On the impact of cross-country imitation on climate change: A game-theoretical analysis★

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ABSTRACT

As far as climate change is concerned, a recurrent question that is asked either at the government or a consumer level is: Why should I make efforts to reduce my CO_2 emission levels whereas the others will not make these efforts? The present paper provides qualitative elements to this question when asked at the government level. More precisely, we assume that each country wants to maximize a tradeoff between an individual benefit brought by emitting CO_2 and an economical damage due to climate change while being influenced by the reduction strategies of the other countries. The influence term is key for the analysis and enables more virtuous or cooperative behavior. Mathematically speaking, the contribution of this paper is: to propose an abstracted model of a complex decision problem; to integrate an abstracted model of climate change in the game of interest; to conduct the complete Nash equilibrium analysis of the proposed game (existence, uniqueness, expression, quantitative analysis); to conduct a detailed numerical analysis to quantify the discussed aspects such as the impact of cross-country imitation on the atmospheric global temperature in 2100.

1. Introduction

Climate is described by a complex geophysical model that interconnects many nonlinear dynamics (CO_2 in the atmosphere and oceans, temperatures in the atmosphere and oceans, radiative forcing, etc). The Conference of Parties (COP) was created to study and to provide solutions for the global warming mitigation, one of the most challenging problems that humanity starts to face today. It is clear nowadays that climate change impacts the ecosystems, economies, and societies worldwide. Given the urgent need for effective mitigation strategies, it is essential to develop analytical frameworks that capture the strategic decision-making processes of various actors involved in reducing greenhouse gas emissions.

Mitigating climate change requires a coordinated effort from all countries, as greenhouse gas emissions from one country affect the global climate dynamics. It is noteworthy that the benefits and costs of mitigation efforts are not evenly distributed among countries. This creates a complex interaction between three key aspects of the problem: (i) the economic trade-offs between ecological policies and industrial productivity, (ii) the environmental impact of CO_2 emissions and mitigation measures, and (iii) the strategic decision-making process of nations, which can be analyzed using game theory.

The strategy of each country to reduce global warming mainly implements a trade-off between the short-term economic loss induced by the ecological policies, the damage induced by the increased CO_2 emissions, and the political pressure of the economic partners. This leads to a situation in which the actors need to take interdependent decisions that have different impacts on all the cost functions. A natural mathematical framework to study these scenarios is game theory. The objective of this paper is to provide an abstracted modeling of the problem in order to provide insights on the complex decision-making process of interconnected countries that optimize their utility functions, and this, by taking into account the dynamics of the global atmospheric temperature and CO_2 concentration.

Climate change has mainly been addressed both from geophysical and economic points of view, and the results mostly provide empirical or ad-hoc strategies [1, 2]. Typical formal economic analyses do not integrate the geophysical aspect of the problem (see [3, 4] where the temperature dynamics are ignored). Models that couple economic aspects and climate science are referred to as integrated assessment models (IAMs). Among the most famous IAMs one can find the DICE model introduced by the Nobel Prize winner W. Nordhaus and his collaborators [5]. The corresponding analyses typically rely on simple climate models that approximate the elaborate and complex geophysical models used by the IPCC (Intergovernmental Panel on Climate Change) [6].

Studying climate change from a game-theoretical point of view has already been considered in the literature [3, 7, 8], where the authors are imposing convexity properties on the utility functions in a static game. The country agreement problem in a non-cooperative game has been studied in [9]. In contrast, in [10], the authors present a dynamic model that effectively simulates the negotiation process leading

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to the 2015 Paris Agreement on climate change. A neat game formulation that includes coalition formation, financial transfers, and cost-sharing was presented in [11]. This work provides insights into the complexity of environmental cooperation, coalition stability, and the design of efficient and stable agreements. Nevertheless, [11] provides a game-theoretical analysis without considering the temperature dynamics. In summary, the literature on climate change is quite rich, but, to our knowledge, there is no formal game-theoretical work where both geophysical aspects and strategic aspects are considered and analyzed formally.

Another line of research investigates opinion dynamics and social interactions in the context of climate change. In [12], the authors introduce a climate model coupled with a Hegselmann-Krause opinion dynamics model. An evolutionary game is used in [13] to understand the interconnectedness of climate dynamics and social networks, especially real-world inequalities. Public debates shaped by incomplete scientific data, highlighting the role of inflexible/stubborn agents in influencing public opinion, are studied in [14]. The influence of media on climate communication is another area of exploration. The analysis of how various media influence public perception of climate change through their coverage, agenda setting, and framing is done in [15]. Furthermore, several studies have explored the dynamics of climate change agreements and the cooperation among countries in addressing this global challenge, e.g., [16].

Recent advancements in decision-making under uncertainty provide new mathematical tools that could enhance climate mitigation models see [17] for more details. While these contributions significantly advance our understanding of strategic interactions in climate negotiations, they do not formally integrate geophysical aspects and strategic decision-making, leaving a critical gap in the literature that this paper aims to address.

Another crucial aspect of climate change mitigation is the role of the agricultural sector, particularly open and protected agricultural practices, as well as livestock management. These sectors significantly contribute to global greenhouse gas emissions (CO_2). Studies of [18, 19] highlight how technological innovations, such as energy-efficient greenhouse systems and sustainable livestock management, can aid in reducing emissions. Furthermore, cross-country adoption of these technologies is influenced by strategic interactions among nations, reinforcing the necessity of a game-theoretic approach to understanding climate policy coordination.

Empirical studies indicate that cross-country imitation significantly influences the diffusion of green technologies and climate policies. Research on environmental regulation and technology diffusion shows that countries tend to adopt new mitigation technologies from nations with similar regulatory frameworks, reinforcing strategic dependencies in climate action [20]. Additionally, climate negotiations involve imitation, persuasion, and dissuasion, shaping cooperative agreements. A stochastic model based on thermodynamics and influence dynamics provides insights into coalition formation and stability during negotiation rounds [21].

By incorporating both the strategic decision-making dynamics of countries and the geophysical aspects of climate change, this paper aims to provide a novel, comprehensive framework bridging these research directions. The formal integration of these factors into a game-theoretic model will offer deeper insights into policy trade-offs, coalition formations, and long-term sustainability strategies.

In this work, we propose a nonlinear climate dynamics controlled by strategies designed to maximize individual utility functions composed of three terms: an individual benefit term, a weighted global damage term, and a term capturing the effect of imitation/agreement with the other players. It is useful to notice that the introduced imitation term can also be interpreted as a contract for emissions reduction or as a penalty for countries that do not align their emissions with a common strategy. This results in the study of a complex network between the players and the climate, yielding a certain trajectory of global warming. An important contribution of this work is the presence of an imitation term. The addition of this term significantly impacts the entire study, influencing the analysis of the game dynamics, the existence and uniqueness of Nash equilibrium (NE), and the numerical analysis (in particular through the presence of the imitation graph). Our key contributions are the following:

- We propose a novel static strategic-form game model, where the utility functions depend on the geophysical state and on the players' actions, and integrate an imitation term.

- We provide a sufficient condition that ensures the existence of at least one pure Nash equilibrium. We also show the uniqueness of the pure NE by proving that the utility functions are diagonally strictly concave (in the sense of Rosen [22]). Moreover, we prove that the game is weighted potential when all the players do not consider the imitation part in their utility functions. This property ensures the unconditional existence of at least one pure Nash equilibrium. We also provide sufficient conditions for the uniqueness of the pure NE in the presence of the imitation term.

- We provide the expression of the corresponding unique pure Nash equilibrium, under a sufficient condition, both with and without considering the imitation term, particularly for the quadratic case functions.

- We numerically evaluate the impact of the shape of the damage function on the strategy design. We also numerically analyze the effects of the imitation function and the interconnection graph topology.

The rest of the paper is organized as follows. Section 2 is dedicated to the problem formulation. The game-theoretical analysis is provided in Section 3, where we study the existence and the uniqueness of the Nash equilibrium in a specific case, and in both scenarios, with and without the imitation term. In addition, we provide the expression of the pure NE for these cases in this section. Numerical simulations illustrate our results in Section 4 and provide several new insights. Conclusions and some perspectives are given in Section 5.

In the sequel, we adopt the following standard notations. Let $\mathbb{R}$ and $\mathbb{R}_{>0}$ denote the sets of real numbers and strictly positive real numbers, respectively. Vectors in $\mathbb{R}^N$ are assumed to be column vectors. The inner product between two vectors $x, y \in \mathbb{R}^N$ is denoted by $\langle x, y \rangle : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, and the Euclidean norm by $\|x\| : \mathbb{R}^N \to \mathbb{R}$. We denote by $\mathrm{I}_N \in \mathbb{R}^{N\times N}$ the identity matrix and by $\mathbb{1}_{N\times N} \in \mathbb{R}^{N\times N}$ the all-ones matrix, where $[\mathbb{1}_{N\times N}]_{ij} = 1$ for all $i, j \in \mathcal{N}$. Given a function $f : \mathbb{R}^N \to \mathbb{R}^N$ and $a \in \mathbb{R}^N$ we denote $\partial_{a_n} f(a) = \partial f(a)/\partial a_n$ the partial derivative of f with respect to a_n. Finally, the sign function $\mathrm{sign} : \mathbb{R} \to \{-1, 0, 1\}$ maps any real number $x \in \mathbb{R}$ to its sign.

2. Problem formulation

Previous studies on international climate cooperation, particularly those of Finus [23] and Tulkens [11], have primarily analyzed coalition formation through game-theoretic frameworks. These models focus on the stability of self-enforcing international environmental agreements (IEAs), emphasizing the role of strategic incentives in cooperation. Finus' models explore cartel formation and partition function games, highlighting the free-rider problem and proposing transfer schemes to enhance coalition stability. Tulkens extends this approach by incorporating cooperative game theory, particularly characteristic function games, to examine cost-sharing mechanisms and negotiation dynamics.

In this article, we study a strategic-form static game defined as $\Gamma = \left(\mathcal{N}, (\mathcal{A}_n)_{n\in\mathcal{N}}, (u_n)_{n\in\mathcal{N}}\right)$, where $\mathcal{N} = \{1, \dots, N\}$ denotes the set of players. Each player $n \in \mathcal{N}$ represents a country that contributes to climate change, and is associated with an action set $\mathcal{A}_n = [e_n^{\min}, e_n^{\max}] \subset \mathbb{R}$, which corresponds to its admissible range of CO_2 emissions. The utility function of player n is denoted by u_n, and a strategy profile is given by $\mathcal{A} = \prod_{n=1}^{N} \mathcal{A}_n$.

The players are assumed to be interconnected via a fixed network, modeled by a weighted directed graph $\left(\mathcal{N}, \mathcal{E}, P\right)$, where $\mathcal{E}$ is the set of edges and P is the matrix of edge weights. For each player n, we denote its set of neighbors in this graph by $\mathcal{V}_n$, i.e., the set of nodes connected to n via outgoing edges.

