Passivity-based tracking control of multiconstraint complementarity Lagrangian systems

Irinel Constantin Morărescu*, Bernard Brogliato#

Abstract

In this study one considers the tracking control problem of a class of nonsmooth fully actuated Lagrangian systems subject to frictionless unilateral constraints. A passivity-based switching controller that guarantees some stability properties of the closed-loop system is designed. A particular attention is paid to transition (impacting) and detachment phases of motion. This paper extends previous works on the topic as it considers multiconstraint n-degree-of-freedom systems.

Index Terms

Lagrangian systems, Complementarity problem, Impacts, Stability, Tracking control, Passivity-based control, Nonsmooth systems.

I. Introduction

The control of mechanical systems subject to unilateral constraints has been the object of many studies in the past fifteen years. Such systems, which consist of three main ingredients (see (1) below) are highly nonlinear nonsmooth dynamical systems. Theoretical aspects of their Lyapunov stability and the related stabilization issues have been studied in [10], [21], [19], [33]. The specific yet important task of the stabilization of impacting transition phases was analyzed and experimentally tested in [18], [30], [31], [34], [35], [36]. From the point of view of

INRIA, BIPOP team-project, Inovallée, 655 avenue de l'Europe, 38330, Montbonnot, France.

*`constantin.morarescu@inrialpes.fr`

`bernard.brogliato@inrialpes.fr`

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tracking control of complementarity Lagrangian systems along general constrained/unconstrained paths, such studies focus on a module of the overall control problem. The problem of robust impact detection with only position measurement received attention in [6]. One of the first works formulating the control of complete robotic tasks via unilateral constraints and complementarity conditions was presented in [17]. In that work the impacts were considered inelastic and the control problem was solved using a time optimal problem. The tracking control problem under consideration, involving systems that undergo transitions from free to constrained motions, and vice-versa, along an infinity of cycles, was formulated and studied in [9] for the 1-dof (degree-of-freedom) case and in [5] for the n-dof case. Both of these works consider systems with only one unilateral frictionless constraint. In this paper we not only consider the multiconstraint case but the results in Section VII relax some very hard to verify conditions imposed in [5] to assure the stability. Moreover the accurate design of the control law that guarantees the detachment from the constraints is formulated and incorporated in the stability analysis for the first time. Considering multiple constraints may be quite important in applications like virtual reality and haptic systems, where typical tasks involve manipulating objects modelled as rigid bodies [12] in complex environments with many unilateral constraints. We note that in the case of a single nonsmooth impact the exponential stability and bounded-input bounded state (BIBS) stability was studied in [26] using a state feedback control law. A study for a multiple degree-of-freedom linear systems subject to nonsmooth impacts can be found in [27]. That approach proposes a proportional-derivative control law in order to study BIBS stability via Lyapunov techniques. Other approaches for the tracking control of nonsmooth mechanical systems can be found in [13], [25], [29], [37] and in [20]. The analysis and control of systems subject to unilateral constraints also received attention in [4].

This paper focuses on the problem of tracking control of complementarity Lagrangian systems [28] subject to frictionless unilateral constraints whose dynamics may be expressed as:

$$\begin{cases} M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) = U + \nabla F(X)\lambda_X \\ 0 \leq \lambda_X \perp F(X) \geq 0, \\ \text{Collision rule} \end{cases} \tag{1}$$

where $X(t) \in \mathbb{R}^n$ is the vector of generalized coordinates, $M(X) = M^T(X) \in \mathbb{R}^{n\times n}$ is the positive definite inertia matrix, $F(X) \in \mathbb{R}^m$ represents the distance to the constraints, $C(X,\dot{X})$

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is the matrix containing Coriolis and centripetal forces, $G(X)$ contains conservative forces, $\lambda_X \in \mathbb{R}^m$ is the vector of the Lagrangian multipliers associated to the constraints and $U \in \mathbb{R}^n$ is the vector of generalized torque inputs. For the sake of completeness we precise that ∇ denotes the Euclidean gradient $\nabla F(X) = (\nabla F_1(X), \dots, \nabla F_m(X)) \in \mathbb{R}^{n\times m}$ where $\nabla F_i(X) \in \mathbb{R}^n$ represents the vector of partial derivatives of $F_i(\cdot)$ w.r.t. the components of X. We assume that the functions $F_i(\cdot)$ are continuously differentiable and that $\nabla F_i(X) \neq 0$ for all X with $F_i(X) = 0$. It is worth to precise here that for a given function $f(\cdot)$ its derivative w.r.t. the time t will be denoted by $\dot{f}(\cdot)$. For any function $f(\cdot)$ the limit to the right at the instant t will be denoted by $f(t^+)$ and the limit to the left will be denoted by $f(t^-)$. A simple jump of the function $f(\cdot)$ at the moment $t = t_\ell$ is denoted $\sigma_f(t_\ell) = f(t_\ell^+) - f(t_\ell^-)$.

Definition 1: A Linear Complementarity Problem (LCP) is a system given by:

$$\begin{cases} \lambda \geq 0 \\ A\lambda + b \geq 0 \\ \lambda^T(A\lambda + b) = 0 \end{cases} \tag{2}$$

which is compactly re-written as

$$0 \leq \lambda \perp A\lambda + b \geq 0 \tag{3}$$

Such an LCP has a unique solution for all b if and only if A is a P-matrix [11].

The admissible domain associated to the system (1) is the closed set Φ where the system can evolve and it is described as follows:

$$\Phi = \{X \mid F(X) \geq 0\} = \bigcap_{1\leq i\leq m} \Phi_i,$$

where $\Phi_i = \{X \mid F_i(X) \geq 0\}$ considering that a vector is non-negative if and only if all its components are non-negative. In order to have a well-posed problem with a physical meaning we consider that Φ contains at least a closed ball of positive radius.

Definition 2: A singularity of the boundary $\partial\Phi$ of Φ is the intersection of two or more codimension one surfaces $\Sigma_i = \{X \mid F_i(X) = 0\}$.

The presence of $\partial\Phi$ may induce some impacts that must be included in the dynamics of the system. It is obvious that $m > 1$ allows both simple impacts (when one constraint is involved) and multiple impacts (when singularities or surfaces of codimension larger than 1 are involved). Let us introduce the following notion of p_ϵ-impact.

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Definition 3: Let $\epsilon \geq 0$ be a fixed real number. We say that a p_ϵ-impact occurs at the instant t if

$$||F_I(X(t))|| \leq \epsilon, \quad \prod_{i \in I} F_i(X(t)) = 0$$

where $I \subset 1, \ldots, m$, $card(I) = p$.

If $\epsilon = 0$ the p surfaces Σ_i, $i \in I$ are stroked simultaneously. When $\epsilon > 0$ the system collides $\partial\Phi$ in a neighborhood of the intersection $\bigcap_{i \in I} \Sigma_i$.

Definition 4: [28], [22] The tangent cone to $\Phi = \{X \mid F_i(X) \geq 0, \, \forall i = 1, \ldots, n\}$ at $q \in \mathbb{R}^n$ is defined as:

$$T_\Phi(q) = \{z \in \mathbb{R}^n \mid z^T \nabla F_i(q) \geq 0, \, \forall i = J(q)\}$$

where $J(q) \triangleq \{i \in \{1, \ldots, n\} \mid F_i(q) \leq 0\}$. When $q \in \Phi \backslash \partial\Phi$ one has $J(q) = \emptyset$ and $T_\Phi(q) = \mathbb{R}^n$. The normal cone to Φ at q is defined as the polar cone to $T_\Phi(\cdot)$:

$$N_\Phi(q) = \{y \in \mathbb{R}^n \mid \forall z \in T_\Phi(q), y^T z \leq 0\}$$

The collision (or restitution) rule in (1), is a relation between the post-impact velocity and the pre-impact velocity. Among the various models of collision rules, Moreau's rule is an extension of Newton's law which is energetically consistent [15] and is numerically tractable [1]. For these reasons throughout this paper the collision rule will be defined by Moreau's relation [28]:

$$\dot{X}(t_\ell^+) = (1+e) \mathop{\arg\min}_{z \in T_\Phi(X(t_\ell))} \frac{1}{2}[z - \dot{X}(t_\ell^-)]^T \times M(X(t_\ell))[z - \dot{X}(t_\ell^-)] - e\dot{X}(t_\ell^-) \tag{4}$$

where $\dot{X}(t_\ell^+)$ is the post-impact velocity, $\dot{X}(t_\ell^-)$ is the pre-impact velocity and $e \in [0, 1]$ is the restitution coefficient. Denoting by T the kinetic energy of the system, we can compute the kinetic energy loss at the impact t_ℓ as [23]:

$$T_L(t_\ell) = -\frac{1-e}{2(1+e)} \Big[\dot{X}(t_\ell^+) - \dot{X}(t_\ell^-)]^T M(X(t_\ell)) \times [\dot{X}(t_\ell^+) - \dot{X}(t_\ell^-)\Big] \leq 0 \tag{5}$$

The collision rule can be rewritten considering the vector of generalized velocities as an element of the tangent space to the configuration space of the system, equipped with the kinetic energy metric. Doing so (see [7] §6.2), the discontinuous velocity components $\dot{X}_{norm}$ and the continuous ones $\dot{X}_{tang}$ are identified. Precisely, $\begin{pmatrix} \dot{X}_{norm} \\ \dot{X}_{tang} \end{pmatrix} = \mathcal{M}\dot{X}$, $\mathcal{M} = \begin{pmatrix} \mathbf{n}^T \\ \mathbf{t}^T \end{pmatrix} M(X)$ where $\mathbf{n} \in \mathbb{R}^m$ represents the m unitary normal vectors $\mathbf{n}_i = \frac{M^{-1}(X)\nabla F_i(X)}{\sqrt{\nabla F_i(X)^T M^{-1}(X) \nabla F_i(X)}}$, $i = 1, \ldots, m$ and $\mathbf{t}$ represents $n - m$ mutually independent unitary vectors $\mathbf{t}_i$ such that $\mathbf{t}_i^T M(X)\mathbf{n}_j = 0, \, \forall i, j$.

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In this case the collision rule (4) at the impact time t_ℓ becomes the generalized Newton's rule $\begin{pmatrix} \dot{X}_{norm}(t_\ell^+) \\ \dot{X}_{tang}(t_\ell^+) \end{pmatrix} = -\eta \begin{pmatrix} \dot{X}_{norm}(t_\ell^-) \\ \dot{X}_{tang}(t_\ell^-) \end{pmatrix}$, $\eta = diag(e_1, \ldots, e_m, 0, \ldots, 0)$ where e_i is the restitution coefficient w.r.t. the surface Σ_i. For the sake of simplicity we consider in this paper that all the restitution coefficients are equal, i.e. $e_1 = \ldots = e_m \triangleq e$.

Remark 1: 1) If $X \in \Sigma_1 \bigcap \Sigma_2$ and the angle $\angle(\Sigma_1, \Sigma_2) \leq \pi$ then in the neighborhood of X one has $\Phi \simeq T_\Phi(X)$.

2) Let $m = 1$. The case $e = 0$ is called a plastic impact and the case $e = 1$ is called an elastic impact. In the first case the normal component of the velocity becomes zero and in the second case the normal component of the velocity changes only its direction and preserves its magnitude. As we can easily see from (5) in the second case there is no loss of kinetic energy at the impact moment.

3) One recalls that we deal with frictionless unilateral constraints. Some frictional contact laws that fit within the nonsmooth mechanic framework (1) can be found in [19].

