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Consensus and influence power approximation in time-varying and directed networks subject to perturbations*

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SUMMARY

The paper focuses on the analysis of multi-agent systems interacting over directed and time-varying networks in presence of parametric uncertainty on the interaction weights. We assume that agents reach a consensus and the main goal of this work is to characterize the contribution that each agent has to the consensus value. This information is important for network intervention applications such as targeted advertising over social networks. Indeed, for an advertising campaign to be efficient, it has to take into account the influence power of each agent in the graph (*i.e.,* the contribution of each agent to the final consensus value). In our first results we analytically describe the trajectory of the overall network and we provide lower and upper bounds on the corresponding consensus value. We show that under appropriate assumptions, the contribution of each agent to the consensus value is smooth both in time and in the variation of the uncertainty parameter. This allows approximating the contribution of each agent when small perturbations affect the influence of each agent on its neighbors. Finally, we provide a numerical example to illustrate how our theoretical results apply in the context of network intervention. Copyright © 0000 John Wiley & Sons, Ltd.

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1. INTRODUCTION

Analysis and control of multi-agent systems have attracted a lot of attention during the last decades. Due to decentralized behavior and control of subsystems (agents), the multi-agent framework has applications in a wide variety of domains such as robotic teams, energy and telecommunication networks, opinion dynamics in social networks, analysis of biological networks, etc. The coherent behavior of the agents in such systems is often described in terms of consensus, *i.e.,* the agents have to reach agreement in some variables of interest.

While many works focus on the consensus in networks with fixed topologies, a number of

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applications lead to networks with time-varying topologies as in [21, 15]. For instance, in robotic fleets, the weight of agents can vary in time according to different priorities such as connectivity preservation, collision avoidance and achievement of a global coordination [26]. In social networks the confidence in individuals varies also in time according to their state evolution. Therefore, the consensus analysis has been extended to networks with time-varying weights [9, 12] but also in networks of clusters that sporadically interact [5, 17]. The conditions for consensus express the need of persistent interactions ensuring the connectivity of the overall network of agents [3]. To the best of our knowledge, in this framework of time-varying interactions, only conditions to achieve consensus have been investigated without considering the characterization of the consensus value in term of influence power of each agent.
An interesting application of consensus algorithms is the analysis and control of opinion dynamics in social networks [8, 16, 11]. Targeted marketing over digital social networks has recently become a key tool in advertising. Finding the major influencers in the social network is an essential ingredient of the process [4]. For opinion dynamics driven by consensus systems, detecting who the key influencers are, boils down to knowing the contribution of each particular agent in forming the consensus value [23]. This contribution can be understood as a measure of centrality in the social network. However, the knowledge that a marketer has on the social network topology usually comes from estimations rather than direct knowledge. This motivates us to assume that the interaction weights are not precisely known and some bounded uncertainties affect them.
In many robotic fleet applications the main objective is consensus or synchronization when interactions take place according to switching topologies [10, 19]. Unlike that, in social networks it is interesting to know who are the main agents that influence the consensus value and have rough information on the value that will be achieved by the system although the interaction weights are continuously evolving and the convergence speed is generally limited. In this context, we will first address the problem of computation of the asymptotic consensus value in terms of initial condition and interaction network. It is noteworthy that, the characterization of consensus value has been provided only in some special cases. Precisely, when the interaction network is balanced (*i.e.,* the adjacency matrix is doubly stochastic), the consensus is always the average of the initial states [20, Theorem 4]. For directed but fixed interactions, the consensus is a weighted average of the initial states and the weights correspond to the components of a given left normalized eigenvector of the Laplacian matrix (see for instance [20]). The consensus value for some particular linear impulsive systems was characterized in [5]. In general, for linear time-varying dynamics, it is known that the asymptotic consensus value is always a convex combination of the states of the agents at any time t. However, it is not trivial to characterize the weights of these states in the convex combination since they depend on the interaction network over time. Consequently, it is interesting to also provide an approximation of the consensus value and of the influence power of each agent when uncertainties (modelled as perturbations) affect the interaction weights.

Contributions of the paper. First, Section 3.1 provides an analytic characterization of the asymptotic consensus value. The weights of the agents in the convex combination describing the consensus value can be seen as the agents contributions to the consensus value, which can also be seen as the overall influence power that agents have over their peers. So, we call the vector formed by these weights the *agent influence power vector* (AIPV) (see also [23]). We show that the AIPV can

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be expressed in terms of the fundamental (state transition) matrix. Moreover, we prove that, when the interaction network satisfies the cut-balance property [9], each *agent influence power* (AIP) is lower bounded by a strictly positive value which we make explicit. In other words, any agent has a nonzero and non vanishing influence power. Consequently, the overall consensus value can be controlled by exogenously influencing any of the agents. Nevertheless, the control is more efficient when the most influent agents are targeted by the exogenous actions.

As mentioned above, in the context of network intervention, it is often more reasonable to assume that only a perturbed version of the interaction network weights is known. The second original contribution concerns the problem of approximation of consensus value and agent influence power for perturbed interaction weights. On one hand, we show that when the interaction weights vary smoothly both in time and in the perturbation parameter ε, the consensus value and the AIPV are also smooth both in time and in ε (Sections 3.2 and 3.3). This result holds only for specific dynamics including the linear time-varying ones considered in this paper. For general linear dynamics, it is only known that the trajectory is smooth point-wise at any fixed given time, however, in some cases, the asymptotic convergence value may fail to be smooth with respect to (*w.r.t.*) ε. For instance, consider the simple linear equation $\dot{y} = -\varepsilon y$, when $\varepsilon > 0$, the solutions converge to 0 for all initial conditions while when $\varepsilon = 0$ the solution is the initial condition. The failure in this example is due to the non-uniformity of the stability property *w.r.t.* parameter ε which is prevented in the sequel thanks to an appropriate assumption.
In Section 4, we apply the results from the previous section to the particular case of additive perturbations which is a standard way to model identification errors in the network intervention application. In such a context, the knowledge a marketer has on the interaction network weights comes from a learning algorithm employed by a marketer to identify the network topology [25]. The perturbation on the interaction weights depicts the error produced by the algorithm. The more information the algorithm collects over time, the smaller the perturbation becomes. We show in Section 4.1 that, when the perturbations vanish sufficiently fast in time, the AIPV reach a limit when time goes to infinity. This is a non trivial result since we also provide an example where this limit does not exist when the perturbation does not respect the fast vanishing assumption although the system itself converges to a consensus (Section 4.2). The practical meaning of this result is the following. If the AIPV converge in time, it is then possible to use its current approximation for a future marketing campaign. The final section (Section 5) shows how our theoretical results apply in the context of network intervention.

The present work is an extension of preliminary version [14]. Here, we add complementary results showing that the AIPV is smooth both *w.r.t.* time and to the perturbation parameter ε, for any kind of smooth interaction weights. Finally, we provide a concrete example of application in the context of intervention in digital social networks.

Notation. The following notation will be used throughout the paper. The sets of non-negative integers, real and non-negative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For a vector v and a matrix A we denote by $\|v\|_\infty = \max_i |v_i|$ and $\|A\|_\infty = \max_i \sum_j |A_{ij}|$ their infinity norms. We denote by $\mathcal{L}_1$ the space of integrable functions on $\mathbb{R}_+$. For a matrix-valued function of time M in $\mathcal{L}_1$, we denote $|||M||| = \int_{t_0}^{+\infty} \|M(s)\|_\infty ds$ where t_0 is a fixed initial time. The transpose of a matrix A is denoted by $A^\top$. By I_k we denote the $k \times k$ identity matrix. $\mathbf{1}_k$ and $\mathbf{0}_k$ are the column vectors of size k having all the components equal 1 and 0, respectively. Vector e_i denotes the i-th canonical

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vector in $\mathbb{R}^n$, *i.e.,* the vector of all zeros but one 1 for its i-th coefficient, for some given $i \in \mathcal{N}$.
A non trivial subset S of a set C, denoted as $S \sqsubset C$, is a non-empty set with $S \subsetneq C$. A *directed path of length* p in a given directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ is a union of directed edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \dots, p-1\}$. The node j is *connected* to node i in a directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ if there exists at least one directed path in $\mathcal{G}$ from i to j.