The aggregate emissions from all players are denoted by $s := \sum_{n=1}^{N} a_n$, where $a_n \in \mathcal{A}_n$ is the action of player n. The total feasible emission range across all players is given by $\mathbb{A} = [E^{\min}, E^{\max}]$, with $E^{\min} = \sum_{n=1}^{N} e_n^{\min}$ and $E^{\max} = \sum_{n=1}^{N} e_n^{\max}$. Thus, for any $a \in \mathcal{A}$, one has $s \in \mathbb{A}$.

We use a_{-n} to denote the vector of actions of all players except player n, and the corresponding set of joint actions is denoted by $\mathcal{A}_{-n}$.

In this paper, we study two scenarios of an imitation game, where we highlight the effect of the connectivity of the graph and the influence of the neighbors on the players' actions. The first one considers that the players are only influenced by the climate dynamics without trying to synchronize their strategy with those of the other players. This scenario is a particular case of a more general imitation game in which the players are directly influenced by the strategy of the others. The first scenario can be simply obtained from the general imitation game by taking the imitation weights equal to zero for all the players.

Each player aims to maximize its own utility function, as a trade-off between its individual benefits, and a weighted global damage, where a part of this trade-off is the imitation cost. The geophysical state is given by a vector $x \in \mathcal{X} \subset \mathbb{R}^p$ where p depends on the choice of climate model (CM). The state is a vector of (relative) temperatures θ given in some boxes and CO_2 concentrations C in some boxes. Indeed, the temperatures are calculated as variations relative to pre-industrial temperatures. Throughout this paper, we employ the CM dynamics from [24] in which $x = (\theta, C)$. In this setup, the atmospheric CO_2 concentration and the (relative) atmospheric temperature can be described as follows:

$$
\begin{aligned}
C_{\mathrm{AT}}(x, a) &= \psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n, \\
\theta_{\mathrm{AT}}(x, a) &= \psi_\theta(x) + b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n\right),
\end{aligned} \tag{1}
$$

where $a = (a_1, ..., a_N)$ is the action profile/vector, $\psi_\theta(x)$, and $\psi_{\mathrm{C}}(x)$, are functions of the climate state, and $b_\theta > 0$, and $b_{\mathrm{C}} > 0$ are given parameters of the climate model.

While the models of Finus and of Tulkens provide valuable insights, they rely on emission-based damage functions, which may not fully capture the delayed and cumulative effects of climate change. In contrast, our approach considers damage as a function of atmospheric temperature, offering a more realistic representation of climate dynamics. Furthermore, we integrate dynamic negotiation processes, where imitation, persuasion, and dissuasion shape coalition formation.

Definition 1. The utility function of a player $n \in \mathcal{N}$ is defined as the difference between their benefits B_n and the sum of the weighted global damage $w_n D$ and a weighted imitation term $\delta_n \mathrm{Imi}_n$. Notice that B_n depends on the action of player n, $w_n D$ represents a projection of the global damage induced by the increase of the atmospheric temperature, and the imitation function Imi_n depends on the overall vector of actions. We also note that the weights w_n and δ_n are positive for all $n \in \mathcal{N}$. For all, $n \in \mathcal{N}$ the utility function is mathematically described by :

$$
u_n(x, a) := B_n\left(a_n\right) - w_n D\left(\theta_{\mathrm{AT}}(x, a)\right) - \delta_n \mathrm{Imi}_n(a). \tag{2}
$$

Note that, strictly speaking, the utility u_n should only depend on the action profile a. However, because of the presence of the climate dynamics, we parameterize u_n by the dynamics state x and use a piecewise static analysis of the dynamical problem at hand. Although a dynamic game model would be more mathematically general, the authors argue that governments would not implement such complex decision-making. Modeling governments' behavior by a piecewise static model is thus considered much more reasonable. In the rest of the paper, we will mention the use of a quadratic case, where the benefits are quadratic in player n's action, and the damages are quadratic in the atmospheric temperature. In this case, for all $n \in \mathcal{N}$, the utility function is given by the following:

$$\begin{aligned} u_n(x,a) = \sum_{i=0}^{2} \beta_{i,n} a_n^i - \omega_n \sum_{i=0}^{2} \gamma_i \theta_{\mathrm{AT}}^i(x,a) \\ - \delta_n e_n^{\max} \sum_{m \in \mathcal{V}_n} e_m^{\max} \left(\frac{a_n}{e_n^{\max}} - \frac{a_m}{e_m^{\max}} \right)^2 . \end{aligned} \tag{3}$$

In the sequel, we denote by $\xi_n = a_n / e_n^{\max}$, the ratio between the player n's action and their maximal emissions. Note that the imitation term is minimal (maximizes the utility) when all the neighbors have the same emission ratio: $\xi_n = \xi_m, \forall n, m \in \mathcal{N}$.

3. Game analysis

In this section, we conduct the analysis of the game proposed in the preceding section for two scenarios, which correspond to the presence/absence of the imitation term. The Nash equilibrium is a reasonable solution concept for the proposed game since governments have the freedom to choose their emission strategies. Therefore, the existence and uniqueness of an equilibrium scenario is a key problem to be investigated. We start with the imitation game scenario noted by (IG), which is the general scenario. Then we study the scenario where players are not influenced by the others through an imitation graph; this is a special case where the influence is ignored by the players, this scenario is denoted by (NIP), for non-influenced players.

3.1. Imitation game scenario (IG)

We start the analysis by providing a sufficient condition related to the convexity of global damages. This condition guarantees the existence of at least one pure Nash equilibrium. Moving forward, we introduce additional sufficient conditions ensuring the uniqueness of the pure Nash equilibrium. Precisely, we will focus on the concavity of the benefits. Finally, we assume a complete interaction graph among players and, in this context, we provide a closed-form expression of the unique interior pure Nash equilibrium. This equilibrium represents a scenario where players strategically optimize emissions, avoiding emission levels at their maximum. The comprehensive exploration of these subsections contributes to a coherent understanding of equilibrium properties in the game.

3.1.1. IG: Existence of a pure Nash equilibrium

In game theory, the Nash equilibrium is known to be an important solution concept. The players aim to maximize their individual utilities while being aware that their decisions significantly influence one another. To prove the existence of a pure Nash equilibrium, we provide in the following proposition, a sufficient condition for the concavity of the utility functions with respect to a_n (see e.g., [25] for continuous quasi-concave games), a concept introduced by [26]. The next result proves that if one has enough severe damage as the temperature rises, then there exists a pure NE.

Proposition 1. *For all $n \in \mathcal{N}$, let the benefit and imitation functions B_n and Imi_n be twice-differentiable functions, with B_n being concave and Imi_n being convex with respect to a_n, and let the damage function be twice-differentiable with respect to θ_{AT}. If the following condition is satisfied,*

$$\frac{D'\left(\theta_{\mathrm{AT}}(x,a)\right)}{D''\left(\theta_{\mathrm{AT}}(x,a)\right)} < b_\theta, \tag{4}$$

then there exists at least one pure NE for the game Γ.

Proof. First, we prove that the utility function u_n of player $n \in \mathcal{N}$, is concave with respect to a_n. Consequently, we calculate the second derivative of u_n with respect to a_n by using the expressions of u_n in Eq (2) and of the atmospheric temperature in Eq (1). For all $n \in \mathcal{N}$ one obtains

$$\begin{aligned} \frac{\partial^2 u_n}{\partial a_n^2}(x,a) = B_n''(a_n) - \delta_n \frac{\partial^2 \mathrm{Imi}_n}{\partial a_n^2}(a) - \\ \frac{\omega_n b_\theta b_{\mathrm{C}}^2 \left[b_\theta D''(\theta_{\mathrm{AT}}(x,a)) - D'(\theta_{\mathrm{AT}}(x,a)) \right]}{\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n \right)^2} . \end{aligned}$$

Since we assume that the benefit function is concave with respect to a_n, the damage function is convex with respect to θ_{AT}, and the imitation function is convex with respect to a_n, from Eq (4) we conclude that the utility function u_n is concave with respect to a_n. Moreover, for all $n \in \mathcal{N}$, the action set $\mathcal{A}_n$ is compact and convex, and u_n is continuous in the profile of actions $a \in \mathcal{A}$. Thus, using Debreu's theorem from [26], there exists at least one pure NE in the game Γ. □

3.1.2. IG: Uniqueness of the pure Nash equilibrium

In this section, our focus is on examining the game's structure to establish the uniqueness of the pure Nash equilibrium. In the following, the Jacobian of $U(x,a) := (u_1(x,a), \ldots, u_N(x,a))$ is $\nabla U(x,a) := [\nabla_1 u_1(x,a), \ldots, \nabla_N u_N(x,a)]^\top$ where $\nabla_m u_n(x,a)$ represents the gradient vector of u_n with respect to a_m, for all $n, m \in \mathcal{N}$. The Hessian matrix $H(x,a)$ is defined as:

$$H(x,a) = \begin{bmatrix} \frac{\partial^2 u_1}{\partial a_1^2}(x,a) & \frac{\partial^2 u_1}{\partial a_1 \partial a_2}(x,a) & \ldots \\ \frac{\partial^2 u_1}{\partial a_2 \partial a_1}(x,a) & \frac{\partial^2 u_1}{\partial a_2^2}(x,a) & \\ \vdots & & \ddots \end{bmatrix}.$$

It is worth noting that $H(x,a)$ is Hermitian due to the following property $\partial_{a_i}\partial_{a_j}u_n(x,a) = \partial_{a_j}\partial_{a_i}u_n(x,a)$, holds for all $n, i, j \in \mathcal{N}$.

To demonstrate the uniqueness of the pure Nash equilibrium, we start by recalling the definition of the Diagonally Strict Concavity (DSC) condition as follows.

Definition 2. Let the utilities $(u_n)_{n\in\mathcal{N}}$ be differentiable functions in $a \in \mathcal{A}$ and concave in a_n. For any $r = (r_1, \dots, r_N)^\top \in \mathbb{R}^N$, we define the pseudogradient associated with the game Γ and parameter r by

$$\gamma_r(a) = \left[r_1\frac{\partial u_1}{\partial a_1}(a), \dots, r_N\frac{\partial u_N}{\partial a_N}(a)\right]^\top .$$

The game Γ satisfies the so-called Diagonally Strict Concavity (DSC) condition if there exists a $r \in \mathbb{R}^N_{>0}$ such that the following holds

$$\forall (a,\tilde{a}) \in \mathcal{A}^2; \; a \neq \tilde{a} : (a-\tilde{a})^\top \left(\gamma_r(a) - \gamma_r(\tilde{a})\right) > 0. \quad (5)$$

Indeed, to prove the uniqueness of the pure NE, we use the following lemma from [22] to find a sufficient condition for ensuring the uniqueness of the pure Nash equilibrium.