The structure of the paper is as follows: in Section II one presents some basic concepts and prerequisites necessary for the further developments. Section III is devoted to the controller design. In Section IV one defines the desired (or ”exogenous”) trajectories entering the dynamics. The desired contact-force that must occur on the phases where the motion is constrained, is explicitly defined in Section V. Section VI focuses on the strategy for take-off at the end of the constraint phases. The main results related to the closed-loop stability analysis are presented in Section VII. One example and concluding remarks end the paper.

The following standard notations will be adopted: $|| \cdot ||$ is the Euclidean norm, $b_p \in \mathbb{R}^p$ and $b_{n-p} \in \mathbb{R}^{n-p}$ are the vectors formed with the first p and the last $n-p$ components of $b \in \mathbb{R}^n$, respectively. $N_\Phi(X_p = 0)$ is the normal cone $N_\Phi(X)$ to Φ at X [28], [22] when X satisfies $X_p = 0$, $\lambda_{min}(\cdot)$ and $\lambda_{max}(\cdot)$ represent the smallest and the largest eigenvalues, respectively.

II. Basic concepts

A. Typical task

In the case $m = 1$ (only one unilateral constraint) the time axis can be split into intervals Ω_k and I_k corresponding to specific phases of motion [9]. Precisely, Ω_{2k} corresponds to free-motion phases ($F(X) > 0$) and Ω_{2k+1} corresponds to constrained-motion phases ($F(X) = 0$).

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Between free and constrained phases the dynamical system always passes into a transition phase I_k containing some impacts. Since the dynamics of the system does not change during the transition between constrained and free-motion phases, in the time domain one gets the following typical task representation:

$$\mathbb{R}^+ = \Omega_0 \cup I_0 \cup \Omega_1 \cup \Omega_2 \cup I_1 \cup \ldots \cup \Omega_{2k} \cup I_k \cup \Omega_{2k+1} \cup \ldots \tag{6}$$

In the case $m \geq 2$ (multiple constraints) things complicate since the number of typical phases increases due to the singularities that must be taken into account. Explicitly, the constrained-motion phases need to be decomposed in sub-phases where some specific constraints are active. Between two such sub-phases a transition phase occurs when the number of active constraints increases. Nevertheless, a typical task can be represented in the time domain as:

$$\begin{gathered} \mathbb{R}^+ = \bigcup_{k\geq 0} \left(\Omega_{2k}^{J_k} \cup I_k^{J_k} \cup \left(\bigcup_{i=1}^{m_k} \Omega_{2k+1}^{J_{k,i}} \right) \right) \\ J_k \subset J_{k,1};\ J_{k+1} \subset J_{k,m_k} \subset J_{k,m_k-1} \subset \ldots J_{k,1} \end{gathered} \tag{7}$$

where the superscript J_k represents the set of active constraints ($J_k = \{i \in \{1,\ldots,m\} \mid F_i(X) = 0\}$) during the corresponding motion phase, and $I_k^{J_k}$ denotes the transient between two Ω_k phases when the number of active constraints increases. When the number of active constraints decreases there is no impact, thus no other transition phases are needed. We note that $J_k = \emptyset$ corresponds to free-motion.

For the sake of simplicity and without any loss of generality[1] we replace $\bigcup_{i=1}^{m_k} \Omega_{2k+1}^{J_{k,i}}$ by $\Omega_{2k+1}^{J'_k}$ where $J_k \subset J'_k$ and $J_{k+1} \subset J'_k$. Therefore the typical task simplifies as:

$$\begin{gathered} \mathbb{R}^+ = \bigcup_{k\geq 0} \left(\Omega_{2k}^{J_k} \cup I_k^{J_k} \cup \Omega_{2k+1}^{J'_k} \right) \\ J_k \subset J'_k, \quad J_{k+1} \subset J'_k \end{gathered} \tag{8}$$

Since the tracking control problem involves no difficulty during the Ω_k phases, *the central issue is the study of the passages between them (the design of transition phases I_k and detachment conditions), and the stability of the trajectories evolving along (8)* (i.e. an infinity of cycles). Throughout the paper, the sequence $\Omega_{2k}^{J_k} \cup I_k^{J_k} \cup \Omega_{2k+1}^{J'_k}$ will be referred to as the cycle k of the

[1]As we shall see in Section VIII we can also consider more complicated tasks without influencing the stability results obtained in the paper.

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system's evolution. For robustness reasons during transition phases I_k we impose a closed-loop dynamics (containing impacts) that mimics somehow the bouncing-ball dynamics (see e.g. [7]).

B. *Stability analysis criteria*

The system (1) is a complex nonsmooth and nonlinear dynamical system which involves continuous and discrete time phases. A stability framework for this type of systems has been proposed in [9] and extended in [5]. This is an extension of the Lyapunov second method adapted to closed-loop mechanical systems with unilateral constraints. Since we use this criterion in the following tracking control strategy it is worth to clarify the framework and to introduce some definitions.

Let us introduce the trajectories playing a role in the dynamics and the design of the controller:

- $X^{nc}(\cdot)$ denotes the desired trajectory of the unconstrained system (i.e. the trajectory that the system should track if there were no constraints). We suppose that $F(X^{nc}(t)) < 0$ for some t, otherwise the problem reduces to the tracking control of a system with no constraints.
- $X_d^*(\cdot)$ denotes the signal entering the control input and playing the role of the desired trajectory during some parts of the motion.
- $X_d(\cdot)$ represents the signal entering the Lyapunov function. This signal is set on the boundary $\partial\Phi$ after the first impact of each cycle.

These signals may coincide on some time intervals as we shall see later.

Next, let us define Ω as the complement in $\mathbb{R}^+$ of $I = \bigcup_{k\geq 0} I_k^{J_k}$ and assume that the Lebesgue measure of Ω, denoted $\lambda[\Omega]$, equals infinity. Consider $x(\cdot)$ the state of the closed-loop system in (1) with some feedback controller $U(X, \dot{X}, X_d^*, \dot{X}_d^*, \ddot{X}_d^*)$.

Definition 5 (Weakly Stable System [5]): The closed loop system is called weakly stable if for each $\epsilon > 0$ there exists $\delta(\epsilon) > 0$ such that $||x(0)|| \leq \delta(\epsilon) \Rightarrow ||x(t)|| \leq \epsilon$ for all $t \geq 0$, $t \in \Omega$. The system is asymptotically weakly stable if it is weakly stable and $\lim_{t\in\Omega,\, t\to\infty} x(t) = 0$. Finally, the practical weak stability holds if there exists $0 < R < +\infty$ and $t^* < +\infty$ such that $||x(t)|| < R$ for all $t > t^*$, $t \in \Omega$.

Consider $I_k^{J_k} \triangleq [\tau_0^k, t_f^k]$ and $V(\cdot)$ such that there exists strictly increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ satisfying the conditions: $\alpha(0) = 0$, $\beta(0) = 0$ and $\alpha(||x||) \leq V(x, t) \leq \beta(||x||)$.

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Definition 6: A transition phase $I_k^{J_k}$ is called finite if it involves a sequence of p_ϵ-impact times $(t_\ell^k)_{0\le \ell\le N}$, $N \le \infty$ with the accumulation point $t_N^k < \infty$ (for the sake of simplicity we shall denote the accumulation point by t_∞^k even if $N < \infty$).

In the sequel, we consider that each cycle k contains only finite transition phases which implies that $e < 1$ (in [3] it is shown that $e = 1$ implies that $t_\infty^k = +\infty$).

The following criterion is inspired from [5], and will be used for studying the stability of system (1).

Proposition 1 (Weak Stability): Assume that the task admits the representation (8) and that

- **a)** $\lambda[I_k^{J_k}] < +\infty, \quad \forall k \in \mathbb{N}$,
- **b)** outside the impact accumulation phases $[t_0^k, t_\infty^k]$ one has $\dot{V}(x(t), t) \le -\gamma V(x(t), t)$ for some constant $\gamma > 0$,
- **c)** $\sum_{\ell \ge 0} \left[V(t_{\ell+1}^{k-}) - V(t_\ell^{k+})\right] \le K_1 V^{p_1}(\tau_0^k)$, $\forall k \in \mathbb{N}$ for some $p_1 \ge 0,\ K_1 \ge 0$,
- **d)** the system is initialized on Ω_0 such that $V(\tau_0^0) \le 1$,
- **e)** $\sum_{\ell \ge 0} \sigma_V(t_\ell^k) \le K_2 V^{p_2}(\tau_0^k) + \xi$, $\forall k \in \mathbb{N}$ for some $p_2 \ge 0,\ K_2 \ge 0$ and $\xi \ge 0$.

If $p = \min\{p_1, p_2\} < 1$ then $V(\tau_0^k) \le \delta(\gamma, \xi)$, $\forall k \ge 1$, where $\delta(\gamma, \xi)$ is a function that can be made arbitrarily small by increasing the value of γ. The system is practically weakly stable with $R = \alpha^{-1}(\delta(\gamma, \xi))$.

Proof: From assumption **(b)** one has

$$V(t_f^k) \le V(t_\infty^k) e^{-\gamma(t_f^k - t_\infty^k)}$$

It is clear that condition **(c)** combined with **(e)** leads to

$$V(t_\infty^k) \le V(\tau_0^k) + K_1 V^{p_1}(\tau_0^k) + K_2 V^{p_2}(\tau_0^k) + \xi$$

Considering $p < 1$, the assumption **(d)** guarantees that $\max\{V(\tau_0^k), V^{p_1}(\tau_0^k), V^{p_2}(\tau_0^k)\} \le V^p(\tau_0^k) \le 1$ and we get

$$\begin{aligned} V(t_f^k) &\le e^{-\gamma(t_f^k - t_\infty^k)} \left[1 + K_1 + K_2 + \xi\right] V^p(\tau_0^k) \\ &\le e^{-\gamma(t_f^k - t_\infty^k)} \left[1 + K_1 + K_2 + \xi\right] \triangleq \delta(\gamma, \xi) \end{aligned}$$

From assumption **(b)** one has $V(\tau_0^{k+1}) \le V(t_f^k)$ and thus $V(\tau_0^k) \le \delta(\gamma, \xi)$, $\forall k \ge 1$. The term $\delta(\gamma, \xi)$ can be made as small as desired increasing either γ or the length of the interval $[t_\infty^k, t_f^k]$. The proof is completed by the relation $\alpha(||x||) \le V(x, t)$, $\forall x,\ t$. ■

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Remark 2: Since the Lyapunov function is exponentially decreasing on the Ω_k phases, assumption **(d)** in Proposition 1 means that the system is initialized on Ω_0 sufficiently far from the moment when the trajectory $X^{nc}(\cdot)$ leaves the admissible domain.

Precisely, the weak stability is characterized by an "almost decreasing" Lyapunov function $V(x(\cdot).\cdot)$ as illustrated in Figure 1.

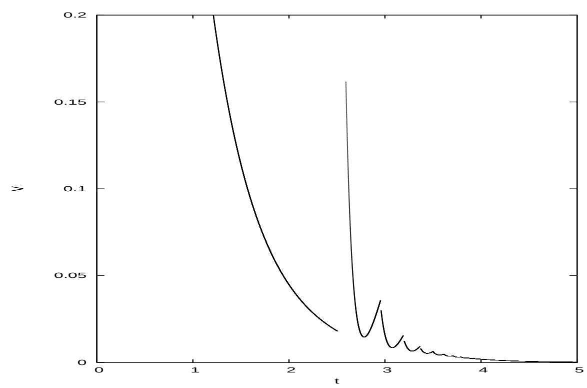

Fig. 1. Typical evolution of the Lyapunov function during one cycle of a weakly stable system.