2. PRELIMINARIES

2.1. Model statement

Let $\mathcal{N} \triangleq \{1, \dots, n\}$ be a set of $n \geq 2$ agents. By abuse of notation we denote both an agent and its index by the same symbol $i \in \mathcal{N}$. Each agent is characterized at time t by a scalar state $x_i(t) \in \mathbb{R}, \forall i \in \mathcal{N}$ that evolves according to the following consensus system

$$\begin{array}{l} \dot{x}_i = \sum_{j=1}^{n} a_{ij}(t, \varepsilon)(x_j - x_i), \\ x_i(t_0) = x_{t_0,i}(\varepsilon), \end{array} \tag{1}$$

where $t_0 \geq 0$ is a fixed given initial time, $x_{t_0,i}(\varepsilon) \in \mathbb{R}$ a given initial state value (that depends on ε) for agent i, $\varepsilon \in I$ is a perturbation parameter with $I \subset \mathbb{R}_+$ a fixed bounded interval. We assume that interval I has a finite supremum ε^*. The functions $a_{ij} : \mathbb{R}_+ \times I \to \mathbb{R}_+$ represent the *interaction weights* and are assumed to be measurable.

It is noteworthy that system (1) is an extension of the well-known DeGroot opinion dynamics model [6] . It simply says that agent's opinion tends to the neighbors ones while interacting. Unlike [6] we consider that the confidence of agents in their neigbors is not constant but time-varying.

We call the solution of system (1) the *trajectory* of the overall system and we denote it by $x(\cdot, t_0, \varepsilon)$ to point out its dependence on the parameters ε and t_0. We say the trajectory asymptotically reaches a consensus when there exists a common agreement value $\alpha(t_0, \varepsilon) \in \mathbb{R}$ such that

$$\lim_{t \to +\infty} x_i(t, t_0, \varepsilon) = \alpha(t_0, \varepsilon), \ \forall i \in \mathcal{N}. \tag{2}$$

When convergence to consensus occurs, we denote

$$x^*(t_0, \varepsilon) = \lim_{t \to +\infty} x(t, t_0, \varepsilon) = \alpha(t_0, \varepsilon)\mathbf{1}_n.$$

For any $\varepsilon \in I$ we denote $A(t, \varepsilon) = [a_{ij}(t, \varepsilon)]_{ij}$ the adjacency matrix of communication weights at time t, $D(t, \varepsilon) = diag(d_{ii}(t, \varepsilon))$ with $d_{ii}(t, \varepsilon) = \sum_{j \in \mathcal{N}} a_{ij}(t, \varepsilon)$, and $L(t, \varepsilon) = D(t, \varepsilon) - A(t, \varepsilon)$ its associated Laplacian matrix. Using the matrix notation, system (1) can be represented as

$$\dot{x}(t, t_0, \varepsilon) = -L(t, \varepsilon)x(t, t_0, \varepsilon), \quad x(t_0, t_0, \varepsilon) = x_{t_0}(\varepsilon). \tag{3}$$

2.2. Framework assumptions

In the following let us introduce some notation and the main hypotheses of this work. We denote

$$c(t, \varepsilon) = \min_{S \sqsubset \mathcal{N}} \sum_{i \in S, j \notin S} a_{ij}(t, \varepsilon), \tag{4}$$

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which is known as the *edge connectivity* [24].

Assumption 1
The interaction weights satisfy the two following properties :

- $c(t,\varepsilon) > 0, \forall t \geq t_0$ and $\int_{t_0}^{+\infty} c(s,\varepsilon)ds = +\infty$.
- cut-balance: there exists $K(\varepsilon) \geq 1$ such that for all non trivial subsets $S \sqsubset \mathcal{N}$, $t \geq t_0$ and $\varepsilon \in I$,

$$\sum_{i\in S, j\notin S} a_{ij}(t,\varepsilon) \leq K(\varepsilon) \cdot \sum_{i\in S, j\notin S} a_{ji}(t,\varepsilon),$$

From opinion dynamics perspective Assumption 1 is interpreted as follows. First, any set of agents has a sufficiently important influence (the integral over time of this influence is infinite meaning that it does not rapidly vanish) on the rest of the network. Second, while a set of individuals influences the rest of the network, the reciprocal is also true *i.e.,* the rest of the network influences the set with a certain amont and the ratio between the two influences does not converge to 0 or ∞.

Proposition 1 (see [14] for proof)
Suppose that Assumption 1 holds. Then, for any fixed $\varepsilon \in I$, and initial time t_0, the trajectory of system (1) reaches consensus (the value generally depends on t_0 and ε).

The speed of convergence to consensus depends on the rate at which $\int_{t_0}^{\top} c(s,\varepsilon)ds$ diverges (see Proposition 4 in [12] for an explicit bound on the convergence speed). The disagreement between agents can be characterized using the distance to consensus :

$$\|x(t,t_1,\varepsilon) - \alpha(t_1,\varepsilon)\mathbf{1}_n\|_\infty.$$

To show that the consensus value is smooth *w.r.t.* parameter ε, we will require the distance to consensus to be integrable, therefore we add the following assumption.

Assumption 2
Suppose that Assumption 1 holds so that consensus takes place. Moreover, suppose that the disagreement converges to 0 uniformly in ε, and is uniformly integrable with convergence characterized by a function $d(t,t_1)$ such that for all $t_1, t \geq 0$ with $t \geq t_1$,

$$\|x(t,t_1,\varepsilon) - \alpha(t_1,\varepsilon)\mathbf{1}_n\|_\infty \leq d(t,t_1)\|x_{t_1} - \alpha(t_1,\varepsilon)\mathbf{1}_n\|_\infty,$$

where x_{t_1} is the initial condition set at time t_1 and

$$\int_{t_1}^{+\infty} d(t,t_1)dt < +\infty \text{ and } d(t,t_1) \leq 2, \forall t_1 \geq t_0.$$

Finally, assume that $\forall \varepsilon \in I,\ K(\varepsilon) \in [1, K^*]$ where $K^* \geq 1$ is a uniform upper bound.

This assumption introduces finite bounds on the convergence speed and on the reciprocity of the influences. For instance, this is satisfied in the context of opinion dynamics since agreement is not instantaneously achieved.

Remark 1
If the uniform bound on $K(\varepsilon)$ is satisfied in Assumption 2, one way to verify the uniform

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integrability of $x(t,t_0,\varepsilon)-\alpha(t_0,\varepsilon)\mathbf{1}_n$ is to suppose there is a uniform lower bound $\underline{c}>0$ such that $\forall s\geq t_0, \forall\varepsilon\in I, c(s,\varepsilon)\geq\underline{c}$, in which case $x(t,t_0,\varepsilon)$ converges uniformly exponentially fast to $\alpha(t_0,\varepsilon)\mathbf{1}_n$ (see Proposition 4 in [12]). A simple way to obtain this property is to make sure that the interaction graph includes a strongly connected subgraph $(\mathcal{N},\mathcal{E}_C)$ in which all interaction weights remain lower bounded : $\forall i,j\in\mathcal{N},(i,j)\in\mathcal{E}_C\Rightarrow a_{ij}\geq\underline{a}$ with $\underline{a}>0$.

Lemma 1
The function d in Assumption 2 can always be chosen such that $d(t,t_1)\leq 2$.

Proof
We have that

$$\|x(t,t_1,\varepsilon)-\alpha(t_1,\varepsilon)\mathbf{1}_n\|_\infty\leq\max_{i,j\in\mathcal{N}}|x_i(t,t_1,\varepsilon)-x_j(t,t_1,\varepsilon)|\leq$$
$$\max_{i,j\in\mathcal{N}}|x_i(t_1,t_1,\varepsilon)-x_j(t_1,t_1,\varepsilon)|\leq 2\|x_{t_1}-\alpha(t_1,\varepsilon)\mathbf{1}_n\|_\infty.$$

where the first inequality comes from the fact that consensus value is in the convex hull of the states of the agents at time t. The second inequality follows from the fact that disagreement is not increasing in consensus algorithms. Finally, the third inequality is a simple application of the triangular inequality for norms. □

From Proposition 1, it is known that the trajectory of system (1) will converge to consensus. It remains to characterize the consensus value $\alpha(t_0,\varepsilon)$. The analysis is carried out for general time-varying, non-symmetric communication weights.

3. PARAMETRIC TIME-VARYING CONSENSUS SYSTEMS

The aim of this section is to study the behavior of the time-varying consensus system (1) parametrized by $\varepsilon\in I$. As recalled below, the consensus value is always a convex combination or weighted average of the current states of the agents. These weights correspond to the respective contributions or importance that the agents have in forming the consensus value. These contributions are called *agent influence power. In general, we show that non-trivial bounds can be obtained on these agent influence powers* provided that the interaction weights satisfy some reciprocity condition. Our main objective is to show that the AIPs and the consensus value vary smoothly both in time and in the perturbation parameter. This will allow us to detect the most influent agents in presence of perturbations, which is useful to efficiently control the overall consensus dynamics. Although we provide rough approximations of the AIPs and of the consensus value, the paper does not focus on the accurate approximation of these values.