Lemma 1. *If for all $a \in \mathcal{A}$, one has*

$$y^\top (H(x,a) + H^\top(x,a))y < 0, \quad \forall y \neq 0, \quad (6)$$

then, the utility functions $(u_1, \dots, u_N)$ are diagonally strictly concave for $a \in \mathcal{A}$.

Proof. Let $a, a' \in \mathcal{A}$, since $\mathcal{A} = \prod_{n=1}^N \mathcal{A}_n$ is convex, then $a(\zeta) = \zeta a + (1-\zeta)a' \in \mathcal{A}$, for any $\zeta \in [0,1]$. Knowing that $H(x,a)$ is the Jacobian of $\nabla U(x,a)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}\zeta}\nabla u(x, a(\zeta)) = H(x, a(\zeta))\frac{\mathrm{d}a(\zeta)}{\mathrm{d}\zeta} = H(x, a(\zeta))(a - a'),$$

or

$$\int_0^1 H(x, a(\zeta))(a-a')\mathrm{d}\zeta = \nabla u(a) - \nabla u(a').$$

Multiplying the preceding by $(a - a')$, yields the following:

$$(a-a')^\top \nabla u(a) + (a'-a)^\top \nabla u(a') =$$
$$\frac{1}{2}\int_0^1 (a-a')^\top \left[H(x,a(\zeta)) + H^\top(x,a(\zeta))\right](a-a')\mathrm{d}\zeta < 0,$$

where to get the strict inequality, we used the assumption in Eq (6): the symmetric matrix $H(x,a(\zeta)) + H^\top(x,a(\zeta))$ is negative definite for all $a \in \mathcal{A}$. $\square$

To establish a sufficient condition for the uniqueness of the pure Nash equilibrium, we provide an instrumental result presented in the following Lemma.

Lemma 2. *Let $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^N$. Then, the maximal eigenvalue of the matrix $M = \mathbf{v}_1\mathbf{v}_2^\top + \left(\mathbf{v}_1\mathbf{v}_2^\top\right)^\top$ is $\mu = \mathbf{v}_1^\top\mathbf{v}_2 + \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|$.*

Proof. Since $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^N$, the matrix $\mathbf{v}_1\mathbf{v}_2^\top$ is of rank 1 which implies that M is at most a rank 2 matrix. Let us note the non-zero eigenvalues of M by μ_1 and μ_2. We have the following,

$$\mu_1 + \mu_2 = \mathrm{Tr}\,(M) = 2\mathbf{v}_1^\top\mathbf{v}_2,$$

and

$$\begin{aligned}
\mu_1^2 + \mu_2^2 &= \mathrm{Tr}\left(M^2\right) \\
&= \mathrm{Tr}\left(\left(\mathbf{v}_1\mathbf{v}_2^\top\right)^2 + \left(\mathbf{v}_2\mathbf{v}_1^\top\right)^2 \mathbf{v}_1\mathbf{v}_2^\top\mathbf{v}_2\mathbf{v}_1^\top + \mathbf{v}_2\mathbf{v}_1^\top\mathbf{v}_1\mathbf{v}_2^\top\right) \\
&= 2\mathrm{Tr}\left(\left(\mathbf{v}_1\mathbf{v}_2^\top\right)^2\right) + (\mathbf{v}_2^\top\mathbf{v}_2)\mathrm{Tr}\left(\mathbf{v}_1\mathbf{v}_1^\top\right) + (\mathbf{v}_1^\top\mathbf{v}_1)\mathrm{Tr}\left(\mathbf{v}_2\mathbf{v}_2^\top\right) \\
&= 2\mathrm{Tr}\left(\mathbf{v}_1\mathbf{v}_2^\top\right)^2 + 2(\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2) \\
&= 2\left((\mathbf{v}_1^\top\mathbf{v}_2)^2 + (\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2)\right).
\end{aligned}$$

Moreover, we also have,

$$\mu_1\mu_2 = \frac{\left(\mu_1+\mu_2\right)^2 - \left(\mu_1^2+\mu_2^2\right)}{2} = (\mathbf{v}_1^\top\mathbf{v}_2)^2 - (\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2).$$

Then by injecting this into the characteristics polynomial of M given by $P_M(X) = (-X)^{N-2}\left(\mu_1 - X\right)\left(\mu_2 - X\right)$, we notice that μ_1 and μ_2 are also the roots of

$$Q(X) = X^2 - 2(\mathbf{v}_1^\top\mathbf{v}_2)X + (\mathbf{v}_1^\top\mathbf{v}_2)^2 - (\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2).$$

And finally, the biggest root of this polynomial $\mu := \max\left(\mu_1, \mu_2\right)$, is given by

$$\begin{aligned}
\mu &= \frac{2(\mathbf{v}_1^\top\mathbf{v}_2) + \sqrt{4(\mathbf{v}_1^\top\mathbf{v}_2)^2 - 4\left((\mathbf{v}_1^\top\mathbf{v}_2)^2 + (\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2)\right)}}{2} \\
&= \mathbf{v}_1^\top\mathbf{v}_2 + \sqrt{(\mathbf{v}_1^\top\mathbf{v}_1)(\mathbf{v}_2^\top\mathbf{v}_2)} = \mathbf{v}_1^\top\mathbf{v}_2 + \|\mathbf{v}_1\|\,\|\mathbf{v}_2\|.
\end{aligned}$$ $\square$

The next result proves that in addition to the condition of severe damage for the existence, a sufficient condition is given on the benefits, to ensure the uniqueness of the pure NE. Indeed, the right-hand side of the condition in Eq (8), is a simple combination of the damage and imitation weights. This limit is shown to be positive in the numerical examples that we took, and then by simply considering a concave benefit function, which is assumed to be true, this condition is satisfied. Eventually, this condition can be interpreted as a threshold on the concavity of the benefit function.

Proposition 2. *Let the benefit functions B_n be twice differentiable with respect to a_n for all $n \in \mathcal{N}$ and the damage function be twice differentiable with respect to $\theta_{\mathrm{AT}} \in \mathbb{R}$. If the two following conditions are satisfied, then the game has a unique pure NE.*

- *The damage function must verify*

$$\frac{D'\left(\theta_{\mathrm{AT}}(x,a)\right)}{D''\left(\theta_{\mathrm{AT}}(x,a)\right)} < b_\theta. \quad (7)$$

- *The benefit functions must verify*

$$B''_{\underline{n}}(a_{\underline{n}}) < 2\left(\sum_{m\in\mathcal{V}_{\underline{n}}} \frac{e_m^{\max}}{e_{\underline{n}}^{\max}} - 1\right)\delta_{\underline{n}} - \sum_{n=1}^{N}\delta_n - N\,\mathrm{sign}\left(H_{2,\overline{n}}(x,a)\right)\delta_{\overline{n}}, \quad \forall n \in \mathcal{N} \tag{8}$$

With the notations $\underline{n} = \arg\max_{n\in\mathcal{N}} H_{1,n}(a_n)$*,* $\overline{n} = \arg\max_{n\in\mathcal{N}} |H_{2,n}(x,a)|$ *and* $H_{2,n}$ *given in the Eq (9).*

Moreover, if for all $n \in \mathcal{N}$*,* $\mathrm{sign}\left(H_{2,n}(x,a)\right) < 0$*, the following additional condition over the damage weights must be verified*

$$w_{\overline{n}} < \frac{1}{N}\sum_{n=1}^{N} w_n.$$

Proof. For all $a \in \mathcal{A}, x \in \mathcal{X}$ let us introduce

$$H_{1,n}(a_n) = B''_n(a_n) - 2\delta_n\left(\sum_{m\in\mathcal{V}_n}\frac{e_m^{\max}}{e_n^{\max}} - 1\right)$$

and

$$H_{2,n}(x,a) = 2\delta_n + \frac{b_\theta b_{\mathrm{C}}^2 w_n (D'(\theta_{\mathrm{AT}}(x,a)) - b_\theta D''(\theta_{\mathrm{AT}}(x,a)))}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^{N} a_n}. \tag{9}$$

We built sequentially $H_1(a) = \mathrm{diag}\left(H_{1,1}(a_1),\ldots,H_{1,N}(a_N)\right)$, $H_2(x,a) = (H_{2,1}(x,a),\ldots,H_{2,N}(x,a))^\top$, the matrix

$$H(x,a) = H_1(a) + H_2(x,a)\mathbb{1}_N^\top,$$

and the Hermitian matrix $M := H(x,a) + H^\top(x,a)$. Applying Weyl's inequality to the largest eigenvalue of M, will lead to the DSC. This will allow us to conclude the uniqueness of the pure Nash equilibrium.

Indeed, $M = 2H_1(a) + H_2(x,a)\mathbb{1}_N^\top + \mathbb{1}_N H_2^\top(x,a)$, we recall that the eigenvalues of $H_1(a)$ are $H_{1,n}(a_n)$, for all $n \in \mathcal{N}$. Using Lemma 2, for $v_1 = H_2(x,a)$, and $v_2 = \mathbb{1}_N$, we get that the maximum eigenvalue of $H_2(x,a)\mathbb{1}_N^\top + \mathbb{1}_N H_2^\top(x,a)$, is given by

$$\mu = H_2^\top(x,a)\mathbb{1}_N + \|H_2(x,a)\|\,\|\mathbb{1}_N\|.$$

Knowing that $\|x\| \le \sqrt{N}\|x\|_\infty = \sqrt{N}\max_{n\in\mathcal{N}}|x_n|$, $\forall x \in \mathbb{R}^N$, it suffices to use the infinity norm instead of the Euclidean norm. Indeed, the infinity norm of $H_2(x,a)$ is given by $\|H_2(x,a)\|_\infty = \max_{n\in\mathcal{N}}|H_{2,n}(x,a)|$ and $\|\mathbb{1}_N\|_\infty = 1$. To find a sufficient condition for the uniqueness of the NE, it suffices to show that:

$$2\max_{n\in\mathcal{N}} H_{1,n}(a_n) + \sum_{n=1}^{N} H_{2,n}(x,a) + N\max_{n\in\mathcal{N}}|H_{2,n}(x,a)| < 0.$$

