École doctorale IAEM Lorraine

Sparsity and Clustering in multi-agent systems

Mémoire de recherche

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Habilitation à Diriger les Recherches
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(Spécialité Automatique)

par

Irinel-Constantin MORĂRESCU

Composition du jury

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Rapporteurs :	Franck PLESTAN	Professeur des Universités, Ecole Centrale de Nantes
	Robert SHORTEN	Professor, University College Dublin and IBM Research, Irlande
	Luca ZACCARIAN	Directeur de Recherche CNRS, LAAS-CNRS, Toulouse
Examinateurs :	Bernard BROGLIATO	Directeur de Recherche INRIA, INRIA Rhône-Alpes
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Mis en page avec la classe thesul.

Remerciements

Rédiger ces remerciements est une tâche à la fois très facile et très difficile. Facile, parce que beaucoup de personnes m'ont soutenu et encouragé au cours de ma formation et de mon travail. Difficile, parce qu'il n'est pas simple de trouver les mots exprimant ma reconnaissance pour toutes ces personnes.

Je commencerais par remercier tous mes professeurs de la Faculté de Mathématiques de l'Université de Bucarest qui m'ont énormément aidé à bâtir la fondation de ma carrière. Sans leur passion pour leur métier, je n'aurai jamais poursuivi la voie de la recherche en mathématiques appliquées et la théorie du contrôle. Mes pensées se dirigent particulièrement vers Gheorghe Grigore, Şerban Strătilă et Ion Colojoară qui ont dirigé mes premiers travaux de recherche.

Je ne pourrai jamais trouver les mots pour exprimer ma reconnaissance et mon amitié pour Silviu-Iulian Niculescu, Antoine Girard et Jamal Daafouz qui m'ont étroitement accompagné dans mon activité de recherche. Ils ont toujours su m'écouter et me stimuler pour maximiser mon potentiel et l'impact de mon travail.

Je remercie respectueusement Janan Zaytoon d'avoir accepté de présider le jury de mon HdR ainsi que Robert Shorten, Franck Plestan et Luca Zaccarian d'avoir accepté le travail de rapporteur de mon manuscrit. Je n'oublierai pas de remercier ici Bernard Brogliato et encore Silviu-Iulian Niculescu et Jamal Daafouz d'avoir accepté le rôle d'examinateur. Ce fut un grand honneur pour moi de soutenir mon HdR devant ce prestigieux jury. Un grand merci à Bernard Brogliato pour son amitié et son aide précieuse pendant mon premier post-doctorat. Son regard sur la recherche a façonné la manière dont je réalise mon travail.

Un chaleureux merci à mes collègues Pierre, Marc, Romain, Samuel, Samson, Steven, Radu, Nicolae, Benoît, José, Charles, Samia, Vineeth, Maya et beaucoup d'autres qui ont rendu ma vie au CRAN très agréable et qui ont contribué à ma recherche au travers de nombreuses discussions enrichissantes. Un grand merci à Christine et Carole qui m'ont accompagné dans toutes les démarches administratives ce qui m'a permis de me concentrer sur mes activités de recherche et d'enseignement. Je suis également reconnaissant envers Wim Michiels, Carlos Canudas de Wit, Mirko Fiacchini, Lucian Buşoniu et Dragan Nešić qui, au-delà de leur amitié, ont contribué à mon développement comme chercheur grâce aux discussions stimulantes et la maîtrise parfaite de leur domaine de recherche. Je tiens également à remercier M. C. Bragagnolo, J. Ben Rejeb, M. Taki Asghar et T. Borzone qui ont accepté de préparer leur thèse sous ma co-direction.

Enfin, je dédie cette habilitation à ma famille qui a soutenu tous mes choix même si certains n'étaient pas évidents. Une très grande reconnaissance pour Adina, mon épouse, Ştefan et Sebastian mes garçons qui enchantent ma vie et qui comprennent la nécessité de mes déplacements et des longues journées passées au bureau.

Sommaire

Résumé

Ce manuscrit est divisé en trois parties. La première est rédigée en français et elle contient un CV détaillé. Je commence par présenter brièvement mes principales activités d'enseignement ainsi que mes principales contributions à la formation proposée à l'ENSEM. Une partie importante résume mon activité de recherche en commençant par les contributions de ma thèse de doctorat et finissant avec les derniers résultats que j'ai obtenus au CRAN. La présentation inclut la liste des étudiants que j'ai encadré en Master et en Thèse. Cette partie contient également les listes complètes des contrats et des publications dans lesquels je suis impliqué.

Pour une meilleure diffusion internationale, les deux dernières parties du manuscrit sont rédigées en anglais.

La deuxième partie de ce manuscrit fournit une sélection de résultats que j'ai obtenus sur le thème des systèmes multi-agents. Le fondement de cette recherche est le fait que les réseaux réels de grande taille sont divisés en plusieurs groupes. La première question qu'on se pose dans ce contexte est liée à la détection des clusters de manière décentralisée. Il est également naturel d'analyser ces réseaux dans la perspective des interactions continues à l'intérieur des clusters et sporadiques/discrets à l'extérieur. Le premier chapitre est consacré à la description des concepts de base et les principaux résultats existants dans la littérature de consensus. Le chapitre 2 présente une classe des dynamiques d'opinion à temps discret décrite par des systèmes multi-agents avec une confiance décroissante. Ce modèle de dynamique d'opinion est appliqué pour résoudre le problème de la détection de communautés dans un graphe. La vitesse de convergence dans un cluster est plus grande et la confiance décroissante nous permet de couper les liens entre différents clusters. Le chapitre 3 traite le problème de la coordination dans les réseaux hétérogènes contenant à la fois des agents linéaires et linéaires-impulsifs. Précisément, nous supposons que plusieurs clusters / communautés existent dans le réseau. Les agents à l'intérieur d'un cluster peuvent interagir de manière continue les uns avec les autres. En plus de cela, quelques agents dans chaque cluster peuvent interagir à certains instants de temps en dehors de leur propre cluster.

Nous terminons la présentation avec une partie contenant certaines de mes perspectives de recherche. Une première direction de recherche de mes travaux futurs est à l'intersection de la théorie des perturbations singulières et l'analyse de systèmes multi-agents. Deux aspects différents seront abordés. Le premier est directement lié aux résultats présentés dans la partie 2. Précisément, la présence de clusters dans le réseau génère, comme souligné auparavant, une séparation d'échelle de temps. En d'autres termes, les agents seront d'accord plus rapidement à l'intérieur d'un cluster que dans l'ensemble du réseau. Le deuxième aspect considère le problème de consensus au sein de réseaux d'agents singulièrement perturbés (*i.e.* dont les composantes de l'état évoluent sur différentes échelles de temps). Un deuxième axe de recherche se concentre sur l'analyse des dynamiques d'opinion dans les réseaux sociaux. Nous proposons un modèle fondé sur les protocoles de consensus qui utilise l'information quantifiée sur l'opinion des voisins. Le principal avantage de ce modèle est la capacité de capturer les comportements oscillatoires d'opinions ainsi que le dissensus dans le réseau.

Mots-clés: Systèmes multi-agents, consensus, commande décentralisée, perturbations singulières.

Abstract

This manuscript is partitioned in three parts. The first one is written in French and contains an extended resume. I start by briefly presenting my main teaching activities as well as my main contributions to the formation proposed at ENSEM. An important part summarizes my research activity starting with the contributions of my Ph.D. thesis and finishing with the last results that I obtained at CRAN. The presentation includes the list of Ph.D. and Master students that I supervised. This part also contains complete lists of grants and publications in which I was involved.

For the sake of accessibility, the last two parts of the manuscript are written in English.

The second part of this work provides a selection of results that I obtained on the topic of multi-agent systems. The basis of this research is the fact that real large-scale networks are partitioned in several clusters. The first question that arises in this context is related to the cluster detection in a decentralized way. It is also natural to analyze these networks from the perspective of continuous interactions inside clusters and sporadic-discrete ones outside. The first chapter is dedicated to the description of the basic concepts and the main results existing in the consensus literature. Chapter 2 presents a class of discrete-time multi-agent systems modeling opinion dynamics with decaying confidence. This opinion dynamics model is applied to address the problem of community detection in graphs. Basically the convergence speed is higher inside clusters and the decaying confidence allows to cut the links between different clusters. Chapter 3 addresses the problem of coordination in heterogeneous networks containing both linear and linear impulsive agents. Precisely we assume that several clusters/communities exist in the network. The agents inside one cluster can continuously interact with each others. On top of this, few agents in each cluster can interact at some discrete time instants outside their own cluster.

We finish the presentation with one part containing some of my research perspectives. A first research direction of my future works is at the intersection of singular perturbation theory and multi-agent systems' analysis. Two different aspects will be addressed. The first one is directly related to the results presented in Part 2. Precisely, the presence of clusters in the network generates, as pointed out before, a time-scale separation. In other words, the agents will agree faster inside a cluster than in the whole network. The second aspect considers the consensus problem in networks of singularly perturbed agents *e.g.* whose state components evolve on different time-scales. A second research direction focusses on the analysis of opinion dynamics in social networks. We propose a consensus like model based on quantized information about the neighbors opinion. The main advantages of this model are that it captures oscillatory behaviors of opinions as well as the dissensus in the network.

Keywords: Multi-agent systems, consensus, decentralized control, singular perturbations.

Première partie

CV détaillé

Chapitre 1

Cursus et expérience professionnelle

Identification

Nom : Morărescu
Prénom : Irinel-Constantin
Nationalité : roumaine et française
Email : constantin.morarescu@univ-lorraine

Situation actuelle

Maître des conférences à l'Université de Lorraine depuis 01/10/2010
Chercheur associé au Centre de Recherche en Automatique de Nancy (CRAN),
Département Contrôle Identification Diagnostic.

Adresse professionnelle : ENSEM
2 avenue de la Forêt de Haye, Vandœuvre-lès-Nancy, 54516
Téléphone : 03 83 59 57 06

Formation

• Thèse en Mathématiques et en les Technologies de l'Information et des Systèmes, respectivement (co-tutelle Université de Bucarest et Université de Technologie de Compiègne), Septembre 2006.
Sujet : Analyse qualitative des systèmes à retards distribués : Méthodologie et Algorithmes
Directeurs de thèse : Ion COLOJOARĂ et Silviu-Iulian NICULESCU

• DEA d'Analyse Complexe et Réelle, Université de Bucarest, Faculté de Mathématiques, Juin 1999
Mémoire : Operateurs affiliés à une algèbre de von Neumann
Encadrent : Şerban STRĂTILĂ

• Maîtrise de Mathématiques, Université de Bucarest, Faculté de Mathématiques, Juin 1997
Mémoire : Méthodes d'Interpolation
Responsable : Gheorghe GRIGORE

• Baccalauréat S, Lycée "Ion Minulescu", Slatina, 1993

Expérience professionnelle

- Postdoctorant CNRS, GIPSA - lab, équipe NeCS, Janvier 2010 - Septembre 2010
- Postdoctorant à l'Université Joseph Fourier, Laboratoire Jean Kuntzmann, équipe CASYS, Janvier 2009- Décembre 2009
- Post-doctorant à INRIA Rhône-Alpes, équipe BIPOP, Mars 2007- Décembre 2008.
- Maître de Conférence, Université "POLITEHNICA" de Bucarest, 2006 -2009
  Activité : Cours et encadrement TD, correction des devoirs, rédaction des sujets pour les examens, correction des examens (39 heures CM et 65 heures TD).
  Disciplines : Analyse Mathématique (Analyse Réelle, Fonctions de plusieurs variables)
- Assistant Universitaire (chargé de TD) Université "POLITEHNICA" de Bucharest, 2001-2006
  Activité : Encadrement TD, correction des devoirs, rédaction des sujets pour les examens, correction des examens (plus de 1000 heures de TD).
  Disciplines : Analyse Mathématique (Analyse Réelle, Fonctions de plusieurs variables), Mathematiques Avancée (Transformation Integrales, Equations aux dérivées partielles d'ordre 2)
- Professeur de Mathématiques, Lycée "MIRCEA VULCĂNESCU", Bucarest, 1998-2000.

Publications

— 19 articles de revues internationales ;
— 4 articles de revues nationales ;
— 8 chapitres de livre ;
— 39 actes de conférences ;

Activité d'encadrement scientifique

Encadrement de thèses

1. Marcos César Bragagnolo - "Commande décentralisée et analyse des systèmes en réseau", 01/11/2012-31/10/2015. Thèse codirigée avec P. Riedinger, taux d'encadrement 50%. Il a publié 3 conférences internationales avec actes et comité de lecture et 1 article à Automatica.
2. Jihene Ben Rejeb - "Analyse et commande décentralisée des systèmes interconnectés à ré-initialisation d'état ", 01/10/2014-30/09/2017. Thèse codirigée avec J.Daafouz, taux d'encadrement 50%. Elle a publié 2 conférences internationales avec actes et 1 article de revue nationale à JESA.
3. Mohamad Taki Asghar - "Commande multivariable pour les procédés sidérurgiques : Application au laminage à froid", 01/01/2015-31/12/2018. Thèse codirigée avec M. Jungers, taux d'encadrement 50%.

Encadrement de stages Master

1. Jihene Ben Rejeb , Master 2 à ESIEE : "Analyse et commande décentralisée des systèmes interconnectés à ré-initialisation d'état", 2014. Stage codirigé avec J.Daafouz, 50%.

