Asymptotic stability of some distributed delay systems: An algebraic approach

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Abstract

This paper focuses on the stability of a class of linear systems including gamma-distributed delay with a gap. More precisely, a *complete characterization of stability regions* is given in the corresponding (delay, mean-delay) parameter-space. *Optimal* delay intervals are explicitly computed. The stabilizing/destabilizing delay effect will be explicitly outlined, and discussed. Several illustrative examples complete the paper. *Copyright IFAC 2006.*

1 Introduction

The stability of dynamical systems in presence of time-delay have been extensively studied for the past 50 years. For a good introduction to the subject, see, for instance, Hale and Verduyn-Lunel (2003); Gu *et al.* (2003); Niculescu (2001) and the references therein. Most of the work that has been done treats delay differential equations with one or a few discrete delays, and it is well-known by now that the

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characterization of stability regions wrt the delays turns to be an NP-hard problem (see, e.g., Toker and Ozbay (1996)). Some insights in the quasipolynomial case including two independent delays can be found in Gu *et al.* (2005).

The problem becomes more difficult in the case when the delays are distributed. Realistic models in this sense can be encountered in modeling the physiological behavior, the population dynamics, and various schemes for controlling objects over networks. In most of the cases, the *overall delay* is defined by a *distributed* delay term (where the kernel is defined by some appropriate gamma-distribution laws in the simplest case, see, for instance, MacDonald (1989), and the references therein), with a *gap*. Such a gap[1] simply describes the *propagation*, which is inherent to most of the dynamical models encountered in biology, and in most of the closed-loop schemes for describing controlling objects over (communication) networks.

To the best of the authors' knowledge, the first population dynamics model including gamma-distributed delays is due to Cushing Cushing (1981), and it received a lot of attention starting with the 80s: Cooke and Grossman (1983); Blythe *et al.* (1985); Boese (1989), to cite only a few. The linearized model (Cooke and Grossman (1983)) simply writes as:

$$\dot{x}(t) = -\alpha x(t) + \beta \int_0^t g(t-\theta)x(\theta)d\theta, \tag{1}$$

under appropriate initial conditions. It is easy to see that a narrow distribution leads to some simple "discrete delay" system of the form $\dot{x}(t) = -\alpha x(t) + \beta x(t-h)$, whose dynamics, and stability are completely known, and understood by now (see, e.g., Hale and Verduyn-Lunel (2003)). Next, if one assumes that the delay kernel is given by the gamma-distribution law:

$$g(\xi) = \frac{a^{n+1}}{n!}\xi^n e^{-a\xi}, \tag{2}$$

the Laplace transform applied to (1), under the definition (2) reduces the stability analysis of (1) to the analysis of some parameter-dependent polynomials of the form:

$$D(s, \bar{\tau}, n) := (s+\alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta = 0, \tag{3}$$

where $\bar{\tau} = \dfrac{n+1}{a}$ denotes the corresponding *mean-delay* value. One of the problem discussed in Cooke and Grossman (1983) was the analysis of the behavior of

[1] By a *gap*, we usually understand a "discrete delay value" added to the corresponding model (see, for instance, the terminology in MacDonald (1989).

the roots of the characteristic equation with respect to the imaginary axis when the mean delay value $\bar{\tau}$, or the exponent n are varying. The main interest of such a study was to compute the stability regions with respect to the corresponding parameters, and to analyze the sensitivity of such regions when the parameters change. Further discussions on this topics can be found in MacDonald (1989).