Remark 3: It is worth to point out the local character of the stability criterion proposed by Proposition 1. This character is firstly given by condition **d)** of the statement and secondly by the synchronization constraints of the control law and the motion phase of the system (see (8) and (11) below).

The practical stability is very useful because attaining asymptotic stability is not an easy task for the unilaterally constrained systems described by (1) especially when $n \geq 2$ and $M(q)$ is not a diagonal matrix (i.e. there are inertial couplings, which is the general case).

C. Dissipativity and tracking versus stabilization

Let us make a parenthesis to highlight the major discrepancy between the trajectory tracking problem and the stabilization problem. To this aim let us first recall that the dynamics in (1) and (4) can be equivalently rewritten as the *measure differential inclusion* [1], [7], [23], [28]:

$$\begin{cases} -M(q(t))dv - [C(X(t), v(t^+))v(t^+) - G(X(t)) + U(t)]dt \in N_{T_\Phi(X(t))}(w(t)) \\ w(t) = \frac{v(t^+)+ev(t^-)}{1+e} \end{cases} \tag{9}$$

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where dv is the differential measure associated with the velocity $v(\cdot)$ that is a right-continuous function of local bounded variation, $v(\cdot)$ is equal almost everywhere to $\dot{X}(\cdot)$, $X(\cdot)$ is absolutely continuous and $X(t)-X(0)=\int_{[0,t]} v(s)ds$. The right-hand-side is the normal cone to the tangent cone, where the cones are as in Definition 4. As shown in [10] and [8, §3.9.4, 6.8.2, 7.2.4], a crucial property for stabilization is that the *cone complementarity problem*:

$$N_{T_\Phi(X(t))}(w(t)) \ni \xi \perp w(t) \in T_\Phi(X(t)) \tag{10}$$

defines a monotone mapping $\xi \mapsto w$, because the two cones $T_\Phi(\cdot)$ and $N_\Phi(\cdot)$ are polar cones [22], and $N_{T_\Phi(X(t))}(\cdot) \subseteq N_\Phi(\cdot)$. This monotonicity property allows one to use dissipativity arguments in an absolute stability framework to derive a Lyapunov function. Let us consider now the tracking control problem. The new (closed-loop) state vector is $(\tilde{X}, \dot{\tilde{X}})$. Therefore the right-hand-side of the closed-loop measure differential inclusion becomes the normal cone $N_{T_\Phi(\tilde{X}(t)+X_d(t))}(\tilde{w}(t)+w_d(t))$, with $w_d(t)=\frac{v_d(t^+)+ev_d(t^-)}{1+e}$. The sets $T_{\Phi_t}(\cdot) \triangleq T_\Phi(\cdot + X_d(t))$ and $N_{T^t_{\Phi_t}}(\cdot) \triangleq N_{T_{\Phi_t}}(\cdot + w_d(t))$ are now time-varying, and the monotonicity property is generally lost. This explains why the trajectory tracking problem is much more intricate than its stabilization counterpart, as even passivity-based controllers generally fail to preserve the passivity of the overall closed-loop system. We shall however call the controller that is designed in the next Section a passivity-based controller, because the closed-loop stability will essentially rely on the use of an energy-like function (see (14) below).

III. Controller design

In order to overcome some difficulties that can appear in the controller definition, the dynamical equations (1) will be expressed in the generalized coordinates introduced by McClamroch & Wang [24]. We suppose that the generalized coordinates transformation holds globally in Φ, which may obviously not be the case in general. However, the study of the singularities that might be generated by the coordinates transformation is out of the scope of this paper. Let us consider $D=[I_m \vdots O] \in \mathbb{R}^{m\times n}$, $I_m \in \mathbb{R}^{m\times m}$ the identity matrix. The new coordinates will be $q=Q(X)\in\mathbb{R}^n$, with $q=\begin{bmatrix} q_1 \\ q_2 \end{bmatrix}$, $q_1=\begin{bmatrix} q_1^1 \\ \vdots \\ q_1^m \end{bmatrix}$ such that $\Phi=\{q \mid Dq \geq 0\}^2$. The tangent

[2]In particular it is implicitly assumed that the function $F_i(\cdot)$ in (1) are linearly independent.

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cone $T_\Phi(q_1 = 0) = \{v \mid Dv \geq 0\}$ is the space of admissible velocities on the boundary of Φ.

The controller used here consists of different low-level control laws for each phase of the system. More precisely, the switching controller can be expressed as

$$T(q)U = \begin{cases} U_{nc} & \text{for } t \in \Omega_{2k}^{\emptyset} \\ U_t^J & \text{for } t \in I_k^J \\ U_c^J & \text{for } t \in \Omega_k^J \end{cases} \tag{11}$$

where $T(q) = \begin{pmatrix} T_1(q) \\ T_2(q) \end{pmatrix} \in \mathbb{R}^{n \times n}$ is full-rank under some basic assumptions (see [24]). The dynamics becomes:

$$\begin{cases} M_{11}(q)\ddot{q}_1 + M_{12}(q)\ddot{q}_2 + C_1(q,\dot{q})\dot{q} + g_1(q) &= T_1(q)U + \lambda \\ M_{21}(q)\ddot{q}_1 + M_{22}(q)\ddot{q}_2 + C_2(q,\dot{q})\dot{q} + g_2(q) &= T_2(q)U \\ q_1^i \geq 0,\, q_1^i \lambda_i = 0,\, \lambda_i \geq 0,\, 1 \leq i \leq m \\ \text{Collision rule} \end{cases} \tag{12}$$

where the set of complementary relations can be written more compactly as $0 \leq \lambda \perp Dq \geq 0$.

In the sequel U_{nc} coincides with the fixed-parameter controller proposed in [16], [32] and the closed-loop stability analysis of the system is based on Proposition 1. First, let us introduce some notations: $\tilde{q} = q - q_d$, $\bar{q} = q - q_d^*$, $s = \dot{\tilde{q}} + \gamma_2 \tilde{q}$, $\bar{s} = \dot{\bar{q}} + \gamma_2 \bar{q}$, $\dot{q}_e = \dot{q}_d - \gamma_2 \tilde{q}$ where $\gamma_2 > 0$ is a scalar gain and q_d, q_d^* represent the desired trajectories defined in the previous section. Using the above notations the controller is given by

$$T(q)U \triangleq \begin{cases} U_{nc} &= M(q)\ddot{q}_e + C(q,\dot{q})\dot{q}_e + G(q) - \gamma_1 s \\ U_t^J &= U_{nc}^J,\, t \leq t_0^k \\ U_t^J &= M(q)\ddot{q}_e + C(q,\dot{q})\dot{q}_e + G(q) - \gamma_1 \bar{s},\,,\, t > t_0^k \\ U_c^J &= U_{nc} - P_d + K_f(P_q - P_d) \end{cases} \tag{13}$$

where $\gamma_1 > 0$ is a scalar gain, $K_f > 0$, $P_q = D^T\lambda$ and $P_d = D^T\lambda_d$ is the desired contact force during persistently constrained motion. It is clear that during Ω_k^J not all the constraints are active and, therefore, some components of λ and λ_d are zero.

In order to prove the stability of the closed-loop system (11)–(13) we will use the following positive definite function:

$$V(t, s, \tilde{q}) = \frac{1}{2} s^T M(q) s + \gamma_1 \gamma_2 \tilde{q}^T \tilde{q} \tag{14}$$

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IV. TRACKING CONTROL FRAMEWORK

In this paper we treat the tracking control problem for the closed-loop dynamical system (11)–(13) with the complete desired path a priori taking into account the complementarity conditions and the impacts. In order to define the desired trajectory let us consider the motion of a virtual and unconstrained particle perfectly following a trajectory (represented by $X^{nc}(\cdot)$ on Figure 2) with an orbit that leaves the admissible domain for a given period. Therefore, the orbit of the virtual particle can be split into two parts, one of them belonging to the admissible domain (inner part) and the other one outside the admissible domain (outer part). In the sequel we deal with the tracking control strategy when the desired trajectory is constructed such that:

(i) when no activated constraints, it coincides with the trajectory of the virtual particle (the desired path and velocity are defined by the path and velocity of the virtual particle, respectively),

(ii) when $p \leq m$ constraints are active, its orbit coincides with the projection of the outer part of the virtual particle's orbit on the surface of codimension p defined by the activated constraints (X_d between A'' and C in Figure 2),

(iii) the desired detachment moment and the moment when the virtual particle re-enters the admissible domain (with respect to $p \leq m$ constraints) are synchronized.

Therefore we have not only to track a desired path but also to impose a desired velocity allowing the motion synchronization on the admissible domain.

The main difficulties here consist of:

- stabilizing the system on $\partial\Phi$ during the transition phases $I_k^{J_k}$ and incorporating the velocity jumps in the overall stability analysis;
- deactivating some constraints at the moment when the unconstrained trajectory re-enters the admissible domain with respect to them;
- maintaining a persistently constrained motion between the moment when the system was stabilized on $\partial\Phi$ and the detachment moment.

Remark 4: The problem can be relaxed considering that we want to track only a desired path like $X^{nc}(\cdot)$ (without imposing a desired velocity on the inner part of the desired trajectory and/or a given period to complete a cycle). In this way the synchronization problem (iii) disappears and we can assume there exists a twice differentiable desired trajectory outside $[t_0^k, t_f^k]$ that assures the detachment when the force control is dropped. In other words, in this case we have to design the desired trajectory only during $I_k^{J_k}$ phases.

A. Design of the desired trajectories

Throughout the paper we consider $I_k^{J_k} = [\tau_0^k, t_f^k]$, where τ_0^k is chosen by the designer as the start of the transition phase $I_k^{J_k}$ and t_f^k is the end of $I_k^{J_k}$. We note that all superscripts $(\cdot)^k$ will refer to the cycle k of the system motion. We also use the following notations:

- t_0^k is the first impact during the cycle k,
- t_∞^k is the accumulation point of the sequence $\{t_\ell^k\}_{\ell \geq 0}$ of the impact instants during the cycle k ($t_f^k \geq t_\infty^k$),
- τ_1^k will be explicitly defined later and represents the instant when the signal $X_d^*(\cdot)$ reaches a given value chosen by the designer in order to impose a closed-loop dynamics with impacts during transition phases,
- t_d^k is the desired detachment instant, therefore the phases $\Omega_{2k+1}^{J'_k}$ can be expressed as $[t_f^k, t_d^k]$.

It is noteworthy that t_0^k, t_∞^k, t_d^k are state-dependent whereas τ_1^k and τ_0^k are exogenous and imposed by the designer. To better understand the definition of these specific instants, in the Figure 2 we simplify the system's motion as follows:

- during transition phases we take into account only the constraints that must be activated $J'_k \setminus J_k$.
- at the end of Ω_{2k+1} phases we take into account only the constraints that must be deactivated $J'_k \setminus J_{k+1}$.

The points A, A', A'' and C in Figure 2 correspond to the moments τ_0^k, t_0^k, t_f^k and t_d^k respectively. We have seen that the choice of τ_0^k plays an important role in the stability criterion given by Proposition 1. On the other hand in Figure 2 we see that starting from A the desired trajectory $X_d(\cdot) = X_d^*(\cdot)$ is deformed compared to $X^{nc}(\cdot)$. In order to reduce this deformation τ_0^k and implicitly the point A must be close to $\partial\Phi$ (see also Figure 4). Further details on the choice of τ_0^k will be given later. Taking into account just the constraints $J'_k \setminus J_{k+1}$ we can identify t_d^k with the moment when $X_d(\cdot)$ and $X^{nc}(\cdot)$ rejoin at C.