2. Van Hoa NGUYEN, Master 2 à l'INPG-ENSE3 : "Etude de la stabilisation d'un système mécanique avec des articulations flexibles sur un environnement rigide", 2010. Stage codirigé avec B. Brogliato, 50%.
3. Florent STEVENIN, Master 2 PSPI : "Traffic model calibration for simulation of the Grenoble South Ring", 2010, Stage codirigé avec C. Canudas de Wit, 50%.
4. Diana STEFAN, Master 2 à l'INPG-ENSE3 : "Analyse des lois de commande pour le trafic routier", 2010. Stage codirigé avec C. Canudas de Wit, 50%.
5. Tran-Anh-Tu NGUYEN, Master 2 à l'INPG-ENSE3 : "Etude analytique et numérique de la stabilisation d'un système mécanique sur un environnement rigide", 2009. Stage codirigé avec B. Brogliato, 50%.

Responsabilités scientifiques

Projets internationaux :

PICS CNRS France-Roumanie :	"Artificial-Intelligence-Based Optimization for the Control of Networked and Hybrid Systems",
période :	2015-2017,
rôle :	porteur,
budget :	15 K€,
équipe française :	J. Daafouz, I.-C. Morărescu, R. Postoyan, J. Ben Rejeb.

PHC Brâncuşi France-Roumanie :	"Artificial-Intelligence-Based Optimization for the Stable and Optimal Control of Networked Systems",
période :	2015-2016,
rôle :	porteur,
budget :	5500 €,
équipe française :	I.-C. Morărescu, R. Postoyan, M.C. Bragagnolo.

PHC Tournesol France-Belgique :	"Qualitative analysis and design of decentralized controllers for multi-agent systems",
période :	2012-2013,
rôle :	porteur,
budget :	2000 €,
équipe française :	M. Jungers, I.-C. Morărescu.

PICS CNRS (France)- KOSEF (South Korea) :	"Physically-Based Collaborative Simulations under Various Network Configurations",
période :	2010-2011,
rôle :	participant,
budget :	15 K€,
consortium :	10 participants.

ECO-NET France-Roumanie :	"Dynamique - Interconnexions - Environnement", no.12645SD, Egide, (founded by CNRS)
période :	2006-2007
rôle :	participant,
budget :	15 K€,
consortium :	12 participants.

Projets nationaux :

ANR :	"Computation Aware Control Systems",
période :	2013-2017,
rôle :	responsable scientifique CRAN ,
budget :	320 K€,
équipe CRAN :	J. Daafouz, M. Jungers, S. Martin, I.-C. Morărescu, R. Postoyan, J. Ben Rejeb.

Projet université - région Lorraine :	"Analysis and control design for interconnected systems",
période :	2012
rôle :	porteur,
budget :	10 K€,
équipe CRAN :	J. Daafouz, M. Jungers, I.-C. Morărescu.

PEPS Mirabelle :	"Truffles' system inference with gene networks",
période :	2014
rôle :	participant,
budget :	17 K€,
équipe CRAN :	S. Martin, I.-C. Morărescu.

Collaborations industrielles :

Thèse CIFRE :	"Commande multivariable pour les procédés sidérurgiques : Application au laminage à froid"
période :	2015-2018,
Entreprise :	ArcelorMittal,
budget :	60 K€,
co-encadrement avec :	M. Jungers

Contrat d'accompagnement stage ingénieur :	"Etude et régulation d'un modèle de laminage d'acier",
période :	mars-septembre 2014
élève ENSEM :	Nassif El Najjar,
Entreprise :	ArcelorMittal,
budget :	10 K€,
équipe CRAN :	M. Jungers, I.-C. Morărescu, P. Riedinger.

Contrat d'accompagnement stage ingénieur :	"Modeling and control of a steel strip in a constant magnetic field",
période :	mars-septembre 2012
élève ENSEM :	Salma Bouraqqadi,
Entreprise :	ArcelorMittal,
budget :	10 K€,
équipe CRAN :	I.-C. Morărescu, P. Riedinger.

Autres responsabilités scientifiques

— **Membre élu** du conseil du pôle scientifique AM2I (Automatique, Mathématiques, Informatique et leurs Interactions) qui regroupe les plus importants laboratoires d'Automatique, Mathématiques et Informatique de la région Lorraine.

— **Membre nommé** au CNU dans la période 2011-2015.

— **Membre nommé** dans la commission de prospectives du CRAN (Centre de Recherche en Automatique de Nancy), 2012-2014.

— **Membre nommé** dans le comité technique TC 1.5 Networked Systems de l'IFAC (International Federation of Automatic Control).

— **Comité de programme :**
  — IFAC Workshop on Time Delay Systems, Sinaia, Roumanie, 1-3 Septembre, 2009 ;

  — KES International Conference on Intelligent Decision Technologies, Athènes, Grèce, 20-22 Juillet, 2011 ;

  — International Conference on Network Games, Control and Optimization, Avignon, 28-30 Novembre, 2012.

— **Chair/co-chair des sessions** à : Conference on Decision and Control, European Control Conference, American Control Conference, DelSys Workshop.

— **Comité de sélection pour une position de Maître de Conférences :**
  — Poste MCF 1341, UJF/GIPSA-lab UMR-CNRS 5216, Section 61, Commande des Systèmes en Réseaux et ses applications, 2011

  — Poste MCF 1433, UHP/CRAN UMR-CNRS 7039, Section 61, Modélisation, simulation et optimisation des systèmes de production, 2012.

  — Poste MCF 4192, UFC/FEMTO-ST UMR-CNRS 6174, Section 61, Automatique,2013.

  — Poste MCF 0356UL/CRAN UMR-CNRS 7039, Section 61, Automatique, 2013.

— **Expert** pour l'évaluation des projets nationaux (ANR) et locaux (université-région Lorraine, Actions Incitatives du CRAN, etc).

— **Raporteur :** IEEE Transactions on Automatic Control, Automatica, Systems and Control Letters, IMA Journal of Mathematical Control and Information, IEEE Transactions on Intelligent transportation Systems, International Journal of Robust and Nonlinear Control, Asian Journal of Control, ...

Chapitre 2

Activités d'enseignement

De Septembre 1998 à Décembre 2009 j'ai enseigné les mathématiques à plusieurs niveaux en utilisant trois langues différentes. J'ai commencé au Lycée "Mircea Vulcănescu" (Bucarest) avec des enseignements destinés aux élèves de Troisième, Seconde, Première et Terminale. Grâce à l'interaction directe avec les élèves et le contrôle continu effectué, cette expérience m'a donné la possibilité de développer mes compétences pédagogiques. A partir d'Octobre 2001 j'ai pris en charge des TD à l'Université "Politehnica" de Bucarest. A partir de Février 2002 j'ai obtenu un poste d'assistant universitaire (chargé de TD) et en Février 2006 je suis devenu par concours Maître de Conférences. Mes activités d'enseignement en mathématiques à l'Université "Politehnica" de Bucarest ont été destinées aux étudiants de première et deuxième année en roumain et en anglais. A partir d'Octobre 2004 à Février 2006 j'ai également pris en charge des TDs au niveau Master en roumain. Dans la période Octobre - Décembre 2009 j'ai enseigné les mathématiques comme vacataire à l'Université Joseph Fourier, Grenoble.

Depuis Octobre 2010 j'occupe un poste de Maître de Conférences à l'Ecole Nationale Supérieure d'Electricité et Mécanique (ENSEM) à Nancy. Dans ce cadre, j'ai proposé :

— un nouveau module en troisième année : PILOTAGE DES SYSTÈMES MULTI-AGENT. Ce module introduit des notions théoriques sur la commande décentralisée et la théorie spectrale de graphes mais il propose aussi une implémentation des algorithmes sur la plate-forme SAMI. Cette plate-forme est composée d'une flottille de robots accompagnée par différents types des capteurs (infrarouge, bluetooth, etc). Les robots sont connectés via un réseau WiFi et la topologie du réseau peut être définie fixe ou dynamique. Précisément, les élèves doivent réaliser des formations spécifiques à l'aide des robots mobiles Khepera III.

— un module de "Commande numérique de systèmes" en deuxième année ENSEM. Ce module se concentre sur le principe de régulation numérique, la modèlisation des convertisseurs analogique-numérique (CAN) et numérique-analogique (CNA), le calcul de la fonction de transfert pour une boucle de régulation numérique ainsi que sur la stabilité et les performances dynamiques de la boucle fermée. La conception des différents types de correcteurs numériques termine ce module.

— un module de "Régulation Numérique" pour la formation par apprentissage qui est montée actuellement à l'ENSEM. Ceci est une version adaptée pour la formation par apprentissage du module "Commande numérique de systèmes" proposé à la formation classique.

— plusieurs sujets de Projet de Fin d'Etudes (PFE) avec implémentation de résultats sur la plate-forme SAMI.

— une partie du module Smart-Grid qui démarrera en septembre 2016 en troisième année de la filière SInergie.

Mes activités d'enseignement à l'ENSEM portent essentiellement sur les domaines de mathématiques et de l'automatique. Même si j'ai une formation initiale en mathématiques, je me suis beaucoup investi dans la préparation des cours magistraux, TPs et TDs d'automatique. Actuellement mes activités d'enseignement en automatique sont prépondérantes. J'interviens dans les TDs et TPs d'automatique première année à l'ENSEM et j'ai pris la responsabilité d'un module "Commande numérique des systèmes" en deuxième année. Toujours en deuxème année, j'interviens dans les cours de "Commande avancée" et "Optimisation dynamique". A l'ENSEM j'ai continué à m'investir dans l'enseignement des différents cours de mathématiques comme : Mathématiques pour l'Ingénieur, Analyse Numérique et Algorithmes, Modèles probabilistes et méthodes stochastiques, etc. Pendant deux ans 2011-2012, 2012-2013 j'ai été le responsable du module Mathématiques pour l'Ingénieur.
Chaque année j'ai encadré deux stages ingénieur, ce qui m'a permis de mieux connaître les vrais préoccupations et besoins scientifiques des industriels.
Mon service a représenté chaque année un volume horaire annuel d'environ 230-260h eq TD et se décline en :

— Mathématiques pour l'Ingénieur 1/ première année

— Modélisation, Signaux, Systèmes/ première année

— Analyse Numérique et Algorithmes 1/ première année

— Automatique/ première année

— Modèles probabilistes et méthodes stochastiques 1/ deuxième année

— Analyse numérique et algorithmes 2/ deuxième année

— Commande numérique des systèmes/ deuxième année

— Commande avancée/ deuxième année

— Commande optimale/ deuxième année

— Pilotage des systèmes multi-agent/ troisième année

Chapitre 3

Thèmes de recherche

3.1 Résumé des travaux de thèse

Thèse de doctorat – Université de Bucarest et Université de Technologie de Compiègne :

Analyse qualitative des systèmes à retards distribués : Méthodologie et Algorithmes

Mots-clés : **Stabilité, Retards distribués, systèmes SISO, Régions de stabilité, Dynamiques de populations, Réseaux de communication, Réseaux de transport**

Mes travaux de thèse ont porté sur l'analyse de stabilité d'une classe des systèmes dynamiques linéaires incluant des retards distribués. Précisément, nous avons développé deux approches complémentaires (géométrique et algébrique) dans le domaine fréquentiel pour l'analyse de stabilité de systèmes dynamiques linéaires avec retards ponctuels ou distribués. Ces approches nous permettent de dériver les régions de stabilité dans l'espace de paramètres défini par les retards. En d'autres termes, nous obtenons des conditions nécessaires et suffisantes pour la stabilité des systèmes appartenant à la classe considérée.

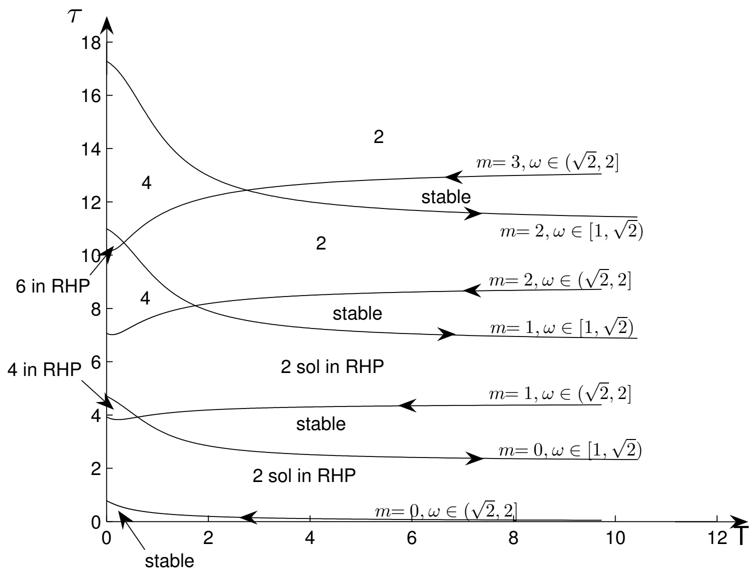
La classe principale des systèmes traités dans mes travaux de thèse est décrite par la dynamique suivante :

$$\dot{x}(t) = -\alpha x(t) - \beta \int_0^\infty x(t-\tau) g(\tau) \mathrm{d}\tau,$$

où α, β sont des paramètres constants du système et g est la densité de probabilité qui donne la distribution du retard. La principale densité de probabilité utilisée dans cette thèse est donnée par *la distribution gamma* avec un retard de propagation :

$$g(\theta) = \begin{cases} 0, & \theta < \tau, \\ \frac{(\theta-\tau)^{n-1} \mathrm{e}^{-\frac{\theta-\tau}{T}}}{T^n (n-1)!}, & \xi \geq \tau, \end{cases} \tag{3.1}$$

où $\tau \geq 0$ est le retard de propagation et $n \in \mathbb{N},\ T > 0$ caractérisent le retard moyen. Mon objectif était de trouver l'ensemble des valeurs du couple (T, τ) garantissant la stabilité du système. Les résultats sont présentés sous la forme d'un partitionnement de l'espace de paramètres comme dans la figure 3.1



Mes travaux de thèse présentent aussi des applications diverses. D'abord, nous avons précisé quelques propriétés qualitatives au sujet de la stabilité de certains modèles en biologie et communication en réseau. Ensuite, nous avons développé une méthode permettant d'étudier un modèle récent incluant un effet de mémoire pour la dynamique des véhicules qui se poursuivent.