3.1. Consensus value

To characterize the consensus value approximation, we heavily rely on the property of the fundamental matrix of system (3). We start by defining this matrix and provide some properties associated with its evolution and limit in time. This will in turn allow us to describe the consensus value.

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For all $\varepsilon \in I$ we define, for all $t, t_1 \geq t_0$ with $t \geq t_1$, the fundamental matrix $\Phi(t, t_1, \varepsilon)$ of system (3) such that

$$x(t, t_0, \varepsilon) = \Phi(t, t_1, \varepsilon) x(t_1, t_0, \varepsilon). \tag{5}$$

It is important to define Φ not only for $t_1 = t_0$ since we will need to study $\lim\limits_{t_1 \to +\infty} \lim\limits_{t \to +\infty} \Phi(t, t_1, \varepsilon)$. The fundamental matrix has the following important property [2] :

Property 1

For all $t_1 \geq t_0$ and $\varepsilon \in I$, the fundamental matrix $\Phi(t, t_1, \varepsilon)$ is invertible, it is independent of the initial state $x_{t_0}(\varepsilon)$ and it is solution of the following system :

$$\begin{cases} \frac{\partial \Phi}{\partial t}(t, t_1, \varepsilon) = -L(t, \varepsilon)\Phi(t, t_1, \varepsilon), \\ \Phi(t_1, t_1, \varepsilon) = I. \end{cases} \tag{6}$$

First, we notice that the fundamental matrix components form convex combinations to obtain the future state in function of past state. Indeed, as a direct consequence of Lemma 6 in [13], for all $t_1, t \geq t_0$ with $t \geq t_1$, for any $i, j \in \mathcal{N}$, weight $\Phi_{ij}(t, t_1, \varepsilon)$ is non-negative and

$$\sum_{j \in \mathcal{N}} \Phi_{ij}(t, t_1, \varepsilon) = 1. \tag{7}$$

To relate the fundamental matrix to the final consensus value, we present he following rather standard result on the limit of the fundamental matrix.

Lemma 2

For any $t_1 \geq t_0$ and $\varepsilon \in I$, matrix $\Phi(t, t_1, \varepsilon)$ has a limit when $t \to +\infty$, which we denote by

$$\Phi^*(t_1, \varepsilon) = \lim_{t \to +\infty} \Phi(t, t_1, \varepsilon).$$

Moreover, there exists $q(t_1, \varepsilon) \in \mathbb{R}^n$ such that

$$\Phi^*(t_1, \varepsilon) = \mathbf{1}_n (q(t_1, \varepsilon))^\top. \tag{8}$$

Similarly to $\Phi(t, t_1, \varepsilon)$, the vector $q(t_1, \varepsilon)$ is independent of the initial state $x(t_0, t_0, \varepsilon) = x_{t_0}(\varepsilon)$.

Reciprocally, vector q can be obtained in function of the fundamental matrix limit as

$$q(t_1, \varepsilon) = \frac{1}{n} (\Phi^*(t_1, \varepsilon))^\top \mathbf{1}_n. \tag{9}$$

The vector $q(t, \varepsilon)$ plays a major role in the rest of the study, in particular it relates the final consensus value $\alpha(t_0, \varepsilon)$ defined in equation (2) to the current states $x(t, t_0, \varepsilon)$ as given in the next instrumental lemma.

Lemma 3

Quantity $q(t, \varepsilon)^\top x(t, t_0, \varepsilon)$ is invariant in time and defines the *consensus value*, *i.e.,* for all $\varepsilon \in I$, for all $t \geq t_0$,

$$q(t, \varepsilon)^\top x(t, t_0, \varepsilon) = \alpha(t_0, \varepsilon).$$

Moreover, the coefficients $q_i(t, \varepsilon)$ are non-negative for all $i \in \mathcal{N}$ and $t \geq t_0$, and sum up to one :

$$q(t, \varepsilon)^\top \mathbf{1}_n = 1. \tag{10}$$

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Lemma 3 precisely characterizes the the consensus value $\alpha(t_0, \varepsilon)$ and shows that it is a convex combination of the current states $x_i(t, t_0, \varepsilon)$ weighted by the coefficients in vector $q(t, \varepsilon)$ (itself related to the limit of the fundamental matrix). Consequently, quantity $q_i(t, \varepsilon)$ corresponds to the contribution of the agent i's state in the final consensus value $\alpha(t_0, \varepsilon)$. Thus, we call q the *agent influence power vector* (AIPV).

Accordingly, knowing bounds on coefficients $q_i(t, \varepsilon)$ will result in approximations for the consensus value $\alpha(t_0, \varepsilon)$. We provide such bounds in the next theorem.

Theorem 1

Under Assumption 1, the components $q_i(t, \varepsilon)$ are positive uniformly bounded *w.r.t.* time t: for all $t \geq t_0$ and $i \in \{1, \ldots, n\},\ n \geq 2$ one has

$$q_i(t, \varepsilon) \in [q_{min}(\varepsilon), q_{max}(\varepsilon)] \subseteq (0, 1),$$

with

$$q_{min}(\varepsilon) = \left(\frac{e^{-K(\varepsilon)}}{n}\right)^{n-1},\ q_{max}(\varepsilon) = 1 - (n-1)q_{min}(\varepsilon).$$

Proof

See Appendix 7.1. □

One can check that $q_{min}(\varepsilon) \in (0, \frac{1}{n})$ and $q_{min}(\varepsilon) \leq q_{max}(\varepsilon)$. Moreover, if Assumption 2 is satisfied, uniform bounds *w.r.t.* ε can be found for $q_{min}(\varepsilon)$ and $q_{max}(\varepsilon)$ by replacing $K(\varepsilon)$ by K^*. It is noteworthy that the lower-bound $q_{min}(\varepsilon)$ in Theorem 1 can be close to 0 if an agent has small influence on the consensus value. However, our result states that under Assumption 1 all the agents have a non-zero contribution to consensus. In other words, the opinion of each individual matters in the overall dynamics since all the components of the AIPV are non-zero. Consequently, one can control the overall consensus value by continuously controlling the state of any agent despite its AIP is very small. It is worth noting here that this result is a generalization of the classic result stating that the left eigenvector associated with the eigenvalue 0 of a fixed Laplacian matrix (of a strongly connected directed graph) has strictly positive components

Remark 2

We note that Theorem 1 does not hold in general without the cut-balance of the interaction network contained in Assumption 1. For instance, if the interaction network is given by a tree, it can be shown that the consensus value is the initial state of the root, irrespective of the initial state of the other agents.

Remark 3

Note that Theorem 1 could have been stated on consensus systems with no dependance on a parameter ε, but we kept this notation to remain consistent over the paper.

Corollary 1

Suppose Assumption 1 holds. Then, the final consensus value of (1) is within the following bounds:

$$\alpha(t_0, \varepsilon) \in [q_{min}(\varepsilon) \cdot \min_{i \in \mathcal{N}} x_{t_0,i}\ ,\ q_{max}(\varepsilon) \cdot \max_{i \in \mathcal{N}} x_{t_0,i}],$$

where $q_{min}(\varepsilon)$ and $q_{max}(\varepsilon)$ are defined in Theorem 1.

As stated in the Introduction, if the network is represented as a fixed directed graph, it is well known that the consensus is a weighted average of the initial opinions and the weight of $x_{t_0,i}$ in

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this average is the i^{th} component of the left eigenvector (q) associated with the eigenvalue 0 of the Laplacian matrix. The result in Corollary 1 is aligned with that since it states that consensus in time varying case is also in between the bounds characterizing the weighted average of the initial opinions with weights defined by the minimum and maximum AIP.

3.2. *Smoothness w.r.t. time of the agent influence power vector*

This section and the next one analyse two other important properties of the AIPV: one is smoothness *w.r.t.* time t and the other is smoothness *w.r.t.* parameter ε. These properties allow quantifying how the final consensus value and the AIPs are influenced by perturbations on the communication weights. For this we need the following supplementary assumption.

Assumption 3

- The functions $a_{ij} : \mathbb{R}_+ \times I \to \mathbb{R}_+$ are twice continuously differentiable and uniformly upper-bounded functions in both variables. We denote the supremum as

$$\bar{a} = \sup_{i,j\in\mathcal{N},t\geq 0,\varepsilon\in I} a_{ij}(t,\varepsilon). \tag{11}$$

- We also assume that its derivatives *w.r.t.* ε are uniformly bounded :

$$\bar{d}_a = \sup_{i,j\in\mathcal{N},t\geq 0,\varepsilon\in I} \left| \frac{\partial a_{ij}}{\partial \varepsilon}(t,\varepsilon) \right|. \tag{12}$$

- The initial condition $x_{t_0,i}(\varepsilon)$ is twice continuously differentiable and uniformly lower and upper-bounded in ε.