Next, Nisbet and Gurney (1983) mention that population dynamics models based on partial differential equations, and reduced for convenience to integro-differential forms are more realistic if the corresponding delay kernel $\hat{g}$ includes some *gap* (see also Blythe *et al.* (1985); MacDonald (1989)), that is if the kernel can be expressed as:

$$\hat{g}(\xi) = \begin{cases} 0, & \xi < \tau \\ \frac{a^{n+1}}{n!}(\xi - \tau)^n e^{-a(\xi-\tau)}, & \xi \geq \tau, \end{cases} \tag{4}$$

for some positive delay values τ. Simple computations prove that the corresponding *mean delay* is defined by $\hat{\tau} = \tau + \dfrac{n+1}{a}$. In this case, the stability analysis becomes more complicated, since the parameter-dependent polynomial $D(s, \bar{\tau}, n)$ in (3) becomes a parameter-dependent quasipolynomial of the form (see, for instance, Blythe *et al.* (1985); Boese (1989)):

$$D(s, \bar{\tau}, \tau, n) \quad := \quad (s + \alpha)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} - \beta e^{-s\tau}. \tag{5}$$

It is important to note that, even for this simple example, some of the first results (Cooke and Grossman (1983); Blythe *et al.* (1985)) concerning its stability analysis includes errors as discussed by Boese (1989).

Recently, it was pointed out that such gamma-distri-buted delays with some gap can be also encountered in the problem of controlling objects over communication networks (Roesch *et al.* (2005)). More explicitly, the overall communication delay in the network is modeled by a gamma-distributed delay with a gap, where the *gap* value corresponds to the minimal *propagation delay* in the network, which is always a strictly positive quantity. Without entering in the details, the stability of the closed-loop system reduces to the stability analysis of the following parameter-dependent quasipolynomial:

$$D(s, \bar{\tau}, \tau, n) \quad := \quad Q(s)\left(1 + s\frac{\bar{\tau}}{n+1}\right)^{n+1} + P(s)e^{-s\tau}, \tag{6}$$

where $P(s)$, $Q(s)$ are polynomials of some appropriate degree. It is quite simple to observe that (5) represents a particular case of (6).

In this paper, we consider systems described by (6), and we shall give a *complete characterization* of the behavior of the roots of (6) in the parameter-space *(gap, mean-delay)*. This paper can be seen as the "dual" of Morărescu *et al.* (2005a), where the characterization of the crossing curves[2] was given using some geometric arguments. More precisely, we shall *explicitly* compute all the "points" $\left(\tau, \dfrac{\bar{\tau}}{n+1}\right)$, for which a change of the number of roots in $\mathbb{C}_+$ will take place, and next for *each* mean-delay value interval, an explicit computation of the corresponding (stability) delay interval can be performed.

The interest of the approach is *twofold*:

- First, the computation of the corresponding delay intervals can be performed relatively easily, and the corresponding algorithm includes a finite number of steps. Furthermore, various interesting *instability* cases can be detected, and the underlying ideas can be applied to various other delay analysis problems;

- Second, the propagation delay (gap τ) can be used as a *design parameter* in the case of controlling objects over communication network. Such an idea was already exploited in the context of constant communication delays (see, e.g., Niculescu (2002)), and to the best of the authors' knowledge, there does not exist any extension in the distributed delay case. In other words, the propagation delay can be used to define a so-called "wait-and-act" strategy similar to the one encountered in synchronisation, and also mentioned in the case of delayed output feedback stabilization problems (Niculescu (2001)), *etc*.

The remaining paper is organized as follows: In Section 2 we briefly present the problem formulation and some prerequisites necessary to develop our (frequency-domain) stability analysis. The main results are presented in Section 3, and illustrative examples are given in Section 4. Some concluding remarks end the paper. For the brevity of the paper, the proofs are omitted, but they can be found in the full version of the paper (Morărescu *et al.* (2005b)).

2 Problem formulation, and preliminaries

Without any loss of generality, the (asymptotic) stability of (5), and (6) is equivalent to:

$$D(s,T,\tau) = Q(s)(1+sT)^n + P(s)\mathrm{e}^{-s\tau} = 0. \tag{7}$$

[2] Crossing curves represent curves in the delay-parameter space for which at least one root of the corresponding characteristic equation lies on the imaginary axis.

for some appropriate pair (T,τ). We will make now the following supplementary *assumptions*: (i) $\deg(Q) = n_q > \deg(P) = n_p$; (ii) $P(0) + Q(0) \neq 0$; (iii) $P(s)$ and $Q(s)$ have no common zeros.

