A hybrid model of opinion dynamics with memory-based connectivity

S. Mariano, I.C. Morărescu, R. Postoyan, L. Zaccarian

***Abstract*— Given a social network where the individuals know the identity of the other members, we present a model of opinion dynamics where the connectivity among the individuals depends on both their current and past opinions. Thus, their interactions are not only based on the present states but also on their past relationships. The model is a multi-agent system with active or inactive pairwise interactions depending on auxiliary state variables filtering the instantaneous opinions, thereby taking the past experience into account. When an interaction is (de)activated, a jump occurs, leading to a hybrid model. The proven stability properties ensure that opinions converge to local agreements/clusters as time grows. Simulation results are provided to illustrate the theoretical guarantees.**

I. Introduction

Motivated by the growing importance of digital social networks, opinion dynamics has received an increasing attention from the control community e.g., [1], [3], [17], [19], [21]. The multi-agent systems formalism is well-suited for modelling these networks, as a node can model the individual's opinion and an edge describes the interaction between two given individuals e.g., [4], [7], [13], [18].

Two main models provide convergence towards local agreement or disagreement patterns. One of them (FJ) [12] essentially filters the consensus dynamics by using the initial opinions of the agents. The idea is that, although individuals influence each other, a major role in the opinion update is played by their culture, belonging to a community (social class, political party, etc), principles and beliefs, as captured by the initial condition of each individual. The second one is the bounded confidence model (HK) described in [16], which formalizes the idea that only individuals with similar opinions actually interact.

Social psychologists agree that both the FJ and HK models are relevant, depending on the context, see [9] for a detailed survey. Nevertheless, as pointed out in [9], opinion dynamics in social networks is a complex phenomenon, whose key features cannot be completely captured by any of these models separately. This has motivated the development of other deterministic models e.g., [1], [4], [10], [19], [21], as well as stochastic models e.g., [6], [18], [25]. It is noteworthy that most of the aforementioned works provide either empirical or rigorous convergence results but stability is eluded in general. The recent work in [11] provides a Lyapunov analysis of the HK model and reveals that only an attractivity property can be guaranteed. This motivated developing in [11] a variant of the HK model where the connectivity depends on adaptive thresholds, instead of fixed ones as in [16]. In this way, a link is activated when the opinions mismatch between two given agents is small compared to the average opinion mismatch with their other neighbours. This strategy ensures stability of the emergent clusters.

This paper is a further step in the direction paved by [11]: we propose a model where connectivity also depends on the past history. With this, we merge the features of the FJ model (where importance is given to the past, the initial conditions) and those of the HK model (the bounded confidence mechanism based on current opinions). It is indeed reasonable to assume that the interactions within a social network do not only depend on the current state but also on the past relationships, when the members are aware of the identity of their neighbours. To this end, we account for the past by linearly filtering the instantaneous (de)activation functions (de)activating a link only when both the adaptive threshold *and* its filtered version reach certain thresholds. Our model uses the hybrid formalism of [15]. A new Lyapunov function is constructed, which guarantees a suitable $\mathcal{KL}$-stability property ensuring asymptotic convergence to opinion clusters. In addition, solutions are proved not to generate Zeno phenomenon and to stop jumping in finite-time. Simulation results illustrate the behaviour of the model and the impact of the filters and their parameters.

The technical proofs of this work emerge from some interesting analogy between the adaptive threshold connectivity as in [11] and the event-triggered control technique of [23]. These two domains - a priori unrelated - have actually much in common in terms of modelling and analysis tools. As a result, the memory-based connectivity proposed in this paper is inspired by the dynamic event-triggering control policy proposed in [14].

Background and problem statement are given in Section II. The main results, including the new hybrid model and its stability analysis, are presented in Section III. Illustrative simulations results are reported in Section IV and Section V concludes the paper.

Notation. $\mathbb{R}$ represents the real numbers, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{R}_{>0} := (0, \infty)$. $|x|$ is the Euclidean norm of vector $x \in \mathbb{R}^n$. Moreover, (x, y) stands for $[x^\top\, y^\top]^\top$. A continuous function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \longrightarrow \mathbb{R}_{\geq 0}$ is of class-$\mathcal{KL}$ ($\beta \in \mathcal{KL}$), if it is non-decreasing in its first argument, non-increasing in its second argument, $\lim_{r \to 0^+} \beta(r, s) = 0$ for each $s \in \mathbb{R}_{\geq 0}$, and $\lim_{s \to \infty} \beta(r, s) = 0$ for each $r \in \mathbb{R}_{\geq 0}$. $U^\circ(x; v) := \limsup_{h \to 0^+,\, y \to x} (U(y + hv) - U(y))/h$ denotes the Clarke generalized directional derivative at x in the direction v of a

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Lipschitz function U [5].