Let $H_{2,n}(x,a)$ be a vector for each $n \in \mathcal{N}$. The index $\overline{n}$ is defined as the unique element in $\mathcal{N}$ that maximizes the absolute value of $H_{2,n}(x,a)$, *i.e.,* ,

$$\overline{n} = \arg\max_{n\in\mathcal{N}}|H_{2,n}(x,a)|.$$

This notation denotes that $\overline{n}$ is the index for which $|H_{2,n}(x,a)|$ is maximal among all indices n in the set $\mathcal{N}$. Since we are interested in the largest eigenvalue of H_1, we denote by $\underline{n}$ the index of the largest $H_{1,n}$:

$$\underline{n} = \arg\max_{n\in\mathcal{N}} H_{1,n}(a_n).$$

Hence,

$$\begin{aligned}
&2H_{1,\underline{n}}(a_{\underline{n}}) + \sum_{n=1}^{N} H_{2,n}(x,a) + N\max_{n\in\mathcal{N}}|H_{2,n}(x,a)| \\
&= 2B''_{\underline{n}}(a_{\underline{n}}) - 4\delta_{\underline{n}}\left(\sum_{m\in\mathcal{V}_{\underline{n}}}\frac{e_m^{\max}}{e_{\underline{n}}^{\max}} - 1\right) + 2\sum_{n=1}^{N}\delta_n \\
&+ \sum_{n=1}^{N} w_n \frac{b_\theta b_{\mathrm{C}}^2\left[D'(\theta_{\mathrm{AT}}(x,a)) - b_\theta D''(\theta_{\mathrm{AT}}(x,a))\right]}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^{N} a_n} \\
&+ N\,\mathrm{sign}\left(H_{2,\overline{n}}(x,a)\right)\Big[2\delta_{\overline{n}} + w_{\overline{n}}\frac{b_\theta b_{\mathrm{C}}^2\left[D'(\theta_{\mathrm{AT}}(x,a)) - b_\theta D''(\theta_{\mathrm{AT}}(x,a))\right]}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^{N} a_n}\Big] \\
&= 2B''_{\underline{n}}(a_{\underline{n}}) - 4\delta_{\underline{n}}\left(\sum_{m\in\mathcal{V}_{\underline{n}}}\frac{e_m^{\max}}{e_{\underline{n}}^{\max}} - 1\right) + 2\sum_{n=1}^{N}\delta_n + \\
&2N\,\mathrm{sign}\left(H_{2,\overline{n}}(x,a)\right)\delta_{\overline{n}} + \left[\sum_{n=1}^{N} w_n + N\,\mathrm{sign}\left(H_{2,\overline{n}}(x,a)\right)w_{\overline{n}}\right] \\
&\times \frac{b_\theta b_{\mathrm{C}}^2\left[D'(\theta_{\mathrm{AT}}(x,a)) - b_\theta D''(\theta_{\mathrm{AT}}(x,a))\right]}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^{N} a_n}
\end{aligned}$$

In conclusion, the sufficient conditions are given by (7) and (8). In addition, if $H_{2,\overline{n}}(x,a) < 0$, the additional condition that has to be satisfied is $w_{\overline{n}} < \sum_{n=1}^{N} w_n/N$. Then, using Lemma 1, we conclude that the payoff functions are diagonally strictly concave. Hence, the game has a unique pure Nash equilibrium. □

3.1.3. IG: Expression of the pure Nash equilibrium

In this section, we will give the expression of the unique pure NE of the game when the utility functions are quadratic strictly concave, given in the Eq (3). Let us denote by $a^* = \left(a_1^*,\ldots,a_N^*\right)$ the Nash equilibrium profile of actions for all the players. We emphasize that the actions of the players are the amount of emissions they emit to maximize their utility. The Nash equilibrium of the game under consideration may be either on the boundary, *i.e.,* , $a_n^* \in \left\{e_n^{\min}, e_n^{\max}\right\}$, for some $n \in \mathcal{N}$, or an interior point $a_n^* \in \left(e_n^{\min}, e_n^{\max}\right), \forall n \in \mathcal{N}$. Our goal is to characterize the interior solution because this means that the emissions are not maximal. We are also interested in pointing out the impact of different elements: the imitation term, the economic coefficients (benefit coefficients and damage coefficients), the impact of the CM parameters, and the radiative forcing.

Assuming that the utility functions $(u_1,\ldots,u_N)$ verify the DSC condition in Eq (5), we get that the Nash equilibrium is unique. In this case, the NE can be found by applying the KKT conditions given as follows:

$$\frac{\partial u_n}{\partial a_n}(x,a) = \overline{\lambda}_n - \underline{\lambda}_n =: \lambda_n, \quad \forall n \in \mathcal{N}, \tag{KKT}$$

with $\overline{\lambda}_n, \underline{\lambda}_n \geq 0$ being the *KKT* multipliers that satisfy $\underline{\lambda}_n^*(a_n^* - e_n^{\min}) = 0$, and $\overline{\lambda}_n^*(a_n^* - e_n^{\max}) = 0$.

To simplify the statement of the next proposition, we will define the following notations. We recall that $s = \sum_{n=1}^N a_n$ and since for a given state $x \in \mathcal{X}$, the atmospheric temperature can be expressed as a function of the sum of actions, *i.e.,* , $\theta_{\text{AT}}(x,a) := \theta_{\text{AT}}(s)$. For all $n \in \mathcal{N}$, let $z_n = 2\left(\beta_{2,n} - \delta_n E^{\max}/e_n^{\max}\right)$ and $\sigma_n = \beta_{1,n} - \lambda_n$ with $\beta_{1,n}, \beta_{2,n}$ and δ_n from (3) and λ_n from (KKT). Last, we define the function h defined for $o \in \mathbb{R}^N$ by

$$h(o) = \frac{\sum_{n=1}^N (o_n/z_n)}{1 + 2\sum_{n=1}^N (\delta_n/z_n)}.$$

Proposition 3. *Let the interaction graph among players be complete. For all $n \in \mathcal{N}$, let u_n be strictly concave and $w_n > 0$. Using the above notations, we assume that $z_n \neq 0$ and $1 + 2\sum_{n=1}^N \delta_n/z_n \neq 0$. If the following holds*

$$2\gamma_2 b_\theta^2 b_{\text{C}}^2 h(w) + \frac{\left(\psi_{\text{C}}(x) + b_{\text{C}} h(\sigma)\right)^2}{8} < 0, \tag{10}$$

then the unique pure NE of the game $a^* = \left(a_1^*, \dots, a_N^*\right)$ is expressed by the following:

$$a_n^* = \frac{w_n}{z_n}\frac{b_\theta b_{\text{C}}\left(\gamma_1 + 2\gamma_2\theta_{\text{AT}}(\tilde{s})\right)}{\psi_{\text{C}}(x) + b_{\text{C}}\tilde{s}} - \frac{2\delta_n}{z_n}\tilde{s} - \frac{\sigma_n}{z_n}, \tag{11}$$

with $\tilde{s}$ being the unique solution of the equation $rs^2 + ps + q = k\ln(rs + v)$, where $r = b_{\text{C}}$, $p = \psi_{\text{C}} + b_{\text{C}} h(\sigma)$, $q = -b_\theta b_{\text{C}}\left(\gamma_1 + 2\gamma_2\psi_\theta(x)\right)h(w)$, $k = 2\gamma_2 b_\theta^2 b_{\text{C}} h(w)$, and $v = \psi_{\text{C}}$.

Proof. We recall the utility function, where the benefit, damage, and imitation functions are quadratic. Using the *KKT* conditions, and noting $s^* = \sum_{n=1}^N a_n^* \in \mathbb{A}$, we get for all $n \in \mathcal{N}$,

$$\beta_{1,n} + 2\beta_{2,n}a_n^* - w_n\left(\gamma_1 + 2\gamma_2\theta_{\text{AT}}(s^*)\right)\frac{\partial\theta_{\text{AT}}}{\partial a_n}(s^*) - \frac{2\delta_n}{e_n^{\max}}\sum_{m\in\mathcal{V}_n}\left(e_m^{\max}a_n^* - e_n^{\max}a_m^*\right) = \lambda_n^*.$$

Assuming a complete graph, the expression can be further simplified by using:

$$\begin{aligned}
&\sum_{m\in\mathcal{V}_n}\left(e_m^{\max}a_n^* - e_n^{\max}a_m^*\right) + e_n^{\max}a_n^* - e_n^{\max}a_n^* \\
&= \sum_{m\in\mathcal{N}, m\neq n}\left(e_m^{\max}a_n^* - e_n^{\max}a_m^*\right) + e_n^{\max}a_n^* - e_n^{\max}a_n^* \\
&= a_n^*\sum_{n=1}^N e_n^{\max} - e_n^{\max}\sum_{n=1}^N a_n^*.
\end{aligned}$$

The KKT condition for each $n \in \mathcal{N}$ takes the form:

$$z_n a_n^* + 2\delta_n s + \sigma_n - w_n\frac{b_\theta b_{\text{C}}\left(\gamma_1 + 2\gamma_2\theta_{\text{AT}}(s)\right)}{\psi_{\text{C}}(x) + b_{\text{C}}s} = 0.$$

Dividing by $z_n = 2\left(\beta_{2,n} - \delta_n E^{\max}/e_n^{\max}\right) \neq 0$ for all $n \in \mathcal{N}$, and summing over n yields:

$$\left(1 + 2\sum_{n=1}^N\frac{\delta_n}{z_n}\right)s + \sum_{n=1}^N\frac{\sigma_n}{z_n} = \sum_{n=1}^N\frac{w_n}{z_n}\frac{b_\theta b_{\text{C}}\left(\gamma_1 + 2\gamma_2\theta_{\text{AT}}(s)\right)}{\psi_{\text{C}}(x) + b_{\text{C}}s}.$$

Furthermore, dividing by $1 + 2\sum_{n=1}^N(\delta_n/z_n) \neq 0$, the KKT conditions can be expressed as:

$$s + h(\sigma) - h(w)\frac{b_\theta b_{\text{C}}\left(\gamma_1 + 2\gamma_2\theta_{\text{AT}}(s)\right)}{\psi_{\text{C}}(x) + b_{\text{C}}s} = 0. \tag{12}$$

Using the expression of θ_{AT} given in Eq (1), the condition given in Eq (12) can be rewritten as $rs^2 + ps + q = k\ln(rs + v)$. By using the results of [27, Lemma 1], we conclude that if the condition in Eq (10) is verified, then there exists at most one solution $\tilde{s}$ to the equation $rs^2 + ps + q = k\ln(rs + v)$ in $\mathbb{A}$. Using the Proposition 1, and assuming that for all $n \in \mathcal{N}$, u_n is concave, then we know that there exists at least one pure NE. Thus, the NE is unique and is given explicitly by the Eq (11). □

Remark on the NE expression. In this context, it is of interest to exhibit several limiting cases for which the expression of the NE is particularly simple. Using the notations provided in Proposition 3 and recalling that $z_n = 2\left(\beta_{2,n} - \delta_n E^{\max}/e_n^{\max}\right)$, we point out that:

$$\lim_{\delta_n\to+\infty} z_n = -\infty \quad \text{and} \quad \lim_{\delta_n\to+\infty}\frac{-\delta_n}{z_n} = \frac{e_n^{\max}}{2E^{\max}}.$$

In a straightforward manner, one obtains that:

$$\lim_{\delta_n\to+\infty} a_n^* = \frac{e_n^{\max}}{E^{\max}}\tilde{s}, \quad \text{for all } n \in \mathcal{N},$$

where $\tilde{s}$ is the unique solution of the equation $rs^2 + ps + q = k\ln(rs + v)$. When δ_n is large enough, one can see that players' actions at the equilibrium represent a proportion of the total maximum emissions, that is proportional to their maximal emissions. They will tend to stop emitting when δ_n tends to infinity. Indeed, $\lim_{\delta_n\to+\infty} h(o) = 0$, for all $o_n < +\infty$. In this case, the equation is rewritten as: $b_{\text{C}}s^2 + \psi_{\text{C}}(x)s = 0$, and $\tilde{s} \in \mathbb{A}$, with $e_n^{\min} = 0$, for all $n \in \mathcal{N}$. In other words, the only solution of this equation is $\tilde{s} = 0$. Thus, at the NE, players stop emitting when faced with sufficiently considerable imitation weights or significant benefit coefficients.