The assumption (i) can be relaxed to $n_q \geq n_p$, but with the supplementary constraint $\lim_{s\to\infty} \dfrac{Q(s)}{P(s)} < 1$ if equality (see Gu *et al.* (2003) for some discussions on retarded, and neutral systems). If assumption (ii) is not satisfied then 0 becomes one zero of (7) for any $(T,\tau) \in \mathbb{R}^2_+$, and therefore (7) can never be stable. Finally, if (iii) is violated there is a common factor $c(s) \neq$ constant such that $P(s) = c(s)P_1(s)$ and $Q(s) = c(s)Q_1(s)$, and the problem can be reduced to the previous case using the pair (P_1, Q_1) instead of (P,Q), etc.

The problem addressed in the sequel can be resumed as follows: *deriving necessary, and sufficient conditions in terms of* (T,τ) *for guaranteeing the asymptotic stability of (7).*

In this sense, the following two quantities will play a major role in the stability study:

1) $card(\mathcal{U})$, where $\mathcal{U}$ is the set of roots of $D(s,T,0) = 0$, situated in the *closed right half plane*, and $\mathrm{card}(\cdot)$ denotes the cardinality (number of elements).

2) $card(\mathcal{S})$, where $S = \{\omega > 0 \mid (1+\omega^2T^2)^n|Q(j\omega)|^2 - |P(j\omega)|^2 = 0\}$.

The characteristic equation (7) is said to be *hyperbolic* at some point (T_0,τ_0) if no root of the characteristic equation lies on the imaginary axis for $T = T_0$, and $\tau = \tau_0$ (see,e.g. Hale *et al.* (1985)). Thus, we have the following result:

Proposition 1 *The system (7) is hyperbolic for all* $(T,\tau) \in \mathbb{R}_+ \times \mathbb{R}_+$ *if and only if:*

$$|Q(j\omega)| > |P(j\omega)|, \qquad \forall \omega \in \mathbb{R}^*, \tag{8}$$

Furthermore, if $card(\mathcal{U}) = 0$ *(*> 0*) for* $T = 0$*, the system is delay-independent stable (unstable).*

Remark 1 *In the stability case, the frequency-sweeping test (8) represents a slight modification of the Tsypkin criterion (see, for instance, Niculescu (2001); Gu* et al. *(2003)), and it gives a simple condition for which* $card\mathcal{S}) = 0$ *for* all *the pairs* (T,τ).

In the sequel, we shall assume that the condition (8) in Proposition 1 does *not hold*. If not, we have stability (or instability) for all the pairs (T,τ), etc. In conclusion, the problem of interest is reduced to analyze the cases when *crossing roots exist*.

Without any loss of generality, assume now that $Q(0) \neq 0$. If not, we get $P(0) = 0$ from (7), which is not possible since it contradicts the assumption (ii). The next step is the characterization of the way the quantities $card(\mathcal{U})$, $card(\mathcal{S})$ depend on the parameter T if $\tau = 0$.

2.1 Quantity $card(\mathcal{U})$

Introduce now the following Hurwitz matrix associated to some polynomial $A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$:

$$H(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \dots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \dots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \dots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \dots & a_{2n_a-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{9}$$

where the coefficients $a_l = 0$, for all $l > n_a$. Next, it is easy to see that $D(s, T, 0)$ can be rewritten as: $D(s, T, 0) = \sum_{k=0}^{n} P_k(s) T^k$, with $P_0(s) = P(s) + Q(s)$, $P_1(s) = sQ(s)$, …, $P_n(s) = s^n Q(s)$ Next introduce the matrix pencil: $\Sigma(\lambda) = det(\lambda U + V)$, with U, V given by:

$$U = \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & H(P_n) \end{bmatrix}, \quad V = \begin{bmatrix} 0 & -I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -I \\ H(P_0) & H(P_1) & \cdots & H(P_{n-1}) \end{bmatrix},$$

where the identity, and the zero-blocks matrices have appropriate dimension, and $H(P_k) \in \mathbb{R}^{(n+n_q)\times(n+n_q)}$ represents the corresponding Hurwitz matrix[3] associated to the polynomial $P_k(s)$ defined above.

