choosing $z = \mathbb{1}_3$ one obtains that $(\Gamma M_{\tau^*})z < z$ is equivalent with

$$\begin{cases} \gamma_{11} + \sqrt{n}\gamma_{13}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} < 1 \\ \gamma_{22} + \sqrt{n}\gamma_{23}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} < 1 \\ \gamma_{31} + \gamma_{32} + \sqrt{n}\gamma_{33}c_\varepsilon \mathrm{e}^{-\lambda_\varepsilon\tau^*} < 1 \end{cases} \tag{34}$$

From (32) we notice that $\gamma_{11} = \gamma_{22}$ and $\gamma_{13} = \gamma_{23}$ thus one of the rows will be neglected from the rest of the analysis since they are equivalent (the sufficient dwell-time τ^* for the convergence of Δ_{r_x} will be the same as for Δ_{r_y}).

We pick for instance the first and the third inequality. The first inequality in (34) has the solution

$$\tau^* > \frac{1}{\lambda_\varepsilon}\ln\frac{\sqrt{n}\gamma_{13}c_\varepsilon}{1-\gamma_{11}}$$

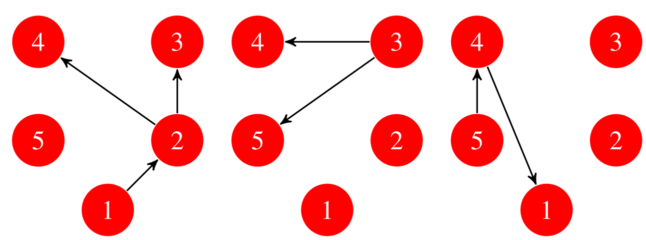
while the second one is solved by

$$\tau^* > \frac{1}{\lambda_\varepsilon}\ln\frac{\sqrt{n}\gamma_{33}c_\varepsilon}{1-\gamma_{31}-\gamma_{32}}.$$

Combining the two conditions above one obtains the first inequality in (11). We still have to prove that $\gamma_{11} < 1$ or $\gamma_{31} + \gamma_{32} < 1$. From Assumption 1 one has $\alpha \in (0,1)$ and thus $\gamma_{11} = 1 - \alpha < 1$. As for the second inequality we use the diagonal dominance in Assumption 3 to show that $\gamma_{31} + \gamma_{32} = 2(1-\beta) < 1$ is verified for all $\beta \in (\frac{1}{2}, 1)$. ■

IV. Numerical Example

In this section we consider a set of 5 robots that have to realize the formation specified by the set of positions $\Pi = ((2,0), (3.90, 1.38), (3.18, 3.62), (0.82, 3.62), (0.10, 1.38))^\top$. The interaction between the agents switches randomly between the ones described by the three graphs represented in Fig. 2 following a switch function $\sigma : \mathcal{T} \to \{1,2,3\}$. The three digraphs topology have been chosen such that the union digraph $\mathcal{G} = \mathcal{G}_1 \bigcup \mathcal{G}_2 \bigcup \mathcal{G}_3$ is strongly connected, in order to comply with Assumption 2.



(a) $\sigma(t_k) = 1$ (b) $\sigma(t_k) = 2$ (c) $\sigma(t_k) = 3$

Fig. 2. Directed graphs used to implement the communication.

To each digraph corresponds a different Laplacian matrix $L_{\sigma(t_k)}$ obtained in the classical way as the difference between the degree matrix and the adjacency matrix of the digraph $L_{\sigma(t_k)} = D_{\sigma(t_k)} - A_{\sigma(t_k)}$. The discrete interaction matrix of the global network $P(t_k) = P_{\sigma(t_k)} = \mathbf{I}_5 - \epsilon L_{\sigma(t_k)}$, where ϵ is a gain chosen as 0.1 to comply with Assumption 1 with $\alpha = 0.1$ and $\beta = 0.8$. Nevertheless, in general it is always possible to find interaction matrices $P(t_k)$ which fulfill the requirements of Assumption 1 picking $\beta \in \left(\frac{1}{2}, 1\right)$ and, for instance, $\alpha = \frac{1-\beta}{1-n}$.

A dwell-time $\tau^* = 4.59$ s has been used as lower bound for the intervals between a reset and the following, where the value of τ^* has been evaluated using expression (11).

The initial positions of the 5 agents are $X_1(0) = (-6, -1)$, $X_2(0) = (-9, -4)$, $X_3(0) = (-6, 3.5)$, $X_4(0) = (0, -2)$ and $X_5(0) = (-11, 5)$ where the coordinates are expressed in meters; the initial heading angle are $\theta_1(0) = -1.5$, $\theta_2(0) = -3$, $\theta_3(0) = -0.9$, $\theta_4(0) = -0.3$ and $\theta_5(0) = 1.5$, all expressed in radians. The values of k and γ for the point stabilization controller of each robot are respectively 0.8 and 1. In Fig. 3 we plot the trajectories of the robots, the sequence of reference positions r_i emphasizing in green the final positions of the agents that realize the formation defined by Π.

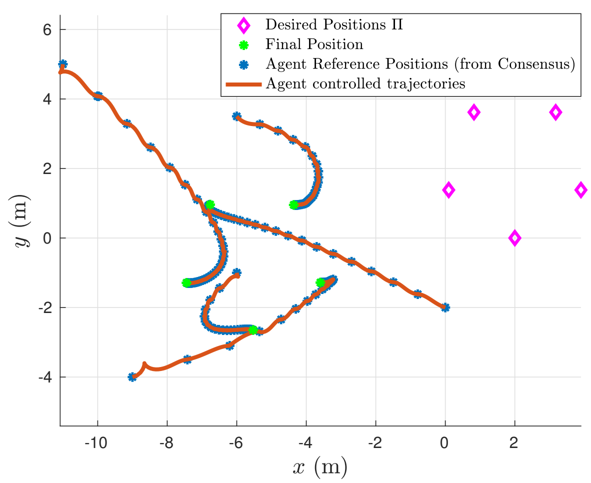


Fig. 3. Robots reaching the pentagon formation.

V. Conclusions

The paper proposed and analyzed a decentralized consensus/formation realization strategy for a fleet of non-holonomic robots. The strategy under consideration requires sporadic interactions between robots. These interactions can be directed and asynchronous. At the interaction instants the robots update their reference based on some relative inter-distance measurements easy to obtained from onboard sensors. The resulting closed-loop dynamics is hybrid and our sufficient stability condition is formulated in term of a minimum dwell-time condition. A numerical example illustrates the theoretical development.

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