On the geometry of stability regions of Smith predictors subject to delay uncertainty

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Abstract

In this paper, we present a geometric method for describing the effects of the *delay induced uncertainty* on the stability of a standard Smith Predictor control scheme. The method consists of deriving the *stability crossing curves* in the parameter space defined by the *nominal delay*, and *delay uncertainty*, respectively. More precisely, we start by computing the *crossing set*, which consists of all frequencies corresponding to all points on the stability crossing curve, and next we give their *complete classification*, including also the explicit characterization of the *directions* in which the zeros cross the imaginary axis. This approach complements existing algebraic stability tests, and it allows some new insights in the stability analysis of such control schemes. Several illustrative examples are also included.

1 Introduction

The stability and control of time-delay systems are subject of recurring interest since the *delay* is inherently present in various applications, from signal propagation in networks to population dynamics (see, for instance, [14, 7] for further references, and examples).

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It is well-known that the presence of delays in a dynamical system often results in poor performance or even in closed-loop instability, especially if the feedback control is designed based on the system without delays. To overcome such problems, various techniques have been proposed in the literature, starting with the well-known Smith predictor in the 1950s [17], and its modifications (see, for instance, the overview of Palmor [16]). The idea behind the Smith predictor is to define an appropriate *interconnection transformation* in the following way: first, finding a controller for the system without delay, and then defining a new compensator for the overall system such that the overall system is equivalent to a closed-loop system for the system without delay coupled with a corresponding *delay* element *outside* of the loop. It is also largely known that such a construction works perfectly when the *delay* is *known exactly* (see, for instance, [16] and the references therein). Further discussions on Smith predictors, and their sensitivity with respect to delay uncertainty can be found in [14].

The aim of this paper is to present some *new interpretations* of the Smith predictors [17] subject to *delay uncertainty*. Such a problem have been treated in the literature since 1980s (see, for instance, [15], [20], [19], [16], and the references therein), and it was reconsidered recently [13], in response to the increasing interest in applications, such as the control of congestions in high-speed networks [12] and the motion synchronization in virtual environments with shared haptics [1].

The approach considered in the paper makes use of some simple geometric idea (triangles inequality), and it is inspired by some recent work devoted to the characterization of the stability crossing curves [8]. The novelty of the results lies in a *simple*, and *easy* to follow *classification* of all the situations when uncertainty on the nominal delay value will induce instability. Unlike [8], the *degenerate cases* are also discussed. The results complements the algebraic characterizations in [14], and [13].

The remaining part of the paper is organized as follows: Section 2 presents the problem formulation and preliminaries. Section 3 discusses the identification of the crossing points, and both regular and degenerate cases are treated. The characterization of the crossing curves (tangent, smoothness, direction of crossing) is presented in Section 4. Two illustrative examples from the control literature are presented in Section 5. Some concluding remarks end the paper.

2 Problem formulation, and preliminaries

Consider a SISO system with a delayed input

$$H(s) \quad = \quad H_0(s)e^{-s\tau}, \tag{2.1}$$

where $H_0(s)$ is a rational transfer function.

Assume that, due to some modeling errors, there exists some *delay uncertainty* Δ on the *nominal delay* value τ_0 satisfying the constraint $\mid \Delta \mid < \delta$, with $\delta > 0$. As a consequence, the real delay τ can be written as $\tau = \tau_0 + \Delta$. Let $C(s)$ be the controller for the system without delay, and let $C_0(s)$ be the corresponding Smith controller for the nominal delay system $H_0(s)e^{-s\tau_0}$

$$C_0(s) \quad = \quad \frac{C(s)}{1 + C(s)H_0(s)(1 - e^{-s\tau_0})}.$$

The presence of delay uncertainty Δ leads to the following closed-loop system

$$H_{cl,\Delta}(s) \quad = \quad \frac{C(s)H_0(s)}{1 + C(s)H_0(s) - C(s)H_0(s)e^{-s\tau_0}(1 - e^{-s\Delta})} e^{-s(\tau_0+\Delta)}. \quad (2.2)$$

For $\Delta = 0$, (2.3) recovers to closed-loop transfer function under the standard Smith predictor:

$$H_{0cl,\Delta}(s) \quad = \quad \frac{C(s)H_0(s)}{1 + C(s)H_0(s)} e^{-s\tau_0}. \quad (2.3)$$

It is important to note that the open-loop transfer function without delay $C(s)H_0(s)$ has an important impact on the *sensitivity* with respect to *delay uncertainty.*

As discussed in [14, 13], the closed-loop stability problem of a system was Smith Predictor with delay uncertainty reduces the following characteristic equation:

$$P(s) + Q(s)\mathrm{e}^{-s\tau_0} - Q(s)\mathrm{e}^{-s(\tau_0+\Delta)} = 0,$$

where τ_0 represents the nominal delay; $P(s)$ and $Q(s)$ are appropriate polynomials depending on the plant without delay and the controller, and $\Delta \geq 0$ is the delay uncertainty.