The following result gives the characterization of $card(\mathcal{U})$ as a function of T, and represent a generalization of some matrix pencil method proposed by Chen (1995) in the context of static output feedback for SISO systems:

Proposition 2 *Let* $0 < \lambda_1 < \lambda_2 < \dots \lambda_h$, *with* $h \leq n + n_q$ *be the real eigenvalues of the matrix pencil* $\Sigma(\lambda) = det(\lambda U + V)$. *Then the system (7) cannot be stable*

[3] The order of P_k is $n_q + k$, for all $k = 0, \dots, n$, and $H(P_k)$ will be constructed as a $(n + n_q) \times (n + n_q)$ matrix by setting the coefficients of high-order terms as zeroes, that is $p_\ell = 0$, for all $\ell > n + k$.

for any $T = \lambda_i$*,* $i = 1, 2, \dots h$*. Furthermore, if there are* r *unstable roots (*$0 \leq r \leq n + n_q$*) for* $T = T^*$*,* $T^* \in (\lambda_i, \lambda_{i+1})$*, then, there are* r *unstable roots for any mean-delay value* $T \in (\lambda_i, \lambda_{i+1})$*. In other words,* $\mathrm{card}(\mathcal{U})$ *remains constant as* T *varies within each interval* $(\lambda_i, \lambda_{i+1})$*. The same holds for the intervals* $(0, \lambda_1)$ *and* (λ_h, ∞)*.*

Proposition 2 allows studying the behavior of $card(\mathcal{S})$ as a function of T. First we have to compute the positive real eigenvalues of Σ, and then the number of unstable roots inside each interval defined by the corresponding eigenvalues. The characterization is complete when computing $\mathcal{U}$ for intermediate values of T.

2.2 Quantity $card(\mathcal{S})$

Based on the arguments, assumptions, and remarks above, we have the following result:

Proposition 3 *If the* $card(\mathcal{S})$ *changes at a value* T^* *then there exists a frequency* $\omega^* > 0$ *such that for* $\omega = \omega^*$ *the following relations hold:*

$$F(\omega, T) = (1 + \omega^2 T^2)^n |Q(j\omega)|^2 - |P(j\omega)|^2 = 0 \tag{10}$$

and

$$\frac{d}{d\omega}\left[\frac{1}{\omega^2}\left(\left|\frac{P(j\omega)}{Q(j\omega)}\right|^{2/n} - 1\right) - T^2\right] = 0 \tag{11}$$

Proof. For any T, F cannot have a root ω where $Q(j\omega) = 0$, because this would imply that also $P(j\omega) = 0$. So that the roots of F coincide with the roots of

$$G(\omega, T) = \frac{1}{\omega^2}\left(\left|\frac{P(j\omega)}{Q(j\omega)}\right|^{2/n} - 1\right) - T^2 = 0 \tag{12}$$

A changes of $card(\mathcal{S})$ at $T = T^*$ implies that $G(\omega, T^*)$ has a root with multiplicity larger than one at some frequency ω^*, i.e.

$$G(\omega^*, T^*) = \frac{\mathrm{d}}{\mathrm{d}\omega}[G(\omega^*, T^*)] = 0.$$

This leads to (10) and (11).

Remark 2 *The equation* $\dfrac{d}{d\omega}[G(\omega^*, T^*)] = 0$ *has a* finite number *of roots. Thus, the quantity* $card(\mathcal{S})$ *changes for a* finite number *of values of* T*.*

As in the previous case, the characterization is complete when computing $\mathcal{S}$ for intermediate values of T.