This Assumption makes sense in the framework of network interventions since the first part assumes bounded influences of each individual on the others while the second part states that influence weights cannot vary infinitely fast when the precision on their values has small variations.

Remark 4

Notice that Assumption 1 and the upper bound $\bar{a}$ on the interaction weights defined in equation (11) does *not* imply a uniform lower bound on the non-zero interaction weights. In fact, to satisfy the lower bound on the edge-connectivity given in the first point of Assumption 1, some weights $a_{ij}(t,\varepsilon)$ may converge to 0 as long as this is compensated by some other stronger weights $a_{ik}(t,\varepsilon)$.

Let $x(t,t_0,\varepsilon) = (x_1(t,t_0,\varepsilon),\ldots,x_n(t,t_0,\varepsilon))^\top \in \mathbb{R}^n$ be the overall solution of the network collecting the states of all the agents. Existence, uniqueness and smoothness of the solution x are given in the following lemma.

Lemma 4 (see [14] for proof)

Let the initial time t_0 and initial condition $x_{t_0}(\varepsilon)$ be given. Under the smoothness and boundedness Assumption 3 on $a_{ij}(t,\varepsilon)$ and $x_{t_0}(\varepsilon)$, for any fixed ε there exists a unique function $x(\cdot,t_0,\varepsilon)$: $[t_0,+\infty) \to \mathbb{R}^n$ whose components satisfy equation (1) for all $t \in [t_0,+\infty)$. Moreover, this function is uniformly bounded in all its arguments and twice continuously differentiable *w.r.t.* both t and ε and its first and second partial derivatives are continuous for all $t \geq t_0$ and $\varepsilon \in I$.

Lemma 5

Suppose Assumption 3 holds. The fundamental matrix $\Phi(t,t_1,\varepsilon)$ associated with the general consensus system (1) is twice continuously differentiable *w.r.t.* t and ε and its partial derivative *w.r.t.* ε evolves according to dynamics

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial \varepsilon}(t,t_1,\varepsilon) = -L(t,\varepsilon)\frac{\partial \Phi}{\partial \varepsilon}(t,t_1,\varepsilon) - \frac{\partial L}{\partial \varepsilon}(t,\varepsilon)\Phi(t,t_1,\varepsilon),$$

with uniform initial condition

$$\frac{\partial \Phi}{\partial \varepsilon}(t_1,t_1,\varepsilon) = 0, \forall t_1 \geq t_0, \varepsilon \in I.$$

Proof
Lemma 4 transfers the smoothness of the interaction weights $a_{ij}(t,\varepsilon)$ to smoothness of the trajectory. Equation (6) describes the trajectory of the fundamental matrix $\Phi(t,t_1,\varepsilon)$. So, applying Lemma 4 to equation (6) shows that $\Phi(t,t_1,\varepsilon)$ is twice continuously differentiable *w.r.t.* t and ε and its first and second partial derivative are continuous for all $t \geq t_0$ and $\varepsilon \in I$. Then, Schwarz's theorem [1] provides

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial \varepsilon}(t,t_1,\varepsilon) = \frac{\partial}{\partial \varepsilon}\frac{\partial \Phi}{\partial t}(t,t_1,\varepsilon).$$

Differentiating the right hand side of system (3) *w.r.t.* ε yields

$$\frac{\partial}{\partial t}\frac{\partial \Phi}{\partial \varepsilon}(t,t_1,\varepsilon) = \frac{\partial}{\partial \varepsilon}\Big(-L(t,\varepsilon)\Phi(t,t_1,\varepsilon)\Big) = -L(t,\varepsilon)\frac{\partial \Phi}{\partial \varepsilon}(t,t_1,\varepsilon) - \frac{\partial L}{\partial \varepsilon}(t,\varepsilon)\Phi(t,t_1,\varepsilon).$$

Moreover, since by definition $\Phi(t_1,t_1,\varepsilon) = I_n$ for all $\varepsilon \in I$, $\frac{\partial \Phi}{\partial \varepsilon}(t_1,t_1,\varepsilon) = 0$. □

Notice that the partial derivative of the Laplacian matrix L *w.r.t.* ε appearing in Lemma 5 inherits from L the right eigenvector $\mathbf{1}_n$ associated with the eigenvalue 0 :

$$\frac{\partial L}{\partial \varepsilon}(t,\varepsilon_0)\mathbf{1}_n = \lim_{\varepsilon \to \varepsilon_0} \frac{L(t,\varepsilon) - L(t,\varepsilon_0)}{\varepsilon - \varepsilon_0}\mathbf{1}_n = 0. \quad (13)$$

This property will be used later on to derive result on the smoothness of the consensus value *w.r.t.* ε (see Theorem 3). We have provided smoothness properties of the fundamental matrix, we now turn to its relation to the consensus value.

Theorem 2
Suppose that Assumptions 1 and 3 are satisfied. Then, $q(t,\varepsilon)$ is continuously differentiable *w.r.t.* time t. Let $t_1 \geq t_0$ and $\varepsilon \in I$, then $\frac{\partial q_i}{\partial t}(t_1,\varepsilon_0)$ is obtained without loss of generality by choosing a specific initial state $x(t_0,t_0,\varepsilon) = x_{t_0}(\varepsilon)$ so that $x(t_1,t_0,\varepsilon) = e_i$, where e_i is the i-th canonical vector. In this case there holds

$$\frac{\partial q_i}{\partial t}(t_1,\varepsilon_0) = -q(t_1,\varepsilon_0)\frac{\partial x}{\partial t}(t_1,\varepsilon_0,t_0).$$

Proof
See Appendix 7.2. □

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3.3. *Smoothness w.r.t. perturbation parameter of the agent influence power vector*

We now turn to the smoothness of q *w.r.t.* parameter ε. This result relies on the continuity *w.r.t.* a parameter of a limit of the solution to a differential equation when $t \to +\infty$.

Theorem 3

Suppose that Assumptions 1, 2 and 3 are satisfied. Then, $q(t,\varepsilon)$ and $\Phi^*(t,\varepsilon)$ are continuously differentiable *w.r.t.* parameter ε. In this case there holds

$$\frac{\partial q}{\partial \varepsilon}(t_1,\varepsilon_0) = \frac{1}{n}\left(\frac{\partial \Phi^*}{\partial \varepsilon}(t_1,\varepsilon_0)\right)^\top \mathbf{1}_n$$

where

$$\frac{\partial \Phi^*}{\partial \varepsilon}(t_1,\varepsilon) = \lim_{t\to+\infty}\int_{t_1}^{\top} G(s,t,t_1,\varepsilon)ds,$$

with

$$G(s,t,t_1,\varepsilon) = -\Phi(t,s,\varepsilon)\frac{\partial L}{\partial \varepsilon}(s,\varepsilon)\Phi(s,t_1,\varepsilon).$$

Moreover,

$$\left\|\tfrac{\partial q}{\partial \varepsilon}(t_1,\varepsilon)\right\|_\infty \quad \leq \quad 2\int_{t_1}^{+\infty} d(s,t_1)\max_{i,j\in\mathcal{N}}\left|\frac{\partial a_{ij}}{\partial \varepsilon}(s,\varepsilon)\right| ds. \tag{14}$$

Theorem 3 is a substantial generalization of Theorem 2 in [14] to the case of general time-varying communication weights depending on a parameter. A direct corollary of Theorem 3 is the following.

Corollary 2

Suppose that Assumptions 1, 2 and 3 hold. Then, for all $t_1 \geq t_0$, $\varepsilon, \varepsilon_0 \in I$,

$$\|q(t_1,\varepsilon) - q(t_1,\varepsilon_0)\|_\infty \leq 2|\varepsilon-\varepsilon_0| \int_{t_1}^{+\infty} d(s,t_1) \max_{i,j\in\mathcal{N},\varepsilon_1\in I}\left|\frac{\partial a_{ij}}{\partial \varepsilon}(s,\varepsilon_1)\right| ds.$$

The corollary comes from applying the mean value theorem to the bound found in Theorem 3. If required, the uniform bound on the derivative of $\partial a_{ij}/\partial\varepsilon$ from equation (12) can be used to obtain a uniform Lipschitz constant. Since, according to Lemma 3, the final consensus value is $\alpha(t_0,\varepsilon) = q(t_0,\varepsilon)^\top x(t_0,t_0,\varepsilon)$, another straightforward corollary to Theorem 3 is the following :

Corollary 3

Suppose that Assumptions 1, 2 and 3 are satisfied. Then the final consensus value $\alpha(t_0,\varepsilon)$ is continuously differentiable *w.r.t.* parameter ε.