1. W. Michiels, I.-C. Morărescu, S.-I. Niculescu - *Consensus problems with distributed delays, with application to traffic flow models*, SIAM Journal on Control and Optimization, Vol. 48, No. 1, pp. 77-101, 2009.

2. I.-C. Morărescu, S.-I. Niculescu - *Stability crossing curves of SISO systems controlled by delayed output feedback*, Dynamics of Continuous Discrete and Impulsive Systems, series B, Vol. 14, No 5, pp. 659-678, 2007.

3. I.-C. Morărescu, S.-I. Niculescu, K. Gu - *Stability Crossing Curves of Shifted Gamma-Distributed Delay Systems*, SIAM Journal on Applied Dynamical Systems, Vol. 6, No 2, pp. 475-493, 2007.

4. I.-C. Morărescu, S.-I. Niculescu, K. Gu - *On the geometry of stability regions of Smith predictors subject to delay uncertainty*, IMA Journal of Mathematical Control and Information, Vol. 24, No 3, pp. 411-423, 2007.

5. I.-C. Morărescu, S.-I. Niculescu, W. Michiels - *Asymptotic stability of some distributed delay systems : An algebraic approach* - International Journal of Tomography & Statistics, Vol 7, No. F07, 128-134, 2007.

3.2 Résumé des travaux de recherche post-doctorale

Post-dostorat – INRIA Rhône-Alpes, Equipe BIPOP :

Commande passive pour la poursuite de trajectoires dans les systèmes Lagrangiens non-réguliers

Mots-clés : **Commande basée sur la passivité, Systèmes non-réguliers, Problème de complémentarité, Impacts, Systèmes Lagrangiens, Stabilité, Poursuite des trajectoires**

Ces travaux portent essentiellement sur le problème de la commande pour assurer la poursuite des trajectoires pour une classe de systèmes Lagrangiens non-réguliers soumis à des contraintes unilatérales sans frottement.

$$\begin{cases} M(X)\ddot{X} + C(X,\dot{X})\dot{X} + G(X) = U + \nabla F(X)\lambda_X \\ 0 \leq \lambda_X \perp F(X) \geq 0, \\ \text{Collision rule} \end{cases} \tag{3.2}$$

où $X(t) \in \mathbb{R}^n$ est le vecteur des coordonnées généralisées, $M(X) = M^T(X) \in \mathbb{R}^{n\times n}$ est la matrice d'inertie, $F(X) \in \mathbb{R}^m$ représente les distances jusqu'aux contraintes, $C(X,\dot{X})$ est la matrice qui contient les forces de Coriolis et centripètes, $G(X)$ contient des forces conservatives, $\lambda_X \in \mathbb{R}^m$ est le vecteur des multiplicateurs de Lagrange associés aux contraintes et $U \in \mathbb{R}^n$ est le vecteur d'entrées.

La tâche typique considérée consiste en une succession des phases de mouvements libres et des phases de mouvements contraints. Une attention particulière est accordée aux phases de transition avec impacts et aux phases de détachement. Une commande basée sur la passivité garantit certaines propriétés de stabilité pour le système en boucle fermée. Des tests numériques réalisés avec la plate-forme SICONOS de l'INRIA montrent l'efficacité de la commande.

Dans un premier temps nous avons considéré des systèmes mécaniques avec des articulations rigides. Les résultats obtenus étendent les travaux précédents sur le sujet car on considère des systèmes avec plusieurs contraintes et n degrés de liberté.

Une deuxième partie de ce travail a été consacrée à l'étude de la commande pour assurer la poursuite des trajectoires pour la même classe de systèmes Lagrangiens non-réguliers avec des articulations flexibles. Les résultats intermédiaires déjà obtenus concernent les systèmes avec plusieurs contraintes et n degrés de liberté soumis à des impacts plastiques.

Cette recherche a été réalisée en collaboration avec B. Brogliato au cours de mon post-doctorat à l'INRIA. J'ai proposé des solutions pour la formulation du cadre général pour le cas multi-contraintes, l'analyse de la classe de systèmes considérés et pour la définition explicite de tous les éléments qui entrent dans la dynamique (signaux exogènes, force de contact).

Il est intéressant de remarquer qu'un cas particulier de dynamique considérée au cours de mon post-doctorat est celle décrite par un système linéaire de complémentarité. Cette dynamique est de plus en plus utilisée pour la modélisation de circuits électriques avec des éléments qui commutent (comme diodes, transistors MOS).

1. I.-C. Morărescu, B. Brogliato - *Trajectory tracking control of multiconstraint complementarity Lagrangian systems*, IEEE Transactions on Automatic Control, Vol **55**, No. 6, 1300-1310, 2010.
2. I.-C. Morărescu, B. Brogliato - *Passivity-based switching control of flexible-joint complementarity mechanical systems*, Automatica, Vol **46** No.1, 160-166, 2010.

Post-dostorat –Laboratoire Jean Kuntzmann, Equipe CASYS :

Problèmes de consensus avec des contraintes sur la vitesse de convergence, détection de communautés dans les réseaux d'agents représentés par graphes

Mots-clés : **Problème de consensus, Systèmes hybrides, Stabilité, Théorie algébrique de graphe, Détection de communautés, Systèmes d'agents dynamiques**

Cette recherche a été réalisée en collaboration avec A. Girard. L'analyse des systèmes multi-agents a reçu une attention croissante ces quinze dernières années. Cela est dû aux applications diverses dans de nombreux domaines tels que le contrôle coopératif de véhicules, le contrôle de congestion dans la communication via réseaux Internet, l'étude de la dynamique d'opinions dans les réseaux sociaux, ou du mouvement de groupes d'individus, les réseaux de capteurs. Dans ces applications, il est généralement souhaitable que les agents du système atteignent un consensus permettant leur coordination. Dans certains cas, une contrainte sur la vitesse de coordination peut être imposée. C'est le problème que nous étudions.
Le modèle peut être interprété en termes de dynamique d'opinion. A chaque pas de temps t, l'agent $i \in V$ reçoit les opinions de ses voisins dans le graphe $\mathcal{G}$. Si l'opinion de i diffère de l'opinion de son voisin j de plus d'un certain niveau de confiance $R\rho^t$, alors i ne prend pas en compte l'opinion de j quand il met à jour sa propre opinion. Le paramètre ρ caractérise la décroissance du voisinage de confiance des agents. L'agent i accorde de manière répétée sa confiance aux voisins dont l'opinion converge suffisamment rapidement vers sa propre opinion. Sous cette contrainte, un consensus global peut être inaccessible et les agents s'accordent localement formant ainsi des communautés.
Nos contributions principales, outre le développement du modèle, sont :
- un résultat de convergence dans des conditions moins restrictives que celles disponibles dans la littérature,
- l'équivalence entre la connectivité asymptotique des nœuds et leur convergence vers la même valeur,
- la prise en compte de délais dans les canaux de communication,
- la caractérisation des communautés en termes de valeurs propres de matrices définissant la dynamique collective,
- l'identification de communautés d'une manière efficace.

1. I.-C. Morărescu, A. Girard - *Opinion dynamics with decaying confidence : application to community detection in graphs*, IEEE Transaction on Automatic Control, Vol. 56, No. 8, 1862 - 1873, 2011.

Post-dostorat –Gipsa-lab, Equipe NeCS :

Estimation et prédiction du trafic sur le périphérique de Grenoble

Mots-clés : **Systèmes hybrides, Stabilité, Trafic péri-urbain**

La dynamique du trafic est exprimée par une équation hyperbolique aux dérivées partielles appelée loi de conservation. La discrétisation du modèle conduit à un système multi-agents avec un réseau qui comporte à la fois des retards et une topologie variable.

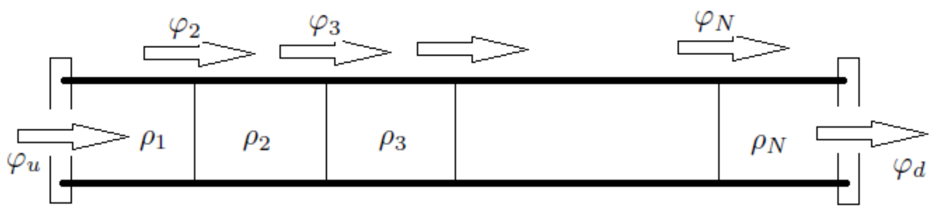


FIGURE 3.1 – Le modèle multi-agent model fondé sur CTM.

Des problèmes très variés qui concernent l'estimation, l'observabilité et la commandabilité de ces systèmes sont encore ouverts. D. Jacquet a proposé en 2006 une commande optimale pour le modèle du trafic. D'une part cette commande est centralisée donc très coûteuse en terme de calcul et de communications, d'autre part elle ne prend pas en compte la simplicité de la loi de commutation. Dans le cadre de mon post-doctorat au sein de l'équipe NeCS nous avons proposé des solutions décentralisées pour le trafic autoroutier. Les objectifs de nos études s'insèraient dans le cadre des projets GTL, HYCON 2, MOCOP :

- l'estimation et l'observabilité du système afin de pouvoir prédire les congestions et de fournir les informations nécessaires en termes de distances et temps de parcours aux utilisateurs ;
- la conception des lois de commande semi-décentralisées qui prennent en compte la topologie du réseau. Celles-ci seront conçues pour limiter les effets des congestions mais aussi les queues sur les rampes d'accès ;
- l'implémentation de résultats pour le trafic du périphérique de Grenoble.

3.3 Résumé des travaux de recherche au CRAN

Mes travaux de recherche au CRAN comportent deux volets. Le premier et plus important concerne l'étude des systèmes multi-agents et englobe le contrôle coopératif et la commande en réseau. Le deuxième porte sur l'analyse et la commande de systèmes subissant des sauts dans leurs états ou leurs dynamiques.

Contrôle coopératif des systèmes multi-agents

Un système multi-agents est un système composé d'un ensemble d'agents (i.e. d'éléments avec une autonomie au moins partielle, une connaissance/perception locale de l'environnement et qui ne sont pas commandés de façon centralisée) interagissant selon certaines règles. La topologie du réseau change au cours du temps en fonction de l'état du système. Par conséquent, la dynamique d'un tel système inclut des comportements continus et discrets. Il s'agit donc de l'étude d'une classe de systèmes hybrides. Un de mes objectifs de recherche est d'exploiter les propriétés topologiques du graphe d'interactions entre les agents afin de donner des conditions algébriques faciles à vérifier pour l'accord global. Je m'intéresse aussi au problème des conditions garantissant une vitesse de convergence au moins égale à une vitesse initialement prévue. Les applications visées sont le contrôle coopératif d'agents (véhicules, flottilles, robots).

Les réseaux et les systèmes en réseau sont omniprésents dans les divers domaines de la science et l'ingénierie. Quelques exemples sont les réseaux biologiques comme les réseaux de gènes, les écosystèmes ; les réseaux technologiques comme les systèmes de capteurs, les systèmes énergétiques à grande échelle, Internet ; les réseaux économiques comme les réseaux de production et de distribution ; les réseaux financiers et les réseaux sociaux comme Facebook, les réseaux de collaboration scientifique, etc. L'analyse de ces systèmes a reçu un intérêt croissant ces dernières décennies. Cela est certainement dû au fait que, dans l'avenir, la mesure, l'estimation, le suivi et la commande des signaux, seront disponibles principalement par le biais de réseaux de communication numérique. Les installations et les dispositifs de contrôle seront répartis dans l'espace. Cela impose naturellement le besoin d'un réseau de communication pour les relier. L'utilisation généralisée des réseaux sans fil pour l'accès Internet à partir de différentes sortes de dispositifs électroniques indique que cela est une tendance qui finira par devenir commun sur les réseaux industriels. Il y a de grandes possibilités de réduction des coûts de câblage et des équipements dédiés à l'utilisation de la communication sans fil pour le contrôle des installations industrielles.

Les contraintes d'économie d'énergie du système à grande échelle en réseau imposent une stratégie de contrôle décentralisé. Afin de maintenir la cohérence globale des actions locales, chaque entité dynamique doit interagir avec certains *voisins*, cela conduit à ce qu'on appelle *contrôle coopératif*. Même si les règles d'interaction sont généralement décrites par des équations simples, le contrôle coopératif est complexe car il englobe l'auto-configuration et la reconfiguration dynamique du réseau de communication, des comportements déterministes et stochastiques, les retards induits par les canaux de communication, etc.