Based on the characteristic equation above, let us introduce the following auxiliary quasipolynomial:

$$D(s, \tau_1, \tau_2) = P(s) + Q(s)\mathrm{e}^{-s\tau_1} - Q(s)\mathrm{e}^{-s\tau_2} = 0. \quad (2.4)$$

It is easy to see that by choosing $\tau_1 = \tau_0$, $\tau_2 = \tau_0 + \Delta$, we completely recover (2.4). Next, let $\mathcal{G} = \{(x, y) \mid 0 \leq x \leq y\}$. Then, obviously, $(\tau_1, \tau_2) \in \mathcal{G}$. The main objective of this paper to study the change of the number of right-hand solutions of (2.4) or equivalently of (2.4) as (τ_1, τ_2) varies $\mathcal{G}$.

Since the main interest lies in identifying the regions of (τ_1, τ_2) in $\mathcal{G}$ such that $D(s, \tau_1, \tau_2)$ is stable (the roots of the characteristic equation are located in $\mathbb{C}_-$), we will exclude some trivial cases, and make the following assumptions on the polynomials P and Q:

Assumption 2.1. *The polynomials P and Q are such that* $\deg(Q) \leq \deg(P)$.

Assumption 2.2. *The polynomial P does not have any roots at the origin, that is $P(0) \neq 0$.*

Assumption 2.3. *The polynomials P and Q do not have common zeros.*

Assumption 2.4. *The polynomials P and Q satisfy the following condition:*

$$\lim_{s\to\infty} \left| \frac{Q(s)}{P(s)} \right| < \frac{1}{2}.$$

For discussions on the implications of these assumptions the readers are referred to [8].

3 Identification of the crossing points

Let $\mathcal{T}$ denote the set of all points of $(\tau_1, \tau_2) \in \mathcal{G}$ such that $D(s, \tau_1, \tau_2)$ has at least one zero on the imaginary axis. Any $(\tau_1, \tau_2) \in \mathcal{T}$ is known as a crossing point. The set $\mathcal{T}$, which is the collection of all crossing points, is known as the stability crossing curves. Let $\mathcal{T}_\omega$ denote the set of all $(\tau_1, \tau_2) \in \mathcal{G}$ such that the quasipolynomial D has at least one zero for $s = j\omega$. Let Ω the set of all $\omega > 0$ for which there exists a pair (τ_1, τ_2) such that $D(j\omega, \tau_1, \tau_2) = 0$. We will refer to Ω as the *crossing set.*

Obviously

$$\mathcal{T} = \{\mathcal{T}_\omega | \omega \in \Omega\}.$$

We will first consider the non-degenerate case satisfying the following assumption:

Assumption 3.1 (Non-degeneracy)**.** *The polynomials P and Q satisfy the following frequency-domain condition:*

$$P(j\omega) \cdot Q(j\omega) \neq 0 \text{ for all } \omega \in \Omega \tag{3.1}$$

The degenerate case will be presented in the later part of the paper.

3.1 Regular cases

In the sequel, we consider

$$h(s) = \frac{Q(s)}{P(s)}$$

and

$$H(s) = 1 + h(s)e^{-s\tau_1} - h(s)e^{-s\tau_2} \tag{3.2}$$

For given τ_1 and τ_2, as long as Assumption 3.1 is satisfied, $D(s, \tau_1, \tau_2)$ and $H(s)$ share all the zeros in a neighborhood of the imaginary axis. Therefore, we may obtain all the crossing points and direction of crossing using $H(s) = 0$ instead of $D(s, \tau_1, \tau_2) = 0$. We may also consider the three terms in $H(s)$ as three vectors in the complex plane, with the magnitudes 1, $|h(s)|$ and $|h(s)|$ respectively. So when we adjust the values of τ_1 and τ_2 in fact we adjust the directions of the vectors represented by the second and the third terms. Equation (3.2) simply means that if we put the first two vectors head to tail then we get the third vector. In other words these vectors form an *isosceles triangle.* These remarks allow us concluding with the following proposition.