II. Background and problem statement

Following [11], we consider a set of individuals $\mathcal{V} := \{1, \ldots, n\}$, also referred to as agents, connected through a social network. The opinions of individuals are modelled by a scalar variable $y_i \in \mathbb{R}$ for any $i \in \mathcal{V}$. The dynamics of opinion y_i, $i \in \mathcal{V}$, depends on the interactions of individual i with its neighbours. We define

$$\mathcal{E}^+ := \{(i,j) \in \mathcal{V} \times \mathcal{V} \mid i < j\} \tag{1}$$

and, for each $(i,j) \in \mathcal{E}^+$, variable a_{ij} defines whether agents i and j interact or not, i.e. whether or not they are neighbours. Thus, a_{ij} is the *connectivity variable* for link (i,j), satisfying

$$a_{ij} = a_{ji} := \begin{cases} 1 & \text{if } i \text{ and } j \text{ interact} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

The graph is undirected as $a_{ij} = a_{ji}$. Interaction changes, if any, are described by a jump of the variable a_{ij}. The corresponding hybrid behavior is well represented with the formalism of [15]. Variables y_i and a_{ij} obey the next continuous-time dynamics between two successive jumps

$$\dot{y}_i = \textstyle\sum_{j=1}^n \varphi_{ij}(y_j - y_i), \qquad \forall i \in \mathcal{V}, \tag{3}$$

$$\dot{a}_{ij} = 0, \qquad \forall (i,j) \in \mathcal{E}^+, \tag{4}$$

where $d_i := 1 + \sum_{j \neq i} a_{ij} \geq 1$ is the degree of agent i augmented by 1, $\varphi_{ij} := \dfrac{a_{ij}}{d_i d_j}$ when $i \neq j$ and $\varphi_{ii} := -\sum_{j \neq i} \varphi_{ij}$. We omit the dependence of d_i and φ_{ij} on the connectivity variables. By construction we have that $\varphi_{ij} = \varphi_{ji}$ and $\sum_{j=1}^n \varphi_{ij} = 0$ for any $i \in \mathcal{V}$. The variable φ_{ij} is such that $\Phi := [-\varphi_{ij}]_{(i,j) \in \mathcal{V}^2}$ defines a normalized Laplacian matrix. Dynamics (4) means that a_{ij} is constant between jumps (along flowing solutions) and that the time-derivative of y_i is given by the weighted average of the opinion mismatch between agent i and its neighbours.

When a jump occurs over the network, i.e. when one of the variables a_{ij} for some $(i,j) \in \mathcal{E}^+$ is updated, solutions obey the following discrete dynamics,

$$\begin{aligned} y_h^+ &= y_h, & \forall h \in \mathcal{V} \\ a_{hk}^+ &= \begin{cases} a_{hk} & \text{if } (h,k) \neq (i,j) \\ 1 - a_{hk} & \text{if } (h,k) = (i,j), \end{cases} & \forall (h,k) \in \mathcal{E}^+ \end{aligned} \tag{5}$$

Dynamics (5) states that the opinions y_i do not change across jumps and that the connectivity variable a_{ij} toggles between 0 and 1 according to (de)activation. It simplifies notation to write the second equation of system (5) as

$$a^+ \;=\; g_{ij}(y,a), \tag{6}$$

where $y := (y_1, \ldots, y_n) \in \mathbb{R}^n$ is the opinions vector, and $a := (a_{12}, \ldots, a_{1n}, a_{23}, \ldots, a_{n-2,n}, a_{n-1,n}) \in \{0,1\}^{\frac{n(n-1)}{2}}$ is the connectivity variables vector.

To complete the model, we present the *memoryless* (de)activation criterion (jump dynamics) for each link between two agents, as proposed in [11, §4]. The adaptive thresholds idea of [11] is that two agents interact when their opinions are close relative to their respective neighbours' opinions (an alternative to the fixed threshold HK model [16]). Roughly speaking, given $(i,j) \in \mathcal{E}^+$:

- *Deactivation.* If $a_{ij} = 1$, link (i,j) is active. Deactivation is then enabled when $\Gamma_{ij}^{\text{off}}(y,a) \leq -\varepsilon$, where $\varepsilon > 0$ is a regularization parameter and $\eta > 0$ is a connectivity parameter, while Γ_{ij}^{off} is defined in (7) at the top of the next page. This means that link (i,j) is cut when y_i and y_j are too far apart, as compared to other neighbours' opinions. Parameter[1] $\varepsilon > 0$ rules out Zeno solutions, i.e. solutions that jump indefinitely in a finite continuous time interval. It is typically set to a small value.
- *Activation.* If $a_{ij} = 0$, link (i,j) is not active. Activation is enabled when $\Gamma_{ij}^{\text{on}}(y,a) \geq \varepsilon$ with Γ_{ij}^{on} defined in (7). The underlying idea is that link (i,j) should be activated when the difference between opinions i and j, namely $|y_i - y_j|$ is small as compared to the average opinion mismatch of agents i and j with their respective neighbours (individuals with relatively close opinions influence each other).

Parameter η determines how big the mismatch $|y_i - y_j|$ needs to be with respect to the average opinions mismatch of agents i and j with their neighbours to (de)activate the link.