3.2. Non-influenced players scenario (NIP)

In the subsequent, we notice that the non-influenced players scenario is a weighted potential game. In this section, we consider that the utility functions are quadratic strictly concave, given in the Eq (3). The following analysis will highlight the potential of this game, leading to the existence of a pure Nash Equilibrium. After that, we provide a sufficient condition for the uniqueness of the pure Nash equilibrium and then provide the expression of the interior pure NE in a case study, where the benefit function is quadratic in player n's action, and the global damage function is quadratic in the atmospheric temperature.

3.2.1. NIP: Existence and uniqueness of a pure Nash equilibrium

A key solution concept for the interactive situation, which involves several players, each aiming to maximize its own utility function, is given by the Nash equilibrium. A Nash equilibrium can be interpreted as a possible forecast for such a situation where decisions are interdependent as they are for the global carbon emission problem. An important property for a game is precisely to know whether it possesses a pure Nash equilibrium. It turns out that, by construction, the game under study always has a pure Nash equilibrium. This is because it belongs to the class of weighted potential games as defined by Monderer and Shapley [28].

Definition 3. A game $\Gamma = \left(\mathcal{N}, (\mathcal{A}_n)_{n\in\mathcal{N}}, (u_n)_{n\in\mathcal{N}}\right)$ is a weighted potential game if and only if there exists a potential function $\phi : \mathcal{A} \mapsto \mathbb{R}$ and $(\delta_n)_{n\in\mathcal{N}}$ a vector of positive weights, such that, for all $n \in \mathcal{N}, a_n, \tilde{a}_n \in \mathcal{A}_n; a_n \neq \tilde{a}_n$ and $a_{-n} \in \mathcal{A}_{-n}$ one has

$$u_n(a_n, a_{-n}) - u_n(\tilde{a}_n, a_{-n}) = \delta_n \left[\phi(a_n, a_{-n}) - \phi(\tilde{a}_n, a_{-n})\right].$$

It can be checked that the following function ϕ is a potential for the considered game with weights $\left(w_n\right)_{n\in\mathcal{N}}$:

$$\phi(x, a) = \sum_{n=1}^{N} \frac{1}{w_n} \sum_{i=0}^{2} \beta_{i,n} a_n^i - \sum_{i=0}^{2} \gamma_i \theta_{\mathrm{AT}}^i(x, a). \tag{13}$$

The previous results yield the existence of at least one pure Nash equilibrium for the quadratic case.

Next, we will provide a necessary condition for uniqueness. In the following, we will consider concave benefit functions and a convex damage function. Indeed, the connection between GDP and emissions is frequently represented through a concave function [23, 11]. Moreover, it is commonly assumed that the damage function is convex since it induces higher damages for higher temperature variations. In economic literature focusing on the consequences of climate change, a quadratic damage convex function is mostly used [5, 29].

Proposition 4. *Let $\gamma_2 > 0$ and for all $n \in \mathcal{N}$, the following inequalities hold*

$$\frac{\gamma_2 b_\theta^2 b_{\mathrm{C}} w_n}{e_n^{\max}} \exp\left[\frac{\gamma_1 + 2\gamma_2 \psi_\theta(x)}{2\gamma_2 b_\theta} - 1\right] - \frac{\beta_{1,n}}{2e_n^{\max}} < \beta_{2,n} < 0. \tag{14}$$

Then, the pure Nash equilibrium is unique and corresponds to all players emitting to the maximum. i.e., $a_n = e_n^{\max}, \forall n \in \mathcal{N}$.

Proof. The game has a unique pure Nash equilibrium if the following is verified

$$\min_{a_n\in\mathcal{A}_n} \frac{B_n'(a_n)}{w_n} > \max_{a_n\in\mathcal{A}_n} \frac{\partial D\left(\theta_{\mathrm{AT}}(x,a)\right)}{\partial a_n}$$

Indeed, in this case, the utility functions are strictly increasing. Then for any player $n \in \mathcal{N}$ and independently of other players' actions, the utility u_n reaches its maximum at $e_n^{\max}$. Thus, the unique pure Nash equilibrium is when all players emit the maximum of possible emissions. Straightforward computation shows that, if $\beta_{2,n} < 0$ (i.e., B_n are concave benefit functions), one has

$$\min_{a_n\in\mathcal{A}_n} B_n'(a_n) = \min_{a_n\in\mathcal{A}_n} \left[\beta_{1,n} + 2\beta_{2,n} a_n\right] = \beta_{1,n} + 2\beta_{2,n} e_n^{\max}.$$

On the other hand,

$$\frac{\partial D\left(\theta_{\mathrm{AT}}(x,a)\right)}{\partial a_n} = \tag{15}$$
$$\frac{b_\theta b_{\mathrm{C}}\left[\gamma_1 + 2\gamma_2\left[\psi_\theta(x) + b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^N a_n\right)\right]\right]}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{n=1}^N a_n}.$$

In this part of the proof, we use the parameters $a, b, c, d, k \in \mathbb{R}$ that are constants, and independent of the problem formulated before. They are used to ease the presentation of the variation of the function. To find the maximum of (15), we consider the case where $\gamma_2 > 0$. Let us find the maximum of the function $f : \mathbb{R} \to \mathbb{R}$ for $a, b, c, k \in \mathbb{R}_+^*, d \in \mathbb{R}$, given for all $z \in \mathbb{R}$ by:

$$f(z) = \frac{k(d + c\ln(a + bz))}{a + bz}.$$

When differentiating f with respect to $z \in \mathbb{R}$, we find the unique root of f' given by $z_0 = (\exp(1 - d/c) - a)/b$. Then simply computing $f'((\exp(-d/c) - a)/b) = kbc/\exp(4 - 2d/c) > 0$, and $f'((\exp(2 - d/c) - a)/b) = -kbc/\exp(4 - 2d/c) < 0$ provides that f' is strictly decreasing. We can conclude that the function f is strictly concave and reaches its maximum at z_0, given by $f(z_0) = kc\exp(d/c - 1)$. Now by using f with $a = \psi_{\mathrm{C}}(x) + b_{\mathrm{C}}\sum_{m=1,m\neq n}^N a_m$, $b = b_{\mathrm{C}}$, $c = 2\gamma_2 b_\theta$, $d = \gamma_1 + 2\gamma_2\psi_\theta(x)$ and $k = b_\theta b_{\mathrm{C}}$, we conclude that the maximum of D' is given by:

$$\max_{a_n\in\mathcal{A}_n} \frac{\partial D\left(\theta_{\mathrm{AT}}(x,a)\right)}{\partial a_n} = 2\gamma_2 b_\theta^2 b_{\mathrm{C}} \exp\left(\frac{\gamma_1 + 2\gamma_2\psi_\theta(x)}{2\gamma_2 b_\theta} - 1\right).$$

After minimizing the benefit variations and maximizing the damage variations, we get that

$$\min_{a_n\in\mathcal{A}_n} \frac{B_n'(a_n)}{w_n} > \max_{a_n\in\mathcal{A}_n} \frac{\partial D\left(\theta_{\mathrm{AT}}(x,a)\right)}{\partial a_n} \Leftrightarrow$$
$$\frac{\beta_{1,n} + 2\beta_{2,n} e_n^{\max}}{w_n} > 2\gamma_2 b_\theta^2 b_{\mathrm{C}} \exp\left(\frac{\gamma_1 + 2\gamma_2\psi_\theta(x)}{2\gamma_2 b_\theta} - 1\right),$$

which is equivalent to the condition given in the Eq (14). □

Proposition 4 states that looking at the short term, all the countries will emit as much as possible as long as the damage function does not have a sufficiently large impact. This can be changed either by considering less optimistic damage functions or looking at the long-term behavior when the atmospheric temperature is higher, which will lead to larger damages.

3.2.2. NIP: Expression of the Nash equilibrium

In this section, the goal is to express the Nash equilibrium actions for the player. The motivation for this is twofold; it makes interpretations much easier (e.g., the impact of radiative forcing or the damage severity level on the behavior of the countries) and it renders the problem of computing the equilibrium very simple to solve. To express the NE, let us assume from now on that ϕ is strictly concave. The pure NE is denoted by $a^* = (a_1^*, \dots, a_N^*)$ where either there exists $n \in \mathcal{N}$ such that $a_n^* \in \{e_n^{\min}, e_n^{\max}\}$, or a^* is an interior NE. In the latter case, the players will tend to reduce their emissions.

We recall that the potential function is defined in the Eq (13) with the atmospheric temperature θ_{AT} given by:

$$\theta_{\mathrm{AT}}(x, a) = \psi_\theta(x) + b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n\right).$$

Proposition 5. *Supposing that ϕ is strictly concave and differentiable, the Nash is the vector $a^* = (a_1^*, \dots, a_N^*)$ that satisfies, for all $n \in \mathcal{N}$, the following N equations*

$$\frac{1}{w_n} B_n'(a_n) - \frac{\partial D\left(\theta_{\mathrm{AT}}(x, a)\right)}{\partial a_n} = \overline{\lambda}_n - \underline{\lambda}_n. \qquad \text{(KKT)}$$

with $\overline{\lambda}_n, \underline{\lambda}_n \geq 0$ with $\underline{\lambda}_n^*(a_n^* - e_n^{\min}) = 0$ and $\overline{\lambda}_n^*(a_n^* - e_n^{\max}) = 0$ being the KKT multipliers with associated constraints.

Proof. The proof is straightforward: ϕ is continuous over $\mathcal{A}$, then there exists a NE, a^*. Moreover, if ϕ is strictly concave, then the NE is unique. Since the constraints are linear, we can apply the KKT conditions. □

In the proposition below, we provide sufficient conditions to express the NE. To do so, we will need the following lemma on the zeros of the equation $rs^2 + ps + q = k \ln(rs + v)$.