4. CONSENSUS VALUE APPROXIMATION UNDER PERTURBED INTERCONNECTIONS

In this section we use the results proven in the previous section to solve the main problem considered in this paper that is the approximation of the AIPV and of the consensus value when the interaction weights are not perfectly known. The results are provided under a supplementary assumption on the behavior of the interaction weights and of the perturbations that affects their variation. We also show that the imposed assumption is necessary since when it does not hold we can built a counterexample.

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4.1. AIPV approximation in presence of vanishing perturbation

In this section, we consider additive perturbations on the influence weights and their effect on the consensus value and AIPV. We focus on dynamics (1) where the interaction weights are of the form

$$a_{ij}(t,\varepsilon) = l_{ij}(t) + \varepsilon m_{ij}(t), \tag{15}$$

where $\varepsilon \in I$ is a small perturbation parameter. Here, we assume either $I = [0, \varepsilon^*)$ or $I = [0, \varepsilon^*]$ with $\varepsilon^* > 0$.

Assumption 4
We assume that $\exists \varepsilon^*$ such that

$$a_{ij}(t,\varepsilon) \geq 0, \ \forall t \geq t_0, \varepsilon \in [0, \varepsilon^*].$$

Moreover, the functions $l_{ij}(t) \geq 0$ and $m_{ij}(t)$ are twice continuously differentiable and uniformly upper-bounded. Moreover, we assume that the upper-bound on $m_{ij}(t)$ is integrable:

$$d_{max} \triangleq \int_{t_0}^{+\infty} \max_{i,j\in\mathcal{N}} m_{ij}(t)dt < +\infty. \tag{16}$$

Finally, as in the previous section, the initial condition $x_{t_0,i}(\varepsilon)$ is twice continuously differentiable and uniformly lower and upper-bounded in ε.

The first part guarantees that the $a_{ij}(t,\varepsilon)$ correspond to the components of a weighted adjacency matrix. The last part of the assumption ensures the smoothness of the interaction weights and implies Assumption 3. In the sequel, we assume that Assumption 4 is verified. In matrix form, system (1) with interaction weights perturbed as in (15) rewrites as

$$\begin{cases} \dot{x}(t,t_0,\varepsilon) = -(L(t) + \varepsilon M(t))x(t,t_0,\varepsilon), \\ \text{with } x(t_0,t_0,\varepsilon) = x_{t_0}(\varepsilon). \end{cases} \tag{17}$$

where matrix $M(t)$ is the Laplacian matrix associated with the adjacency matrix $(m_{ij}(t))_{ij}$.

The next result provides a characterization of the convergence speed of $q(t_1,\varepsilon)$ when the parameter $\varepsilon \to 0$. Consequently, one also obtains a characterization of the convergence speed of $\alpha(t_0,\varepsilon)$ when the parameter $\varepsilon \to 0$.

Proposition 2
Suppose that Assumptions 1, 2 and 4 holds. For all $\varepsilon \in [0, \varepsilon^*]$ the following holds

$$\|q(t_1,\varepsilon) - q(t_1,0)\|_\infty \quad \leq \quad 4\varepsilon \int_{t_1}^{+\infty} \max_{i,j\in\mathcal{N}} |m_{ij}(s)|\, ds,$$

and in particular,

$$\forall t_1 \geq 0, \lim_{\varepsilon\to 0} q(t_1,\varepsilon) = q(t_1,0).$$

Consequently,

$$|\alpha(t_0,\varepsilon) - \alpha(t_0,0)| \leq 4\varepsilon\delta(\varepsilon) \int_{t_0}^{+\infty} \max_{i,j\in\mathcal{N}} |m_{ij}(s)|\, ds,$$

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where $\delta(\varepsilon) = \|x_{t_0}(\varepsilon) - x_{t_0}(0)\|$.

Proof

First notice that $\frac{\partial a_{ij}}{\partial \varepsilon}(s,\varepsilon) = m_{ij}(s)$. As a consequence, we can apply Corollary 2 to show that

$$\|q(t_1,\varepsilon) - q(t_1,0)\|_\infty \leq 2\varepsilon \int_{t_1}^{+\infty} d(s,t_1) \max_{i,j\in\mathcal{N}} |m_{ij}(s)|\, ds \leq 4\varepsilon \int_{t_1}^{+\infty} \max_{i,j\in\mathcal{N}} |m_{ij}(s)|\, ds,$$

where we have used $d(s,t_1) \leq 2$ according to Assumption 2. The last part of the statement becomes a direct corollary of Lemma 3. □

Corollary 4

Under Assumptions 1, 2 and 4 we have

$$\lim_{t_1\to+\infty} q(t_1,\varepsilon) = \lim_{t_1\to+\infty} q(t_1,0), \forall\varepsilon \in [0,\varepsilon^*].$$

In the context of targeted advertising, the time-convergence of the AIPV q given by Corollary 4 means that the contribution of each agent will undergo little evolution after some time. This enables a marketer to use its present approximation of the agent influence powers for its subsequent campaigns. The time-convergence of q may seem natural. However, it is not always true for general consensus systems of type (3), even in instances where the systems converges to consensus. This fact is illustrated via an example in Section 4.2.

4.2. *Example of oscillating q*

The aim of this section is to show that systems which involve uniformly varying weights with *non vanishing* perturbation, unlike the ones described in Section 4.1, may lead to a non converging vector q. Let us study a system of type (1) with two agents. The interaction weights have the following evolution in time

$$\begin{cases} a_{12}(t,\varepsilon) = \frac{1}{2}(1+\varepsilon\cos(\eta t)), \\ a_{21}(t,\varepsilon) = \frac{1}{2}(1-\varepsilon\cos(\eta t)), \end{cases} \tag{18}$$

where $\varepsilon \in [0,1)$ and $\eta \geq 0$ are constant parameters. In other words, we assume that the confidence of each agent in its neighbor oscillates around $1/2$ which may be interpreted as passive and active phases of each individual. This translates the fact that each AIP become periodically dominant in the computation of the final agreement. This system satisfies Assumption 1 with finite $K(\varepsilon) = \frac{1+\varepsilon}{1-\varepsilon} \geq 1$, and as a consequence converges to consensus (see Proposition 1). However, the system does not satisfy the condition (16) in Assumption 4, required for Proposition 2 to apply. Setting $x_1(0) = 1$ and $x_2(0) = 0$ and integrating equation (1), we can show that

$$x_1(t,0,\varepsilon) - x_2(t,0,\varepsilon) = e^{-t},$$

from which, by integration, exact solutions for x_1 and x_2 can be obtained as :

$$x_1(t,0,\varepsilon) = \alpha(0,\varepsilon) + \frac{e^{-t}}{2}\left(1 + \frac{\varepsilon(\cos(\eta t) - \eta\sin(\eta t))}{\eta^2+1}\right),$$

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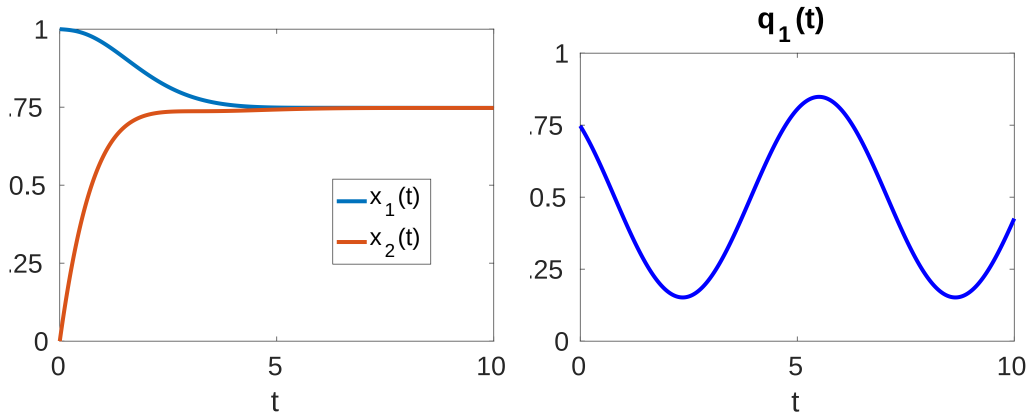

Figure 1. (Left) trajectory of 2-agent system (1) for interaction weights defined in equation (18) with $\varepsilon = 0.9$, $\eta = 1$. (Right) first component $q_1(t, \varepsilon)$ of the convex combination providing the asymptotic consensus value α in function of the current state $x(t, 0, \varepsilon)$.

where the final consensus value is

$$\alpha(0, \varepsilon) = \frac{1}{2}\left(1 - \frac{\varepsilon}{\eta^2 + 1}\right).$$

For this case, the convex combination q can be obtained using $q_1 + q_2 = 1$ and $q(t, \varepsilon)^\top x(t, 0, \varepsilon) = \alpha(0, \varepsilon)$ so that

$$q_1(t, \varepsilon) = 1 - \frac{x_1(t, 0, \varepsilon) - \alpha(0, \varepsilon)}{e^{-t}} = \frac{1}{2}\left(1 - \frac{\varepsilon(\cos(\eta t) - \eta \sin(\eta t))}{\eta^2 + 1}\right),$$

which does not converge despite asymptotic consensus for x_1 and x_2 as long as $\varepsilon, \eta \neq 0$ as shown in Fig. 1: the contribution of x_1 and x_2 for the asymptotic consensus value keeps oscillating over time.