Proposition 3.1. *For some $(\tau_1, \tau_2) \in \mathcal{G}$, $H(s)$ has an imaginary zero $s = j\omega$, $\omega \neq 0$ if and only if*

$$|h(j\omega)| \geq \frac{1}{2}. \tag{3.3}$$

Proof. The relation (3.3) is straightforward from the geometric point of view: a triangle can be formed by three line segments with arbitrary orientation if and only if the length of any one side does not exceed the sum of the other two sides. In the case of an isosceles triangle this condition simply becomes: the sum of the equal

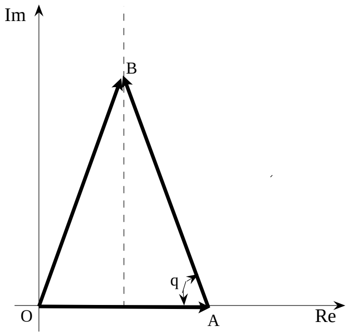


Figure 3.1: Triangle formed by 1, $h(s)e^{-s\tau_1}$ and $h(s)e^{-s\tau_2}$

sides exceed the other side. Notice also that $\angle[h(s)e^{-j\omega\tau_l}]$, $l = 1, 2$ can take any value by adjusting τ_l, $l = 1, 2$. $\square$

Due to the symmetry and Assumption 2.2, we only need to consider positive ω. So Ω is the set of all $\omega > 0$ satisfying (3.3). Also, for a given $\omega \in \Omega$ we may find all the pairs (τ_1, τ_2) satisfying $H(j\omega) = 0$ as follows:

$$\tau_1 = \tau_1^{u\pm}(\omega) = \frac{\angle h(j\omega) + (2u-1)\pi \pm q}{\omega}, \tag{3.4}$$
$$u = u_0^{\pm}, u_0^{\pm} + 1, u_0^{\pm} + 2, ...$$
$$\tau_2 = \tau_2^{v\pm}(\omega) = \frac{\angle h(j\omega) + 2v\pi \mp q}{\omega}, \tag{3.5}$$
$$v = v_0^{\pm}(u), v_0^{\pm}(u) + 1, v_0^{\pm}(u) + 2, ...$$

where $q \in [o, \pi]$ is the internal angle of triangle in Figure 3.1 which can be calculated by the cosine law as

$$q(j\omega) = \cos^{-1}\left(\frac{1}{2|h(\omega)|}\right) \tag{3.6}$$

and u_0^+, u_0^- are the smallest integers (may be dependent on ω) such that the corresponding values $\tau_1^{u_0^+ +}$, $\tau_1^{u_0^- -}$ are nonnegative, and v_0^+ and v_0^- are integers dependent on u such that $\tau_2^{v_0^+ +} \geq \tau_1^{u+}$, $\tau_2^{v_0^- -} > \tau_1^{u-}$ are satisfied. The position in Figure 3.1 corresponds to $(\tau_1^{u+}, \tau_2^{v+})$ and the mirror image about the real axis corresponds to $(\tau_1^{u-}, \tau_2^{v-})$. If we define $\mathcal{T}_{\omega,u,v}^+$ and $\mathcal{T}_{\omega,u,v}^-$ as the singletons $(\tau_1^{u+}(\omega), \tau_2^{v+}(\omega))$ and $(\tau_1^{u-}(\omega), \tau_2^{v-}(\omega))$ respectively, then we can characterize $\mathcal{T}_\omega$ as follows:

$$\mathcal{T}_\omega = \left(\bigcup_{u \geq u_0^+, v \geq v_0^+} \mathcal{T}_{\omega,u,v}^+\right) \bigcup \left(\bigcup_{u \geq u_0^-, v \geq v_0^-} \mathcal{T}_{\omega,u,v}^-\right)$$

Proposition 3.2. *The crossing set Ω consists of a finite number of intervals of finite length including the cases which may violate Assumption 3.1.*

Proof. First, one can easily observe that the number of points in Ω violating (3.1) is always finite. In conclusion, we only need to show that the set of all points satisfying (3.3) consists of a finite number of intervals of finite length. Since

$$|h(j\omega)| = \frac{1}{2} \Leftrightarrow |P(j\omega)| = 2|Q(j\omega)|$$

is a polynomial equation of variable ω^2, it always has a finite number of positive solutions. Therefore the solution of (3.3) consists of a finite number of intervals.

Next, due to the Assumption 2.4, any sufficiently large ω violates the condition (3.3). Therefore, the lengths of all intervals are *finite*. $\square$

In what follows we will denote these intervals as Ω_1, $\Omega_2, ..., \Omega_N$ and without loss of generality we may suppose that the intervals are ordered such that for any $\omega_1 \in \Omega_{k_1}$, $\omega_2 \in \Omega_{k_2}$, $k_1 < k_2$ we have $\omega_1 < \omega_2$.