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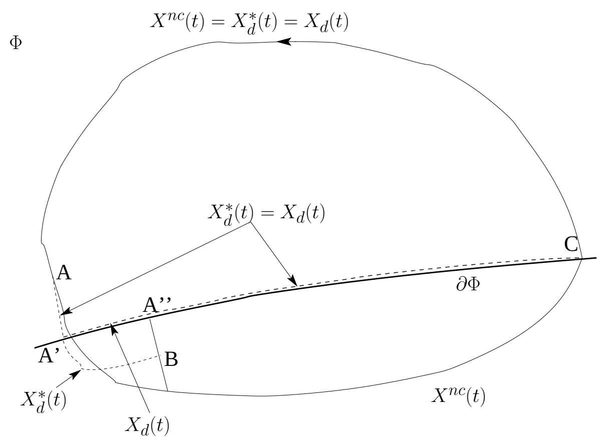

Fig. 2. The closed-loop desired trajectory and control signals

B. Design of $q_d^*(\cdot)$ and $q_d(\cdot)$ on the phases $I_k^{J_k}$

During the transition phases the system must be stabilized on $\partial\Phi$. Obviously, this does not mean that all the constraints have to be activated (i.e. $q_1^i(t) = 0$, $\forall i = 1, \ldots, m$). Let us consider that only the first p constraints (eventually reordering the coordinates) define the border of Φ where the system must be stabilized. On $[\tau_0^k, t_0^k)$ we define $q_d^*(\cdot)$ as a twice differentiable signal such that $q_d^*(\cdot)$ approaches a given point in the normal cone $N_\Phi(q_p = 0)$ on $[\tau_0^k, \tau_1^k]$. Precisely, we define $q_d^*(\cdot)$ such as:

- during a small period $\delta > 0$ chosen by the designer the desired velocity becomes zero preserving the twice differentiability of $q_d^*(\cdot)$. For instance we can use the following definition:
$$q_d^*(t) = q^{nc}\left(\tau_0^k + \frac{(t-\tau_0^k-\delta)^2(t-\tau_0^k)}{\delta^2}\right), t \in [\tau_0^k, \tau_0^k+\delta]$$
which means $q_d^*(\tau_0^k+\delta) = q_d^*(\tau_0^k) = q^{nc}(\tau_0^k)$, $\quad \dot{q}_d^*(\tau_0^k+\delta) = 0$ and $\dot{q}_d^*(\tau_0^k) = \dot{q}^{nc}(\tau_0^k)$
- choosing $\varphi > 0$ and denoting $t' = \frac{t-(\tau_0^k+\delta)}{\tau_1^k-(\tau_0^k+\delta)}$, the components $\left(q_d^i\right)^*$, $i = 1, \ldots, p$ of $(q_d^*)_p$ are defined as:
$$\left(q_d^i\right)^*(t) = \begin{cases} a_3(t')^3 + a_2(t')^2 + a_0, & t \in [\tau_0^k+\delta, \min\{\tau_1^k; t_0^k\}] \\ -\varphi V^{1/3}(\tau_0^k), & t \in (\min\{\tau_1^k; t_0^k\}, t_f^k] \end{cases} \tag{15}$$
where $V(\cdot)$ is defined in (14) and with the coefficients given by:
$$\begin{aligned} a_3 &= 2[(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)] \\ a_2 &= -3[(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)] \\ a_0 &= (q^i)^{nc}(\tau_0^k) \end{aligned} \tag{16}$$

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- all the other components of $q_d^*(\cdot)$ are frozen:

$$\left(q_d^*\right)_{n-p}(t) = q_{n-p}^{nc}(\tau_0^k), \quad t \in (\tau_0^k + \delta, t_f^k] \tag{17}$$

The rationale behind the choice of $q_d^*(\cdot)$ is on one hand to assure a robust stabilization on $\partial\Phi$, mimicking the bouncing-ball dynamics; on the other hand to enable one to compute suitable upper-bounds that will help using Proposition 1 (hence $V^{1/3}(\cdot)$ terms in (15) with $V(\cdot)$ in (14)).

Remark 5: 1) Straightforward computations show that $q_d^*(\cdot)$ satisfies the following relations.

$$\left(q_d^i\right)^*(\tau_1^k) = -\varphi V^{1/3}(\tau_0^k), \quad \left(\dot{q}_d^i\right)^*(\tau_1^k) = 0, \, i = 1, \ldots, p$$

2) Two different situations are possible. The first is given by $t_0^k > \tau_1^k$ (see Figure 3) and we shall prove that in this situation all the jumps of the Lyapunov function in (14) are negative. The second situation was pointed out in [5] and is given by $t_0^k < \tau_1^k$. In this situation the first jump at t_0^k in the Lyapunov function may be positive. It is noteworthy that $q_d^*(\cdot)$ will then have a jump at the time t_0^k since $(q_d^i)^*(t_0^{k+}) = -\varphi V^{1/3}(\tau_0^k)$, $\forall i = 1, \ldots, p$ (see (15)).

In order to limit the deformation of the desired trajectory $q_d^*(\cdot)$ w.r.t. the unconstrained trajectory $q^{nc}(\cdot)$ during the I_k phases (see Figures 2 and 3), we impose in the sequel

$$||q_p^{nc}(\tau_0^k)|| \leq \psi \tag{18}$$

where $\psi > 0$ is chosen by the designer. It is obvious that a smaller ψ leads to smaller deformation of the desired trajectory and to smaller deformation of the real trajectory as we shall see in Section VIII. Nevertheless, due to the tracking error, ψ cannot be chosen zero. We also note that $||q_p^{nc}(\tau_0^k)|| \leq \psi$ is a practical way to choose τ_0^k.

During the transition phases I_k we define $\left(q_d\right)_{n-p}(t) = \left(q_d^*\right)_{n-p}(t)$. Assuming a finite accumulation period, the impact process can be considered in some way equivalent to a plastic impact. Therefore, $\left(q_d\right)_p(\cdot)$ and $\left(\dot{q}_d\right)_p(\cdot)$ are set to zero on the right of t_0^k.

V. Design of the desired contact force during constraint phases

For the sake of simplicity we consider the case of the constraint phase Ω_k^J, $J \neq \emptyset$ with $J = \{1, \ldots, p\}$. Obviously a sufficiently large desired contact force P_d assures a constrained movement on Ω_k^J. Nevertheless at the end of the Ω_{2k+1}^J phases a detachment from some surfaces Σ_i has to take place. It is clear that a take-off implies not only a well-defined desired trajectory but also some small values of the corresponding contact force components. On the other hand,

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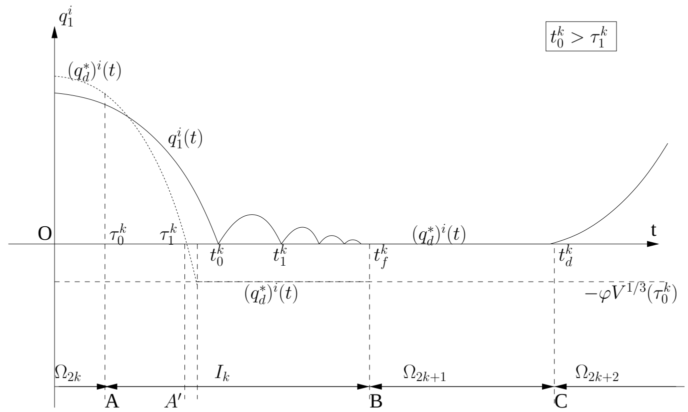

Fig. 3. The design of q_{1d}^* on the transition phases I_k

if the components of the desired contact force decrease too much a detachment can take place before the end of the Ω_k^J phases which can generate other impacts. Therefore we need a lower bound of the desired force which assures the contact during the Ω_k^J phases.

Dropping the time argument, the dynamics of the system on Ω_k^J can be written as

$$\begin{cases} M(q)\ddot{q} + F(q,\dot{q}) = U_c + D_p^T \lambda_p \\ 0 \le q_p \perp \lambda_p \ge 0 \end{cases} \tag{19}$$

where $F(q,\dot{q}) = C(q,\dot{q})\dot{q} + G(q)$ and $D_p = [I_p \vdots O] \in \mathbb{R}^{p\times n}$. On Ω_k^J the system is permanently constrained which implies $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. In order to assure these conditions it is sufficient to have $\lambda_p > 0$.

In the following let us denote $M^{-1}(q) = \begin{pmatrix} [M^{-1}(q)]_{p,p} & [M^{-1}(q)]_{p,n-p} \\ [M^{-1}(q)]_{n-p,p} & [M^{-1}(q)]_{n-p,n-p} \end{pmatrix}$ and $C(q,\dot{q}) = \begin{pmatrix} C(q,\dot{q})_{p,p} & C(q,\dot{q})_{p,n-p} \\ C(q,\dot{q})_{n-p,p} & C(q,\dot{q})_{n-p,n-p} \end{pmatrix}$ where the meaning of each component is obvious.

Proposition 2: On Ω_k^J the constraint motion of the closed-loop system (19),(11),(13) is assured if the desired contact force is defined by

$$\begin{aligned} (\lambda_d)_p \triangleq \beta - \frac{\bar{M}_{p,p}(q)}{1+K_f} \big([M^{-1}(q)]_{p,p} C_{p,n-p}(q,\dot{q}) + \\ [M^{-1}(q)]_{p,n-p} C_{n-p,n-p}(q,\dot{q}) + \gamma_1 [M^{-1}(q)]_{p,n-p}\big) s_{n-p} \end{aligned} \tag{20}$$

where $\bar{M}_{p,p}(q) = ([M^{-1}(q)]_{p,p})^{-1} = \left(D_p M^{-1}(q) D_p^T\right)^{-1}$ is the inverse of the Delassus' matrix and $\beta \in \mathbb{R}^p$, $\beta > 0$.