3.3.1 Réduction de dimension pour les systèmes interconnectés à grande échelle

Les réseaux et les systèmes en réseau sont omniprésents dans les divers domaines de la science et l'ingénierie. Une façon de surmonter les difficultés induites par la grande dimension de ces systèmes consiste à développer des modèles approximatifs de faible dimension. Dans ce contexte, avec Romain Postoyan, j'ai proposé une méthodologie d'approcher la dynamique d'un système multi-agents à grande échelle par un système multi-agents de dimension inférieure. D'abord, on regroupe les nœuds dans des communautés qui forment les agents du système réduit. Ensuite, nous associons une dynamique scalaire appropriée à chaque communauté. L'idée principale est de rapprocher la trajectoire de chaque nœud par la trajectoire de sa communauté. Les liens entre différentes communautés sont calculés en considérant des combinaisons linéaires des forces de liaison entre les éléments de chaque collectivité. Enfin, les conditions initiales sont sélectionnées pour garantir la cohérence asymptotique du modèle réduit avec le système d'origine. Il faut souligner que notre approche est flexible que l'utilisateur est libre de choisir la dimension du système de réduit.

3.3.2 Synchronisation des oscillateurs non-linéaires couplés en présence des retards gamma-distribués

Dans ce travail réalisé avec Wim Michels et Marc Jungers, on déduit des conditions pour la synchronisation des oscillateurs non-linéaires couplés affectés par des retards distribués dans les interconnexions. Les retards distribués sont caractérisés par un noyau de distribution gamma avec un retard de transport. L'approche est basée sur l'analyse de la stabilité des équilibres synchronisés dans l'espace des paramètres (délais, gain) et la caractérisation de la structure des solutions émanant de bifurcations. Les résultats sont appliqués à des réseaux des oscillateurs de Lorenz couplés. En particulier, il est montré que, indépendamment de la topologie du réseau, pour des gains de couplage suffisamment grands, la distribution du retard a un effet stabilisateur sur l'équilibre synchronisé.

3.3.3 Planification optimiste pour le consensus

Un défi important dans les systèmes multi-agents est le consensus. L'objectif des agents est de synchroniser certaines variables d'intérêt contrôlées en utilisant seulement un graphe partiel de communication et variable dans le temps. Avec Lucian Buşoniu, j'ai proposé une approche d'accord basée sur la planification optimiste (OP), un algorithme de commande prédictive qui trouve les actions quasi-optimales de contrôle pour toutes les dynamiques non-linéaires et toutes les fonctions récompense (coût). A chaque étape, chaque agent utilise OP pour résoudre un problème de contrôle local avec des récompenses qui expriment les objectifs consensuels. Les agents voisins se synchronisent en échangeant leurs comportements prévus dans un ordre prédéfini. En raison de sa généralité, le consensus OP peut s'adapter à toute dynamique d'agents et, en changeant la fonction de récompense, à une variété d'objectifs de consensus. L'OP consensus est démontrée pour un accord en vitesse (flocking) avec un graphe de communication variant dans le temps. Dans ce contexte, il conserve la connectivité mieux qu'un algorithme classique. On montre aussi le consensus dans un réseau de bras robotiques avec des dynamiques non-linéaires avec et sans leader.

Analyse et commande de systèmes non-réguliers

3.3.4 Cadre numérique pour la commande optimale des systèmes non-linéaires commutés soumis aux contraintes

Avec Pierre Riedinger, j'ai abordé le problème de la mise en œuvre numérique d'une loi de contrôle optimal pour une classe de systèmes non-linéaires affines commutés soumis aux contraintes. Afin de bien résoudre le problème, on introduit un système relaxé et on explique le lien entre la solution de ce système et la solution du système initial. L'une des principales difficultés est alors liée au fait que la solution optimale est généralement singulière. Nous montrons, en utilisant des variables d'écart, qu'un ensemble de contraintes de complémentarité peut être utilisé pour prendre en compte la singularité de la solution. Le problème de contrôle optimal est alors reformulé comme un problème d'optimisation sous contraintes sur les systèmes hamiltoniens et résolu par une méthode directe. Cette formulation ne nécessite pas de connaissance *a priori* sur la structure (régulière / singulière) de la solution. En outre, les contraintes d'état sont incluses. Des simulations numériques pour différents types de convertisseurs de puissance en mode de conduction continue et discontinue, illustrent l'efficacité de la méthode proposée.

3.3.5 Conditions nécessaires et suffisantes pour la stabilité de systèmes linéaires impulsifs

Dans ce travail réalisé avec Mirko Fiacchini nous avons proposé des conditions facilement implémentable numériquement pour la stabilité exponentielle globale d'un système linéaire impulsif. Plus précisément nous avons supposé que les sauts dans l'état du système arrivent quasi-périodiquement. Dans un premier temps le système a été réécrit comme une inclusion différentielle discrète. La stabilité de celle ci a été ensuite caractérisée par des conditions fondées sur la théorie des ensembles pour des systèmes linéaires incertaines avec la matrice d'état appartenant à un ensemble non-convexe. Les conditions nécessaires et suffisantes pour la stabilité de systèmes linéaires impulsifs sont données en termes d'existence d'une fonction de Lyapunov polyhédrale. Une méthode constructive pour tester l'existence de cette fonction est aussi proposée. La complexité numérique de cette méthode est similaire à celle utilisée pour les systèmes linéaires avec un paramètre incertain avec la matrice d'état appartenant à un polytope.

3.4 Travaux de recherche avec les thésards et post-doctorants

3.4.1 Thèse M.C. Bragagnolo : Commande décentralisée et analyse des systèmes en réseau

La plupart des réseaux de grande taille présente une partition en clusters. Pour réaliser le consensus dans ce type de réseaux, une stratégie spécifique de commande doit être développée. Ceci était l'objectif de la thèse de M.C. Bragagnolo que j'ai co-encadrée avec P. Riedinger. Nous avons étudié un réseau d'agents groupés en clusters. Ils interagissent de manière continue à l'intérieur de chaque cluster. En plus, chaque cluster contient un leader qui interagit de manière discrète avec les leaders d'autres clusters. Les contributions principales de cette thèse sont :

- la caractérisation de la valeur de consensus en fonction de la condition initiale, la topologie d'interaction à l'intérieur de chaque cluster et la topologie d'interaction entre leaders.
- l'analyse de stabilité et de vitesse de convergence dans le type de réseaux considéré.
- la synthèse de la topologie d'interaction entre leaders assurant la convergence vers une valeur de consensus *a priori* définie.
- l'implantation de cette stratégie pour la réalisation d'une formation à l'aide d'une flotte de robots mobiles partitionnée en clusters.

Cette thèse a donné lieu à la publication d'un article Automatica, 3 actes de conférences (ACC2014, CDC2015, ENOC2014) et 2 chapitres de livre. Un deuxème article Automatica est en cours de soumission.

3.4.2 Master J. Ben Rejeb : Loi événnementielle décentralisée pour le consensus en réseaux partitionnés en clusters

Ce stage de Master poursuit la direction de recherche de la thèse de M. Bragagnolo. Son objectif était de proposer une loi événnementielle pour l'activation des interactions entre les leaders des différents clusters. Les agents d'un même cluster évoluent selon une dynamique continue vers un accord local. La loi événementielle proposée permet de ramener ces accords locaux vers un consensus global en assurant la persistance d'interactions entre les leaders et éviter le phénomène de Zéno. Un intérêt majeur de cette stratégie est d'optimiser les interactions entre les leaders et réduire les transmissions inutiles qui nécessitent généralement un apport énergétique important.
Ce stage a conduit à la publication d'un article de revue nationale (JESA) et un acte de conférence internationale - ECC2015.

3.4.3 Thèse J. Ben Rejeb : Commande décentralisée et analyse des systèmes singulièrement perturbés en réseau

Ce travail est en cours de réalisation. L'objectif est d'obtenir des conditions suffisantes pour la stabilité et la synchronisation des systèmes linéaires singulièrement perturbés en présence des commutations et des sauts des états des systèmes. Dans un premier temps, nous avons étudié la synchronisation des systèmes linéaires singulièrement perturbés sans saut ou commutation. Nous avons montré que ce problème peut être reformulé comme un problème de stabilisation simultanée de plusieurs systèmes linéaires avec une structure particulière. Finalement nous avons montré que la stabilisation simultanée de ces systèmes est équivalente à une condition de commandabilité d'un seul système linéaire.

3.4.4 Post-doc M. Fiacchini : Conditions LMI pour la préservation d'une topologie d'interaction

La préservation de la connectivité du réseau d'interconnexion est un ingrédient essentiel pour le contrôle décentralisé. Dans un travail récent, je propose avec Mirko Fiacchini des conditions LMI (linear matrix inequalities) pour la préservation de la topologie du réseau d'interactions dans une flottille d'agents avec une sensitivité limitée. La reconfiguration du réseau est évitée tant que les états des agents respectent une condition algébrique. Notre synthèse garantit que cette condition est respectée même s'ils ont une connaissance limitée du système et leurs actions décentralisées sont perturbées par la dynamique des autres agents. A part la préservation du réseau, une coordination globale peut être réalisée en ajoutant des contraintes supplémentaires dans la conception de la commande. La différence principale par rapport aux travaux existant dans la littérature est que notre méthode permet de maintenir un lien entre deux agents même si on contrôle juste une partie de l'état et pas l'état entier.
Ce travail a mené à la publication d'un article IEEE Transactions on Automatic Control et un acte de conférence - CDC2012.

Chapitre 4

Liste de publication

Articles de revues internationales :

1. I.-C. Morărescu, S. Martin, A. Girard, A. Muller-Gueudin - *Coordination in networks of linear impulsive agents*, IEEE Transactions on Automatic Control, to appear, 2016.
2. M. Fiacchini, I.-C. Morărescu - *Constructive necessary and sufficient condition for the stability of quasi-periodic linear impulsive systems*, IEEE Transactions on Automatic Control, to appear, 2016.
3. M.C. Bragagnolo, I.-C. Morărescu, J. Daafouz, P. Riedinger - *Reset strategy for consensus in networks of clusters*, Automatica, Vol. 65, 53-63, 2016.
4. I.-C. Morărescu, W.Michiels, M. Jungers - *Effect of a distributed delay on relative stability of diffusely coupled systems, with application to synchronized equilibria*, International Journal of Robust and Nonlinear Control, Vol 26, No. 7, 1565-1582, 2016.
5. I. Boussaada, I.-C. Morărescu, S.-I. Niculescu - *Inverted Pendulum Stabilization : Characterization of Codimension-Three Triple Zero Bifurcation Via Multiple Delayed Proportional Gains*, Systems and Control Letters, Vol. 82, 1-9, 2015.
6. L. Busoniu, I.-C. Morărescu - *Topology-preserving flocking of nonlinear agents using optimistic planning*, Control Theory and Technology, special issue on Learning and Control in Cooperative Multi-agent Systems, Vol. 13, No. 1, pp. 70-81, 2015.
7. L. Buşoniu, I.-C. Morărescu - *Consensus for black-box nonlinear agents using optimistic optimization*, Automatica, Vol. 50, No. 4, pp. 1201-1208, 2014.
8. M. Fiacchini, I.-C. Morărescu - *Convex conditions on decentralized control for graph topology preservation*, IEEE Transactions on Automatic Control, Vol. 59, No.6, 1640-1645, 2014.
9. I.-C. Morărescu, M. Fiacchini - *LMI conditions for topology preservation : applications to multi-agent tasks*, Journal of Control Engineering and Technology Vol. 4, No. 3, 183-191, 2014
10. P. Riedinger, I.-C. Morărescu - *A numerical framework for optimal control of switched input affine nonlinear systems subject to path constraint*, Mathematics and Computers in Simulation, Vol.95, pp. 63-77, 2013.
11. I.-C. Morărescu, M. Jungers - *Comments on : "$\mathcal{H}_\infty$ control design for time-delay linear systems : a rational transfer function based approach"*, European Journal of Control, Vol. 18, No. 5, 2012.
12. I.-C. Morărescu, A. Girard - *Opinion dynamics with decaying confidence : application to community detection in graphs*, IEEE Transaction on Automatic Control, Vol. 56, No. 8, 1862 - 1873, 2011.

13. I.-C. Morărescu, B. Brogliato - *Trajectory tracking control of multiconstraint complementarity Lagrangian systems*, IEEE Transactions on Automatic Control, Vol **55**, No. 6, 1300-1310, 2010.
14. I.-C. Morărescu, B. Brogliato - *Passivity-based switching control of flexible-joint complementarity mechanical systems*, Automatica, Vol **46** No.1, 160-166, 2010.
15. W. Michiels, I.-C. Morărescu, S.-I. Niculescu - *Consensus problems with distributed delays, with application to traffic flow modèles*, SIAM Journal on Control and Optimization, Vol. 48, No. 1, pp. 77-101, 2009.
16. I.-C. Morărescu, S.-I. Niculescu - *Stability crossing curves of SISO systems controlled by delayed output feedback*, Dynamics of Continuous Discrete and Impulsive Systems, series B, Vol. 14, No 5, pp. 659-678, 2007.
17. I.-C. Morărescu, S.-I. Niculescu, K. Gu - *Stability Crossing Curves of Shifted Gamma-Distributed Delay Systems*, SIAM Journal on Applied Dynamical Systems, Vol. 6, No 2, pp. 475-493, 2007.
18. I.-C. Morărescu, S.-I. Niculescu, K. Gu - *On the geometry of stability regions of Smith predictors subject to delay uncertainty*, IMA Journal of Mathematical Control and Information, Vol. 24, No 3, pp. 411-423, 2007.
19. I.-C. Morărescu, S.-I. Niculescu, W. Michiels - *Asymptotic stability of some distributed delay systems : An algebraic approach* - International Journal of Tomography & Statistics, Vol 7, No. F07, 128-134, 2007.