3 Stability analysis

For the sake of simplicity, assume that all the roots of F are *simple*. Notice that this condition is satisfied for almost all T. Next, we need to explicitly compute the sensitivity of the roots with respect to the delay parameter τ when crossing the imaginary axis, that is, in other words, the *delay crossing direction*. We have the following result:

Theorem 1 *The characteristic equation has a root $j\omega$ on the imaginary axis for some τ_0 if and only if $\omega \in \mathcal{S}$. Furthermore, for $\omega \in \mathcal{S}$, the set of corresponding values of τ where $card(\mathcal{U})$ changes is given by*[4]

$$\mathcal{T}_\omega = \left\{ \frac{1}{\omega} \left[-jLog \frac{P(j\omega)}{(1+j\omega T)^n Q(j\omega)} + 2k\pi \right] \geq 0, \qquad k \in \mathbb{Z} \right\} \tag{13}$$

When increasing the delay, the corresponding crossing direction of characteristic roots is towards instability (stability) when $F'(\omega) > 0 (< 0)$.

The above theorem combined with the continuous dependence of the characteristic roots with respect to the delay, allows to say that $\mathcal{T} = \bigcup_{\omega \in \mathcal{S}} \mathcal{T}_\omega$ makes a partitions of the τ-delay space ($\mathbb{R}_+$) into intervals in which the *number of roots in the open right half plane is constant.* Such an argument will be used in developing our stability region characterization.

3.1 Small delays

3.1.1 Robustness stability issues

First, assume that the system free of delays is asymptotically stable ($\tau, T = 0$), that is $card(\mathcal{U}) = 0$ with $T = 0$, and that the frequency-sweeping condition (8) does not hold. Then Theorem 1, combined with the Propositions 2, and 3 give a simple way to compute the first delay-intervals guaranteeing stability:

Proposition 4 *Under the assumption $card(\mathcal{U}) = 0$ for the system free of delays, the system (7) is asymptotically stable for all the pairs (T, τ), with $0 \leq T < T^*$, where T^* is the smallest positive generalized eigenvalue of Σ, and $\tau \in [0, \tau^*)$, where τ^* is given by:*

$$\tau^* = \min_{\omega \in \mathcal{S}(T)} \{\mathcal{T}_\omega(T)\} \tag{14}$$

as a function of T, for all $T \in [0, T^*)$.

[4] Here, Log denotes the principal value of the logarithm. Consequently when $|z| = 1$, $\text{Log}(z) = j \arg(z)$

In other words, Proposition 4 defines the explicit dependence of the stability boundary in (T, τ) space bounded by the corresponding OT, and $O\tau$-axis, and by the curve $\tau(T)$, defined as a function of T, for all $T \in [0, T^*)$. The case $T = 0$ gives the standard first delay-interval bound (see, e.g. Niculescu (2001)). Using the terminology of Gu *et al.* (2003), we derive the corresponding *delay margins* in $OT\tau$ parameter-space.

3.1.2 Delay-induced stability/instability

Assume now that the system free-of-delays ($\tau = 0$, $T = 0$) is *unstable*. We start by presenting various cases in which the *gap*, seen as a free-parameter cannot have a stabilizing effect. We have the following results:

Proposition 5 *If the $card(\mathcal{U})$ is an odd number then the stability of the system cannot be obtain increasing the time delay τ.*

Proposition 6 *If $card(\mathcal{S}) \in \{0, 1\}$ then the stability of the system cannot be obtain increasing the time delay τ.*

The first case, when the delay gap τ may induce stability in the system by increasing its value appears when $card(\mathcal{S}) \in \{2, 3\}$. More precisely, we have the following result:

Proposition 7 *If $card(\mathcal{S}) \in \{2, 3\}$ then the stability of the system can be obtain increasing the time delay τ, if and only if:*
1. $card(\mathcal{U}) = 2$
2. $\tau_- < \tau+$, *where* $\begin{cases} \tau_- = \min \bigcup\limits_{\omega \in \mathcal{S},\, F'(\omega) < 0} \mathcal{T}_\omega \\ \tau_+ = \min \bigcup\limits_{\omega \in \mathcal{S},\, F'(\omega) > 0} \mathcal{T}_\omega \setminus \{0\} \end{cases}$
In this case, for all delay values $\tau \in (\tau_-, \tau_+)$ the system is stable.