As a result, the overall hybrid model is given by

$$\begin{aligned} \begin{bmatrix} \dot{y} \\ \dot{a} \end{bmatrix} &= \begin{bmatrix} -\Phi y \\ 0 \end{bmatrix}, & (y,a) \in C_{\text{inst}} \\ \begin{bmatrix} y^+ \\ a^+ \end{bmatrix} &\in \left[\bigcup_{\substack{(y,a) \in D_{ij,\text{inst}} \\ (i,j) \in \mathcal{E}^+}} \begin{matrix} y \\ g_{ij}(y,a) \end{matrix} \right], & (y,a) \in D_{\text{inst}}, \end{aligned} \tag{8a}$$

where we recall that $\Phi = [-\varphi_{ij}]_{(i,j) \in \mathcal{V}^2}$ and $\mathcal{E}^+$ is in (1), and $\mathbb{X}_{\text{inst}} := \mathbb{R}^n \times \{0,1\}^{\frac{n(n-1)}{2}}$, and

$$\begin{aligned} D_{ij,\text{inst}}^{\text{on}} &:= \left\{ (y,a) \in \mathbb{X}_{\text{inst}} \mid a_{ij} = 0,\ \Gamma_{ij}^{\text{on}}(y,a) \geq \varepsilon \right\} \\ D_{ij,\text{inst}}^{\text{off}} &:= \left\{ (y,a) \in \mathbb{X}_{\text{inst}} \mid a_{ij} = 1,\ \Gamma_{ij}^{\text{off}}(y,a) \leq -\varepsilon \right\}, \end{aligned} \tag{8b}$$

$D_{\text{inst}} := \bigcup_{(i,j) \in \mathcal{E}^+} D_{ij,\text{inst}}^{\text{on}} \cup D_{ij,\text{inst}}^{\text{off}}$, and $C_{\text{inst}} := \overline{\mathbb{X} \setminus D_{\text{inst}}}$. The main stability result of [11] is to prove that all maximal solutions to (7), (8) are complete and eventually continuous (i.e, they perform a finite number of jumps) and all enjoy a desirable global asymptotic stability property for the following set $\mathcal{A}_{\text{inst}}$, measured by the following function ω_0,

$$\begin{aligned} \mathcal{A}_{\text{inst}} &:= \{(y,a) \in \mathbb{X} \mid a_{ij}(y_i - y_j)^2 = 0,\ \forall (i,j) \in \mathcal{E}^+\}, \\ \omega_0(y,a) &:= \min_{(z,a) \in \mathcal{A}_{\text{inst}}} |y - z|. \end{aligned} \tag{9}$$

Since ω_0 is not a Euclidean norm in the extended (y,a) space (because a is fixed when defining ω_0 in (9)), we deem it more appropriate to use in this paper the following notion of $\mathcal{KL}$-stability, which combines the approach in [24] with the $\mathcal{KL}$ results in [2, §3.5].

Definition 1: Let $\omega : \mathbb{R}^{n_q} \to \mathbb{R}_{\geq 0}$ be continuous. A hybrid system is *$\mathcal{KL}$-stable with respect to ω* if there exists

[1] Constant ε is the same for every link of the network in [11], however the results do hold mutatis mutandis when it is link dependent, i.e. when we have different $\varepsilon_{ij} > 0$ for each $(i,j) \in \mathcal{V}^2$.

$$\begin{aligned}\Gamma_{ij}^{\text{on}}(y,a) &:= \sum_{\ell\neq i,\ \ell\neq j}\Big[(d_j+1)\varphi_{i\ell}(y_i-y_\ell)^2+(d_i+1)\varphi_{j\ell}(y_j-y_\ell)^2\Big]-\left(1+\frac{\eta^2}{d_id_j}\right)(y_i-y_j)^2\\ \Gamma_{ij}^{\text{off}}(y,a) &:= \sum_{\ell\neq i,\ \ell\neq j}\Big[\frac{d_ja_{i\ell}}{(d_i-1)d_\ell}(y_i-y_\ell)^2+\frac{d_ia_{j\ell}}{(d_j-1)d_\ell}(y_j-y_\ell)^2\Big]-\left(1-\frac{\eta^2}{d_id_j}\right)(y_i-y_j)^2\end{aligned} \tag{7}$$

$\beta \in \mathcal{KL}$ such that all maximal solutions ϕ are complete and satisfy $\omega(\phi(t,j)) \le \beta(\omega(\phi(0,0)), t+j)$ for all $(t,j) \in \operatorname{dom}\phi$. □

It is proven in the text beneath [11, Lemma 5] that system (8) is $\mathcal{KL}$-stable with respect to ω_0. Due to the structure of $\mathcal{A}_{\text{inst}}$ where $a_{ij}(y_i-y_j)^2 = 0$, this property means that solutions asymptotically form clusters [11, Section 4.3].

A possible criticism of the result of [11] summarized above is that the connectivity variables a_{ij} are only based on the instantaneous opinions mismatch, see (8b). If two agents had or had not been in agreement for a long time, their current interaction status is not affected by the past. The main contribution of this paper is to introduce a novel model with memory-based connectivity features. In the next section, we formalize this intuition via a new hybrid model where the past memory is captured by additional state variables. For this model, we will prove a generalization of the above mentioned $\mathcal{KL}$-stability property.

III. Memory-based connectivity

A. Hybrid model

We define the connectivity between agents i and j, for $(i,j) \in \mathcal{E}^+$, using Γ_{ij}^{on} or Γ_{ij}^{off} in (7), but also based on a new *memory state variable* $\theta_{ij} \in \mathbb{R}$ that is a filtered version of the instantaneous threshold criterion reviewed in Section II. Loosely speaking, θ_{ij} reflects the history of the interaction between agents i and j.