Lemma 3. *For $k, p, q \in \mathbb{R}$, $r, v \in \mathbb{R}_{>0}$, and for all $s \in \mathbb{R}$ such that $rs + v > 0$, the following equation in s:*

$$rs^2 + ps + q = k \ln(rs + v),$$

- *has at most one solution if $rk + (2v - p)^2/8 < 0$,*
- *has at most two solutions if $rk + (2v - p)^2/8 = 0$,*
- *has at most three solutions if $rk + (2v - p)^2/8 > 0$.*

When ϕ is strictly concave, we can apply the KKT conditions to find the unique pure NE. Let us assume that ϕ is continuous on $\mathcal{A}$, so there exists a pure NE, denoted by a^*. Applying the Proposition 5, we have for every $n \in \mathcal{N}$, a^* verifies the conditions in the Eq (KKT). This leads to sufficient conditions for the unique interior NE given in the following proposition.

Proposition 6. *We assume that ϕ is strictly concave with $\beta_{2,n} \neq 0, \forall n \in \mathcal{N}$, and denote $\Lambda_n := (w_n(\underline{\lambda}_n - \overline{\lambda}_n) - \beta_{1,n})/(2\beta_{2,n})$. If*

$$b_\theta^2 b_{\mathrm{C}}^2 \gamma_2 \sum_{n=1}^{N} \frac{w_n}{\beta_{2,n}} + \frac{\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} \Lambda_n\right)^2}{8} < 0, \quad (16)$$

then the unique pure NE of the game Γ is given by $a^* = (a_1^*, \dots, a_N^*)$, where for all $n \in \mathcal{N}$,

$$a_n^* = \Lambda_n + \frac{w_n b_\theta b_{\mathrm{C}} \left[\gamma_1 + 2\gamma_2 \psi_\theta(x) + 2\gamma_2 b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \tilde{S}\right)\right]}{2\beta_{2,n}\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \tilde{S}\right)},$$

where $\tilde{S}$ is the unique solution of the equation $rs^2 + ps + q = k \ln(rs + v)$, with $r = b_{\mathrm{C}}$, $p = \psi_{\mathrm{C}}(x) - b_{\mathrm{C}} \sum_{n=1}^{N} \Lambda_n$, $k = b_\theta^2 b_{\mathrm{C}} \gamma_2 \sum_{n=1}^{N} \beta_{2,n}/w_n$, $v = \psi_{\mathrm{C}}(x)$, and $q = -\psi_{\mathrm{C}}(x) \sum_{n=1}^{N} \Lambda_n - \sum_{n=1}^{N} w_n b_\theta b_{\mathrm{C}} \left(\gamma_1 + 2\gamma_2 \psi_\theta\right)/(2\beta_{2,n})$.

Proof. Using Proposition 5, for all $n \in \mathcal{N}$ one has $\underline{\lambda}_n - \overline{\lambda}_n =$

$$\frac{1}{w_n}\left(\beta_{1,n} + 2\beta_{2,n} a_n^*\right) - \left[\gamma_1 + 2\gamma_2 \theta_{\mathrm{AT}}(x, a^*)\right] \frac{\partial \theta_{\mathrm{AT}}}{\partial a_n}(x, a^*).$$

Dividing by $\beta_{2,n} \neq 0$ and using the notation Λ_n introduced in the statement, we get that $\forall n \in \mathcal{N}$, one has $a_n^* = \Lambda_n +$

$$\frac{w_n b_\theta b_{\mathrm{C}} \left[\gamma_1 + 2\gamma_2 \psi_\theta(x) + 2\gamma_2 b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n^*\right)\right]}{2\beta_{2,n}\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} \sum_{n=1}^{N} a_n^*\right)}.$$

Using the sum notation s as before and summing the above equation over $n \in \mathcal{N}$ yields $s = \sum_{n=1}^{N} \Lambda_n +$

$$\sum_{n=1}^{N} \frac{w_n}{2\beta_{2,n}} \frac{b_\theta b_{\mathrm{C}} \left[\gamma_1 + 2\gamma_2 \psi_\theta(x) + 2\gamma_2 b_\theta \ln\left(\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} s\right)\right]}{\psi_{\mathrm{C}}(x) + b_{\mathrm{C}} s},$$

which can be re-written as $rs^2 + ps + q = k \ln(rs + v)$ with r, p, q, v, and k given in the statement above. By using the results of the lemma 3, we conclude that if the condition in the Eq (16) is verified, then there exists at most one solution $\tilde{S}$ of the equation $rs^2 + ps + q = k \ln(rs + v)$ in $\left[E^{\min}, E^{\max}\right]$. Thus, the NE is unique. □

4. Numerical analysis

In this section, we aim to illustrate our theoretical findings numerically, emphasizing the role played by the imitation term in the proposed model. We go even further with the numerical analysis by considering scenarios that have not been theoretically addressed. It is noteworthy that the numerical analysis can be further improved by using the flexible code that we have developed. It enables us to manipulate

Player	$e_n^{\max}$ ($GtCO_2$/y)	$GDP_n^{\max}$ (10^9\$)	w_n	r_n	e_n
China	22	14630	1.1847	0.7	3
USA	14	19290	1.1941	1	-1
EU	8	13890	1.1248	2	-5
India	6	2500	0.9074	0.5	-4
Russia	4	1420	1.2866	1.25	-5
AOC	10	11640	1.1847	0.45	2.5

Table 1
Parameter values for each player in the imitation game as of 2020.

various parameters such as climate model attributes, benefit, damage, and imitation functions. Our simulations involve a setting with $N = 6$ players, and the parameter details are outlined in Table 1, where "AOC" collectively refers to all other countries. In our game, we are using the climate structure of IAMs, which incorporates geophysical connections that involves various powers influencing environmental change. The visualizations are derived using the carbon cycle data from [30] and temperature dynamics from [31], recognized for their proximity to IPCC results [32, 33].

We specifically focus on a scenario where the benefit function adopts a sigmoid shape, and damages follow a quadratic and rescaled pattern, *i.e.,* ,

$$u_n(x,a) = \mathrm{GDP}_n^{\max}\left(B_n(a_n) - w_n\left[D\left(\theta_{\mathrm{AT}}(x,a)\right)\right]^{\alpha}\right) - \delta_n \mathrm{GDP}_n^{\max} e_n^{\max} \sum_{m\in\mathcal{V}_n} e_m^{\max}\left(\xi_n - \xi_m\right)^2,$$

where the sigmoid benefit function is given by :

$$B_n(a_n) = \frac{f_n(a_n) - f_n(e_n^{\min})}{f_n(e_n^{\max}) - f_n(e_n^{\min})},$$

with $e_n^{\min} = 0, \forall n \in \mathcal{N}$, the logistic function is given by :

$$f_n(a_n) = \frac{1}{1+\exp(-r_n(a_n - e_n))}.$$

The global damage, quadratic function, is given by :

$$D\left(\theta_{\mathrm{AT}}(x,a)\right) = 1 + \theta_{\mathrm{AT}}(x,a) + 2\theta_{\mathrm{AT}}^2(x,a),$$

and α represents the power of the damages and measures the severity level of climate change on the economics, we used $\alpha = 2.5$.

The utility function, denoted as $u_n(x,a)$, incorporates an imitation term influenced by the actions of neighboring players. To study the effect of the imitation part on the decision of the players, we consider that the imitation weight is a function of time, represented by $\delta_n(t)$, that exponentially increases with respect to time. This illustrates the influence of neighboring countries on the utility optimization of each player. We highlight that the influence weights are given for a player $n \in \mathcal{N}$ and for a neighbor $m \in \mathcal{V}_n$, by $1/(e_n^{\max} e_m^{\max})$.

The influence term between two countries is inversely proportional to the product of their maximum emissions. This means that countries with lower emissions exert more influence on others, as their emissions have a stronger impact on the negotiations. This can be interpreted in a network theory context, where countries' emissions act as nodes, and influence is determined by their emissions levels. A higher emissions product weakens the influence between two countries, suggesting a global interdependence where larger emitters have less influence on each other, while smaller emitters can have a more significant impact. This reciprocal influence reflects the mutual dependencies in climate negotiations.

4.1. Network structure: global vs grouped influence

The static game model for climate change mitigation is recurrently applied every five years from 2020 until 2100. We consider two types of network structure: a complete graph and a network containing two groups of countries that do not influence each other. In both configurations, we make a numerical analysis by varying the influence weights. In the second scenario, we consider two coalitions. We assume that EU, USA, and AOC form one group while China, Russia, and India form the other. The interconnection topology is directed: China does not receive any influence, while AOC does not influence any player. The associated network structures are plotted in Figure 1.

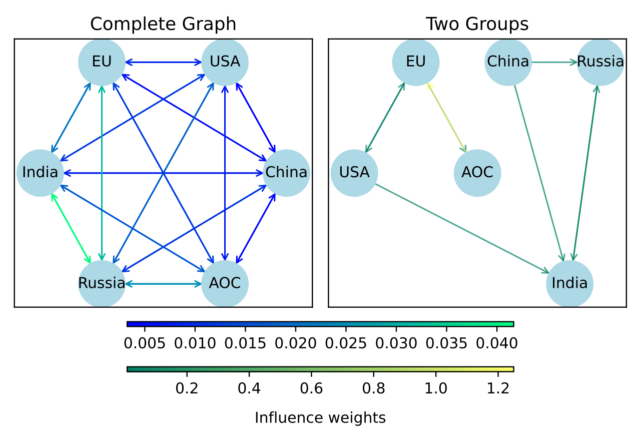

Figure 1: Illustration of the two assumed cross-country influence/imitation network topologies.

Our goal is to numerically compare the players' strategies and the temperature increase in the two scenarios. In Figure 2, the solid line represents the first scenario (complete graph) while the dashed line represents the second scenario (network containing two groups that do not influence each other). We plot the temperature increase corresponding to the emissions at NE, and we observe that global cooperation leads to better behavior, where the temperature increase will stay below 1.5°C, the limit stated in the Paris Agreement.

In the following, we address the scenario in which EU adopts a decarbonization strategy. EU stops emissions progressively, modeled by an exponentially decreasing function, reaching zero CO_2 emissions in 2050. In response,

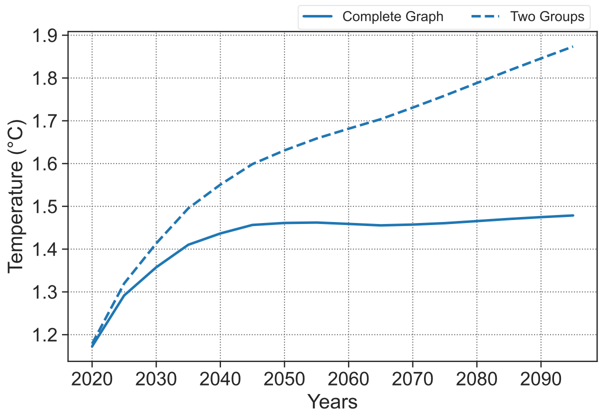

Figure 2: Temperature (excess) increase due to CO_2 emissions at Nash Equilibrium for the two different imitation graph structures. The dashed line represents a two-group network, while the solid line represents a complete graph scenario.