Remark 3.1. *If (3.3) is satisfied for $\omega = 0$ and sufficiently small positive value of ω then we will take 0 as the left end of Ω_1, and $\Omega_1 = (0, \omega_1^r]$.*

Any other end point ω^* must satisfy $|P(j\omega^*)| = 2|Q(j\omega^*)|$, which corresponds to the limiting case when internal angle q of the triangle is 0. In this case, we obtain $\overrightarrow{OB} = -\overrightarrow{AB}$ on the real axis.

We will not restrict $\angle h(j\omega)$ to be within the 2π range, but make it a continuous function of ω within each Ω_k. Thus, for each fixed u, v and k, (3.4) and (3.5) describe two continuous curves denoted as $\mathcal{T}_{u,v}^{k+}$ and $\mathcal{T}_{u,v}^{k-}$ respectively. We should keep in mind that, for some u, v and k, part or the entire curve $\mathcal{T}_{u,v}^{k+}$ (or $\mathcal{T}_{u,v}^{k-}$) may be outside of the range $\mathcal{G}$, and therefore, may not be physically meaningful.

The collection of all the points in $\mathcal{T}$ corresponding to Ω_k may be expressed as

$$\begin{aligned} \mathcal{T}^k &= \bigcup_{u=-\infty}^{\infty} \bigcup_{v=-\infty}^{\infty} \left[(\mathcal{T}_{u,v}^{k+} \cup \mathcal{T}_{u,v}^{k-}) \cap \mathcal{G} \right] \\ &= \bigcup_{\omega \in \Omega_k} \mathcal{T}_\omega \end{aligned} \tag{3.7}$$

Obviously

$$\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k$$

Our previous discussions allow us concluding:

Proposition 3.3. *The end points of Ω_k must be in one of the following situation:*

Type 1. *It satisfies the equation $|h(x)| = \dfrac{1}{2}$.*

Type 2. *It equals 0.*

If one end of Ω_k is of type 1 then $q = 0$ and $\mathcal{T}^{k+}_{u,v}$ is connected with $\mathcal{T}^{k-}_{u,v}$ at this end. So, if both ends of Ω_k are of type 1 we get $\mathcal{T}^k$ is a series of closed curves.

Obviously, only the left end of Ω_1 can be of type 2. In this case, as $\omega \to 0$, both τ_1 and τ_2 approach ∞. In fact, $\mathcal{T}^{1+}_{u,v}$ and $\mathcal{T}^{1-}_{u,v}$ approach ∞ with asymptotes with slopes of

$$m^{\pm}_{u,v} = \frac{\tau_2^{v\pm}}{\tau_1^{u\pm}} = \frac{\angle h(0) + 2v\pi \mp q(0)}{\angle h(0) + (2u-1)\pi \pm q(0)}$$

where $q(0)$ is evaluated using (3.6).

In the sequel, we will say that an interval is of type 11 if both ends are of type 1, and Ω_1 is of type 21 if his left end is 0. Therefore, the crossing set Ω consists of a finite number of intervals of type 11, with the possibility of the first interval Ω_1 of type 21.

In conclusion, based on the remarks above, we have:

Proposition 3.4. *The set $\mathcal{T}^k$ consists of a series of curves belonging to one of the following categories:*

A) *A series of closed curves (Ω_k is of type 11)*

B) *A series of open ended curves with both ends approaching ∞ (Ω_k is of type 21)*

We continue this section with some illustrative examples regarding the above discussion and characterization.

Example 3.1. (**type 11**) *Consider a system with*

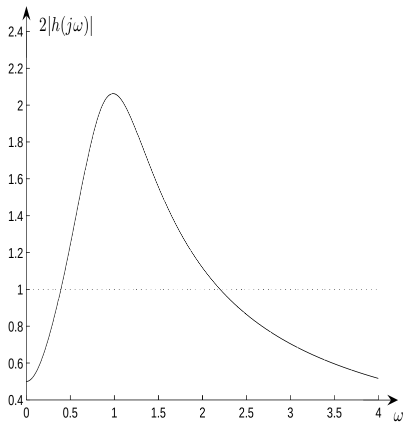
$$h(s) = \frac{4s+1}{4(s^2+s+1)} \tag{3.8}$$

Figure 3.2 (left) plots $2|h(j\omega)|$ against ω. The crossing set can be easily identified from this figure. It contains one interval $\Omega_1 = [0.39, 2.21]$ of type 11. Correspondingly $\mathcal{T}^1$ consists of a series of closed curves *as illustrated in Figure 3.2 (right).*

Example 3.2. (**type 21**) *Consider a system with*

$$h(s) = \frac{s+\sqrt{2}}{2s^3+s^2+8s+1} \tag{3.9}$$

Figure 3.3 (left) plots $|h(j\omega)|$ against ω. The crossing set Ω can be easily identified from the Figure 3.3, and it contains two intervals: $\Omega_1 = (0, 0.364]$ of type 21, and $\Omega_2 = [1.673, 2.198]$ of type 11. Figure 3.3 (right) illustrates $\mathcal{T}^1$ corresponding to Ω_1. It consists of a series of open ended curves*.*