More precisely, for each $(i,j) \in \mathcal{E}^+$, the flow dynamics for θ_{ij} is selected as

$$\begin{aligned}\dot\theta_{ij} &= -\beta_{ij}\theta_{ij} + (1-a_{ij})\Gamma_{ij}^{\text{on}}(y,a) + a_{ij}\Gamma_{ij}^{\text{off}}(y,a)\\ &=: f_{\theta,ij}(y,a,\theta_{ij}),\end{aligned} \tag{10}$$

where $\beta_{ij} > 0$ are tunable parameters associated to how fast each agent "forgets" the past, and Γ_{ij}^{on} and Γ_{ij}^{off} are given in (7). When link (i,j) is active, $a_{ij} = 1$ according to (2) and $\dot\theta_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{off}}(y,a)$ in view of (10). Hence, variable θ_{ij} filters $\Gamma_{ij}^{\text{off}}(y,a)$, which is indeed the right term to be monitored for deciding whether link (i,j) should be deactivated, see Section II. Conversely, when link (i,j) is not active, $a_{ij} = 0$ and $\dot\theta_{ij} = -\beta_{ij}\theta_{ij} + \Gamma_{ij}^{\text{on}}(y,a)$ so that $\Gamma_{ij}^{\text{on}}(y,a)$ is filtered to infer whether or not the link should be activated.

Parameters β_{ij} in (10) represent how "nostalgic" each pair of agents are with respect to their common past. When β_{ij} is very large, the past is not given much credit and, as $\beta_{ij} \to \infty$, we recover the criterion of Section II. Conversely, when β_{ij} is small, the past values of Γ_{ij}^{off} or Γ_{ij}^{on} matter more, as compared to the instantaneous ones.

When a jump occurs, i.e. when a link is (de)activated, the memory variable θ_{ij} is unchanged, namely $\theta_{ij}^+ = \theta_{ij}$ for each $(i,j) \in \mathcal{E}^+$. The proposed memory-based (de)activation policy then intuitively generalizes the one of Section II:

- *Activation.* If $a_{ij} = 0$, link (i,j) is not active. Activation is enabled when $\Gamma_{ij}^{\text{on}}(y,a) \ge \varepsilon$ and θ_{ij} is non-negative. Parameter ε plays the same role as in Section II, preventing Zeno solutions, see footnote 1 on page 2.
- *Deactivation.* If $a_{ij} = 1$, link (i,j) is active. Deactivation is then enabled when $\Gamma_{ij}^{\text{off}}(y,a) \le -\varepsilon$ and θ_{ij} is non-positive. The rationale is similar to the previous case.

The mechanism described above can be written in a compact form extending the memoryless model (8). Introducing

$$\begin{aligned}&\theta := (\theta_{12},\dots,\theta_{1n},\theta_{23},\dots,\theta_{n-2,n},\theta_{n-1,n}) \in \mathbb{R}^{\frac{n(n-1)}{2}}\\ &x := (y,a,\theta) \in \mathbb{X}_{\text{mem}} := \mathbb{R}^n \times \{0,1\}^{\frac{n(n-1)}{2}} \times \mathbb{R}^{\frac{n(n-1)}{2}},\end{aligned}$$

the memory-based hybrid model is given by

$$\begin{aligned}\begin{bmatrix}\dot y\\ \dot a\\ \dot\theta\end{bmatrix} &= f(x) := \begin{bmatrix}-\Phi y\\ 0\\ f_\theta(y,a,\theta)\end{bmatrix}, & x \in C_{\text{mem}}\\ \begin{bmatrix}y^+\\ a^+\\ \theta^+\end{bmatrix} &\in g(x) := \begin{bmatrix}y\\ \displaystyle\bigcup_{\substack{(y,a)\in D_{ij,\text{mem}}\\ (i,j)\in\mathcal{E}^+}} g_{ij}(y,a)\\ \theta\end{bmatrix}, & x \in D_{\text{mem}},\end{aligned} \tag{11a}$$

with $x := (y,a,\theta)$, with

$$\begin{aligned}&D_{\text{mem}} := \bigcup_{(i,j)\in\mathcal{E}^+} D_{ij,\text{mem}}^{\text{on}} \cup D_{ij,\text{mem}}^{\text{off}}, \quad C_{\text{mem}} := \overline{\mathbb{X}_{\text{mem}} \backslash D_{\text{mem}}}\\ &D_{ij,\text{mem}}^{\text{on}} := \Big\{(y,a,\theta) \in \mathbb{X}_{\text{mem}} \mid a_{ij} = 0,\ \Gamma_{ij}^{\text{on}} \ge \varepsilon,\ \theta_{ij} \ge 0\Big\}\\ &D_{ij,\text{mem}}^{\text{off}} := \Big\{(y,a,\theta) \in \mathbb{X}_{\text{mem}} \mid a_{ij} = 1,\ \Gamma_{ij}^{\text{off}} \le -\varepsilon,\ \theta_{ij} \le 0\Big\}.\end{aligned} \tag{11b}$$

System (11) satisfies the hybrid basic conditions of [15, As. 6.5], in view of the definition of the flow and jump maps and the flow and jump sets. Then, from [15, Thm 6.30], it is (nominally) well-posed, namely its solutions satisfy a desirable sequential compactness property.

B. Main stability result

We establish here a $\mathcal{KL}$-stability property for (11) generalizing the one established for (8) at the end of Section II. To this end, function ω_0 in (9) is generalized to

$$\omega(x) := \omega_0((y,a)) + \sum_{(i,j)\in\mathcal{E}^+}(1-a_{ij})\max\{0,\theta_{ij}\}, \tag{12}$$

for any $x \in \mathbb{X}_{\text{mem}}$ which incorporates the memory variable θ.