Articles de revues nationales :

1. J. Ben Rejeb, I.-C. Morărescu, J. Daafouz. *Stratégies événementielles de réinitialisation pour un consensus dans les systèmes multi-agents*, Journal Européen de Systèmes Automatisés, to appear.
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Deuxième partie

Clustering and consensus in Multi-agent systems : Theory and Applications

This part provides a selection of results that I obtained on the topic of multi-agent systems. The basis of this research is the fact that real large-scale networks are partitioned in several clusters. The first question that arises in this context is related to the cluster detection in a decentralized way. It is also natural to analyze these networks from the perspective of continuous interactions inside clusters and sporadic-discrete ones outside. The first chapter is dedicated to the description of the basic concepts and the main results existing in the consensus literature. The rest of this part describes some of my contributions to the field of analysis and control design for multi-agent systems.

Chapter 2 presents a class of discrete-time multi-agent systems modeling opinion dynamics with decaying confidence. This opinion dynamics model is applied to address the problem of community detection in graphs. The main advantage of this methodology is that it runs in a decentralized manner. We consider three examples of networks, and compare the communities we detect with those obtained by existing algorithms based on modularity optimization. We show that our opinion dynamics model not only provides an appealing approach to community detection but that it is also effective.

Chapter 3 addresses the problem of coordination in heterogeneous networks containing both linear and linear impulsive agents. Precisely we assume that several clusters/communities exist in the network. The agents inside one cluster can continuously interact with each others. On top of this, few agents in each cluster can interact at some discrete time instants outside their own cluster. We first consider time-invariant interconnections inside clusters and synchronous activation of inter-cluster links. In this framework we are able to characterize the consensus value in terms of initial conditions, intra-cluster and inter-cluster interaction topologies. This allows as to design inter-cluster topology in order to reach an a priori fixed consensus. In a second time we extend the study to time-varying intra-cluster interactions and asynchronous activation of inter-cluster links. The study initially considers that inter-cluster links are activated following some time-dependent conditions. We finish this part by proposing some event triggering rules for the activation of the inter-cluster interactions. Numerical simulations shows that in some cases these event-triggering rules out-performs the time-triggering ones.

This part is based on the following publications :

1. I.-C. Morărescu, A. Girard - *Opinion dynamics with decaying confidence : application to community detection in graphs*, IEEE Transaction on Automatic Control, Vol. 56, No. 8, 1862 - 1873, 2011.
2. M.C. Bragagnolo, I.-C. Morărescu, J. Daafouz, P. Riedinger - *Reset strategy for consensus in networks of clusters*, Automatica, to appear.
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4. J. Ben Rejeb, I.-C. Morărescu, J. Daafouz. Stratégies événementielles de réinitialisation pour un consensus dans les systèmes multi-agents, Journal Européen de Systèmes Automatisés, to appear.

Chapitre 1

Basic concepts and preliminaries

1.1 Introduction

"A multi-agent system (MAS) is a computerized system composed of multiple interacting intelligent agents within an environment." Each agent is characterized by the following features :

— "**Autonomy :** the agents are at least partially independent, self-aware, autonomous ."
— "**Local view :** generally no agent has a full global view of the system, or the system is too complex for an agent to make practical use of such knowledge."
— **Decentralization :** there is no agent designated to control the whole group, the intelligence is distributed to each member and the actions are designed locally.

The analysis of multi-agent systems received an increasing interest in the past decades. In such systems, a set of agents interact according to simple local rules in order to achieve some global coordinated behavior. The most widely studied problem is certainly the consensus or agreement problem where each agent in the network maintains a value and repetitively averages its value with those of its neighbors, resulting in all the agents in the network reaching asymptotically a common value. Conditions ensuring consensus have been established by various authors including [35, 8, 48, 59] (see [55] for a survey). More recently, there have been several works providing estimations of the rate of convergence towards the consensus value [57, 3, 75].

Classical consensus algorithms are designed for homogeneous agents with linear dynamics, for which the behavior is well understood. In this setting, the literature considers fixed and time-varying communication topologies [59, 48], directed or undirected graphs [55, 60], synchronous or asynchronous information exchange [72, 21], delayed or immediate transmissions [56, 47], etc. Consensus approaches also exist for nonlinear agent dynamics, such as second-order systems with nonlinear acceleration dynamics [67, 76], nonholonomic robots [69], and Euler-Lagrange dynamics [46]. These works usually require an explicit mathematical model of the agents, the form of which is exploited to derive tailored control laws (often via Lyapunov synthesis). A more general approach considering requiring only input-output map has been presented in [14].

It has been emphasized that controlling interconnected systems in a decentralized manner [35, 56, 59] has advantages related to the computation and communication cost reduction. On the other hand the changes of the network topology may hamper the global coordination goal. To avoid this, recent works have been oriented towards the connectivity preservation of the interconnection graph of mobile networks [15, 74]. In [42] the authors compute a robust connected spanning subgraph which allows the highest degree of freedom for the agents position and find the initial states (position and velocities) assuring the graph preservation.

Studies concerning real networks revealed that the topology of interactions in communication, social or biological systems presents a cluster/community structure [58, 10, 28]. In order to detect these communities, different algorithms are available in the literature [54, 39, 49]. A consequence of the presence of decoupled clusters in the network is that consensus/synchronization cannot be reached and different local agreements are obtained [49, 71].

Notation. The following standard notation will be used throughout the paper. The sets of nonnegative integers, real and nonnegative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}_+$, respectively. For a vector x we denote by $\|x\|$ its Euclidian norm. The transpose of a matrix A is denoted by $A^\top$. Given a symmetric matrix $A \in \mathbb{R}^{n\times n}$, notation $A > 0$ ($A \geq 0$) means that A is positive (semi-)definite. By I_k we denote the $k \times k$ identity matrix. $\mathbb{1}_k$ and $\mathbf{0}_k$ are the column vectors of size k having all the components equal 1 and 0, respectively. We also use $x(t_k^-) = \lim_{t\to t_k, t\leq t_k} x(t)$. Throughout the manuscript we say that the LMI : $A > 0$ is satisfied on the subspace $\mathcal{K}$ if and only if $x^\top Ax > 0$ for all $x \in \mathcal{K}$.

1.2 Graph theory

In the following, a MAS is represented by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where the vertex set $\mathcal{V}$ represents the set of agents and the edge set is a relation $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ representing the interactions between agents. The vertex set $\mathcal{V}$ is finite and for simplicity is identified with $\{1, \ldots, n\}$. In some situations the graph is *undirected* meaning that $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$, while in some others the graph is *directed*. In both cases the relation $\mathcal{E}$ is considered anti-reflexive meaning that $(i, i) \notin \mathcal{E}$. Throughout the rest of this manuscript a digraph means a directed graph.
Other useful concepts are introduced in the following definition.

Definition 1 *A* **directed/undirected path of length** *p in a given digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a union of directed/undirected edges $\bigcup_{k=1}^{p}(i_k, j_k)$ such that $i_{k+1} = j_k$, $\forall k \in \{1, \ldots, p-1\}$.*
The node j is **connected** *with node i in the directed/undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if there exists at least a directed/undirected path in $\mathcal{G}$ from i to j (i.e. $i_1 = i$ and $j_p = j$).*
A **strongly connected digraph** *is such that any two distinct elements are connected. A* **strongly connected component** *of a digraph is a maximal subset of $\mathcal{V}$ such that any of its two distinct nodes are connected.*
We say node i is a parent of node j in the digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ if $(i, j) \in \mathcal{E}$. A **directed tree** *is a directed subgraph in which there exists a single node without parents called* **root** *while all the others have exactly one parent. The length of a directed tree is the length of its longest path. A* **directed spanning tree** *of a digraph is a directed tree that connects all the nodes of the graph. For a given graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, the subgraph* **induced** *by a subset of nodes $\mathcal{U} \subseteq \mathcal{V}$ is the graph $\big(\mathcal{U}, \mathcal{E} \cap (\mathcal{U} \times \mathcal{U})\big)$.*

Throughot the rest of the manuscript we consider a system consisting of n agents. Several matrices are associated with a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Among them, of particular interest are the Laplacian and Perron matrices. A weighted Laplacian matrix $L \in \mathbb{R}^{n\times n}$ is defined as :

$$\begin{cases} L_{(i,j)} = 0, \text{ if } (i,j) \notin \mathcal{E} \\ L_{(i,j)} < 0, \text{ if } (i,j) \in \mathcal{E},\ i \neq j \\ L_{(i,i)} = -\sum_{j\neq i} L_{i,j}, \forall i = 1, \ldots, n \end{cases}. \tag{1.1}$$

We note that, regardless the interconnection topology (i.e. the set $\mathcal{E}$), $\mathbb{1}_n$ is a right eigenvector of L associated with the eigenvalue 0.

On the other hand, a weighted Perron matrix $P \in \mathbb{R}^{n\times n}$ is a row stochastic matrix defined by :

$$\begin{cases} P_{i,j} = 0, \text{ if } (i,j) \notin \mathcal{E} \\ P_{i,j} > 0, \text{ if } (i,j) \in \mathcal{E},\ i \neq j \\ P_{i,i} = 1 - \sum_{j\neq i} P_{i,j}, \forall i = 1,\dots,n \end{cases}. \tag{1.2}$$

Therefore, $\mathbb{1}_n$ is a right eigenvector of P associated with the eigenvalue 1.

1.3 Consensus protocols

From now on, to the vertex $i \in \mathcal{V}$ one assigns a state x_i which may be a scalar or a vector. To avoid extensive usage of Kronecker products we suppose in the following that $x_i \in \mathbb{R}$. As noted in the Introduction, the most studied problem related to MAS is the consensus. The objective is that all agents reach asymptotically a common state value x^* by applying decentralized controls based on local information.

1.3.1 Continuous agent dynamics

Classical consensus algorithms are designed for MAS with linear agent dynamics. The simplest MAS is defined by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each agent is an integrator :

$$\dot{x}_i(t) = u_i(t) \tag{1.3}$$

The classical consensus protocol used in this case is based on the weighted Laplacian matrix L associated with the graph :

$$u_i(t) = \sum_{j\neq i} L_{i,j}\big(x_i(t) - x_j(t)\big) \tag{1.4}$$

leading to a collective closed-loop dynamics described by :

$$\dot{x}(t) = -Lx(t), \quad x(t) = (x_1(t), \dots, x_n(t))^\top. \tag{1.5}$$

The solution of (1.5) is

$$x(t) = \exp(-Lt)x(0),$$

which means that consensus value, if it exists, is explicitly defined by the value of $\lim_{t\to\infty} \exp(-Lt)$. The following result provides guaranty for consensus achievement and defines the consensus value as a function of initial condition and network topology.

Theorem 1 *Assume $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a strongly connected digraph with Laplacian L. Let v be the left eigenvector of L associated with the eigenvalue 0 (e.g. $v^\top L = \mathbf{0}_n$) normalized such that $v^\top \mathbb{1}_n = 1$. Then*

$$\lim_{t\to\infty} \exp(-Lt) = \mathbb{1}_n v^\top \in \mathbb{R}^{n\times n}$$

meaning that

$$\lim_{t\to\infty} x(t) = x^* \mathbb{1}_n \text{ with } x^* = v^\top x(0).$$

Remark 1 *Theorem 1 applies in the case of undirected graphs too. In this case* $v = \frac{1}{n}\mathbb{1}_n$ *meaning that* x^* *is the average of the initial values of the agents' state. Moreover, when the graph is undirected the second smallest eigenvalue* λ_2 *of* L *gives a lower bound on the convergence speed :*

$$\|x(t) - x^*\mathbb{1}_n\| \leq \exp(-\lambda_2 t)\|x(0) - x^*\mathbb{1}_n\|$$

In the following we extend the framework above in two directions. The first one considers adaptive feedback gains corresponding to variation of the weights in the interconnection topology. The second one focuses on more general agent dynamics.
When the network topology is time-varying, the underlined graph associated with the MAS is also time-varying. Let this graph be $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$ and $L(t)$ be the associated Laplacian matrix. The closed-loop dynamics (1.3)-(1.4) of each agent becomes :

$$\dot{x}_i(t) = \sum_{j\neq i} L_{i,j}(t)\big(x_i(t) - x_j(t)\big) \tag{1.6}$$

which can be written in the collective dynamics form as :

$$\dot{x}(t) = -L(t)x(t). \tag{1.7}$$

In order to guarantee that system (1.7) admits a unique solution we further impose that for all $i, j \in \{1, \dots, n\}$ the functions $L_{i,j}$ are measurable functions of time (see Theorem 54 in [65]).
In the following it is useful to introduce the weighted adjacency matrix whose elements are

$$a_{i,j} = \begin{cases} -L_{i,j} & \text{if } i \neq j \\ 0 & \text{if } i = j \end{cases} \tag{1.8}$$

One of the most general consensus results [31, 43] for MAS with dynamic interaction topology make use of two assumptions. The first one is related to the topology of interactions and it writes as follows.

Assumption 1 (Persistent connectivity) *The graph* $(\mathcal{V}, \bar{\mathcal{E}})$ *is strongly connected, where*

$$\bar{\mathcal{E}} = \left\{(j,i) \in \mathcal{V}\times\mathcal{V} \mid \int_0^{+\infty} a_{i,j}(s)ds = +\infty\right\}.$$

The second one impose a reciprocity of the interactions. Roughly it says that any group of agents can influence another group only if the second one influence back the first. This idea is formulated as "Cut-balanced weights" in [31] and as "Slow divergence of reciprocal interaction weights" in [43].