Remark 3 *One can conclude that in the previous case is sufficiently to investigate the first crossing in order to check the stabilizability in the delay. When one determines the stability by numerically computations the Proposition 4 is very useful because we can stop the computations after the first root crossing.*

In the case $card(\mathcal{S}) = 2$, the set of all stabilizing delay values can be expressed analytically:

Corollary 1 *Assume that the following conditions are satisfied*

1. $card(\mathcal{S}) = 2$

2. $card(\mathcal{U}) = 2$

3. $\tau_- < \tau+$

Then all the stabilizing delay values are defined by $\tau \in (\underline{\tau}_k, \overline{\tau}_k), \quad k = 0, 1, ..., k_m$*, where*

$$\underline{\tau}_k = \tau_- + \frac{2k\pi}{\omega_-}, \quad \overline{\tau}_k = \tau_+ + \frac{2k\pi}{\omega_+}$$

and k_m *is the largest integer for which* $\underline{\tau}_k < \overline{\tau}_k$*, which can be explicitly expressed as*

$$k_m = \max_{l \in \mathbb{Z}} \left\{ l < \frac{\omega_- \omega_+}{\omega_+ - \omega_-} \cdot \frac{\tau_+ - \tau_-}{2\pi} \right\} \tag{15}$$

3.2 General case

Based on the results, and the remarks above, we have the following

Proposition 8 *Assume that* $card(\mathcal{S}) = 2p$ *or* $card(\mathcal{S}) = 2p + 1$*, with* $p \geq 1$ *and* $card(\mathcal{U}) > 2p$*. Then there does not exist any gap* $\tau > 0$ *such that (7) becomes asymptotically stable.*

Define now the following quantities:

$$n_+(\tau) = \sum_{\omega \in \mathcal{S}_+,\ F'(\omega) > 0} \text{card} \left\{ \mathcal{T}_\omega \cap (0, \tau] \right\}, \tag{16}$$

$$n_-(\tau) = \sum_{\omega \in \mathcal{S}_+,\ F'(\omega) < 0} \text{card} \left\{ \mathcal{T}_\omega \cap [0, \tau] \right\}, \tag{17}$$

for some positive $\tau > 0$. Furthermore, introduce the sets $\mathcal{T}^+$, and $\mathcal{T}^-$, which represent a partition of $\mathcal{T}$ in function of the sign of the derivative F' evaluated at the corresponding crossing frequency, that is:

$$\mathcal{T}^+ = \bigcup_{\omega \in \mathcal{S}_+,\ F'(\omega) > 0} \mathcal{T}_\omega \setminus \{0\},$$

$$\mathcal{T}^- = \bigcup_{\omega \in \mathcal{S}_+,\ F'(\omega) < 0} \mathcal{T}_\omega.$$

Based on the conditions and the notations above, we conclude with the following result:

Proposition 9 *For a given* T *the system with characteristic equation (7) is asymptotically stable if and only if the following conditions are satisfied:*

1. *$card(\mathcal{U}(\mathcal{T}))$ is a strictly positive even integer and the following inequality holds: $card(\mathcal{U}(\mathcal{T})) \leq card(\mathcal{S}(\mathcal{T}))$*

2. *there exists at least one gap value $\tau^* \in \mathcal{T}$, such that: $n_-(\tau^*) = n_+(\tau^*) + card(\mathcal{U}(\mathcal{T}))$.*

Then all gap values $\tau \in (\tau^*, \tau_+^*)$, with $\tau_+^* = \min\{\mathcal{T}^+ \cap (\tau^*, +\infty)\}$ guarantee the asymptotic stability.