Since ω_0 is continuous, then ω is continuous too on $\mathbb{X}_{\text{mem}}$. Our main result below ensures the $\mathcal{KL}$-stability property for model (11) with respect to ω, as well as properties of the hybrid time domains of its solutions. The proof of Theorem 1 is based on a novel hybrid Lyapunov function characterized in the next section.

Theorem 1: All maximal solutions to system (11) are complete and eventually continuous. For each maximal solution x, there exists $x^\star \in \mathbb{X}_{\text{mem}}$ such that $x(t,j) \to x^\star$ as $t+j \to \infty$. Moreover, system (11) is $\mathcal{KL}$-stable with respect to ω in (12).

Since the second term in (12) is non-negative, the convergence to zero of $\omega(x)$ established in Theorem 1 immediately implies that $\omega_0((y,a)) \to 0$. As result, Theorem 1 ensures that opinions converge to clusters as time grows. In addition, $(1-a_{ij})\max\{0,\theta_{ij}\}$ converges to zero for all $(i,j) \in \mathcal{E}^+$, namely the memory variable θ_{ij} associated to individuals belonging to different clusters is not positive. The asymptotic behavior of solutions is clarified in the following corollary, which is an immediate consequence of eventual continuity of solutions (a eventually settles to a clustering pattern) and convergence of solutions (opinions settle to constant values that coincide within each cluster because $\omega_0((y,a)) \to 0$).

Corollary 1: Each maximal solution of (11) converges to a clustering pattern with constant and equal opinions in each cluster.

C. Lyapunov function and proof of Theorem 1

Consider the following candidate Lyapunov function,

$$U(x) := V(x) + \gamma \sum_{(i,j)\in\mathcal{E}^+} (1-a_{ij})\max\{0,\theta_{ij}\}, \qquad (13)$$

for each $x = (y,a,\theta) \in \mathbb{X}_{\text{mem}}$, $\gamma > 0$ to be selected, and

$$V(x) := \frac{1}{2} y^\top \Phi y = \frac{1}{4} \sum_{(i,j)\in\mathcal{V}^2} \varphi_{ij}(y_i - y_j)^2. \qquad (14)$$

The second equality above arises from the Dirichlet form [8, Prop. 1.9] and the definition of Φ after (4). Function V is the Lyapunov function used in [11, eqn. (18)], while the second term in (13) accounts for the new dynamics θ. The next proposition states key properties of U.

Proposition 1: Given system (11), there exist $\gamma > 0$ in (13) and $c_1, c_2, c_F, c_J > 0$ such that the following holds.

(i) U is locally Lipschitz on $\mathbb{X}_{\text{mem}}$ and satisfies $c_1\omega(x) \le U(x) \le c_2\omega(x)$ for all $x \in \mathbb{X}_{\text{mem}}$.

(ii) For all $x \in C_{\text{mem}}$, $U^\circ(x; f(x)) \le -c_F U(x)$.

(iii) For all $x \in D_{\text{mem}}$, and $v \in g(x)$, $U(v) - U(x) \le -c_J$.

Proof. We prove the three items one by one.

Proof of item (i). Function U is locally Lipschitz on $\mathbb{X}_{\text{mem}}$ in view of its definition in (13). According to [11, eqn. (19)], there exist $\tilde{c}_1, \tilde{c}_2 > 0$ such that for any $(y,a) \in \mathbb{X}_{\text{inst}}$, $\tilde{c}_1\omega_0((y,a))^2 \le V(x) \le \tilde{c}_2\omega_0((y,a))^2$. As a result, by the definition of ω in (12), we obtain the inequality in (i) with $c_1 := \min\{\tilde{c}_1, \gamma\}$ and $c_2 := \max\{\tilde{c}_2, \gamma\}$.

Proof of item (ii). Given any $x = (y,a,\theta) \in C_{\text{mem}}$, introduce the sets (below, for simplicity, the dependence on x is sometimes omitted)

$$\begin{array}{lcl} \mathcal{E}^c_{>0}(x) & := & \{(i,j) \in \mathcal{E}^+ \mid a_{ij} = 0 \text{ and } \theta_{ij} > 0\}, \\ \mathcal{E}^c_{=0}(x) & := & \{(i,j) \in \mathcal{E}^+ \mid a_{ij} = 0 \text{ and } \theta_{ij} = 0\}, \\ \mathcal{E}^c_{\Gamma}(x) & := & \{(i,j) \in \mathcal{E}^+ \mid \Gamma^{\text{on}}_{ij}(y,a) \ge 0\}. \end{array} \qquad (15)$$