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Proof: First, we notice that the second relation in (19) implies on Ω_k^J (see [14])

$$0 \leq \ddot{q}_p \perp \lambda_p \geq 0 \Leftrightarrow 0 \leq D_p \ddot{q} \perp \lambda_p \geq 0. \tag{21}$$

From (19) and (13) one easily gets:

$$\ddot{q} = M^{-1}(q)\big[-F(q,\dot{q}) + U_{nc} + (1+K_f)D_p^T(\lambda - \lambda_d)_p\big]$$

Combining the last two equations we obtain the following LCP with unknown λ_p:

$$\begin{aligned} 0 \leq D_p M^{-1}(q)\big[-F(q,\dot{q}) + U_{nc} - (1+K_f)D_p^T\,(\lambda_d)_p\big] \\ +(1+K_f)D_p M^{-1}(q) D_p^T \lambda_p \perp \lambda_p \geq 0 \end{aligned} \tag{22}$$

Since $(1+K_f)D_p M^{-1}(q) D_p^T > 0$ and hence is a P-matrix, the LCP (22) has a unique solution and one deduces that $\lambda_p > 0$ if and only if

$$\begin{aligned} &\frac{\bar{M}_{p,p}(q)}{1+K_f} D_p M^{-1}(q)\big[U_{nc} - F(q,\dot{q}) - (1+K_f)D_p^T\,(\lambda_d)_p\big] < 0 \\ &\Leftrightarrow (\lambda_d)_p > \frac{\bar{M}_{p,p}(q)}{1+K_f} D_p M^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] \\ &\Leftrightarrow (\lambda_d)_p = \beta + \frac{\bar{M}_{p,p}(q)}{1+K_f} D_p M^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] \end{aligned} \tag{23}$$

with $\beta \in \mathbb{R}^p$, $\beta > 0$. Since $U_{nc} - F(q,\dot{q}) = M(q)\ddot{q}_e - C(q,\dot{q})s - \gamma_1 s$, $(\ddot{q}_e)_p = 0$ and $s_p = 0$, (23) rewrites as (20) and the proof is finished. It is noteworthy that

$$\begin{aligned} \lambda_p &= -\frac{\bar{M}_{p,p}(q)}{1+K_f} D_p M^{-1}(q)\big[U_{nc} - F(q,\dot{q}) - (1+K_f)D_p^T\,(\lambda_d)_p\big] \\ &= (\lambda_d)_p - \frac{\bar{M}_{p,p}(q)}{1+K_f} D_p M^{-1}(q)\big[U_{nc} - F(q,\dot{q})\big] = \beta \end{aligned}$$

■

Remark 6: The control law used in this paper with the design of λ_d described above leads to the following closed-loop dynamics on Ω_k^J.

$$\begin{cases} M_{p,n-p}(q)\dot{s}_{n-p} + C_{p,n-p}(q,\dot{q})s_{n-p} = (1+K_f)(\lambda - \lambda_d)_p \\ M_{n-p,n-p}(q)\dot{s}_{n-p} + C_{n-p,n-p}(q,\dot{q})s_{n-p} + \gamma_1 s_{n-p} = 0 \\ q_p = 0, \quad \lambda_p = \beta. \end{cases}$$

It is noteworthy that the closed-loop dynamics is nonlinear and therefore, we do not use the feedback stabilization proposed in [24].

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VI. Strategy for take-off at the end of constraint phases Ω^J_{2k+1}

We have discussed in the previous sections the necessity of a trajectory with impacts in order to assure the robust stabilization on $\partial\Phi$ in finite time and, the design of the desired trajectory to stabilize the system on $\partial\Phi$. Now, we are interested in finding the conditions on the control signal U_c^J that assure the take-off at the end of constraint phases Ω^J_{2k+1}. As we have already seen before, the phase Ω^J_{2k+1} corresponds to the time interval $[t_f^k, t_d^k)$. The dynamics on $[t_f^k, t_d^k)$ is given by (19) and the system is permanently constrained, which implies $q_p(\cdot) = 0$ and $\dot{q}_p(\cdot) = 0$. Let us also consider that the first r constraints ($r < p$) have to be deactivated. Thus, the detachment takes place at t_d^k if $\ddot{q}_r(t_d^{k+}) > 0$ which requires $\lambda_r(t_d^{k-}) = 0$. The last $p - r$ constraints remain active which means $\lambda_{p-r}(t_d^{k-}) > 0$.

To simplify the notation we drop the time argument in many equations of this section. We decompose the LCP matrix (which is the Delassus' matrix multiplied by $1 + K_f$) as:

$$(1 + K_f)D_p M^{-1}(q)D_p^T = \begin{pmatrix} A_1(q) & A_2(q) \\ A_2(q)^T & A_3(q) \end{pmatrix}$$

with $A_1 \in \mathbb{R}^{r\times r}, A_2 \in \mathbb{R}^{r\times(p-r)}$ and $A_3 \in \mathbb{R}^{(p-r)\times(p-r)}$

Proposition 3: For the closed-loop system (19),(11),(13) the passage when the number of active constraints decreases from p to $p - r$ (with $r < p$), is possible if

$$\begin{pmatrix} (\lambda_d)_r\,(t_d^k) \\ (\lambda_d)_{p-r}\,(t_d^k) \end{pmatrix} = \begin{pmatrix} \left(A_1 - A_2 A_3^{-1} A_2^T\right)^{-1}\left(b_r - A_2 A_3^{-1} b_{p-r}\right) - C_1 \\ C_2 + A_3^{-1}\left(b_{p-r} - A_2^T\,(\lambda_d)_r\right) \end{pmatrix} \tag{24}$$

where

$$b_p \triangleq b(q, \dot{q}, U_{nc}) \triangleq D_p M^{-1}(q)[U_{nc} - F(q, \dot{q})] \geq 0$$

and $C_1 \in \mathbb{R}^r$, $C_2 \in \mathbb{R}^{p-r}$ such that $C_1 \geq 0$, $C_2 > 0$.

Proof: From (13) and (19) one gets

$$\ddot{q}_p(t) = b_p + (1 + K_f)D_p M^{-1}(q)D_p^T(\lambda - \lambda_d)$$

Therefore the LCP (21) rewrites as:

$$0 \leq \begin{pmatrix} \lambda_r \\ \lambda_{p-r} \end{pmatrix} \perp \begin{pmatrix} b_r + A_1(\lambda - \lambda_d)_r + A_2(\lambda - \lambda_d)_{p-r} \\ b_{p-r} + A_2^T(\lambda - \lambda_d)_r + A_3(\lambda - \lambda_d)_{p-r} \end{pmatrix} \geq 0 \tag{25}$$

Under the conditions $\lambda_r = 0$ and $\lambda_{p-r} > 0$ one has

$$0 \leq \lambda_{p-r} \perp b_{p-r} - A_2^T(\lambda_d)_r + A_3(\lambda - \lambda_d)_{p-r} \geq 0$$

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with the solution

$$\lambda_{p-r} = -A_3^{-1}\left(b_{p-r} - A_2^T(\lambda_d)_r - A_3(\lambda_d)_{p-r}\right) \tag{26}$$

Thus $\lambda_{p-r} > 0$ is equivalent to

$$(\lambda_d)_{p-r} > A_3^{-1}\left(b_{p-r} - A_2^T\,(\lambda_d)_r\right)$$

which leads to the second part of definition (24). Furthermore, replacing $(\lambda_d)_{p-r}$ in (26) we get $\lambda_{p-r} = C_2$ and $b_r + A_1(\lambda - \lambda_d)_r + A_2(\lambda - \lambda_d)_{p-r} \geq 0$ yields the first part of definition (24). To conclude, the solution of the LCP (25) is $\lambda_p = \begin{pmatrix} 0 \\ C_2 \end{pmatrix} \in \mathbb{R}^p$ and $(\lambda_d)_p$ is defined by (24). ■

Proposition 4: The closed-loop system (19),(11),(13) is permanently constrained on $[t_f^k, t_d^k)$ and a smooth detachment is guaranteed on $[t_d^k, t_d^k + \epsilon)$ (ϵ is a small positive real number chosen by the designer) if

(i) $(\lambda_d)_p\,(\cdot)$ is defined on $[t_f^k, t_d^k)$ by (24) where C_1 is replaced by $C_1(t - t_d^k)$.

(ii) On $[t_d^k, t_d^k + \epsilon)$

$$q_d^*(t) = q_d(t) = \begin{pmatrix} q_r^*(t) \\ q_{n-r}^{nc}(t) \end{pmatrix},$$

where $q_r^*(\cdot)$ is a twice differentiable function such that

$$\begin{aligned} q_r^*(t_d^k) = 0, &\quad q_r^*(t_d^k + \epsilon) = q_r^{nc}(t_d^k + \epsilon), \\ \dot{q}_r^*(t_d^k) = 0, &\quad \dot{q}_r^*(t_d^k + \epsilon) = \dot{q}_r^{nc}(t_d^k + \epsilon) \end{aligned} \tag{27}$$

and $\ddot{q}_r^*(t_d^{k+}) = a > \max\left(0,\, -A_1(q)\,(\lambda_d)_r\,(t_d^{k-})\right)$.

Proof: (i) The uniqueness of solution of the LCP (21) guarantees that (20) and (24) agree if $C_1 < 0$. In other words, replacing C_1 by $C_1(t - t_d^k)$ in (24) we assure a constrained motion on $[t_f^k, t_d^k)$ and the necessary conditions for detachment on $[t_d^k, t_d^k + \epsilon)$.
(ii) Obviously (27) is imposed in order to assure the twice differentiability of the desired trajectory. Finally, straightforward computations show that

$$\sigma_{\ddot{q}_r(t_d^k)} = \ddot{q}_r^*(t_d^{k+}) + A_1(q)\,(\lambda_d)_r\,(t_d^{k-})$$

which means that the detachment is guaranteed and no other impacts occur when the desired acceleration satisfies $\ddot{q}_r^*(t_d^{k+}) > \max\left(0,\, -A_1(q)\,(\lambda_d)_r\,(t_d^{k-})\right)$. ■

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VII. Closed-loop stability analysis

In the case $\Phi = \mathbb{R}^n$, the function $V(t,s,\tilde{q})$ in (14) can be used in order to prove the closed-loop stability of the system (12), (13) (see for instance [8]). In the case studied here ($\Phi \subset \mathbb{R}^n$) the analysis becomes more complicated.

To simplify the notation $V(t,s(t),\tilde{q}(t))$ is denoted as $V(t)$. In order to introduce the main result of this paper we make the next assumption, which is verified in practice for dissipative systems.

Assumption 1: The controller U_t in (13) assures that all the transition phases are finite (see Definition 6) and the accumulation point t_∞^k is smaller than t_d^k for all $k \in \mathbb{N}$.

It is worth to precise that during the stabilization on the intersection of p surfaces Σ_i we do not know which one and how many surfaces are stroked. However we assume that all the impacts are p_ϵ-impacts in the sense of Definition 3. The proofs of the next two results are in the Appendix.

Lemma 1: Consider the closed-loop system (11)-(13) with $(q_d^*)_p(\cdot)$ defined on the interval $[\tau_0^k, t_0^k]$ as in (15)-(17). Let us also suppose that condition **b)** of Proposition 1 is satisfied. The following inequalities hold:

$$\begin{aligned} ||\tilde{q}(t_0^{k-})|| \leq \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}},\ ||s(t_0^{k-})|| \leq \sqrt{\frac{2V(\tau_0^k)}{\lambda_{min}(M(q))}} \\ ||\dot{\tilde{q}}(t_0^{k-})|| \leq \left(\sqrt{\frac{2}{\lambda_{min}(M(q))}} + \sqrt{\frac{\gamma_2}{\gamma_1}}\right) V^{1/2}(\tau_0^k) \end{aligned} \tag{28}$$

Furthermore, if $t_0^k \leq \tau_1^k$ one has

$$\begin{aligned} ||(q_d)_p(t_0^{k-})|| \leq \epsilon + \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} \\ ||(\dot{q}_d)_p(t_0^{k-})|| \leq K + K' V^{1/6}(\tau_0^k) \end{aligned} \tag{29}$$

where ϵ is the real constant fixed in Definition 3 and $K,\ K' > 0$ are some constant real numbers that will be defined in the proof (see Appendix A).

We now state the main result of this paper.

Theorem 1: Let Assumption 1 hold, $e \in [0,1)$ and $(q_d^*)_p$ defined as in (15)-(17). The closed-loop system (11)-(13) initialized on Ω_0 such that $V(\tau_0^0) \leq 1$, satisfies the requirements of Proposition 1 and is therefore practically weakly stable with the closed-loop state $x(\cdot) = [s(\cdot), \tilde{q}(\cdot)]$ and $R = \sqrt{e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)/\rho}$ where $\rho = \min\{\lambda_{min}(M(q))/2; \gamma_1\gamma_2\}$ and K_1, K_2 are defined in the proof (see Appendix B).