4 Illustrative examples

Several examples are considered (see Morărescu *et al.* (2005b), the full version of the paper). For the sake of brevity, we present only two simple examples: the Cushing equation, and a second-order system, respectively.

Example 1 (linearized Cushing equation with a gap) *In this example we apply the above method for the Cushing linearized equation $(s+a)(1+sT)^n + be^{-s\tau} = 0$, $a > 0$, $b < 0$. First it is easy to remark that $(s+a)(1+sT)^n + b$ has at least one (strictly) unstable root if and only if $a + b < 0$. Consider the case $n = 1$, that is the polynomial $F(\omega, T)$ is given by:*

$$\begin{aligned} F(\omega, T) &= (\omega^2 + a^2)(1 + \omega^2 T^2) - b^2 \\ &= \omega^4 T^2 + \omega^2(a^2 T^2 + 1) + a^2 - b^2. \end{aligned} \tag{18}$$

For $a^2 - b^2 \geq 0$ ($a + b \neq 0$) we have $card(\mathcal{S}) = 0$, that is no crossing with respect to the imaginary axis for all T (see Proposition 1), while for $a^2 - b^2 < 0$ we have $card(\mathcal{S}) = 1$. According to the results of the previous section, the stability of the Cushing equation can be delay-independent stable (unstable), function of the sign *of $a + b$ for all (T, τ) if $card(\mathcal{S}) = 0$. If the system is not delay-independent stable (unstable), Proposition 4 will give the corresponding delay-intervals for which stability is preserved under the assumption of asymptotic stability for some mean-delay intervals (in T) given by Proposition 2, etc.*

Example 2 (second-order system) *Consider the following second-order system:*

$$P(s) = -s, \quad Q(s) = s^2 + 2 \tag{19}$$

Simple computations prove that $Q(s)(1 + sT) + P(s)$ has two unstable roots. So that $card(\mathcal{U}) = 2$.

The characteristic equation of the closed-loop system is given by

$$(s^2+2)(1+sT)-se^{-s\tau}=0 \tag{20}$$

and polynomial $F(\omega,T)$ by

$$\begin{aligned} F(\omega,T) &= (2-\omega^2)^2+(1+\omega^2T^2)-\omega^2 \\ &= \omega^6T^2+\omega^4(1-4T^2)+\omega^2(4T^2-5)+4. \end{aligned}$$

So we need to find how many positive roots has the following equation:

$$x^3T^2+x^2(1-4T^2)+x(4T^2-5)+4 \tag{21}$$

First it is easy to see that the previous equation has at least one real negative solution because $x_1x_2x_3=-\frac{4}{T^2}<0$ (where x_1, x_2, x_3 are the solutions of the equation (21)). Computing the discriminant and the Hurwitz determinants of the equation (21) we find $card(\mathcal{S})=\begin{cases} 2 & T>\frac{1}{2} \\ 0 & T\le\frac{1}{2} \end{cases}$. According to the result of the previous section a necessary condition for asymptotic stability of the closed-loop system is given by

$$T>\frac{1}{2} \tag{22}$$

Furthermore, for T satisfying (22) the existence of a stability region in the delay parameter is determined by the condition $\tau_-<\tau_+$.

Summarizing,we have:

Proposition 10 *The system (19) is asymptotically stable if and only if $T>\frac{1}{2}$ and in addition $\tau_-<\tau_+$, where:*

$$\tau_-=\min\bigcup_{\omega\in\mathcal{S},\,F'(\omega)<0}\frac{1}{\omega^2T},\qquad \tau_+=\min\bigcup_{\omega\in\mathcal{S},\,F'(\omega)>0}\frac{1}{\omega^2T}$$

A stability region is defined by the pair (T,τ), where $T>\frac{1}{2}$, and $\tau\in(\tau_-(T),\tau_+(T))$.

5 Concluding remarks

This paper addressed the stability problem of a class of of linear systems including distributed delays with a gap. A characterization of stability regions in the (mean-delay,gap) parameter-space has been proposed. Illustrative examples complete the presentation.

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