According to [20, Prop. 1.1], in view of (10), (11), (13) and the definition of f_θ, we have that

$$\begin{aligned} U^\circ(x; f(x)) &= \langle \nabla V(x), f(x) \rangle \\ &\quad + \gamma \textstyle\sum_{(i,j)\in\mathcal{E}^c_{>0}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right) \\ &\quad + \gamma \textstyle\sum_{(i,j)\in\mathcal{E}^c_{=0}} \max\left\{0, -\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right\} \\ &= \langle \nabla V(x), f(x) \rangle + \gamma \textstyle\sum_{(i,j)\in\mathcal{E}^c_{>0}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right) \\ &\quad + \gamma \textstyle\sum_{(i,j)\in(\mathcal{E}^c_{=0}\cap\mathcal{E}^c_{\Gamma})} \Gamma^{\text{on}}_{ij}(y,a) \\ &= \langle \nabla V(x), f(x) \rangle + \gamma \textstyle\sum_{(i,j)\in\widehat{\mathcal{E}}} \left(-\beta_{ij}\theta_{ij} + \Gamma^{\text{on}}_{ij}(y,a)\right), \end{aligned}$$

where $\widehat{\mathcal{E}}(x) := \mathcal{E}^c_{>0}(x) \cup (\mathcal{E}^c_{=0}(x) \cap \mathcal{E}^c_{\Gamma})(x))$. Note that [11, eqn. (20)] holds because variables (y,a) obey the same flow dynamics as in (8) and (11). Then $\langle \nabla V(x), f(x) \rangle \le -\tilde{c}_F V(x)$ for some $\tilde{c}_F > 0$. Consequently,

$$U^\circ(x; f(x)) \le -\tilde{c}_F V(x) - \gamma \sum_{(i,j)\in\widehat{\mathcal{E}}} \beta_{ij}\theta_{ij} + \gamma \sum_{(i,j)\in\widehat{\mathcal{E}}} \Gamma^{\text{on}}_{ij}(y,a).$$

The expression of Γ^{on}_{ij} in (7), together with (14) yields

$$\begin{aligned} &\sum_{(i,j)\in\widehat{\mathcal{E}}} \Gamma^{\text{on}}_{ij}(y,a) = \sum_{(i,j)\in\widehat{\mathcal{E}}} \Bigg(-\left(1 + \frac{\eta^2}{d_i d_j}\right)(y_i - y_j)^2 \\ &+ \sum_{\ell\neq i,\ \ell\neq j} \left((d_j+1)\varphi_{i\ell}(y_i - y_\ell)^2 + (d_i+1)\varphi_{j\ell}(y_j - y_\ell)^2\right) \Bigg) \\ &\le n \sum_{(i,j)\in\widehat{\mathcal{E}}} \sum_{\ell\neq i,\ \ell\neq j} \left(\varphi_{i\ell}(y_i - y_\ell)^2 + \varphi_{j\ell}(y_j - y_\ell)^2\right) \\ &\le n \sum_{(i,\ell)\in\mathcal{V}^2} \varphi_{i\ell}(y_i - y_\ell)^2 + n \sum_{(j,\ell)\in\mathcal{V}^2} \varphi_{j\ell}(y_j - y_\ell)^2 \\ &= 4nV(x) + 4nV(x) = 8nV(x), \end{aligned} \qquad (16)$$

which can be substituted in the preceding inequality to get

$$U^\circ(x; f(x)) \le -(\tilde{c}_F - 8n\gamma)V(x) - \gamma \sum_{(i,j)\in\widehat{\mathcal{E}}} \beta_{ij}\theta_{ij}. \qquad (17)$$

Now, $\theta_{ij} = (1-a_{ij})\max\{0,\theta_{ij}\}$ for $(i,j) \in \widehat{\mathcal{E}} \subset \mathcal{E}^c_{>0} \cup \mathcal{E}^c_{=0}$, and $(1-a_{ij})\max\{0,\theta_{ij}\} = 0$ for $(i,j) \in \mathcal{E}^+ \backslash \widehat{\mathcal{E}}$, because either $a_{ij} = 1$ or $\theta_{ij} \le 0$ for those edges. Then,

$$U^\circ(x; f(x)) \le -c_F U(x), \qquad (18)$$

where $c_F = \min\{\tilde{c}_F - 8n\gamma, \beta\}$ and $\beta := \min_{(i,j)\in\mathcal{E}^+} \beta_{ij} > 0$, which implies item (ii) with any $\gamma \in (0, \frac{\tilde{c}_F}{8n})$.

Proof of item (iii). From (11b), for each $x \in D_{\text{mem}}$ and $v \in g(x)$, there exists $(i,j) \in \mathcal{E}^+$ such that $x \in D^{\text{on}}_{ij,\text{mem}} \cup D^{\text{off}}_{ij,\text{mem}}$, and $a^+ \in g_{ij}(y,a)$. Since $D^{\text{on}}_{ij,\text{mem}}$ and $D^{\text{off}}_{ij,\text{mem}}$ are disjoint, then two cases may occur: case "on" and case "off" below.

Case "on": $x \in D^{\text{on}}_{ij,\text{mem}}$. In this case, $a_{ij} = a_{ji} = 0$, $\Gamma^{\text{on}}_{ij}(y,a) \ge \varepsilon$ and $\theta_{ij} \ge 0$. Since link (i,j) is activated at this jump, $a_{ij}^+ = a_{ji}^+ = 1$ and from (5), (6), the other connectivity variables, as well as all the memory variables in view of (11), remain constant across the jump. Consequently,

$$U(\upsilon) - U(x) = V(\upsilon) - V(x) - \gamma \max\{0, \theta_{ij}\}. \qquad (19)$$

Following the proof [11, Lemma 5], we get $V(\upsilon) - V(x) = \frac{1}{2(d_i+1)(d_j+1)} \left(-\Gamma_{ij}^{\text{on}}(y,a) - \frac{\eta^2}{d_i d_j}(y_i - y_j)^2 \right)$.
Thus, (19) yields

$$U(\upsilon) - U(x) \leq \frac{1}{2 d_i^+ d_j^+} \left(-\varepsilon - \frac{\eta^2}{d_i d_j}(y_i - y_j)^2 \right),$$

where $d_i^+ = d_i + 1$ and $d_j^+ = d_j + 1$. Taking $c_J \in \left(0, \frac{\varepsilon}{2n^2}\right]$, characterizing the least possible decrease with n agents, we prove item (iii) for the case "on".