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VIII. ILLUSTRATIVE EXAMPLE

A. A planar two-link rigid-joint manipulator with one constraint

The main issues of the control scheme proposed in this paper are first emphasized simulating the behavior of a planar two-link rigid-joint manipulator in presence of one unilateral constraint. The admissible domain is the upper half plane $y \geq 0$ and the unconstrained desired trajectory $q^{nc}(\cdot)$ is given by a circle that violates the constraint. Precisely, the end effector must follow a half-circle, stabilize on the constraint ($y = 0$) and move on the constraint until the point where the circle $q^{nc}(\cdot)$ re-enters the admissible domain. The lengths l_1, l_2 of the manipulator's links are set to $0.5m$, and their masses m_1, m_2 are set to $1kg$. The impacts are imposed using the parameter $\varphi = 100$ in (15)-(16). The numerical simulations are done with the Moreau's time-stepping algorithm of the SICONOS software platform [2]. The choice of a time-stepping algorithm was mainly dictated by the presence of accumulations of impacts which render the use of event-driven methods difficult [1]. A further reason to choose the SICONOS software platform for the simulation of the complementarity systems consist of its capability to automatically and quickly solve LCPs [3].

Let us set $e = 0.7$, $\gamma_1 = 8$, $\gamma_2 = 7$, 10 seconds the period of each cycle and 30 seconds the final simulation time. First, let us point out (Figure 4 (left)) the influence of ψ (i.e. the choice of τ_0^k) on the degree of deformation of the real trajectory w.r.t. the desired unconstrained one. As we have pointed out in Section IV the deformation gets smaller when $\psi > 0$ decreases. It is noteworthy that the tangential approach corresponding to $\psi = 0$ lacks of robustness and is unreliable due to the nonzero initial tracking errors.

For $\psi = 0.01$ in Figure 4 (right) we illustrate the trajectory of the system during the stabilization on $\partial\Phi = \{(x, y) \mid y = 0\}$. The switches of the controller during the first cycle are depicted in Figure 5 (left). Clearly since the velocity jumps, the controller jumps as well.

The Figure 5 (right) presents the variation of the contact force λ. Precisely, one sees that λ remains 0 during the free motion phases and it points out each impact during transients (better seen on I_0 since the impacts are more violent). The contact force λ is designed as a decreasing

[3]The control scheme proposed in the paper may require to solve an LCP of dimension $m \approx 10$ (reasonable in some control applications). But this requires a specific solver since the usual "hybrid" methods must treat 2^m cases and quickly become inefficient [1].

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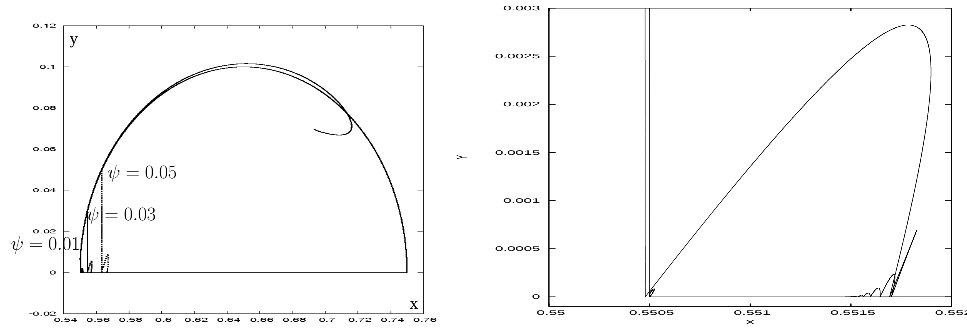

Fig. 4. Left: The influence of ψ on the real trajectory's deformation for controller's gains set to $\gamma_1 = 8$, $\gamma_2 = 7$. Right: The trajectory of the end-effector on the transition phases when $\psi = 0.1$.

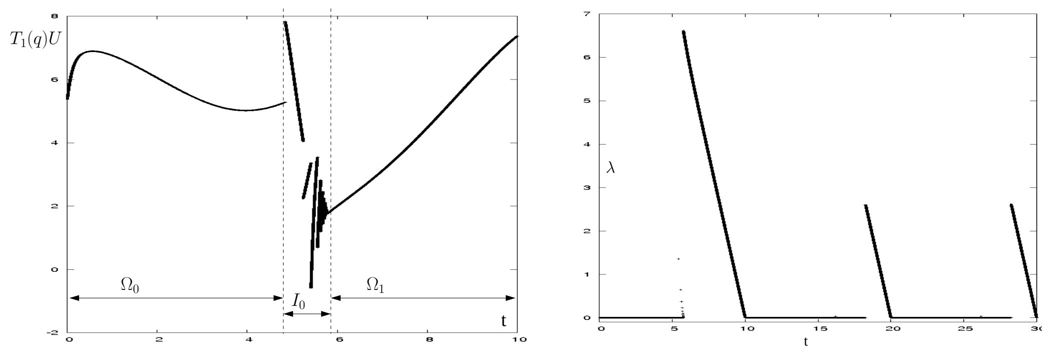

Fig. 5. Left: The switching controller on the first cycle; Right: Variation of the contact force λ

linear function during constrained motion phases Ω_{2k+1} in order to allow a smooth detachment at the end of these phases. It is worth to mention that the magnitude of λ depends indirectly on $V(\tau_0^k)$. Precisely, when $V(\tau_0^k)$ approaches zero the system tends to a tangential stabilization on $\partial\Phi$ which implies larger values of t_0^k and consequently smaller length of $[t_f^k, t_d^k]$ and smaller magnitude of the contact force measured by λ (see Proposition 4).

Figure 6 shows that the tracking error described by the Lyapunov function rapidly decreases and remains close to 0. In other words the practical weak stability is guaranteed. On the zoom made in Figure 6 one can also observe the behavior of $V(\cdot)$ during the stabilization on $\partial\Phi$, that is an almost decreasing function.

B. The influence of the time-step on the closed-loop dynamics

In the next tables we summarize some numerical results when $\psi = 0.01$, $e = 0.7$, $\gamma_1 = 35$, $\gamma_2 = 20$. The period of each cycle is set to 5 seconds and the final simulation time rest

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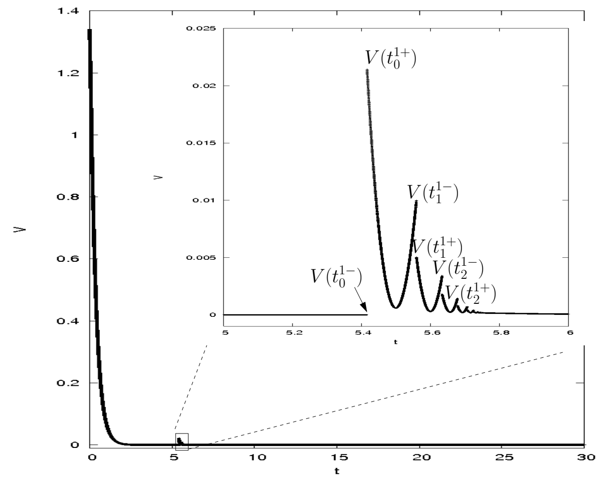

Fig. 6. Variation of the Lyapunov function for $\gamma_1 = 8, \gamma_2 = 7$; Zoom: Variation of the Lyapunov function during the phase I_0

30 seconds. First, one can see that the length of the transition phase I_0 with respect to the time-step h does not vary significantly when the time-step decreases. Let us also denote by CPU the computing time necessary for the simulation (using an Intel(R) Core(TM)2 CPU 6300 1.86GHz) of one cycle.

h	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$
$\lambda[I_0]$	0.945	0.9536	0.9525	0.9523
CPU	$1.5s$	$11.2s$	$111.3s$	$1072.2s$

The evolutions of the number of impacts n_i w.r.t. the restitution coefficient e and the time-step h are quite different. As expected, n_i becomes larger when the restitution coefficient increases. Also, one can see that the accumulation of impacts can be captured with a higher precision when the time-step becomes smaller.

$e \setminus h$	$10^{-3}s$	$10^{-4}s$	$10^{-5}s$	$10^{-6}s$
0.2	$n_i = 3$	$n_i = 5$	$n_i = 6$	$n_i = 8$
0.5	$n_i = 6$	$n_i = 9$	$n_i = 12$	$n_i = 16$
0.7	$n_i = 9$	$n_i = 16$	$n_i = 23$	$n_i = 29$
0.9	$n_i = 23$	$n_i = 40$	$n_i = 64$	$n_i = 81$
0.95	$n_i = 32$	$n_i = 67$	$n_i = 108$	$n_i = 161$

However, a larger number of captured impacts does not change the global behavior of the simulated system and the transition phase ends almost in the same moment when h varies, see

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$\lambda[I_0]$ in the first table. Such results are not surprizing in view of the fact that the numerical method converges [1, Theorem 10.7].

In conclusion, reliable simulations with a reasonable CPU time can be performed with the Moreau's time-stepping scheme of the SICONOS *platform, with a time-step $h = 10^{-4}s$.*

C. A planar two-link rigid-joint manipulator with two constraints

In the sequel we introduce another constraint into the previous dynamics. Precisely we impose an admissible domain $\Phi = \{(x, y) \mid y \geq 0,\, 0.7 - x \geq 0\}$. Let us also consider an unconstrained desired trajectory given by the circle $\{(x, y) \mid (x-0.7)^2+y^2 = 0.5\}$ that violates both constraints. In other words, the two-link planar manipulator must track a quarter-circle; stabilize on and then follow the line $\Sigma_1 = \{(x, y) \mid y = 0\}$; stabilize on the intersection of Σ_1 and $\Sigma_2 = \{(x, y) \mid x = 0.7\}$; detach from Σ_1 and follow Σ_2 until the unconstrained circle re-enters Φ and finally take-off from Σ_2 in order to repeat the previous steps. It is noteworthy that the task presented above is not of type (8) since after a constraint phase (when the end-effector is attached to Σ_1) follows a transition phase containing 2_0-impacts according to Definition 3 (the system must be stabilized on the intersection $\Sigma_1 \cap \Sigma_2$) instead of a detachment. However, the manipulator can accomplish the task under consideration and any other task that use as ingredients: impacting transitions, take-off from some surfaces Σ_i even remaining attached to other ones, constraint and free-motion phases without imposing a specific succession of this phases.

Let us consider in this case that a cycle is $\Omega_{2k} \cup I_k^1 \cup \Omega_{2k+1}^1 \cup I_k^2 \cup \Omega_{2k+1}^2$ where Ω_{2k} is the free-motion phase and I_k^i, Ω_{2k+1}^i are the impacting transients and the constrained phases associated to the surface Σ_i. The numerical values used for the dynamical model are again $l_1 = l_2 = 0.5m$, $I_1 = I_2 = 1kg.m^2$, $m_1 = m_2 = 1kg$ and the restitution coefficient $e_n = 0.7$. The impacts are imposed by $\varphi = 100$ in (15) (16) and the beginning of transition phases are defined using $\psi = 0.05$ in (18). We impose a period of 10 seconds for each cycle and we simulate the dynamics during 6 cycles. Setting the controller gains $\gamma_1 = 15$, $\gamma_2 = 15$ we see in Figure 7 (left) that the desired trajectory is accurately followed. The same conclusion can be deduced looking at the variation of the Lyapunov function plotted in Figure 7 (right). In this case we have imposed a constant contact-force λ_1 during the motion on the surface Σ_1 (see Figure 8 (left)) and a decreasing contact-force, that allows a smooth detachment, during the motion on Σ_2 (see Figure 8 (right)).