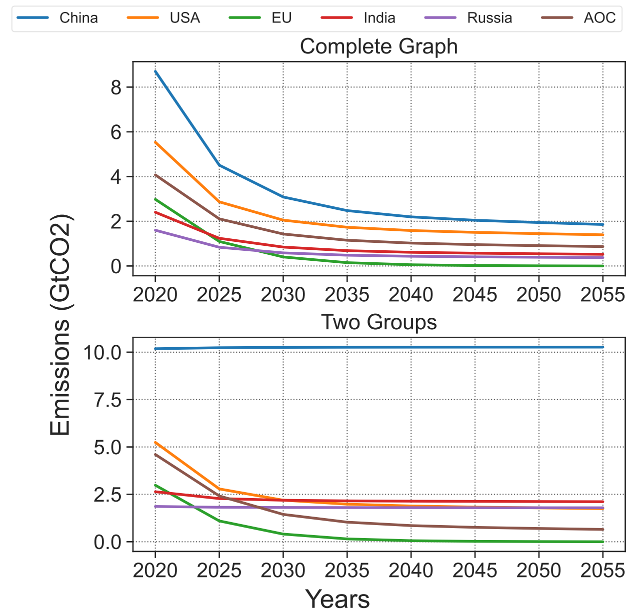

Figure 3: CO_2 emissions at NE in different scenarios with varying network structures. In a complete graph scenario, players collectively converge to 0 $GtCO_2$, aligning with the EU's strategy to stop emissions by 2050. The two groups' scenario presents distinct emission behaviors. China and Russia continue emitting at maximum levels, while India shows a slight reduction. Meanwhile, the USA and AOC converge to the EU's emission reduction strategy. Colors and line shapes are explained in Figure 2.

other players adjust their strategies influenced by the EU decision through the imitation term. The subsequent figures illustrate the trajectories followed by the players' CO_2 emissions under each of the two different network topologies introduced in Figure 1. We can notice the influence of the imitation term in mitigating the temperature increases. This influence is more important in the case of the complete graph, where all players influence each other. We note that in the first scenario, all the players have an exponential decrease of the CO_2 emissions (Figure 3 top). In this scenario, every player emits less than 2 $GtCO_2$ by 2055.

In the second scenario one can see a decrease in the emissions of the AOC which is only and directly influenced by EU, the USA roughly preserves the same behaviour while the other 3 players perform worse due to the influence of China that keeps its CO_2 emission level to 10 $GtCO_2$ (Figure 3 bottom). It is noteworthy that all the players that are directly or indirectly influenced by EU decrease their emissions to less than 2.5 $GtCO_2$.

Another important insight revealed by these simulations is that the imitation term significantly contributes to mitigating the short-term utility increase resulting from emissions. Indeed, in [27], we have shown that the short-term benefits increase faster than the damages as long as the atmospheric temperature is not very high. Adding the imitation term helps to better compensate for this short-term increase in the benefits. When forming groups, the decision will be made by the uninfluenced country, and if they decide to act virtuously, the other countries can indeed change their behaviours and decrease their emissions too.

Table 2 emphasizes the synchronization of emission ratios at Nash Equilibrium with players' maximal emissions. It is another way to highlight the impact of the imitation term on emission reductions across all players. This synchronization represents the consensus among countries, and since countries have different economies and sizes, we cannot compare their emissions but rather this ratio.

	Complete Graph			Two Groups		
	2025	2050	2100	2025	2050	2100
China	0.4	0.1	0.06	0.46	0.47	0.47
USA	0.4	0.1	0.08	0.37	0.13	0.11
EU	0.4	0.003	0	0.37	0.003	0
India	0.4	0.1	0.06	0.44	0.35	0.34
Russia	0.4	0.1	0.07	0.47	0.45	0.44
AOC	0.4	0.1	0.06	0.46	0.08	0.04

Table 2
Synchronization of the ratio of players' emissions at the NE over their maximal emissions, *i.e.*, , $a_n^*/e_n^{\max}$.

It is noteworthy that in the complete graph case, the players will roughly synchronize at the same ratio. On the other hand, when the graph is directed and not strongly connected, the differences in the ratios become apparent. The presence of groups can be observed through the different behavior of the corresponding ratio δ_n. Precisely, the players in the second group preserve relatively high ratios while the ones in the first group have a clear decrease in it. In summary, our study underscores the crucial role of imitation in influencing player behavior and favoring cooperative emission reduction strategies, thereby contributing to global climate change mitigation. We notice that a country between two coalitions will be influenced by a virtuous player, even if in the second group, they are encouraged to continue emitting.

4.2. Network structure: EU and Russia as independent influencers

In this subsection, we explore an alternative network structure to further investigate the dynamics of climate change mitigation strategies. Unlike the previously considered scenarios, we configure the network such that EU and Russia act as independent influencers. Specifically, these two countries do not receive any influence from other players, but they influence all the remaining countries. Meanwhile, the other players continue to influence each other. Moreover, the EU's strategy remains the same: they decrease their emissions exponentially to reach zero emissions by 2050 and stop emitting thereafter. On the other hand, the strategy of Russia is to continue emitting to the maximum.

The configuration of this network is represented in Figure 4. The directed edges from EU and Russia to the remaining players indicate their influential roles, while the interconnections among the other countries represent the influence they exert on each other.

4.2.1. Network Topology

The network topology is designed as follows:

- The USA and EU do not receive any influence from other players.
- The USA and EU influence all other countries in the network.
- The remaining players (China, India, Russia, AOC) influence each other and are influenced by the USA and EU.

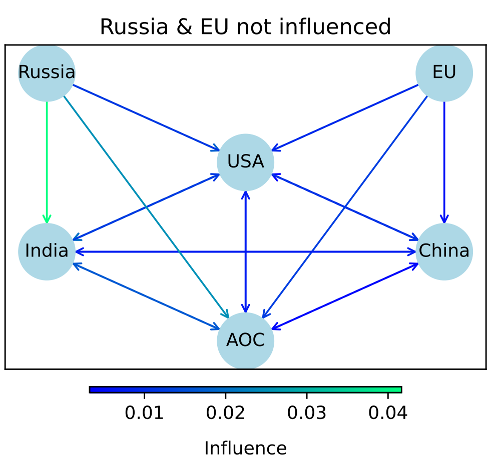

Figure 4: Illustration of a network where USA and Russia are independent influencers.

To evaluate the impact of this alternative network structure on climate change mitigation, we are varying the influence weights among the players, we represent three different functions that describe the variation of the weights with time. Our focus remains on understanding the strategies adopted by each player and the resulting temperature increase. Especially that, EU continues to pursue its decarbonization strategy until achieving zero emissions by 2050, while the USA intends to continue emitting at the same level as in 2020. We model the dynamic of having two opposing forces: one prioritizing climate concerns and the other seeking to emit and increase profits without considering climate damage, to observe their effects on countries' behaviors in this scenario. In other words, we consider two influencers: the first advocates for climate action, aiming to halt emissions in line with IPCC and COP recommendations, while the second prioritizes profit maximization.

In Figure 5, we present the outcomes of our numerical simulations in this case. The solid line corresponds to the temperature increase in the scenario with the alternative network structure, while the dashed line represents the baseline scenario with a complete graph. One can see that as the influence weights become important, the temperature rise is limited. Furthermore, when countries are interconnected, the temperature remains lower compared to the scenario where the USA and Russia act as independent influencers. In this case, the temperature reaches 1.5 °C for $\delta_n(t) = e_n^{\max} \exp(t/20)$. This highlights the effectiveness of global cooperation by being all connected, compared to the scenario where the USA and Russia play distinct roles. Indeed, in the connected network, we naturally achieve better results, as the country aiming to emit to the maximum, like Russia here, will not be able to do so in the complete graph due to the influence of the EU. Even with exponential influence weights, the outcomes in the complete graph are superior.

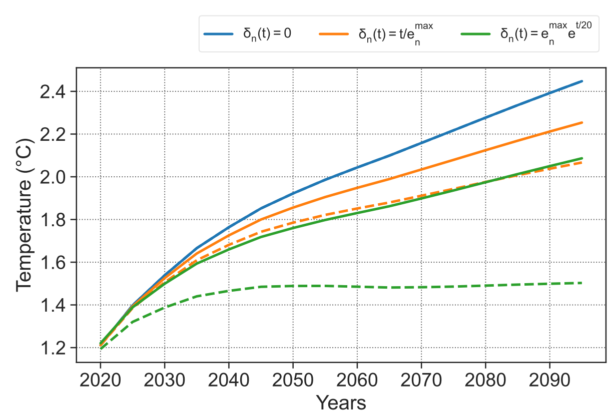

Figure 5: Temperature increase due to CO_2 emissions at Nash equilibrium under different influence/imitation graph structures. The dashed line represents the complete graph scenario, while the solid line represents the case where the USA and Russia are independent influencers.

Next, we explore the trajectories of CO_2 emissions for each player under this alternative network structure. We focus on the behaviors of EU and Russia, which, being independent influencers, might exhibit distinct emission patterns compared to other players.

This alternative scenario provides valuable insights into the role of key players and the effectiveness of different network structures in mitigating climate change. In Figure 6, we represent, for each player, the ratio of emissions at Nash Equilibrium over their maximal emissions, considering $\delta_n(t) = t/e_n^{\max}$. The ratio of EU decreases exponentially, reaching 0 by 2040. Meanwhile, Russia maintains a constant ratio of 1. The ratios of China, USA, India, and AOC converge to the ratio of 0.4. Since EU and Russia do not have an imitation part in their utility, their ratios result from the maximization of their trade-off between benefits and weighted global damage. Moreover, China, USA, India, and AOC are more influenced by EU because EU has the largest maximal emissions, and the imitation/influence weights depend on the influencer's maximal emissions. The synchronization of ratios occurs between those of the two leading countries. The followers' ratios lie between the two, aiming to mitigate the penalty caused by the influence of the two opposing powers. It follows that as long as players are strongly influenced by others, they are forced to follow the result of the average between actions weighted by maximum emissions.