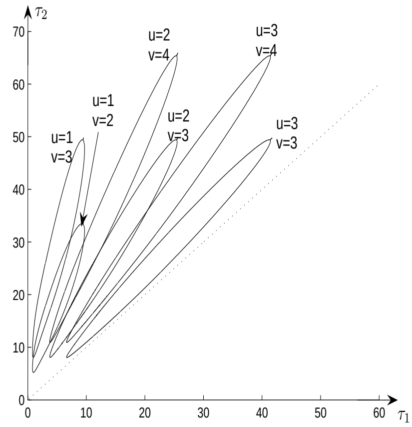
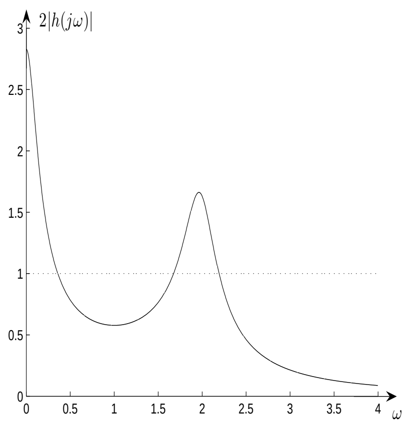


Figure 3.2: Left: The crossing set for the system (3.8) Right: Some crossing curves of system (3.8) are plotted



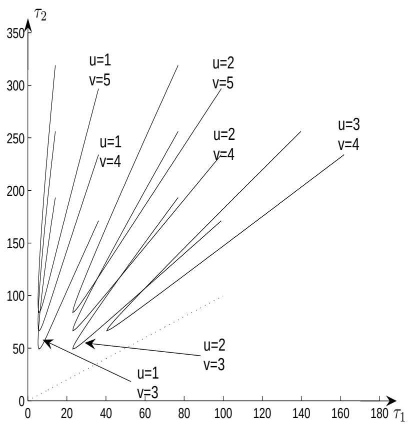


Figure 3.3: Left: The crossing set for the system (3.9) Right: Some crossing curves of system (3.9) are plotted

3.2 Degenerate cases

In the following we focus on the cases that are not satisfying the assumption 3.1. Obviously, the interesting case is $P(j\omega) = 0$ has *at least one* positive *solution.* We can easily state the following:

Remark 3.2. . *For* $\omega^* \neq 0$ *satisfying* $Q(j\omega^*) = 0$ *it is obvious that increasing of* τ_1 *and/or* τ_2 *has no effect regarding stability of the system. We note also that* $h(j\omega^*) = 0$ *imply that* $h(j\omega) < \dfrac{1}{2}$*, for all* ω *in a neighborhood of* ω^**.*

Next, we assume that $P(j\omega^*) = 0$ for $\omega^* \neq 0$. Using Assumption 2.3 we get $Q(j\omega^*) \neq 0$ and therefore $\lim_{\omega\to\omega^*} |h(j\omega)| = \infty$. So that, $|h(j\omega)| > \frac{1}{2}$ for all ω in a neighborhood of ω^*. This means that Ω contains one interval of type $[\omega^l, \omega^*]$ and one of type $[\omega^*, \omega^r]$. It is clear that the first interval is open to the left if $\omega^l = 0$.

In conclusion, we have the following result:

Proposition 3.5. *For $\omega^* \neq 0$ satisfying $P(j\omega^*) = 0$, $\mathcal{T}_{\omega^*}$ consists of the solutions of*

$$\omega^*\tau_2 = \omega^*\tau_1 + 2m\pi,\ m \in \mathbb{Z} \text{ in } \mathcal{G}.$$

and

$$\begin{aligned}
\lim_{\omega\to\omega^*} q(j\omega) &= \frac{\pi}{2};\\
\lim_{\omega\to\omega^*} \tau_1(\omega) &= \frac{2\angle Q(j\omega^*) + (4u - 2 \pm 1)\pi}{2\omega^*};\\
\lim_{\omega\to\omega^*} \tau_2(\omega) &= \frac{2\angle Q(j\omega^*) + (4v \mp 1)\pi}{2\omega^*}.
\end{aligned}$$

Proof. Straightforward computations. □

Remark 3.3. *We have the following properties:*

1) $\mathcal{T}_{\omega^*}$ consists of an infinite number of straight lines of slope 1 of equal distance.

2) $\lim_{\omega\to\omega^*} \tau_2(\omega) - \lim_{\omega\to\omega^*} \tau_1(\omega) = 2m\pi,\ m \in \mathbb{Z}$

Let $\Omega_k = [\omega^l, \omega^*]$ and $\Omega_{k+1} = [\omega^*, \omega^r]$.