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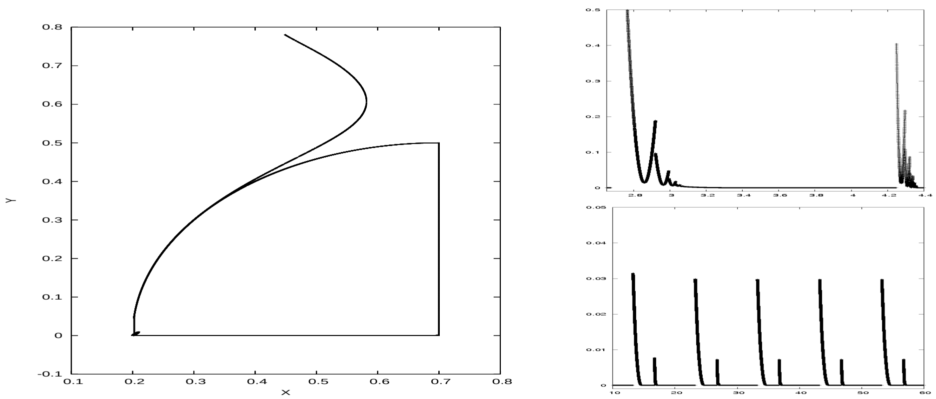

Fig. 7. Left: The trajectory of the system during 6 cycles; Right-Up: Zoom on the transition phases during the first cycle; Right-Down: The variation of the Lyapunov function on the last 5 cycles.

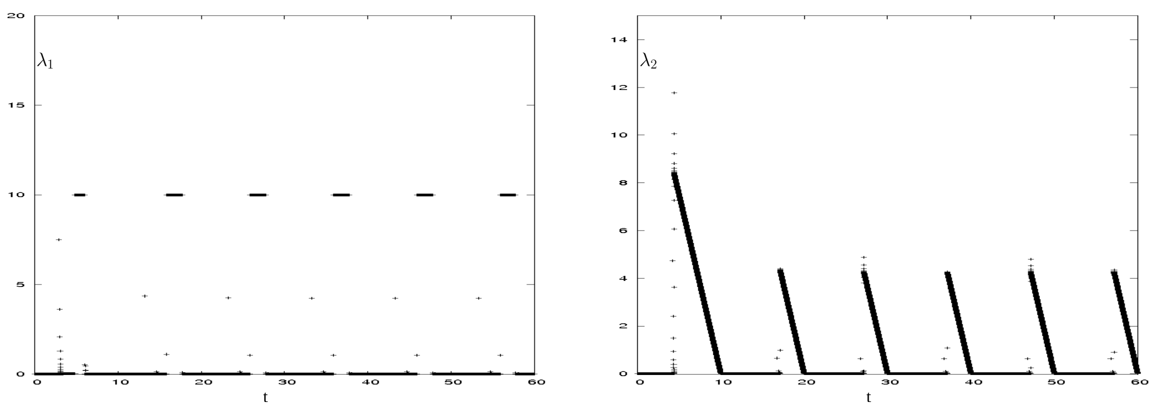

Fig. 8. Left: Variation of the contact force during the motion on Σ_1; Right: Variation of the contact force during the motion on Σ_2

IX. Conclusions

In this paper we have proposed a methodology to study the tracking control of fully actuated Lagrangian systems subject to multiple frictionless unilateral constraints and multiple impacts. The main contribution of the work is twofold: first, it formulates a general framework and second, it provides a complete stability analysis for the class of systems under consideration. It is noteworthy that even in the simplest case of only one frictionless unilateral constraint the paper already presents some notable improvements with respect to the existing works. Precisely, the stability analysis result is significantly more general than those presented in [5] and [9] and, each element entering the dynamics (desired trajectory, contact force) is explicitly defined.

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Numerical simulations are done with the SICONOS software platform [1], [2] in order to illustrate the results.

APPENDIX

A. Proof of Lemma 1

From (14) we can deduce on one hand that

$$V(t_0^{k-}) \geq \gamma_1\gamma_2||\tilde{q}(t_0^{k-})||^2$$

and on the other hand

$$V(t_0^{k-}) \geq \frac{1}{2}s(t_0^{k-})^T M(q(t_0^{k-}))s(t_0^{k-})$$

Since condition **b)** of Proposition 1 is satisfied one has $V(\tau_0^k) \geq V(t_0^{k-})$ and the first two inequalities in (28) become trivial. Let us recall that $s(t) = \dot{\tilde{q}}(t) + \gamma_2\tilde{q}(t)$ which implies $||\dot{\tilde{q}}(t_0^{k-})|| \leq ||s(t_0^{k-})|| + \gamma_2||\tilde{q}(t_0^{k-})||$. Combining this with the first two inequalities in (28) we derive the third inequality in (28).

For the rest of the proof we assume that $t_0^k \leq \tau_1^k$. Therefore $(q_d)_p(t_0^{k-}) = (q_d^*)_p(t_0^k)$. It is clear that

$$||(q_d)_p(t_0^{k-})|| \leq ||\tilde{q}_p(t_0^{k-})|| + ||q_p(t_0^k)||$$

Taking into account that t_0^k is a p_ϵ-impact (which means $||q_p(t_0^k)|| \leq \epsilon$), the first inequality in (29) becomes obvious.

Let us denote $t'_k = \frac{t_0^k - \tau_0^k - \delta}{\tau_1^k - \tau_0^k - \delta} \in [0,1]$. We recall here that τ_0^k was chosen such that $||q_p^{nc}(\tau_0^k)|| \leq \psi$. From (15), (16) and the first inequality in (29), for $i = 1, \ldots, p$ one has

$$q_d^i(t_0^{k-}) = \left[(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)\right]\left(2(t'_k)^3 - 3(t'_k)^2\right) + (q^i)^{nc}(\tau_0^k) \leq \epsilon + \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}$$

It follows that

$$3(t'_k)^2 - 2(t'_k)^3 \geq \frac{(q^i)^{nc}(\tau_0^k) - \epsilon - \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}$$

For $t > 0$ one has $2t - t^2 \geq 3t^2 - 2t^3$, therefore

$$2t'_k - (t'_k)^2 \geq \frac{(q^i)^{nc}(\tau_0^k) - \epsilon - \sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}}}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}$$

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which means that

$$(1-t'_k)^2 \leq \frac{\sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} + \varphi V^{1/3}(\tau_0^k) + \epsilon}{(q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k)}$$

Straightforward computations lead to

$$|\dot{q}_d^i(t_0^{k-})| = \frac{6((q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k))}{\tau_1^k - \tau_0^k - \delta}\left(t'_k - (t'_k)^2\right)$$

Since $t'_k - (t'_k)^2 \leq 1 - t'_k$ and from (18) one has $(q^i)^{nc}(\tau_0^k) \leq \psi$, one arrives at

$$\begin{aligned}
|\dot{q}_d^i(t_0^{k-})| &\leq \frac{6((q^i)^{nc}(\tau_0^k) + \varphi V^{1/3}(\tau_0^k))}{\tau_1^k - \tau_0^k - \delta}(1-t'_k) \\
&\leq \frac{6}{\tau_1^k - \tau_0^k - \delta}\sqrt{(\psi + \varphi V^{1/3}(\tau_0^k))\left(\sqrt{\frac{V(\tau_0^k)}{\gamma_1\gamma_2}} + \varphi V^{1/3}(\tau_0^k) + \epsilon\right)} \\
&= \frac{6}{\tau_1^k - \tau_0^k - \delta}\sqrt{\psi\epsilon + (\psi\varphi + \epsilon\varphi)V^{1/3}(\tau_0^k) + \varphi^2 V^{2/3}(\tau_0^k) + \frac{\varphi V^{5/6}(\tau_0^k) + \psi V^{1/2}(\tau_0^k)}{\sqrt{\gamma_1\gamma_2}}}
\end{aligned}$$

Since $V(\tau_0^k) < 1$ (thus $V^{p_1}(\tau_0^k) > V^{p_2}(\tau_0^k)$ for $p_1 < p_2$) one obtains

$$\begin{aligned}
|\dot{q}_d^i(t_0^{k-})| &\leq \frac{6}{\tau_1^k - \tau_0^k - \delta} \times \sqrt{\psi\epsilon + \left[\left(\frac{1}{\sqrt{\gamma_1\gamma_2}} + \varphi\right)(\varphi + \psi) + \epsilon\varphi\right] V^{1/3}(\tau_0^k)} \\
&\leq \frac{6\sqrt{\psi\epsilon}}{\tau_1^k - \tau_0^k - \delta} + \frac{6\sqrt{\left(\frac{1}{\sqrt{\gamma_1\gamma_2}} + \varphi\right)(\varphi + \psi) + \epsilon\varphi}}{\tau_1^k - \tau_0^k - \delta} V^{1/6}(\tau_0^k)
\end{aligned}$$

Therefore, the second inequality in (29) holds with

$$K = \frac{6\sqrt{p\psi\epsilon}}{\tau_1^k - \tau_0^k - \delta}, \quad K' = \frac{6\sqrt{p}}{\tau_1^k - \tau_0^k - \delta}\sqrt{\left(\frac{1}{\sqrt{\gamma_1\gamma_2}} + \varphi\right)(\varphi + \psi) + \epsilon\varphi}$$

B. *Proof of Theorem 1*

First we observe that conditions **a)** and **d)** of Proposition 1 hold when the hypothesis of the Theorem are verified. Thus in order to prove Theorem 1 it is sufficient to verify the conditions **b)**, **c)** and **e)** of Proposition 1. It is noteworthy that during the transition phases $I_k^{J_k}$ some p_ϵ–impact occurs (according to (8) we have $J'_k = \{1, \ldots, p\}$). This means that we do not know which and how many of them are the constraints touched at each contact. However, in the neighborhood of the desired stabilization point situated on a surface of codimension p, only the corresponding p constraints enter the dynamics. In the sequel we shall also use the function

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$V_1(t,s) = \frac{1}{2}s(t)^T M(q)s(t)$.

b) Using that $\dot{M}(q) - 2C(q,\dot{q}))$ is a skew-symmetric matrix, straightforward computations show that on $\mathbb{R}_+ \setminus \bigcup_{k\geq 0}[t_0^k, t_f^k]$ the time derivative of the Lyapunov function is given by

$$\dot{V}(t) = -\gamma_1 s^T s + 2\gamma_1\gamma_2 \tilde{q}^T\dot{\tilde{q}} = -\gamma_1||\dot{\tilde{q}}||^2 - \gamma_1\gamma_2^2||\tilde{q}||^2$$

On the other hand

$$V(t) \leq \frac{\lambda_{max}(M(q))}{2}||s||^2 + \gamma_1\gamma_2||\tilde{q}||^2 \leq \gamma^{-1}[\gamma_1||\dot{\tilde{q}}||^2 + \gamma_1\gamma_2^2||\tilde{q}||^2]$$

where

$$\gamma^{-1} = \max\left\{\lambda_{max}(M(q))\frac{1+2\gamma_2}{2\gamma_1}; \frac{\lambda_{max}(M(q))(\gamma_2+2)+2\gamma_1}{2\gamma_1\gamma_2}\right\} > 0$$

Therefore $\dot{V}(t) \leq -\gamma^{-1}V(t)$ on $\mathbb{R}_+ \setminus \bigcup_{k\geq 0}[t_0^k, t_f^k]$.