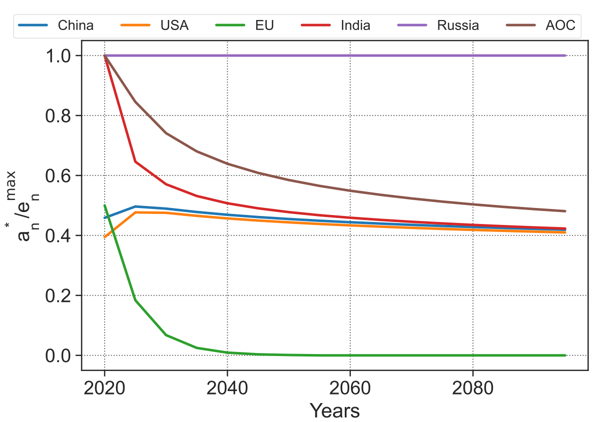

Figure 6: Synchronization of players' emission ratios at the Nash equilibrium over their maximum emissions. The strategy of the EU is an exponential decrease in emissions, reaching zero by 2050. USA and Russia act as independent influencers in the network, considering $\delta_n(t) = t/e_n^{\max}$.

In the following Table 3, we provide additional data for various weight functions, supporting our analysis of the synchronization of ratios. In this table, t denotes the number of repetitions of the game. Indeed, when $t = 0$, it corresponds to the initial iteration of the game, yielding the Nash equilibrium in 2025. As t increments by 5, each step represents an additional shot of the game, reaching the Nash equilibrium in 2050 when $t = 5$, and finally, reaching the Nash equilibrium in 2100 when $t = 15$. Moreover, the primary three columns specify the influence weight functions used in this study.

To summarize, we present the Table 3 showing different scenarios with various imitation weights. The first three columns present the ratios in the case of the game without imitation, where all players maintain the same ratios except for EU, which decides to reduce its emissions. For linear weights over time, the ratios synchronize and converge to 0.4, and for heavier weights, such as exponential ones, synchronization occurs around 0.3. Naturally, EU and Russia maintain the same ratios for different weight functions because they are not influenced. This suggests that even within a group where one player emits at maximum and encourages others to do the same, a consensus towards a smaller and more favorable ratio is reached compared to the coalition network consensus involving both groups.

4.2.2. Network structure: Non-influenced players

In this numerical analysis part, we illustrate the case where the benefit function is quadratic in a_n, and the damages are quadratic and re-scaled, *i.e.,*

$$u_n(x,a) = \mathrm{GDP}_n^{\max}\left[\frac{2a_n}{e_n^{\max}} - \left[\frac{a_n}{e_n^{\max}}\right]^2 - w_n D\left(\theta_{\mathrm{AT}}(x,a)\right)^{\alpha}\right],$$

where α represents the power of the damages and measures the severity level of climate change on the economy. The static game is played repetitively every five years until 2100 while updating $e_n^{\max}$ and $\mathrm{GDP}_n^{\max}$ at each iteration of the game. Indeed, the game in the simulations is considered to be a repeated game where the players decide their strategy at every step of the game, independently of the history and the next steps. For more information, refer to [34].

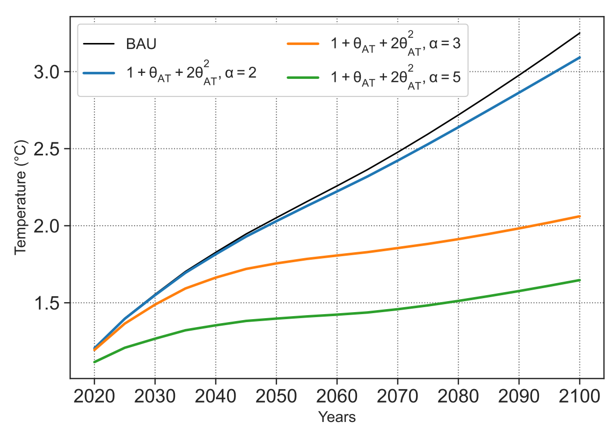

Figure 7: The increase of the forecast temperature due to the CO_2 emissions at NE in different scenarios for the non-influenced players case.

It is interesting to note that higher α induces higher damages and consequently lower CO_2 emissions and smaller increases in the temperature. For large α (e.g., $\alpha = 5$), China, the USA, EU, and AOC reduce their emissions until they completely stop emitting (see Figure 4). The temperatures in 2100 range from around +3.2°C for low damages, resembling a Business-as-Usual (BAU) scenario, to +1.6°C for high damages. These temperature levels are in line with the projections of the IPCC [32] and correspond to the emission trajectories of the countries. To prevent the over-warming of the planet by 2100, we need to revise the modeling of the

	$\delta_n(t) = 0$			$\delta_n(t) = t/e_n^{\max}$			$\delta_n(t) = e_n^{\max} \exp(t/20)$		
	2025	2050	2100	2025	2050	2100	2025	2050	2100
Chine	0.459	0.46	0.46	0.459	0.461	0.419	0.671	0.349	0.342
USA	0.394	0.396	0.396	0.394	0.45	0.41	0.671	0.349	0.342
EU	0.5	0.003	0	0.5	0.003	0	0.5	0.003	0
Inde	1	1	1	1	0.49	0.423	0.674	0.353	0.344
Russie	1	1	1	1	1	1	1	1	1
AOC	1	1	1	1	0.609	0.481	0.677	0.359	0.348

Table 3
Synchronization of the ratio of players' emissions at the NE over their maximal emissions, *i.e.*, , $a_n^*/e_n^{\max}$, when USA and Russia act as independent influencers.

Player	China	USA	EU	India	Russia	AOC
$\alpha = 1$	/	/	/	/	/	/
$\alpha = 2$	/	/	/	/	/	/
$\alpha = 3$	2065	2075	/	/	/	2080
$\alpha = 4$	2020	2020	2045	/	/	2025
$\alpha = 5$	2020	2020	2020	/	/	2020
$\alpha = 6$	2020	2020	2020	/	/	2020
$\alpha = 7$	2020	2020	2020	/	/	2020
$\alpha = 8$	2020	2020	2020	2095	/	2020
$\alpha = 9$	2020	2020	2020	2085	/	2020
$\alpha = 10$	2020	2020	2020	2080	/	2020

Table 4
Time at which the countries stop emitting versus α (which measures the economic damage due to climate change). The symbol / means no stopping.

economic damages and change the strategies accordingly. The idea is that the players are not able to see the real damage caused by their emissions since the damage functions are underestimating it, and this is the reason why the players are not interested in stopping emitting. In short, if the damages are not significant, the players will continue to emit excessively, even emit at maximum, so the atmospheric temperature will continue to rise rapidly.

Since quadratic damages, which are commonly used in the literature, are too optimistic and yield lower temperatures than those currently observed, the question arises: What level of damages would induce the players to stop emitting? The Table 4 presents the time at which countries stop emissions based on different values of α.

Table 4 shows that if the damages are not significant, *i.e.,* α is small, the CO_2 emissions of the players will not stop before 2100. Low damage hampers the cooperation recommended by the IPCC. When α is large enough, the NE strategies of the players are to stop emitting as soon as possible. Except for Russia, which continues to emit no matter how big the damage, and for India, which stops emitting only when $\alpha \geq 8$. This can be explained by the fact that the benefits of India and Russia are still very big compared with the corresponding loss. The product $w_n \mathrm{GDP}_n^{\max}$ has to be increased for these countries in order to stop their emissions. This is also because Russia has the smallest maximum emissions, so when other countries reduce emissions, the damage decreases and becomes less than their benefits, thus increasing Russia's utility. Therefore, as player n's emissions affect other countries, being a nation with high emissions, player n can incentivize countries with low maximum emissions to continue emitting when player n reduces its emissions, as they will incur lower damages.

5. Conclusion

We have conducted the complete Nash equilibrium analysis of a static game model designed to mathematically model the behavior of governments as far as their carbon emissions are concerned. We have considered a game in which the players maximize their utility, a tradeoff between socio-economic benefits for emitting and economic damage due to climate change, in the presence of imitation among countries. In contrast with the existing literature, the game integrates a climate dynamics, which is nonlinear. The choice of the Nash equilibrium as the game solution concept illustrates the importance of aligning individual interests with collective decarbonization strategies. Why should a country make efforts knowing that others are not doing the same? In fact, we demonstrate the effect of influence: it is in a country's interest to act virtuously because it will influence others and lead them to follow its example.

Our investigation went through the existence and uniqueness of the equilibrium, highlighting the evolution of carbon emissions by major emitters from 2020 to 2100. We have exploited this model by using data provided by the IPCC. Our simulations revealed a promising trend: players tend to reduce emissions when playing across multiple stages using imitation terms in the utility functions. We used utility functions that depend on general parameters like the GDP and measured the influence of atmospheric temperature on emission patterns, in addition to the influence of players on each other.

The observed synchronization in emissions refers to a notable alignment or coordination in the emission-related actions taken by different players in the modeled scenario. This coordination is influenced by the network of connections among the players. In simpler terms, the decisions made by one player regarding emission reduction have a domino effect on its neighboring players.

Specifically, when a player unilaterally decides to reduce its emissions, the model suggests that this decision triggers a response from the surrounding players. These neighboring players, influenced by various factors such as mutual influence or strategic considerations, tend to follow a similar path of reducing emissions. This observed behavior highlights the interconnected nature of the decision-making process in the context of emissions reduction strategies, where individual choices can affect the network of players.

The numerical analysis provides several insights, for instance, it is seen that to reach the Paris Agreement on climate (namely, maintain the average atmospheric temperature excess below 2 degrees), the damage induced by climate change has to be significant enough. This constitutes a sufficient condition under which governments will spontaneously reduce their emissions. Depending on the severity level of the damage (which is measured by the exponent α), governments are incited to stop emitting CO_2 , and it is shown to be possible to (roughly) forecast a time at which a country stops emitting. The obtained times are typically higher than values claimed publicly (e.g., 2050). We studied different scenarios with different graph structures, finding a consistent trend: the more connected the graph, the stronger the inclination of players to reduce their emissions. This effect becomes particularly important when at least one player decides to stop emitting. Our results underscore the imperative for collaboration or cooperation among players. Notably, cooperation is most evident when players are connected, influencing each other's decisions (e.g., through explicit international agreements or through social networks). Policy implications include the need to incentivize cooperation through mechanisms like international treaties, emission reduction targets, and financial incentives. By fostering interdependence and mutual influence, such policies can effectively encourage collective action in addressing global climate challenges.

As an extension of this article, introducing the opinion dynamics of players becomes a pivotal consideration. The incorporation of opinion dynamics into the game framework has the potential to significantly impact the weights within the connected network. This addition not only enhances the complexity of the possible planning game but also introduces a dynamic element that makes the study more challenging and inherently interesting. The evolving nature of opinions can contribute to a richer analysis of the game's dynamics, providing insights into the adaptive strategies of players over time. Recent developments in decision-making under uncertainty offer new mathematical frameworks that could improve climate mitigation models. Exploring how these methodologies intersect with game-theoretic climate models represents a promising direction for future research.

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