In this case, using Proposition 3.3, we get $\mathcal{T}_{u,v}^{k+}$ is connected with $\mathcal{T}_{u,v}^{(k+1)+}$ and $\mathcal{T}_{u,v}^{k-}$ is connected with $\mathcal{T}_{u,v}^{(k+1)-}$ at the end corresponding to ω^*.

Using (3.7) and Remark 3.3, we obtain that each crossing curve in $\mathcal{T}_{u,v}^{k\pm}$ consists of an union of one straight line of slope 1 and the curve corresponding to $\Omega_k \setminus \{\omega^*\}$.

From Remark 3.3, we deduce that one end of the curve corresponding to $\Omega_k \setminus \{\omega^*\}$ is on the line in $\mathcal{T}_{\omega^*}$ which correspond to the pair (u, v).

In the sequel, we shall say that ω^* is an end point of type 0 if $P(j\omega^*) = 0$.

Example 3.3. (**type 20 and 01**) *Consider a system with*

$$h(s) = \frac{s + 2}{s^2 + 2} \tag{3.10}$$

Figure 3.4 (left) plots $\frac{1}{2|h(j\omega)|}$ against ω. The crossing set Ω contains two intervals: $\Omega_1 = (0, \sqrt{2}]$ of type 20, and $\Omega_2 = [\sqrt{2}, 3.046]$ of type 01. Figure 3.4 (right) plots $\mathcal{T}_{3,4}^{2\pm}$ which are two curves of type 01.

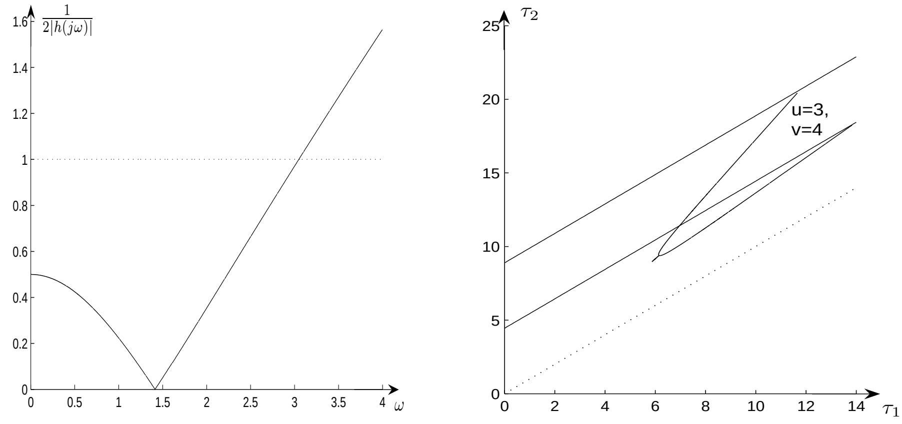


Figure 3.4: Left: The crossing set for the above system Right: $\mathcal{T}_{3,4}^{2\pm}$ of this system

4 Crossing curves, characteristic roots behavior

In this section, we discuss some qualitative aspects regarding the crossing curves and the characteristic roots. More precisely, we study the smoothness of the crossing curves, and we give the characterization of the way the roots cross the imaginary axis.

4.1 Tangent and Smoothness

For a given k, we will discuss the smoothness of the curves in $\mathcal{T}^k$ and thus $\mathcal{T} = \bigcup_{k=1}^{N} \mathcal{T}^k$. In this part, we use an approach based on the implicit function theorem.

For this purpose, we consider τ_1 and τ_2 as implicit functions of $s = j\omega$ defined by (2.4).

Next, for a given k, as s moves along the imaginary axis within Ω_k, $(\tau_1, \tau_2) = (\tau_1^{u\pm}(\omega), \tau_2^{v\pm}(\omega))$ moves along $\mathcal{T}^k$.

For a given $\omega \in \Omega_k$, let

$$\begin{aligned}
R_0 &= Re\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\
&= \frac{1}{\omega} Re\left\{\left[h'(j\omega) - \tau_1 h(j\omega)\right] e^{-j\omega\tau_1} + \left[\tau_2 h(j\omega) - h'(j\omega)\right] e^{-j\omega\tau_2}\right\}, \\
I_0 &= Im\left(\frac{j}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial s}\right)_{s=j\omega} \\
&= \frac{1}{\omega} Im\left\{\left[h'(j\omega) - \tau_1 h(j\omega)\right] e^{-j\omega\tau_1} + \left[\tau_2 h(j\omega) - h'(j\omega)\right] e^{-j\omega\tau_2}\right\},
\end{aligned}$$