5. APPLICATION TO NETWORK INTERVENTION

To close this paper, we explain how our theoretical results apply in the context of network intervention. In order that the discussion remains easy to follow, we use a simple network of $n = 15$ agents connected based on the fixed directed graph from Fig. 2 below.

We assume that a marketer wants to advertise a service/product over the network. The agents' states represent their opinions or level of desire toward this product. If chosen between $[0, 1]$, agents' states can also be seen as the probability that an agent will buy the product. As explained in [23], the revenue of the marketer is proportional to the number of agents favoring this product. Opinions are assumed to evolve according to the consensus dynamics (1). Following the marketing strategy proposed in [18], the marketer targets certain agents to modify their opinions and potentially affect the rest of the network indirectly over time. To maximize its revenue, the marketer must first identify the key influencers in the network. As shown in [18], these are precisely the ones who have the greatest agent influence power $q_i(t, \varepsilon)$. In Section 3.1, we have made explicit that the agent influence

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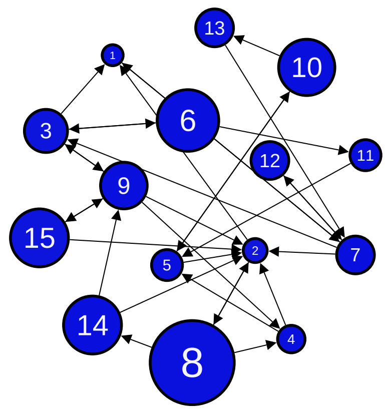

Figure 2. Graph representing the network topology. The size of each node is proportional to the corresponding agent influence power

power vector q can be computed if the interaction weights involved in the opinion dynamics are known. However, it is often more realistic to suppose that the values of the interaction weights have been estimated by a learning algorithm employed by a marketer to identify the network topology such as the one developed in [25]. As a consequence, a relevant model for the estimated interaction weights is the one given by the perturbed weights (15) from Section 4 where ε corresponds to a measure of accuracy of the learning algorithm. Since the revenue is related to an appropriate identification of the agent influence powers, the following question arises : how do identification errors on the interaction weights impact the approximation of the agent influence powers ? This is precisely the information that Proposition 2 provides giving the distance between the true AIPV when $\varepsilon = 0$ and the estimated one. To illustrate how the perturbation level ε affects the AIPV, we show in Fig. 3 the AIPV for different levels of perturbation. Precisely we consider that $l_{ij} \in \{0, 1\}$ according to the graph in Fig. 2 while $m_{ij} \in [0, 1]$ is randomly chosen and remains fixed. Then we take $\varepsilon \in \{0,\ 0.1,\ 0.5,\ 1\}$. As proven by our main results, the AIPV depends smoothly on parameter ε so that when ε is closer to 0 the AIPV is better approximated. Consequently, a company that wants to perform targeted advertising will improve its revenue by investing in a learning algorithm providing a more accurate AIPV (see [25]).

6. CONCLUSION

In this paper, we have studied linear consensus systems with directed and time-varying interactions. A first focus was on characterizing the asymptotic consensus value of the system. The consensus value can be expressed in terms of the contribution of each agent in the final consensus that have been called the *agent influence powers* (AIPs). Knowing the AIPs enable to identify the key influencers in a network. A second focus was to provide approximations for the AIPs. Under the cut-balance assumption on the interaction weights, the AIP are strictly positive meaning that each agent plays a non vanishing role in the asymptotic consensus value. We have also shown that the AIPs vary smoothly *w.r.t.* additive perturbations on the interaction weights. Finally, we have demonstrated that

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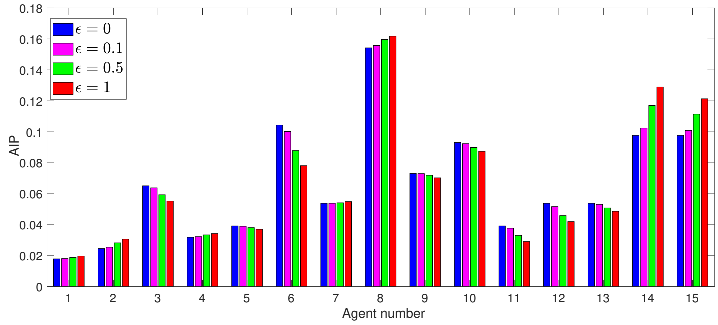

Figure 3. Comparison of the estimations of agent influence powers (AIP) q_i for various levels of perturbation ε. For each agent, the leftmost bar represents the true agent influence power obtained when interaction weights are completely known ($\varepsilon = 0$). On the opposite, the rightmost bar is the estimated consensus share when the uncertainty on the interaction weights is highest ($\varepsilon = 1$).

for $\mathcal{L}_1$ additive perturbation, the AIPs converge asymptotically in time. On the contrary, when this assumption is not satisfied, we have provided a counter-example where the AIPs keep oscillating over time while the agents converge to a consensus. Finally, we have provided an interpretation of the theoretical results in the context of network intervention such as targeted marketing over digital social networks.

7. APPENDIX

7.1. *Proof of Theorem 1*

First let $\varepsilon \in I$. Since ε is fixed throughout this Appendix section, we drop the ε notation altogether. In this Appendix, we provide intermediate results: Lemma 6, Lemma 7 and Corollary 5 which allow to obtain Theorem 1. Let $t \geq t_0$ and $S \sqsubset \mathcal{N}$. Denote $\tau_S(t)$ the first time after t for which the cumulated influence send by group S exceeds 1 :

$$\tau_S(t) = \min \left\{ \tau \geq t | \int_t^\tau \sum_{i \in \mathcal{N} \setminus S,} \sum_{j \in S} a_{ij}(s) ds = 1 \right\}.$$

The first point in Assumption 1 guarentees that $\tau_S(t)$ is finite.

Lemma 6
Let $t \geq t_0$. For all $S \sqsubset \mathcal{N}$, there exists $i \in \mathcal{N} \setminus S$ such that for all $j \in S \cup \{i\}$,

$$\sum_{h \in S} \Phi_{jh}(t, \tau_S(t)) \geq \eta,$$

where $\eta = \exp(-K)/n$.

Proof
We define an artificial trajectory u satisfying system (3) over $[t, \tau_S(t)]$ with initial states $u_h(t) = 1$ for $h \in S$ and $u_i(t) = 0$ for $i \in \mathcal{N} \setminus S$ (this idea was also exploited in Lemma 6 and Remark 3 in [13]). Then, for all $s \in [t, \tau_S(t)]$, and $j \in \mathcal{N}$, the trajectory remains in the convex hull of the

initial states: $u_j(s) \in [0,1]$. By equation (5), for all $j \in \mathcal{N}$,

$$u_j(\tau_S(t)) = \sum_{h\in\mathcal{N}} \Phi_{jh}(t,\tau_S(t))u_h(t) = \sum_{h\in S} \Phi_{jh}(t,\tau_S(t)).$$

It remains to show there exists $i \in \mathcal{N} \setminus S$ such that

$$u_i(\tau_S(t)) \geq \eta, \tag{19}$$

which we do in two steps. First we start by showing that for all $s \in [t, \tau_S(t)]$ and $h \in S$,

$$u_h(s) \geq e^{-K}. \tag{20}$$

The second step is to prove that equation (19) holds by contradiction.