c) By definition

$$V(t_{\ell+1}^{k-}) - V(t_\ell^{k+}) = V_1(t_{\ell+1}^{k-}) - V_1(t_\ell^{k+}) + \gamma_1\gamma_2[(\tilde{q}^T(t_{\ell+1}^{k-}))\tilde{q}(t_{\ell+1}^{k-}) - (\tilde{q}^T(t_\ell^{k+}))\tilde{q}(t_\ell^{k+})] \tag{30}$$

On the other hand, straightforward computations show that

$$V_1(t_{\ell+1}^{k-}) - V_1(t_\ell^{k+}) = \int_{(t_\ell^k, t_{\ell+1}^k)} \dot{V}_1(t)\mathrm{d}t = \gamma_1\gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t - \gamma_1\int_{(t_\ell^k, t_{\ell+1}^k)} s(t)^T s(t)\mathrm{d}t \tag{31}$$

Furthermore,

$$\int_{(t_\ell^k, t_{\ell+1}^k)} s(t)^T s(t)\mathrm{d}t = \int_{(t_\ell^k, t_{\ell+1}^k)} ||\dot{\tilde{q}}(t)||^2 + \gamma_2^2||\tilde{q}(t)||^2\mathrm{d}t + \gamma_2[(\tilde{q}^T(t_{\ell+1}^{k-}))\tilde{q}(t_{\ell+1}^{k-}) - (\tilde{q}^T(t_\ell^{k+}))\tilde{q}(t_\ell^{k+})] \tag{32}$$

Therefore, inserting successively (32) in (31) and (31) in (30) one arrives at

$$V(t_{\ell+1}^{k-}) - V(t_\ell^{k+}) \leq \gamma_1\gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t \tag{33}$$

In the sequel let us denote by $S(q)$ the sum of all the components of the vector q. Taking into account the definition (15) and the fact that $(q_d)_p$ and $(\dot{q}_d)_p$ are set to zero at t_0^{k+} one obtains

$$\int_{(t_\ell^k, t_{\ell+1}^k)} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t = -\varphi V^{1/3}(\tau_0^k)\cdot\left(\int_{(t_\ell^k, t_{\ell+1}^k)} S(\dot{q}_p(t))\mathrm{d}t + \gamma_2\int_{(t_\ell^k, t_{\ell+1}^k)} S(q_p(t))\mathrm{d}t\right)$$

Since $\varphi\gamma_2 V^{1/3}(\tau_0^k) \geq 0$ and $S(q_p(t)) \geq 0$ it follows that

$$\int_{t_\ell^k}^{t_{\ell+1}^k} s_p^T(t)(q_d^*)_p(t)\mathrm{d}t \leq \varphi V^{1/3}(\tau_0^k)[S(q_p(t_\ell^k)) - S(q_p(t_{\ell+1}^k))]$$

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Thus

$$\sum_{\ell\geq 0}\left[V(t_{\ell+1}^{k-})-V(t_\ell^{k+})\right]\leq \gamma_1\gamma_2\varphi V^{1/3}(\tau_0^k)S(q_p(t_0^k))\leq \gamma_1\gamma_2\varphi V^{1/3}(\tau_0^k)\sqrt{3}||q_p(t_0^k)||$$

Recalling that t_0^k is an p_ϵ-impact which means that $||q_p(t_0^k)||\leq\epsilon$ one obtains

$$\sum_{\ell\geq 0}\left[V(t_{\ell+1}^{k-})-V(t_\ell^{k+})\right]\leq K_1 V^{p_1}(\tau_0^k)$$

where $K_1=\sqrt{3}\gamma_1\gamma_2\varphi\epsilon>0$ and $p_1=\frac{2}{3}$.

e) First, let us compute the Lyapunov function's jumps at the instants t_ℓ^k, $\ell\geq 1$. Using the continuity of the real trajectory $q(\cdot)$ and the definition of the desired trajectory $q_d(\cdot)$ on the I_k phases (i.e. $q_d(t_\ell^{k+})=q_d(t_\ell^{k-})$, $\dot q_d(t_\ell^{k+})=0=\dot q_d(t_\ell^{k-})$) one gets

$$\begin{aligned}\sigma_V(t_\ell^k)&=V(t_\ell^{k+})-V(t_\ell^{k-})=\gamma_1\gamma_2\sigma_{||\tilde q||^2}(t_\ell^k)+\frac{s^T(t_\ell^{k+})M_\ell s(t_\ell^{k+})-s^T(t_\ell^{k-})M_\ell s(t_\ell^{k-})}{2}\\&=T_L(t_\ell^k)+\gamma_2\tilde q(t_\ell^k)^T M_\ell\sigma_{\dot q}(t_\ell^k)\end{aligned}\qquad(34)$$

where M_ℓ^k denotes the inertia matrix $M(q(t_\ell^k))$ and T_L is the kinetic energy loss at the impact time t_ℓ^k.

From equation (5) one has $T_L(t_\ell^k)\leq 0$ and equation (34) becomes $\sigma_V(t_\ell^k)\leq\gamma_2\tilde q(t_\ell^k)^T M_\ell\sigma_{\dot q}(t_\ell^k)$. Let us recall that $M_\ell\sigma_{\dot q}(t_\ell^k)$ is the percussion vector (see [7]). On the other hand in the X coordinates the percussion vector can be expressed as $\nabla F(X)\lambda$. Writing the latter in the generalized coordinates introduced in Section III one obtains $M_\ell\sigma_{\dot q}(t_\ell^k)=D^T\lambda$. In other words the generalized coordinates introduced in Section III coincide with the so called quasi-coordinates and the vector $\dot q_{tang}$ is in fact $\dot q_{n-m}$ (i.e. $\sigma_{\dot q}(t_\ell^k)=\begin{pmatrix}\sigma_{\dot q_m}(t_\ell^k)\\ \mathbf{0}_{n-m}\end{pmatrix}$ where $\mathbf{0}_{n-m}$ denotes the $n-m$ vector with all its components equal zero). Therefore

$$\sigma_V(t_\ell^k)\leq\gamma_2\tilde q(t_\ell^k)^T M_\ell\sigma_{\dot q}(t_\ell^k)=\gamma_2 q_p(t_\ell^k)^T\lambda_p=0\qquad(35)$$

where we have used $(q_d)_p(t_\ell^{k+})=0=(q_d)_p(t_\ell^{k-})$ and the last equality is stated using the complementarity relation entering the dynamics.

The Lyapunov function's jump corresponding to the first impact of each cycle can be computed as:

$$\sigma_V(t_0^k)=V(t_0^{k+})-V(t_0^{k-})=\gamma_1\gamma_2\sigma_{||\tilde q||^2}(t_0^k)+\frac{s^T(t_0^{k+})M_0 s(t_0^{k+})-s^T(t_0^{k-})M_0 s(t_0^{k-})}{2}\qquad(36)$$

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- It is clear that $t_0^k > \tau_1^k$ implies $q_d(t_0^{k+}) = q_d(t_0^{k-})$ and $\dot{q}_d(t_0^{k+}) = 0 = \dot{q}_d(t_0^{k-})$. Thus, the computations for t_ℓ^k, $\ell \geq 1$ hold also for t_0^k.
- If $t_0^k \leq \tau_1^k$ one has $(q_d)_p(t_0^{k-}) \neq (q_d)_p(t_0^{k+}) = 0$ and $(\dot{q}_d)_p(t_0^{k-}) \neq (\dot{q}_d)_p(t_0^{k+}) = 0$. Then the initial jump of each cycle is given by:

$$\begin{aligned}\sigma_V(t_0^k) =& T_L(t_0^k) + \dot{q}_d(t_0^{k-})^T M_0 \dot{q}(t_0^{k-}) + \frac{\gamma_2^2}{2}\left(\tilde{q}(t_0^{k+})^T M_0 \tilde{q}(t_0^{k+}) - \tilde{q}(t_0^{k-})^T M_0 \tilde{q}(t_0^{k-})\right) \\ &+ \gamma_2\left(\dot{q}(t_0^{k+})^T M_0 \tilde{q}(t_0^{k+}) - \dot{\tilde{q}}(t_0^{k-})^T M_0 \tilde{q}(t_0^{k-})\right) - \frac{1}{2}\dot{q}_d(t_0^{k-})^T M_0 \dot{q}_d(t_0^{k-})\end{aligned} \tag{37}$$

Since $T_L(t_0^k) \leq 0$ the equation (37) rewrites as:

$$\begin{aligned}\sigma_V(t_0^k) \leq& \lambda_{max}(M(q))\Big[\gamma_2\left(||(\dot{q}_d)_p(t_0^{k-})|| \cdot ||\tilde{q}(t_0^{k-})|| + ||\dot{q}(t_0^{k-})|| \cdot ||(q_d)_p(t_0^{k-})||\right) + \frac{1}{2}||(\dot{q}_d)_p(t_0^{k-})||^2 \\ &+ \frac{\gamma_2^2}{2}\left(||q_p(t_0^k)||^2 + ||\tilde{q}_p(t_0^{k-})||^2 + 2||(q_d)_p(t_0^{k-})|| \cdot ||\tilde{q}_{n-p}(t_0^{k-})||\right) + ||(\dot{q}_d)_p(t_0^{k-})|| \cdot ||\dot{q}(t_0^{k-})||\Big]\end{aligned} \tag{38}$$

Obviously $||\dot{q}(t_0^{k-})|| = ||\dot{\tilde{q}}(t_0^{k-}) + (\dot{q}_d)_p(t_0^{k-})||$ and Lemma 1 combined with $V(\tau_0^k) < 1$ yields

$$||\dot{q}(t_0^{k-})|| \leq K + \left(\sqrt{\frac{2}{\lambda_{min}(M)}} + \sqrt{\frac{\gamma_2}{\gamma_1}} + K'\right) V^{1/6}(\tau_0^k)$$

Therefore

$$\sigma_V(t_0^k) \leq K_2 V^{p_2}(\tau_0^k) + \xi$$

where $p_2 = \frac{1}{6}$, $\xi = \frac{3}{2}K^2 + \gamma_2 \epsilon K + \frac{\gamma_2^2\epsilon^2}{2}$ and

$$\begin{aligned}K_2 = \lambda_{max}(M(q))&\left[3KK' + \frac{3}{2}(K')^2 + \frac{\gamma_2}{2}\sqrt{\frac{\gamma_2}{\gamma_1}} + \sqrt{\frac{2\gamma_2}{\lambda_{min}(M(q))\gamma_1}} + \frac{2\gamma_2}{\gamma_1}\right. \\ &\left. + (K'+K)\left(3\sqrt{\frac{\gamma_2}{\gamma_1}} + \sqrt{\frac{2}{\lambda_{min}(M(q))}}\right) + \epsilon\gamma_2\left(2\sqrt{\frac{\gamma_2}{\gamma_1}} + \sqrt{\frac{2}{\lambda_{min}(M(q))}} + K'\right)\right]\end{aligned}$$

Defining $\alpha : \mathbb{R}_+ \mapsto \mathbb{R}_+$, $\alpha(\omega) = \rho\omega^2$ we get $\alpha(0) = 0$ and $\alpha(||[s(t), \tilde{q}(t)]||) \leq V(t, s, \tilde{q})$. Thus, Proposition 1 also yields

$$R = \alpha^{-1}(e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)) = \sqrt{e^{-\gamma(t_f^k - t_\infty^k)}(1 + K_1 + K_2 + \xi)/\rho}$$

which ends the proof.

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