and

$$
\begin{aligned}
R_l &= Re\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Re\left(h(j\omega)e^{-j\omega\tau_l}\right), \\
I_l &= Im\left(\frac{1}{s}\frac{\partial D(s,\tau_1,\tau_2)}{\partial \tau_l}\right)_{s=j\omega} = (-1)^{l-1} Im\left(h(j\omega)e^{-j\omega\tau_l}\right),
\end{aligned}
$$

for $l = 1, 2$. Then, since $D(s, \tau_1, \tau_2)$ is an analytic function of s, τ_1 and τ_2, the implicit function theorem indicates that the tangent of $\mathcal{T}^k$ can be expressed as

$$
\begin{pmatrix} \frac{\mathrm{d}\tau_1}{\mathrm{d}\omega} \\ \frac{\mathrm{d}\tau_2}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \tag{4.1}
$$

provided that

$$
R_1 I_2 - R_2 I_1 \neq 0. \tag{4.2}
$$

It follows that $\mathcal{T}_k$ is smooth everywhere except possibly at the points where either (4.2) is not satisfied, or when

$$
\frac{\mathrm{d}\tau_1}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau_2}{\mathrm{d}\omega} = 0. \tag{4.3}
$$

From the above discussions, we can conclude:

Proposition 4.1. *The curve T^k is smooth everywhere except possibly at the points corresponding to $s = j\omega$ a multiple solution of (2.4).*

Proof. Straightforward computation. More details can be found in [8] for a more general case. □

4.2 Direction of crossing

Next, we will discuss the direction in which the solutions of (2.4) cross the imaginary axis as (τ_1, τ_2) deviates from the curve $\mathcal{T}^k$. We will call the direction of the curve that corresponds to increasing ω the *positive direction.* We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left.*

To establish the direction of crossing we need to consider τ_1 and τ_2 as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}^k$ along the positive direction is $\left(\frac{\partial \tau_1}{\partial \omega}, \frac{\partial \tau_2}{\partial \omega}\right)$, the normal to $\mathcal{T}^k$ pointing to the left hand side of the positive direction is $\left(-\frac{\partial \tau_2}{\partial \omega}, \frac{\partial \tau_1}{\partial \omega}\right)$. Corresponding to a pair of complex conjugate solutions of (2.4) crossing the imaginary axis along the horizontal direction, (τ_1, τ_2) moves along the direction $\left(\frac{\partial \tau_1}{\partial \sigma}, \frac{\partial \tau_2}{\partial \sigma}\right)$. So, if

$$
\left(\frac{\partial \tau_1}{\partial \omega}\frac{\partial \tau_2}{\partial \sigma} - \frac{\partial \tau_2}{\partial \omega}\frac{\partial \tau_1}{\partial \sigma}\right)_{s=j\omega} > 0, \tag{4.4}
$$

then a pair of complex conjugate solutions of (2.4) crosses the imaginary axis to the right half plane,as (τ_1, τ_2) moves from the region on the right to the region on the left, i.e. the region on the left of $\mathcal{T}^k$ gains two solutions on the right half plane. If the inequality (4.4) is reversed then the region on the left of $\mathcal{T}^k$ has two fewer right half plane solutions as compared to the region on the right. Similar to (4.1) we can express

$$\begin{pmatrix} \frac{\mathrm{d}\tau_1}{\mathrm{d}\sigma} \\ \frac{\mathrm{d}\tau_2}{\mathrm{d}\sigma} \end{pmatrix}_{s=j\omega} = \frac{1}{R_1 I_2 - R_2 I_1} \begin{pmatrix} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{pmatrix}. \tag{4.5}$$

Using this, we arrive at the following:

Proposition 4.2. *Let $\omega \in (\omega_k^l, \omega_k^r)$ and $(\tau_1, \tau_2) \in T^k$ such that $j\omega$ is a simple solution of (2.4) and $D(j\omega', \tau_1, \tau_2) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (τ_1, τ_2) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then, as (τ_1, τ_2) crosses $\mathcal{T}$ from the region on the right to the region on the left, a pair of solutions of (2.4) cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Proof. Easy computation shows that

$$\left(\frac{\partial \tau_1}{\partial \omega} \frac{\partial \tau_2}{\partial \sigma} - \frac{\partial \tau_2}{\partial \omega} \frac{\partial \tau_1}{\partial \sigma} \right)_{s=j\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_1 I_2 - R_2 I_1)^2}$$

Therefore (4.4) can be written as $R_2 I_1 - R_1 I_2 > 0$. $\square$

5 Illustrative examples

In this paragraph we reconsider two examples already treated in the literature.