Assumption 2 (Cut-balanced weights) *There exists* $K \geq 1$ *such that for all non-empty proper subset* $\mathcal{S} \subset \mathcal{V}$ *and for all time* $t \geq 0$*,*

$$K^{-1}\sum_{i\in\mathcal{S}, j\in\mathcal{V}\setminus\mathcal{S}} a_{j,i}(t) \leq \sum_{i\in\mathcal{S}, j\in\mathcal{V}\setminus\mathcal{S}} a_{i,j}(t) \leq K \sum_{i\in\mathcal{S}, j\in\mathcal{V}\setminus\mathcal{S}} a_{j,i}(t)$$

The "Slow divergence of reciprocal interaction weights" basically requires that the ratio

$$\mathbf{r}(t) = \max_{s\in[0,t]}\max_{i,j\in\mathcal{V}} \frac{a_{i,j}(s)}{a_{j,i}(s)}$$

does not grow too fast.
It is noteworthy that neither cut-balanced generalizes slow reciprocity divergence nor slow reciprocity divergence generalizes cut-balanced.

Theorem 2 *If Assumption 1 and Assumption 2 (or "Slow divergence of reciprocal interaction weight") hold, then system (1.7) reaches a consensus.*

Let us now consider more general agent dynamics. We start by a MAS whose agents have identical linear dynamics given by :

$$\dot{x}_i(t) = Ax_i(t) + Bu_i(t), \quad i \in \{1, \ldots, n\}, \tag{1.9}$$

where, for all $i \in \{1, \ldots, n\}$ one has $x_i \in \mathbb{R}^{n_x}$, $u_i \in \mathbb{R}^{n_u}$, $A \in \mathbb{R}^{n_x \times n_x}$ and $B \in \mathbb{R}^{n_x \times n_u}$.
A controller using the local information is defined as :

$$u_i(t) = K \sum_{j \neq i} L_{i,j}(x_i - x_j), \quad i \in \{1, \ldots, n\}, \tag{1.10}$$

where $K \in \mathbb{R}^{n_u \times n_x}$ is a controller gain allowing to modulate the coupling strengths.
Denoting by x the concatenation of the vectors $x_1, \ldots, x_n$ one obtains a closed-loop collective dynamics described by :

$$\dot{x}(t) = \left[I_n \otimes A + \left(I_n \otimes BK\right)(L \otimes I_{n_x})\right] x(t), \tag{1.11}$$

The main result is this context states that the consensus is achieved if we ensure the stability of a single agent with the same dynamics , modified by only a scalar they takes values according to the eigenvalues of L (see [22] for details). When the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ representing the MAS is undirected this result can be simply stated as :

Theorem 3 *Let $\mathcal{G}$ be an undirected and connected graph whose Laplacian L has the eigenvalues $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_n$. System (1.9) achieves consensus if there exists K defining the controller (1.2) that simultaneously renders Hurwitz the matrices $A - \lambda_i BK$, $i \in \{2, \ldots, n\}$. Moreover, if the pair (A, B) is controllable then K ensuring consensus of (1.9) always exists.*

Consensus and synchronization of agents with more general nonlinear dynamics have been also considered in the literature. The results require either passivity properties [66, 4] or Lipschitz like conditions [76, 67, 14].

1.3.2 Discrete agent dynamics

In the manuscript we also make use of some existing results on consensus of agents with discrete dynamics. For the sake of completeness, in the following we briefly recall these results.
Let us first consider a MAS represented by a fixed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where each agent has a discrete dynamics given by :

$$x_i(t+1) = u_i(t), \tag{1.12}$$

where $(t)_{t \in \mathbb{N}}$ is an increasing sequence of time instants.
The classical consensus protocol used in this case is based on the weighted Perron matrix P associated with the graph :

$$u_i(t) = \sum_{j=1}^{n} P_{i,j} x_j(t) \tag{1.13}$$

leading to a collective closed-loop dynamics described by :

$$x(t+1) = Px(t), \quad x(t_k) = (x_1(t), \dots, x_n(t))^\top. \tag{1.14}$$

It is clear that the soluteion of (1.14) is

$$x(t) = P^t x(0)$$

which means that consensus value, if it exists, is explicitly defined by the value of $\lim_{k\to\infty} P^k$. The following standard result can be formulated.

Theorem 4 *Assume $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is a fixed digraph with a weighted Perron matrix P. Let v be the left eigenvector of P associated with the eigenvalue 0 (e.g. $v^\top P = v^\top$) normalized such that $v^\top \mathbb{1}_n = 1$. Then (1.14) reaches consensus asymptotically if and only if $\mathcal{G}$ contains a spanning tree. Moreover*

$$\lim_{t\to\infty} P^t = \mathbb{1}_n v^\top \in \mathbb{R}^{n\times n}$$

meaning that

$$\lim_{t\to\infty} x(t) = x^* \mathbb{1}_n \ \textit{with}\ x^* = v^\top x(0).$$

Remark 2 *In the case of undirected graphs $v = \frac{1}{n}\mathbb{1}_n$ meaning that x^* in Theorem 4 is the average of the initial values of the agents' state. Moreover, when the graph is undirected, the second biggest eigenvalue λ_2 of P gives a lower bound on the convergence speed :*

$$\|x(t) - x^*\mathbb{1}_n\| \leq \lambda_2^k \|x(0) - x^*\mathbb{1}_n\|$$

When the interaction topology dynamically changes in time (i.e. the MAS is described by a time-varying graph $\mathcal{G}(t) = (\mathcal{V}, \mathcal{E}(t))$), sufficient conditions for consensus are given in [48, 59, 8]. In this framework (1.14) becomes :

$$x(t+1) = P(t)x(t), \quad x(t) = (x_1(t), \dots, x_n(t))^\top. \tag{1.15}$$

Consensus in (1.15) requires the following hypothesis.

Assumption 3 (Minimal influence) *There exists a constant $\alpha \in (0,1)$ such that, for all $t \in \mathbb{N}$, $P_{i,i}(t) \geq \alpha$ and, if $P_{i,j}(t) \neq 0$ and $(i,j) \in \mathcal{E}$ then $P_{i,j}(t) \geq \alpha$.*

Remark 3 *Assumption 11 guarantees a minimal influence of one agent on the others at each interaction time instant t.*

Assumption 4 (Connectivity) *For all $t \in \mathbb{N}$ the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}^\infty = \bigcup_{s\geq t} \mathcal{E}(s))$ has a spanning tree.*

The previous assumption ensures an infinite number of activations for a sufficiently large number of links. These practically guarantees the transmission of information all over the vertex set.

Assumption 5 (Bounded intercommunication intervals) *If $(i,j) \in \mathcal{E}^\infty$ then there exists $N \in \mathbb{N}^*$ such that for all $t \in \mathbb{N}$,*

$$(i,j) \in \mathcal{E}(t) \cup \mathcal{E}(t+1) \cup \ldots \cup \mathcal{E}(t+N)$$

Assumption 5 enforce assumption 4 stating that a spanning tree exists in the union of any finite union of $N+1$ graphs describing consecutive interconnection topologies of (1.15). This assumption is required in order to obtain a convergence with a geometric decay rate. This can be relaxed to increasing and unbounded N when we search only asymptotic convergence. However we have to take care to the increasing rate of N as pointed out in [51] for instance.

The main result in this framework is stated as follows.

Theorem 5 *Let a MAS whose evolution is described by* (1.15) *and such that Assumptions 11, 4 and 5 hold. Then, system* (1.15) *asymptotically achieves a consensus. If the interaction pattern is symmetric (i.e.* $(i,j) \in \mathcal{E}(t) \Leftrightarrow (j,i) \in \mathcal{E}(t)$*) then Assumptions 11 and 4 are sufficient to guarantee that* (1.15) *asymptotically achieves a consensus.*

This chapter introduced the theoretical concepts allowing to follow the rest of the presentation. It also presented a brief summary of existing results on consensus.The remaining of this Part will provide some of my contributions in the domain of analysis and control of multi-agent systems. Chapter 2 describes a time varying discrete dynamics with decaying confidence that can be used in the community detection problem. The consensus problem in networks of clusters is considered in Chapter 3.

Chapitre 2

Opinion Dynamics with Decaying Confidence : Community Detection

This section closely follows the presentation in [49]. Its objective is to define a model for the opinion dynamics that takes into account the loss of patience. Even if the model resemble to a consensus protocol, it often provides only local agreements. We first analyze the relation between asymptotic agreement of a subset of agents and the fact that they are asymptotically connected. We show that under suitable assumptions, these are actually equivalent (i.e. communities correspond to asymptotically connected component of the network) except for a set of initial opinions of measure 0. Second, apply our opinion dynamics model to address the problem of community detection in graphs.

2.1 Model Description

We study a discrete-time multi-agent model. Along this section we consider the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ describing the set of agents and the set of interactions that can be activated at a given time-instant. Each agent $i \in \mathcal{V}$ has a *state/opinion* modelled by a real number $x_i(t) \in \mathbb{R}$. Initially, agent i has an opinion $x_i(0) = x_i^0$ independent from the opinions of the other agents. Then, at every time step, the agents update their opinion by taking a weighted average of its opinion and opinions of other agents :

$$x_i(t+1) = \sum_{j=1}^{n} p_{ij}(t) x_j(t) \tag{2.1}$$

with the coefficients $p_{ij}(t)$ satisfying

$$\forall i, j \in \mathcal{V},\ (p_{ij}(t) \neq 0 \iff j \in \{i\} \cup N_i(t)) \tag{2.2}$$

where $N_i(t)$ denotes the *confidence neighborhood* of agent i at time t :

$$N_i(t) = \big\{ j \in \mathcal{V} |\ ((i,j) \in \mathcal{E}) \wedge \big(|x_i(t) - x_j(t)| \leq R\rho^t\big)\big\} \tag{2.3}$$

with $R > 0$ and $\rho \in (0, 1)$ model parameters.

Remark 4 *It is noteworthy that the confidence neighborhoods $N_i(t)$ and the coefficients $p_{ij}(t)$ actually depend also on the opinions $x_1(t), \ldots, x_n(t)$. For the sake of simplicity and in order to reduce the length of the equations we keep the notations $p_{ij}(t)$ and $N_i(t)$ pointing out just the variation in time of these quantities.*

We make the following additional assumptions :

Assumption 6 (Stochasticity) *For $t \in \mathbb{N}$, the coefficients $p_{ij}(t)$ satisfy*

(a) $p_{ij}(t) \in [0,1]$, *for all* $i, j \in \mathcal{V}$.

(b) $\sum_{j=1}^{n} p_{ij}(t) = 1$, *for all* $i \in \mathcal{V}$.

This model can be interpreted in terms of opinion dynamics. At each time step t, agent $i \in \mathcal{V}$ receives the opinions of its neighbors in the graph $\mathcal{G}$. If the opinion of i differs from the opinion of its neighbor j more than a certain threshold $R\rho^t$, then i does not give confidence to j and does not take into account the opinion of j when updating its own opinion. The parameter ρ characterizes the confidence decay of the agents. Agent i gives repetitively confidence only to neighbors whose opinion converges sufficiently fast to its own opinion. This model can be interpreted in terms of negotiations where agent i requires that, at each negotiation round, the opinion of agent j moves significantly towards its opinion in order to keep negotiating with j.

Remark 5 *We assume in this paper that $\rho \in (0,1)$. However, let us remark that for $\rho = 1$ (there is no confidence decay), with a complete graph $\mathcal{G}$ (every agent talks with all the other agents), and with coefficients $p_{ij}(t)$ given for all $j \in \{i\} \cup N_i(t)$ by*

$$p_{ij}(t) = \frac{1}{1 + d_i(t)} \text{ with } d_i(t) = \sum_{j \in N_i(t)} 1$$

our model would coincide with Krause model of opinion dynamics with bounded confidence [38, 29, 9].

Our first result states that the opinion of each agent converges to some limit value :

Proposition 1 *Under Assumption 6 (Stochasticity), for all $i \in \mathcal{V}$, the sequence $(x_i(t))_{t \in \mathbb{N}}$ is convergent. We denote x_i^* its limit. Furthermore, we have for all $t \in \mathbb{N}$,*

$$|x_i(t) - x_i^*| \leq \frac{R}{1-\rho}\rho^t. \tag{2.4}$$

Proof : Let $i \in \mathcal{V}$, $t \in \mathbb{N}$, we have from (2.1), Assumption 6 and (2.2)

$$\begin{aligned} |x_i(t+1) - x_i(t)| &= \left| \left(\sum_{j=1}^{n} p_{ij}(t) x_j(t) \right) - x_i(t) \right| \\ &= \left| \sum_{j=1}^{n} p_{ij}(t)(x_j(t) - x_i(t)) \right| \\ &= \left| \sum_{j \in N_i(t)} p_{ij}(t)(x_j(t) - x_i(t)) \right| \\ &\leq \sum_{j \in N_i(t)} p_{ij}(t) |x_j(t) - x_i(t)| \end{aligned}$$

Then, it follows from equation (2.3) that

$$|x_i(t+1) - x_i(t)| \leq \sum_{j \in N_i(t)} p_{ij}(t) R \rho^t$$

Finally, Assumption 6 gives for all $t \in \mathbb{N}$

$$|x_i(t+1) - x_i(t)| \leq (1 - p_{ii}(t))R\rho^t \leq R\rho^t.$$

Let $t \in \mathbb{N}$, $\tau \in \mathbb{N}$, then

$$|x_i(t+\tau) - x_i(t)| \leq \sum_{k=0}^{\tau-1} |x_i(t+k+1) - x_i(t+k)| \leq \sum_{k=0}^{\tau-1} R\rho^{t+k}$$

Therefore,

$$|x_i(t+\tau) - x_i(t)| \leq \frac{R}{1-\rho}\rho^t(1-\rho^\tau) \leq \frac{R}{1-\rho}\rho^t \tag{2.5}$$

which shows, since $\rho \in (0,1)$, that the sequence $(x_i(t))_{t\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$. Therefore, it is convergent. Equation (2.4) is obtained from (2.5) by letting τ go to $+\infty$.