Step 1. Denote $\underline{h}(s) \in S$ and $\underline{u}(s)$ such that

$$\underline{u}(s) = u_{\underline{h}}(s) = \min_{h\in S} u_h(s).$$

In [9], it has been proven that for almost all time $s \in [t, \tau_S(t)]$,

$$\dot{\underline{u}}(s) = \sum_{i\in\mathcal{N}} a_{\underline{h}i}(s)(u_i(s) - \underline{u}(s)).$$

Moreover, $a_{\underline{h}i}(s) \geq 0$. Also, by definition of $\underline{u}$, $(u_i(s) - \underline{u}(s)) \geq 0, \forall i \in S$, so these terms can be ignored from the sum. Finally recall all states remain in the convex hull of the initial conditions : $\forall i \in \mathcal{N} \setminus S, u_i(s) \geq 0$. As a consequence, using Assumption 1, we have

$$\dot{\underline{u}}(s) \geq -\sum_{i\in\mathcal{N}\setminus S} a_{\underline{h}i}(s)\underline{u}(s) \geq -\left(\sum_{i\in\mathcal{N}\setminus S}\sum_{h\in S} a_{hi}(s)\right)\underline{u}(s) \geq -K\left(\sum_{i\in\mathcal{N}\setminus S}\sum_{h\in S} a_{ih}(s)\right)\underline{u}(s).$$

From this linear differential inequality, a comparison theorem provides a lower bound on $\underline{u}(s)$. Using the definition of $\tau_S(t)$, this lower bound becomes $\underline{u}(s) \geq e^{-K}\underline{u}(t) = e^{-K}$, where $\underline{u}(t) = 1$ comes from the choice of initial condition $u_h(t) = 1$ for $h \in S$. Equation (20) is now granted.

Step 2. Assume, for the purpose of showing contradiction, that for all $i \in \mathcal{N} \setminus S$ and $s \in [t, \tau_S(t)]$,

$$u_i(s) < \eta = \frac{e^{-K}}{n} < e^{-K}. \tag{21}$$

Equations (20) and (21) mean that groups S and $\mathcal{N} \setminus S$ remain separated by the distinct thresholds $\frac{e^{-K}}{n}$ and e^{-K} so that

$$\underline{u}(s) - \max_{i\in\mathcal{N}\setminus S} u_i(s) \geq e^{-K} - \frac{e^{-K}}{n} = \frac{(n-1)e^{-K}}{n} \triangleq \gamma.$$

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Denote $\Sigma(s) = \sum_{i\in\mathcal{N}\setminus S} K^{-i}u_i(s) \geq 0$. Then, the cut-balance Assumption 1 allows applying [12, Lemma 9] to obtain :

$$\dot{\Sigma}(s) \quad \geq \quad \gamma \sum_{i\in\mathcal{N}\setminus S} \sum_{h\in S} a_{ih}(s).$$

Since $\Sigma(t) = 0$, by integration, we obtain

$$\Sigma(\tau_S(t)) \geq \gamma \int_t^{\tau_S(t)} \sum_{i\in\mathcal{N}\setminus S} \sum_{h\in S} a_{ih}(s)ds \geq \gamma,$$

where the definition of $\tau_S(t)$ has been used for the last inequality. Denote $j \in \arg\max_{i\in\mathcal{N}\setminus S}\{K^{-i}u_i(\tau_S(t))\}$. Then,

$$K^{-j}u_j(\tau_S(t)) \geq \frac{\Sigma(\tau_S(t))}{n-1} \geq \frac{\gamma}{n-1} \geq \frac{e^{-K}}{n} = \eta.$$

Using , and $K \geq 1$, we obtain

$$u_j(\tau_S(t)) \geq \eta \text{ with } j \in \mathcal{N} \setminus S,$$

which contradicts equation (21) and shows that equation (19) holds and the lemma is proven. □

Lemma 7

Let $p, d \in \{1, \dots, n\}$ such that $p < d$. Let $S_p \sqsubset \mathcal{N}$ with $|S_p| = p$ and $t_p \geq t_0$. Then, there exists a growing sequence of sets $S_{p+1} \dots, S_d \sqsubset \mathcal{N}$ of cardinality $|S_b| = b$ such that for all $b \in \{p, \dots d-1\}$, and $S_b \subseteq S_{b+1}$, which verifies that for all $b \in \{p+1, \dots d\}$,

$$\forall j \in S_b, \sum_{h\in S_p} \Phi_{jh}(t_p, t_b) \geq \eta^{b-p}, \tag{22}$$

where $\eta = e^{-K}/n$ and $t_b = (\tau_{S_{b-1}} \circ \dots \circ \tau_{S_p})(t_p)$ with $\circ$ standing for the composition of functions.

Proof

We prove the lemma by induction on b. For $b = p+1$, the equation (22) is obtained as a direct consequence of Lemma 6 with $S := S_p$ and $S_{p+1} = S \cup \{i\}$. Assume that equation (22) is true for some $b \in \{p+1, \dots d-1\}$. We apply Lemma 6 with $S := S_b$ and $t := t_b$ to obtain the existence of an element $i' \in \mathcal{N} \setminus S_b$ such that for all $j \in S_b \cup \{i'\}$,

$$\sum_{h\in S_b} \Phi_{i'h}(t_b, t_{b+1}) \geq \eta. \tag{23}$$

Denote $S_{b+1} = S_b \cup \{i'\}$. Let $j \in S_{b+1}$. To prove (22) for $b := b+1$, notice that by definition of the fundamental matrix,

$$\sum_{h\in S_p} \Phi_{jh}(t_p, t_{b+1}) = \sum_{h\in S_p} \sum_{l\in S_b} \Phi_{jl}(t_b, t_{b+1})\Phi_{lh}(t_p, t_b).$$

Applying inequality (23) and then equation (22), we obtain

$$\sum_{h \in S_p} \Phi_{jh}(t_p, t_{b+1}) = \eta \sum_{h \in S_p} \Phi_{lh}(t_p, t_b) \geq \eta \cdot \eta^{b-p} = \eta^{(b+1)-p}.$$

□

Corollary 5

Let $t \geq 0$. There exists a finite time $t' \geq t$ such that for all $r, j \in \mathcal{N}$,

$$\forall s \geq t', \Phi_{jr}(t, s) \geq (\exp(-K)/n)^{n-1}.$$

Proof

Let $r \in \mathcal{N}$. Applying Lemma 7 with $p = 1$, $d = n$ provides the existence of $\tau \geq t$ such that for all $h \in \mathcal{N}$,

$$\Phi_{hr}(t, \tau) \geq \eta^{n-1}.$$

Let $s \geq \tau$ and $h \in \mathcal{N}$. The fundamental matrix definition provides

$$\Phi_{jr}(t, s) = \sum_{h \in \mathcal{N}} \Phi_{jh}(\tau, s) \Phi_{hr}(t, \tau) \geq \eta^{n-1} \sum_{h \in \mathcal{N}} \Phi_{jh}(\tau, s) = \eta^{n-1}.$$

Taking the largest τ for all $r \in \mathcal{N}$ allows to conclude. □

Proof of Theorem 1: We now derive the lower bound q_{min}. Since for all $i, j \in \mathcal{N}$, $\Phi_{ij}(t, +\infty) = q_j^k(t)$, Corollary 5 when $s \to +\infty$ directly provides

$$q_j^k(t) \geq (\exp(-K)/n)^{n-1} \geq q_{min}.$$

We now turn to the upper bound q_{max}. We have

$$q_i(t) \quad = \quad 1 - \sum_{j \in \mathcal{N} \setminus \{i\}} q_j(t) \leq 1 - (n-1) q_{min}.$$

7.2. *Proof of Theorem 2*

Let $\varepsilon \in I$ and $t_1 \geq t_0$. Note that by Property 1 and Lemma 2, the fundamental matrix $\Phi(t, s, \varepsilon)$ is invertible and both this matrix and the AIPV $q(t, \varepsilon)$ are independent of the choice of initial condition $x_{t_0}(\varepsilon)$. So, without loss of generality, we assume the initial condition $x_{t_0}(\varepsilon) = (\Phi(t_1, t_0, \varepsilon))^{-1} e_i$, which, by equation (5), implies $x(t_1, t_0, \varepsilon) = e_i$. We first show that $q(t, \varepsilon)$ is continuous *w.r.t.* t in (t_1, ε). For readability, we momentarily drop the ε dependency for q and the ε, t_0 dependency for x. Let $t \geq t_0$. On one hand, recalling the choice $x(t_1) = e_i$ and using the invariance of $q^T x$ given by Lemma 3, we have

$$\begin{aligned} q_i(t_1) \quad &= \quad q(t_1)^\top x(t_1) = q(t)^\top x(t) = q_i(t) x_i(t) + \sum_{j \neq i} q_j(t) x_j(t) \\ &= \quad q_i(t) x_i(t) + \sum_{j \neq i} q_j(t)(x_j(t) - x_j(t_1)), \end{aligned}$$

where we used for $j \neq i$, $x_j(t_1) = 0$. On the other hand, since $x_i(t_1) = 1$,

$$q_i(t) = q_i(t)x_i(t) - q_i(t)(x_i(t) - 1) = q_i(t)x_i(t) - q_i(t)(x_i(t) - x_i(t_1)).$$