5.1 Neutral system example

The first example considers a system of neutral type treated in [13], but using a different approach:

$$P(s) = (k_1 k_2 + 1)s + (a + k_1), \quad Q(s) = k_1(k_2 s + 1).$$

The authors in [13] assume $a > 0$ and $(a+k_1)/(k_1 k_2+1) > 0$, fact which guarantees internal stability of the closed-loop system. The so-called “practical stability” criterion is given by Assumption 2.4 which simply states $\left| \frac{k_1 k_2}{1 + k_1 k_2} \right| < \frac{1}{2} \Leftrightarrow -1/3 < k_1 k_2 < 1$.

For $a = 1$, $k_1 = 2$, $k_2 = 1/4$ we get $\Omega = (0, 2.37]$, and, in conclusion, Ω consists of only one interval of type 21. Correspondingly, $\mathcal{T}$ consists of a series of open ended curves with both ends approaching infinity, conclusion which is exactly the same to the one derived in [13], but using a different argument.

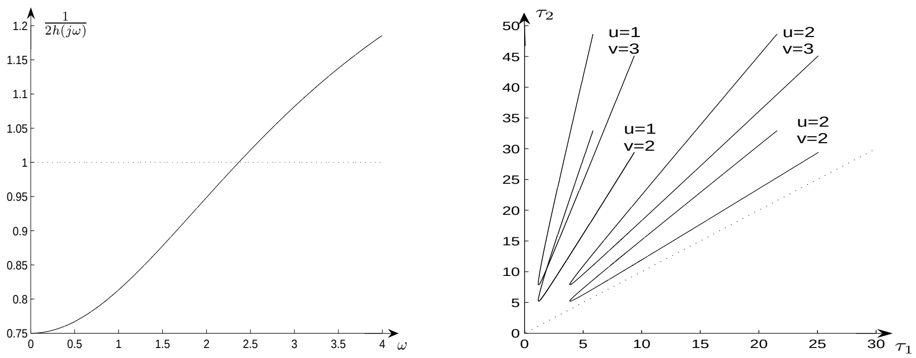


Figure 5.1: Left: The crossing set for the above system Right: Some crossing curves of this system

5.2 Smith predictor in virtual environments

The second example refers to the application of Smith predictor in motion synchronisation in virtual environments with shared haptics, and large time delays as presented, and discussed in [1]. Without entering in the details, the synchronisation scheme makes use of an appropriate feedback controller to compensate the state-error between geographically separated sites.

One of the problems discussed in [1] was the *robustness* of the scheme with respect to perturbations in time delays, and the analysis was performed by characterizing the crossing roots with respect to the imaginary axis. The method proposed in our paper gives a different, and complementary point of view with respect to such a robustness analysis problem.

More precisely, one of the models considered in [1] is given by:

$$P(s) = (s^2 + 2s + 2)^2, \quad Q(s) = -(2s + 2)^2.$$

Obviously, the system is of retarded type, so the practical stability conditions mentioned above are automatically satisfied.

The crossing set Ω consists of *only* one interval of type 21, $\Omega = (0, 2.9]$. This means that the crossing curves have the shape as presented in the figure below. Again, we obtain the same crossing curves and stability regions.

6 Concluding remarks

This paper focused on the stability crossing curves for a class of delay systems controlled by a Smith predictor, subject to some *uncertainty* in the delay. More precisely, the particular form of the closed-loop system allowed us an easy derivation of the stability crossing curves (crossing set characterization, direction of crossing,

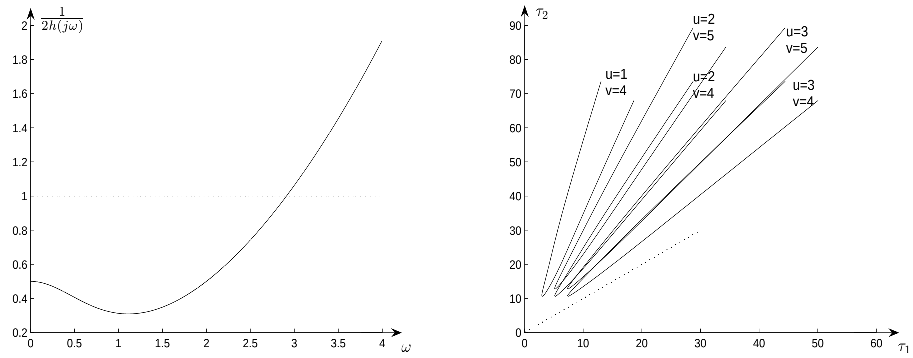


Figure 5.2: Left: The crossing set for the above system Right: Some crossing curves of this system

smoothness). Regular, and degenerate cases have been both treated. Various illustrative examples completed the presentation.

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