Case "off": $x \in D_{ij,\text{mem}}^{\text{off}}$. In this case, $a_{ij} = a_{ji} = 1$, $\Gamma_{ij}^{\text{off}} \leq -\varepsilon$ and $\theta_{ij} \leq 0$. Link (i,j) is deactivated at this jump and $a_{ij}^+ = a_{ji}^+ = 0$. The other connectivity variables and the memory variables remain constant. Then (19) holds again and following again the proof of [11, Lemma 5], we deduce that $V(\upsilon) - V(x) = \frac{1}{2 d_i d_j} \left(\Gamma_{ij}^{\text{off}}(y,a) - \frac{\eta^2}{d_i d_j}(y_i - y_j)^2 \right)$.
Thus, (19) yields

$$U(\upsilon) - U(x) \leq \frac{1}{2 d_i d_j} \left(-\varepsilon - \frac{\eta^2}{d_i d_j}(y_i - y_j)^2 \right).$$

Selecting $c_J \in \left(0, \frac{\varepsilon}{2n^2}\right]$, item (iii) holds in case "off".
Thus, item (iii) holds with $c_J := \frac{\varepsilon}{2n^2}$. ■

Proof of Theorem 1. To prove that maximal solutions to (11) are complete, we invoke [15, Prop. 6.10]. First, the viability condition is satisfied in view of the system definition. Secondly, $g(D_{\text{mem}}) \subset C_{\text{mem}} \cup D_{\text{mem}}$. Moreover, using $W(x) = y^\top y$, we have $\langle \nabla W(x), f(x) \rangle = -2 y^\top \Phi y \leq 0$, therefore the y components are bounded. Also the memory variables θ are bounded, because they are constant across jumps and the components of its flow map are exponentially stable filters with integrable inputs. Consequently, maximal solutions do not escape in finite time and [15, Prop. 6.10] establishes their completeness. Eventual continuity follows from the fact that the decrease of U across jumps in item (iii) of Proposition 1 is constant at each jump and that U does not increase on flows in item (ii) of Proposition 1, therefore any solution jumping forever would eventually lead to a negative $U(x)$, contradicting item (i) of Proposition 1. About convergence of solutions, state a settles due to eventual continuity, y settles too because $\langle \nabla W(x), f(x) \rangle = -2 y^T \Phi y = 0$ and symmetry of Φ implies $-2\dot{y} = \Phi y = 0$, finally θ converges too because it is a linear filter with a converging input.

Let us now prove the $\mathcal{KL}$ bound on the solutions. Since the conditions of Proposition 1 are analogous to those of [11, Lemma 5], we can proceed as in [11, eq. (23)] to obtain

$$\begin{aligned} \omega(x(t,j)) &\leq \beta(\omega(x(0,0)), t, j) \\ &:= \frac{c_2}{c_1} \mathrm{e}^{c_F \cdot t} \left(1 - \min\left\{ 1, \frac{c_J}{c_2 \omega(x(0,0))} \right\} \right)^j \omega(x(0,0)), \end{aligned}$$

establishing a class $\mathcal{KLL}$ bound [2], which is easily transformed into a class $\mathcal{KL}$ bound $\omega(x(t,j)) \leq \bar{\beta}(\omega(x(0,0)), t + j)$ constructing $\bar{\beta}$ from β as in [2, Lemma 6.1]. ■

IV. Simulations

We consider $n = 15$ agents, $\varepsilon = 0.01$ and $\eta = 3$ in (7). The initial topology is an Erds-Rnyi random graph, with probability p of having an interconnection between each node pair (i,j), while the initial values of $y_i, i \in \{1, \ldots, n\}$ are selected randomly in the interval $[0,1]$. The result of simulations[2] done using model (8) is reported in Fig. 1.

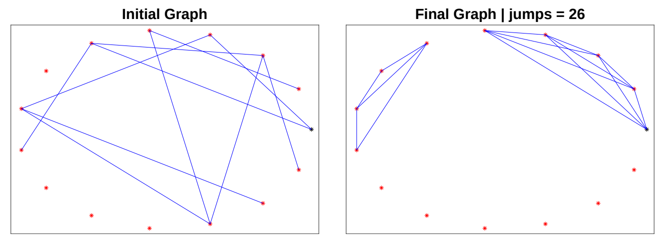

Fig. 1. Initial and final topologies for given $y(0,0)$, with $p = 0.1$. Nodes have been sorted counterclockwise to clearly visualize the clusters appearing in the final topology.

We then study model (11), the impact of the choices of β_{ij} and the initial values of θ_{ij} on the evolution of the opinions, for the same $y(0,0)$ as in Fig. 1. In particular, we take for $\beta_{ij} = \beta$ with $\beta \in \{0.1, 50\}$, and $\theta_{ij}(0,0) = \theta^o$ with $\theta^o \in \{0, 0.01, 1\}$ for $a_{ij}(0,0) = 1$ and $\theta^o = 0$ otherwise, as well as the case where θ^o takes random values in $[-1, 0)$, for all $(i,j) \in \mathcal{E}^+$. The final graphs are depicted in Fig. 2 and 3.