Subtracting the two previous equations, and then using that for $j \in \mathcal{N}$, $q_j(t) \in [0, 1]$,

$$|q_i(t) - q_i(t_1)| \leq \sum_{j \in \mathcal{N}} q_j(t)|x_j(t) - x_j(t_1)| \leq \sum_{j \in \mathcal{N}} |x_j(t) - x_j(t_1)| \leq \|x(t) - x(t_1)\|_1.$$

We obtain

$$|q_i(t) - q_i(t_1)| \leq \|x(t) - x(t_1)\|_1.$$

From Lemma 4, x is continuous *w.r.t.* t, accordingly, we obtain the continuity of q_i with regard to t. Since by Lemma 4, x is twice continuously differentiable *w.r.t.* time t, for any fixed ε and $t \geq t_0$, there exists a function $d_1 \in \mathcal{C}^1(\mathbb{R}_+)$ such that

$$\frac{x(t) - x(t_1)}{(t - t_1)} = d_1(t) \quad \text{and} \quad d_1(t_1) = \frac{\partial x}{\partial t}(t_1). \tag{24}$$

First using $x(t_1) = e_i$ and then the time-invariance of $q^T x$ from Lemma 3, we have

$$q_i(t_1) - q_i(t) = (q(t_1) - q(t))^\top x(t_1) = q(t)^\top (x(t) - x(t_1)).$$

Dividing by $(t - t_1)$ and using equation (24), one obtains

$$\frac{q_i(t_1) - q_i(t)}{(t - t_1)} = q(t)^T d_1(t).$$

Coming back to the original notation that points out the dependence on ε, since $d_1 \in \mathcal{C}^1(R_+)$ and q is continuous *w.r.t.* t we obtain that $q \in \mathcal{C}^1(R_+)$ and taking $t \to t_1$, it comes

$$\frac{\partial q_i}{\partial t}(t_1, \varepsilon) = -q(t_1, \varepsilon)\frac{\partial x}{\partial t}(t_1, t_0, \varepsilon) \text{ when } x(t_1, t_0, \varepsilon) = e_i.$$

7.3. Proof of Theorem 3

We first provide a classical result.

Lemma 8 (Variation of parameters [2])
Consider the non-homogeneous system

$$\begin{cases} \dot{y}(t) = B(t)y(t) + u(t), \\ y(t_1) = y_1, \end{cases}$$

where $y(t), u(t) \in \mathbb{R}^n, B(t) \in \mathbb{R}^{n \times n}$, with $B(t)$ and $u(t)$ are piece-wise continuous. The solution to this system is

$$y(t) = \Psi(t, t_1)y(t_1) + \int_{t_1}^{t} \Psi(t, s)u(s)ds,$$

where $\Psi(t, t_1)$ is the fundamental matrix associated with the homogeneous system $\dot{y}(t) = B(t)y(t)$.

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Proof of Theorem 3

According to Lemma 5, $\frac{\partial \Phi}{\partial \varepsilon}(t, t_1, \varepsilon)$ follows a non-homogeneous linear differential system with 0 as uniform initial condition. The corresponding homogeneous equation is the consensus system (3) and accordingly it has fundamental matrix $\Phi(t, t_1, \varepsilon)$ as defined in equation (5). Using the Variation of the constants formula (Lemma 8) with 0 as the initial condition, we obtain

$$\frac{\partial \Phi}{\partial \varepsilon}(t, t_1, \varepsilon) = \int_{t_1}^{t} G(s, t, t_1, \varepsilon) ds,$$

with

$$G(s, t, t_1, \varepsilon) = -\Phi(t, s, \varepsilon)\frac{\partial L}{\partial \varepsilon}(s, \varepsilon)\Phi(s, t_1, \varepsilon) = -\Phi(t, s, \varepsilon)\frac{\partial L}{\partial \varepsilon}(s, \varepsilon)(\Phi(s, t_1, \varepsilon) - \Phi^*(t_1, \varepsilon)) \quad (25)$$

where for the last equality, we used that $\frac{\partial L}{\partial \varepsilon}(s, \varepsilon)$ has left eigenvector $\mathbf{1}_n$ associated with eigenvalue 0 (see (13)) and the definition of $\Phi^*(t_1, \varepsilon)$ as a rank-1 matrix (see (8)). We now show that $\frac{\partial}{\partial \varepsilon}$ and $\lim_{t\to+\infty}$ can be interchanged. To do so, we prove that : i) $G(s, t, t_1, \varepsilon)$ is continuous in ε ; ii) Φ^* can be expressed in function of G ; and iii) $G(s, t, t_1, \varepsilon)$ is uniformly dominated by an integrable function so that Lebesgue's Dominated Convergence Theorem can be applied to obtain $\frac{\partial \Phi^*}{\partial \varepsilon}$.
i) According to the first equality in equation (25), the smoothness assumptions on $a_{ij}(t, \varepsilon)$ and Lemma 4, we have that $G(s, t, t_1, \varepsilon)$ is continuous in all its arguments.
ii) By Fubini's Theorem [7], using the continuity of $G(s, t, t_1, \varepsilon)$,

$$\begin{aligned}\Phi(t, t_1, \varepsilon) &= \Phi(t, t_1, \varepsilon_0) + \int_{\varepsilon_0}^{\varepsilon} \frac{\partial \Phi}{\partial \varepsilon}(t, \eta, t_1) d\eta = \Phi(t, t_1, \varepsilon_0) + \int_{\varepsilon_0}^{\varepsilon}\int_{t_1}^{t} G(s, t, t_1, \eta) ds d\eta \\ &= \Phi(t, t_1, \varepsilon_0) + \int_{t_1}^{t} f(s, t, t_1, \varepsilon) ds,\end{aligned}$$

where $f(s, t, t_1, \varepsilon) \triangleq \int_{\varepsilon_0}^{\varepsilon} G(s, t, t_1, \eta) d\eta$. Taking the limit,

$$\Phi^*(t_1, \varepsilon) = \Phi^*(t_1, \varepsilon_0) + \int_{t_1}^{+\infty} f(s, t, t_1, \varepsilon) ds, \quad (26)$$

with $\frac{\partial f}{\partial \varepsilon}(s, t, t_1, \varepsilon) = G(s, t, t_1, \varepsilon)$.
iii) Regarding the domination, we use the second equality in equation (25) to bound $\|G(s, t, t_1, \varepsilon)\|$. First, using equation (7), $\|\Phi(t, s, \varepsilon)\|_\infty = 1$. Moreover, by definition of the Laplacian matrix,

$$\left\|\frac{\partial L}{\partial \varepsilon}(s, \varepsilon)\right\|_\infty \leq 2n \max_{i,j\in\mathcal{N}} \left|\frac{\partial a_{ij}}{\partial \varepsilon}(s, \varepsilon)\right| \quad (27)$$

and using equation (12), $\|\frac{\partial L}{\partial \varepsilon}(s, \varepsilon)\|_\infty \leq 2n\bar{d}_a$. Secondly, since all components of Φ and Φ^* are in $[0, 1]$ and sum to one (see equation (7)), the initial disagreement satisfies

$$\|\Phi(t_1, t_1, \varepsilon) - \Phi^*(t_1, \varepsilon)\|_\infty \leq 1.$$

Then, according to equation (6), $\Phi(s, t_1, \varepsilon)$ satisfies the consensus dynamics and therefore, Assumption 2 shows that $\Phi(s, t_1, \varepsilon)$ converges to $\Phi^*(t_1, \varepsilon)$ at a speed characterized by

$$\|\Phi(s, t_1, \varepsilon) - \Phi^*(t_1, \varepsilon)\|_\infty \leq d(s, t_0).$$

To summarize, equation (25) shows that we have

$$\|G(s, t, t_1, \varepsilon)\|_\infty \quad \leq \quad 2n\bar{d}_a d(s, t_1).$$

According to Assumption 2, the right hand side of the previous inequality is integrable. As a consequence, by the Lebesgue's Dominated Convergence Theorem [22], $\int_{t_1}^{\top} G(s, t, t_1, \varepsilon) ds$ converges uniformly in ε when $t \to +\infty$ and $\lim_{t \to +\infty} \int_{t_1}^{\top} f(s, t, t_1, \varepsilon) ds$ is continuously differentiable *w.r.t.* ε for $\varepsilon \in I$ and following equation (26), so is $\Phi^*(t, t_1, \varepsilon)$ and moreover, $\frac{\partial \Phi^*}{\partial \varepsilon}(t_1, \varepsilon) = \lim_{t \to +\infty} \int_{t_1}^{t} G(s, t, t_1, \varepsilon) ds.$ □

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