The previous proposition allows us to complete the interpretation of our opinion dynamics model. The agents try to reach an agreement with the constraint that the consensus value must be approached no slower than $O(\rho^t)$. Under that constraint, global agreement may not be attainable and the agents may only reach local agreements. We refer to the sets of agents that asymptotically agree as communities.

Definition 2 *Let $i, j \in \mathcal{V}$, we say that agents i and j asymptotically agree, denoted $i \sim^* j$, if and only if $x_i^* = x_j^*$.*

It is straightforward to verify that $\sim^*$ is an equivalence relation over $\mathcal{V}$.

Definition 3 *A community $C \subseteq \mathcal{V}$ is an element of the quotient set $\mathscr{C} = \mathcal{V}/\sim^*$.*

Let us remark that the community structure is dependent on the initial distribution of opinions and the rule used to define the coefficients $p_{ij}(t)$. In the following, we shall provide some insight on the structure of these communities. But first, we need to introduce some additional notations.

We define the set of *interactions at time t*, $\mathcal{E}(t) \subseteq \mathcal{V} \times \mathcal{V}$ as

$$\mathcal{E}(t) = \left\{(i,j) \in \mathcal{E} |\ |x_i(t) - x_j(t)| \leq R\rho^t\right\}.$$

Let us remark that $(i,j) \in \mathcal{E}(t)$ if and only if $j \in N_i(t)$. The *interaction graph at time t* is then $\mathcal{G}(t) = (V, \mathcal{E}(t))$. Let us notice that Remark 4 applies also to $\mathcal{E}(t)$ and $\mathcal{G}(t)$.

Definition 4 *For a set of agents $I \subseteq \mathcal{V}$, the subset of edges of $\mathcal{G}$ connecting the agents in I is $\mathcal{E}_I = \mathcal{E} \cap (I \times I)$. Let $\mathcal{E}' \subseteq \mathcal{E}_I$ be a symmetric relation over I, then the graph $\mathcal{G}' = (I, \mathcal{E}')$ is called a subgraph of $\mathcal{G}$. If $I = \mathcal{V}$, then the graph $\mathcal{G}' = (\mathcal{V}, \mathcal{E}')$ is called a spanning subgraph of $\mathcal{G}$. The set of spanning subgraphs of $\mathcal{G}$ is denoted $\mathcal{S}(\mathcal{G})$. For all $t \in \mathbb{N}$, $\mathcal{G}(t) \in \mathcal{S}(\mathcal{G})$. Let us remark that the set $\mathcal{S}(\mathcal{G})$ is finite : it has $2^{|\mathcal{E}|/2}$ elements (because we only consider symmetric relations) where $|\mathcal{E}|$ denotes the number of elements in $\mathcal{E}$. Given a partition of the agents $\mathcal{P} = \{I_1, \ldots, I_p\}$, we define the set of edges $\mathcal{E}_\mathcal{P} = \bigcup_{I\in\mathcal{P}} \mathcal{E}_I$ and the spanning subgraph of $\mathcal{G}$, $\mathcal{G}_\mathcal{P} = (\mathcal{V}, \mathcal{E}_\mathcal{P})$. Essentially, $\mathcal{G}_\mathcal{P}$ is the spanning subgraph of $\mathcal{G}$ obtained by removing all the edges between agents belonging to different elements of the partition $\mathcal{P}$. An interesting such graph is the graph of communities $\mathcal{G}_\mathscr{C} = (\mathcal{V}, \mathcal{E}_\mathscr{C})$ where :*

$$\mathcal{E}_\mathscr{C} = \{(i,j) \in \mathcal{E} |\ i \sim^* j\}.$$

The set of connected components of $\mathcal{G}'$ is denoted $\mathcal{K}(\mathcal{G}')$. Let us remark that $\mathcal{K}(\mathcal{G}')$ is a partition of $\mathcal{V}$.

We define the vectors of opinions $x(t) = (x_1(t), \dots, x_n(t))^\top$ and of initial opinions $x^0 = (x_1^0, \dots, x_n^0)^\top$. The dynamics of the vector of opinions is then given by

$$x(t+1) = P(t)x(t)$$

where $P(t)$ is the row stochastic matrix with entries $p_{ij}(t)$. For a set of agents $I \subseteq \mathcal{V}$, with $I = \{v_1, \dots, v_k\}$, we define the vector of opinions $x_I(t) = (x_{v_1}(t), \dots, x_{v_k}(t))^\top$. Given a $n \times n$ matrix A with entries a_{ij}, we define the $k \times k$ matrix A_I whose entries are the $a_{v_i v_j}$. In particular, $P_I(t)$ is the matrix with entries $p_{v_i v_j}(t)$. Let us remark that $P_I(t)$ is generally not row stochastic. However, if $I \subseteq \mathcal{V}$ is a subset of agents such that no agent in I is connected to an agent in $\mathcal{V} \setminus I$ in the graph $\mathcal{G}(t)$, then it is easy to see that

$$x_I(t+1) = P_I(t)x_I(t)$$

and $P_I(t)$ is an aperiodic row stochastic matrix. Moreover, if I is a connected component of $\mathcal{G}(t)$ then $P_I(t)$ is irreducible.

The following sections are devoted to the analysis of the community structure of the network of agents.

2.2 Asymptotic Connectivity and Agreement

In this section, we explore the relation between communities and asymptotically connected components of the network. Let us remark that the set of edges $\mathcal{E}$ can be classified into two subsets as follows :

$$\mathcal{E}^f = \{(i,j) \in \mathcal{E} | \; \exists t_{ij} \in \mathbb{N}, \; \forall s \geq t_{ij}, \; (i,j) \notin \mathcal{E}(s)\}$$

and

$$\mathcal{E}^\infty = \{(i,j) \in \mathcal{E} | \; \forall t \in \mathbb{N}, \; \exists s \geq t, \; (i,j) \in \mathcal{E}(s)\}.$$

Intuitively, an edge (i,j) is in $\mathcal{E}^f$ if the agents i and j stop interacting with each other in finite time. $\mathcal{E}^\infty$ consists of the interactions between agents that are infinitely recurrent. It is clear that $\mathcal{E}^f \cap \mathcal{E}^\infty = \emptyset$ and $\mathcal{E} = \mathcal{E}^f \cup \mathcal{E}^\infty$. Also, since $\mathcal{E}$ and thus $\mathcal{E}^f$ is a finite set, there exists $T \in \mathbb{N}$ such that

$$\forall (i,j) \in \mathcal{E}^f, \; \forall s \geq T, \; (i,j) \notin \mathcal{E}(s). \tag{2.6}$$

Let us remark that the sets $\mathcal{E}^f$ and $\mathcal{E}^\infty$ and the natural number T generally depend on the vector of initial opinions x^0. We define the graph $\mathcal{G}^\infty = (\mathcal{V}, \mathcal{E}^\infty)$.

Definition 5 *Let $i, j \in \mathcal{V}$, we say that agents i and j are asymptotically connected if and only if i and j are connected in $\mathcal{G}^\infty$ (i.e. there exists a path in $\mathcal{G}^\infty$ joining i and j). We say that they are asymptotically disconnected if they are not asymptotically connected.*

Proposition 2 *Under Assumption 6 (Stochasticity), if two agents $i, j \in \mathcal{V}$ are asymptotically connected then they asymptotically agree.*

Proof : Suppose $(i,j) \in \mathcal{E}^\infty$. From the definition of $\mathcal{E}^\infty$ there exists a strictly increasing sequence of non-negative integers $(\tau_k)_{k\in\mathbb{N}}$ such that for all $k \in \mathbb{N}$, $(i,j) \in \mathcal{E}(\tau_k)$. Then, for all $k \in \mathbb{N}$, $|x_i(\tau_k) - x_j(\tau_k)| \leq R\rho^{\tau_k}$. Since $\rho \in (0,1)$ and $\lim_{k\to\infty} \tau_k = +\infty$ and one gets $\lim_{k\to\infty} x_i(\tau_k) = \lim_{k\to\infty} x_j(\tau_k)$. On the other hand, the sequences $x_i(t)$ and $x_j(t)$ are convergent, which ensures that

$$x_i^* = \lim_{t\to\infty} x_i(t) = \lim_{k\to\infty} x_i(\tau_k) = \lim_{k\to\infty} x_j(\tau_k) = \lim_{t\to\infty} x_j(t) = x_j^*$$

The result in the proposition then follows from the transitivity of equality and the definition of asymptotic connectivity.

Remark 6 *The notion of asymptotic connectivity has already been considered in several works (including [35, 8, 48]) for proving consensus in multi-agent systems. Actually, the previous proposition could be proved using Theorem 3 in [48]. However, for the sake of self-containment, we preferred to provide a simpler proof of the result that uses the specificities of our model.*

The converse result of Proposition 2 is much more challenging : it is clear that it cannot hold for all initial conditions. Indeed, if all the initial opinions x_i^0 are identical, then it is clear that the agents asymptotically agree even though some of them may be asymptotically disconnected which would be the case if the graph $\mathcal{G}$ is not connected. Therefore, we shall prove that the converse result holds for almost all initial conditions. In this paragraph, we will need additional assumptions in order to be able to prove this result. The first one is the following :

Assumption 7 (Invertibility and graph to matrix mapping) *The sequence of matrices $P(t)$ satisfy the following conditions :*

(a) For all $t \in \mathbb{N}$, $P(t)$ is invertible.

(b) For all $t \in \mathbb{N}$, $P(t) = P(\mathcal{G}(t))$ where $P(\mathcal{G}')$ is the matrix associated to a graph $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$.

The first assumption is quite strong and we notice that it is not verified by the original Krause model. However, it can be enforced, for instance, by choosing $p_{ii}(t) > 1/2$ for all $i \in \mathcal{V}$, for all $t \in \mathbb{N}$, in that case $P(t)$ is a strictly diagonally dominant matrix and therefore it is invertible. The second assumption states that $P(t)$ only depends on the graph $\mathcal{G}(t)$. From the first assumption, $P(\mathcal{G}')$ must be invertible. Then, we can define for all $t \in \mathbb{N}$, the following set of matrices :

$$\mathcal{Q}_t = \Big\{ P(\mathcal{G}_0)^{-1} P(\mathcal{G}_1)^{-1} \dots P(\mathcal{G}_{t-1})^{-1} | \; \mathcal{G}_k \in \mathcal{S}(\mathcal{G}), 0 \leq k \leq t-1 \Big\}. \tag{2.7}$$

Let us remark that since $\mathcal{S}(\mathcal{G})$ is finite, the set $\mathcal{Q}_t$ is finite : it has at most $2^{t \times |\mathcal{E}|/2}$ elements.

We shall now prove the converse result of Proposition 2 in two different cases.

2.2.1 Average preserving dynamics

We first assume that the opinion dynamics preserves the average of the opinions :

Assumption 8 (Average preserving dynamics) *For all $t \in \mathbb{N}$, for all $j \in \mathcal{V}$, $\sum_{i=1}^{n} p_{ij}(t) = 1$.*

This assumption simply means that the matrix $P(t)$ is doubly stochastic. It is therefore average preserving : the average of $x(t)$ is equal to the average of $x(t+1)$. Also, if $I \subseteq \mathcal{V}$ is a subset of agents such that no agent in I is connected to an agent in $\mathcal{V} \setminus I$ in the graph $\mathcal{G}(t)$, it is easy to show that $P_I(\mathcal{G}(t))$ is average preserving.

We now state the main result of the section :

Theorem 6 *If the matrices $P(t)$ satisfy Assumptions 6 (Stochasticity), 7 (Invertibility and graph to matrix mapping) and 8 (Average preserving dynamics), for almost all vectors of initial opinions x^0, two agents $i, j \in \mathcal{V}$ asymptotically agree if and only if they are asymptotically connected.*

Proof : The if part of the theorem is a consequence of Proposition 2. To prove the only if part, let us define the following set

$$\mathcal{W} = \Big\{ (I, J) | \; (I \subseteq \mathcal{V}) \wedge (I \neq \emptyset) \wedge (J \subseteq \mathcal{V}) \wedge (J \neq \emptyset) \wedge (I \cap J = \emptyset) \Big\}.$$

Since $\mathcal{V}$ is a finite set, it is clear that $\mathcal{W}$ is finite (it has less than 2^{2n} elements). For all $(I, J) \in \mathcal{W}$, let $|I|$ and $|J|$ denote the number of elements of I and J respectively. We define the vector of $\mathbb{R}^n$, c_{IJ} whose coordinates $c_{IJ,k} = 1/|I|$ if $k \in I$, $c_{IJ,k} = -1/|J|$ if $k \in J$, and $c_{IJ,k} = 0$ otherwise. We define the $(n-1)$-dimensional subspace of $\mathbb{R}^n$:

$$H_{IJ} = \left\{ x \in \mathbb{R}^n |\; c_{IJ} \cdot x = \sum_{i \in I} x_i/|I| - \sum_{j \in J} x_j/|J| = 0 \right\}.$$

Finally, let us define the subset of $\mathbb{R}^n$:

$$X^0 = \bigcup_{t \in \mathbb{N}} \left(\bigcup_{(I,J) \in \mathcal{W}} \left(\bigcup_{Q \in \mathcal{Q}_t} QH_{IJ} \right) \right)$$

where $\mathcal{Q}_t$ is the set of matrices defined in (2.7). Since $\mathcal{W}$ is a finite set and for all $t \in \mathbb{N}$, $\mathcal{Q}_t$ are finite sets, X^0 is a countable union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$. Therefore X^0 has Lebesgue measure 0.