We can see that the opinions converge to fixed agreement values in each cluster, in agreement with Corollary 1. In general these clusters are different from those obtained with model (8). There is only one situation where the obtained clusters are the same as in Fig. 1: when $\theta^o = 0$, see Fig. 3. This can be explained by the fact that the opinions converge quickly to cluster formations and θ has no impact on it.

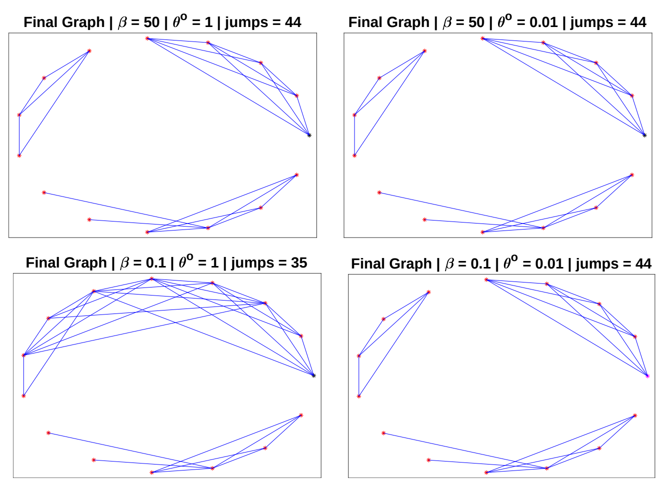

Fig. 2. Four different final topologies for different couples of $(\beta, \theta(0,0))$.

[2]The simulations have been carried out using the Matlab toolbox [22].

When comparing Fig. 2 and 3, we note that positive values of θ tend to preserve the exiting initial interconnections, leading to larger clusters. This suggests that agents remember their mutual relationships with each other. On the other hand, negative values generate clusters made of fewer agents in general, see Fig. 2 compared to Fig. 3.

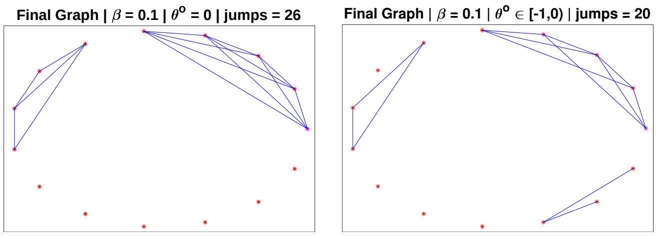

Fig. 3. Different final topologies for different, non positive $\theta(0,0)$.

The evolutions of y and θ as functions of the continuous-time t are depicted in Fig. 4 for $\beta = 0.1$ and $\theta^o = 1$. Two clusters appear as time grows, in agreement with the corresponding plot in Fig. 2. When agents i and j are in the same cluster, θ_{ij} converges to 0 just as $\Gamma_{ij}^{\text{off}}(y,a)$ does in this case. When agents i and j are not in the same cluster, θ_{ij} converges to the same values of $\Gamma_{ij}^{\text{on}}(y,a)/\beta_{ij}$ in view of (10). This ratio, $\Gamma_{ij}^{\text{on}}(y,a)/\beta_{ij}$, can take any constant value in $(-\infty,\varepsilon]$ in view of (11b). Hence, in some cases we have $\theta_{ij}(t,j) \to 0$ as $t+j \to \infty$ although agents i and j are not in the same cluster. In this context, to distinguish agents from the same clusters, it is more relevant to monitor σ_{ij} defined as:

$$\sigma_{ij} := (2a_{ij}-1)\frac{|\theta_{ij}|}{|(1-a_{ij})\Gamma_{ij}^{\text{on}}(y,a)+a_{ij}\Gamma_{ij}^{\text{off}}(y,a)|}. \quad (20)$$

At steady-state, σ_{ij} either converges to $1/\beta_{ij}$ if agents i and j belong to the same cluster, or to $-1/\beta_{ij}$ otherwise. The influence of β_{ij} is clear here: the smaller β_{ij}, the more important the past is and the bigger σ_{ij}, and vice versa.

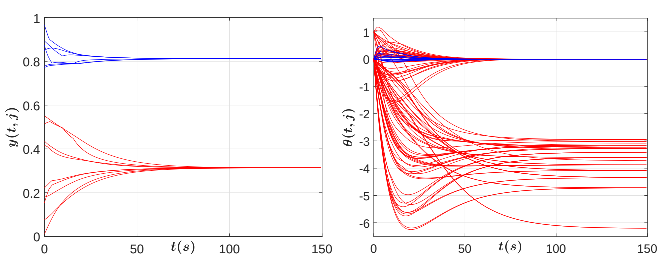

Fig. 4. Values of y (left) and θ (right) for $\beta = 0.1$ and $\theta^o = 1$. Different colors have been used for different clusters in the final topology.

V. Conclusions

We have presented a hybrid model of opinion dynamics where the connectivity among individuals takes into account both the present and the past values of the opinions of the respective individuals. We believe that the idea of taking into account the past when defining connectivity in opinion dynamics is appealing and relevant, and that it has been largely unexplored so far.

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