Let $x^0 \in \mathbb{R}^n$ be a vector of initial opinions, let us assume that there exist two agents $i, j \in \mathcal{V}$ that asymptotically agree but are asymptotically disconnected. Let us show that necessarily, x^0 belongs to the set X^0. Let I and J denote the connected components of $\mathcal{G}^\infty$ containing i and j respectively. Since i and j are asymptotically disconnected, $I \cap J = \emptyset$, therefore $(I, J) \in \mathcal{W}$. Let T be defined as in equation (2.6) (i.e. $\mathcal{E}(t) \subseteq \mathcal{E}^\infty$, $\forall t \geq T$), since no agent in I is connected to an agent outside of I in $\mathcal{G}^\infty$ (and hence in $\mathcal{G}(t)$ for $t \geq T$), we have that for all $t \geq T$, $x_I(t+1) = P_I(\mathcal{G}(t))x_I(t)$. Moreover, $P_I(\mathcal{G}(t))$ is average preserving. Therefore, for all $t \geq T$, the average of $x_I(t)$ is the same as the average of $x_I(T)$. From Proposition 2, all agents in I asymptotically agree, then the limit value is necessarily the average of $x_I(T)$. Therefore $x_i^* = (\mathbf{1}_{|I|} \cdot x_I(T))/|I|$ where $\mathbf{1}_{|I|}$ denote the $|I|$-dimensional vector with all entries equal to 1. A similar discussion gives that $x_j^* = (\mathbf{1}_{|J|} \cdot x_J(T))/|J|$. Since i and j asymptotically agree, we have $(\mathbf{1}_{|I|} \cdot x_I(T))/|I| = (\mathbf{1}_{|J|} \cdot x_J(T))/|J|$. This means that $x(T) \in H_{IJ}$ and therefore

$$x^0 = P(\mathcal{G}(0))^{-1} P(\mathcal{G}(1))^{-1} \ldots P(\mathcal{G}(T-1))^{-1} x(T) \in \bigcup_{Q \in \mathcal{Q}_T} QH_{IJ}$$

which leads to $x^0 \in X^0$.

Hence, in the case of average preserving dynamics, asymptotic connectivity is equivalent to asymptotic agreement for almost all vectors of initial opinions. We shall now prove a similar result under different assumptions.

2.2.2 Fast convergence assumption

We now replace the average preserving assumption by another assumption. From Proposition 1, we know that the opinion of each agent converges to its limit value no slower than $O(\rho^t)$. This is an upper bound, numerical experiments show that in practice the convergence to the limit value is often slightly faster than $O(\rho^t)$. This observation motivates the following assumption.

Assumption 9 (Fast convergence) *There exists $\underline{\rho} < \rho$ and $M \geq 0$ such that for all $i \in \mathcal{V}$, for all $t \in \mathbb{N}$,*

$$|x_i(t) - x_i^*| \leq M \underline{\rho}^t.$$

Remark 7 *The previous assumption always holds unless there exists $i \in \mathcal{V}$ such that*

$$\limsup_{t \to +\infty} \frac{1}{t} \log(|x_i(t) - x_i^*|) = \log(\rho).$$

It should be noted that unlike Assumptions 6 (Stochasticity), 7 (Invertibility and graph to matrix mapping) and 8 (Average preserving dynamics), it is generally not possible to check a priori whether Assumption 9 holds. However, numerical experiments tend to show that in practice, it does.

The previous assumption allows us to state the following result :

Lemma 1 *Under Assumptions 6 (Stochasticity) and 9 (Fast convergence), there exists $T' \in \mathbb{N}$ such that for all $t \geq T'$, $\mathcal{G}(t) = \mathcal{G}^\infty$. Moreover, $\mathcal{G}^\infty = \mathcal{G}_{\mathscr{C}}$.*

Proof : We shall prove the lemma by showing that there exists $T' \in \mathbb{N}$ such that for all $t \geq T'$, $\mathcal{E}(t) \subseteq \mathcal{E}^\infty \subseteq \mathcal{E}_{\mathscr{C}} \subseteq \mathcal{E}(t)$. Firstly, let $T_1 \geq T$ where T is defined as in equation (2.6), then for all $t \geq T_1$, $\mathcal{E}(t) \subseteq \mathcal{E}^\infty$. Secondly, let $(i,j) \in \mathcal{E}^\infty$, then agents i and j are asymptotically connected. From Proposition 2, it follows that i and j asymptotically agree. Therefore, $(i,j) \in \mathcal{E}_{\mathscr{C}}$. Thirdly, let $(i,j) \in \mathcal{E}_{\mathscr{C}}$, then $x_i^* = x_j^*$ and for all $t \in \mathbb{N}$

$$|x_i(t) - x_j(t)| \leq \quad |x_i(t) - x_i^*| + |x_i^* - x_j^*| + |x_j(t) - x_j^*| \leq \quad |x_i(t) - x_i^*| + |x_j(t) - x_j^*|$$

From Assumption 9, we have for all $t \in \mathbb{N}$,

$$|x_i(t) - x_j(t)| \leq 2M\underline{\rho}^t.$$

Since $\underline{\rho} < \rho$, there exists $T_2 \in \mathbb{N}$, such that for all $t \geq T_2$, $2M\underline{\rho}^t \leq R\rho^t$. Then, for all $t \geq T_2$, $(i,j) \in \mathcal{E}(t)$. Let $T' = \max(T_1, T_2)$, then for all $t \geq T'$, $\mathcal{E}(t) = \mathcal{E}^\infty = \mathcal{E}_{\mathscr{C}}$ and thus $\mathcal{G}(t) = \mathcal{G}^\infty = \mathcal{G}_{\mathscr{C}}$.

The previous result states that after a finite number of steps, the graph of interactions between agents remains always the same. Then, we can state a result similar to Theorem 6 :

Theorem 7 *Under Assumptions 6 (Stochasticity), 7 (Invertibility and graph to matrix mapping) and 9 (Fast convergence), for almost all vectors of initial opinions x^0, two agents $i, j \in \mathcal{V}$ asymptotically agree if and only if they are asymptotically connected.*

Proof : The if part of the theorem is a consequence of Proposition 2. To prove the only if part, let us define the following set associated to a spanning subgraph $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$:

$$\mathcal{W}(\mathcal{G}') = \Big\{(I,J)|\ (I \subseteq \mathcal{V}) \wedge (J \subseteq \mathcal{V}) \wedge (I \neq J) \wedge (I \in \mathcal{K}(\mathcal{G}')) \wedge (J \in \mathcal{K}(\mathcal{G}'))\Big\}.$$

Since $\mathcal{V}$ is a finite set, it is clear that $\mathcal{W}(\mathcal{G}')$ is finite (it has less than 2^{2n} elements). Let $(I,J) \in \mathcal{W}(\mathcal{G}')$, $I = \{v_1, \ldots, v_{|I|}\}$, $J = \{w_1, \ldots, w_{|J|}\}$. Since I and J are connected components of $\mathcal{G}'$, we have that $P_I(\mathcal{G}')$ and $P_J(\mathcal{G}')$ are aperiodic irreducible row stochastic matrices. Let $e_I(\mathcal{G}')$ and $e_J(\mathcal{G}')$ be the left Perron eigenvectors of $P_I(\mathcal{G}')$ and $P_J(\mathcal{G}')$, respectively :

$$e_I(\mathcal{G}')^\top P_I(\mathcal{G}') = e_I(\mathcal{G}')^\top \text{ and } e_I(\mathcal{G}') \cdot \mathbf{1}_{|I|} = 1$$

and

$$e_J(\mathcal{G}')^\top P_J(\mathcal{G}') = e_J(\mathcal{G}')^\top \text{ and } e_J(\mathcal{G}') \cdot \mathbf{1}_{|J|} = 1.$$

We define the vector of $\mathbb{R}^n$, c_{IJ} whose coordinates are given by $c_{IJ,v_k} = e_{I,k}$ if $v_k \in I$, $c_{IJ,w_k} = -e_{J,k}$ if $w_k \in J$ and $c_{IJ,k} = 0$ if $k \in \mathcal{V} \setminus (I \cup J)$. We define the $(n-1)$-dimensional subspace of $\mathbb{R}^n$:

$$H_{IJ}(\mathcal{G}') = \left\{x \in \mathbb{R}^n |\ c_{IJ}(\mathcal{G}') \cdot x = 0\right\}.$$

Finally, let us define the subset of $\mathbb{R}^n$:

$$X^0 = \bigcup_{t\in\mathbb{N}} \left(\bigcup_{\mathcal{G}'\in\mathcal{S}(\mathcal{G})} \left(\bigcup_{(I,J)\in\mathcal{W}(\mathcal{G}')} \left(\bigcup_{Q\in\mathcal{Q}_t} QH_{IJ}(\mathcal{G}') \right)\right)\right) \tag{2.8}$$

where $\mathcal{Q}_t$ is the set of matrices defined in (2.7). $\mathcal{S}(\mathcal{G})$ is a finite set and for all $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$, $\mathcal{W}(\mathcal{G}')$ is a finite set. Moreover for all $t \in \mathbb{N}$, $\mathcal{Q}_t$ is a finite set. Then, X^0 is a countable union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$. Therefore X^0 has Lebesgue measure 0.

Let $x^0 \in \mathbb{R}^n$ be a vector of initial opinions, let us assume that there exist two agents $i, j \in \mathcal{V}$ that asymptotically agree but are asymptotically disconnected. Let us show that necessarily, x^0 belongs to the set X^0. Let I and J denote the connected components of $\mathcal{G}^\infty$ containing i and j respectively. Since i and j are asymptotically disconnected, $I \neq J$, therefore $(I, J) \in \mathcal{W}(\mathcal{G}^\infty)$. Since I is a connected component of $\mathcal{G}^\infty$, it follows from Lemma 1 that for all $t \geq T'$, $x_I(t+1) = P_I(\mathcal{G}^\infty)x_I(t)$. Moreover, $P_I(\mathcal{G}^\infty)$ is an aperiodic irreducible row stochastic matrix and from the Perron-Frobenius Theorem (see e.g. [63]), it follows that 1 is a simple eigenvalue of $P_I(\mathcal{G}^\infty)$ and all other eigenvalues of $P_I(\mathcal{G}^\infty)$ have modulus strictly smaller than 1. Therefore,

$$\lim_{t\to+\infty} x_I(t) = (e_I(\mathcal{G}^\infty) \cdot x_I(T'))\mathbf{1}_{|I|}$$

and $x_i^* = e_I(\mathcal{G}^\infty) \cdot x_I(T')$. A similar discussion gives that $x_j^* = e_J(\mathcal{G}^\infty) \cdot x_J(T')$. Since i and j asymptotically agree, we have $e_I(\mathcal{G}^\infty) \cdot x_I(T') = e_J(\mathcal{G}^\infty) \cdot x_J(T')$. This means that $x(T') \in H_{I,J}(\mathcal{G}^\infty)$ and therefore

$$x^0 = P(\mathcal{G}(0))^{-1}P(\mathcal{G}(1))^{-1}\dots P(\mathcal{G}(T'-1))^{-1}x(T') \in \bigcup_{Q\in\mathcal{Q}_{T'}} QH_{IJ}(\mathcal{G}^\infty)$$

which leads to $x^0 \in X^0$.

In the following, under Assumptions 6 (Stochasticity), 7 (Invertibility and graph to matrix mapping) and 9 (Fast convergence), we show that an algebraic characterization of communities can be given in terms of eigenvalues of the matrix associated to the graph of communities $P(\mathcal{G}_{\mathscr{C}})$.

2.3 Algebraic Characterization of Communities

Let $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$, let $I \subseteq \mathcal{V}$ be a subset of agents such that no agent in I is connected to an agent in $V \setminus I$ in the graph $\mathcal{G}'$, then $P_I(\mathcal{G}')$ is a row stochastic matrix. Let $\lambda_1(P_I(\mathcal{G}')), \dots, \lambda_{|I|}(P_I(\mathcal{G}'))$ denote the eigenvalues of $P_I(\mathcal{G}')$ with $\lambda_1(P_I(\mathcal{G}')) = 1$ and

$$|\lambda_1(P_I(\mathcal{G}'))| \geq |\lambda_2(P_I(\mathcal{G}'))| \geq \dots \geq |\lambda_{|C|}(P_I(\mathcal{G}'))|.$$

Let $C \in \mathscr{C}$, then no agent in C is connected to an agent in $\mathcal{V} \setminus C$ in the graph $\mathcal{G}_{\mathscr{C}}$. The following theorem gives a characterization of the communities in terms of the eigenvalues $\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))$ for $C \in \mathscr{C}$.

Theorem 8 *Under Assumptions 6 (Stochasticity), 7 (Invertibility and graph to matrix mapping) and 9 (Fast convergence), for almost all vectors of initial opinions x^0, for all communities $C \in \mathscr{C}$, such that $|C| \geq 2$,*

$$|\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))| < \rho.$$

Proof : Let us consider a spanning subgraph $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$, let $I = \{v_1, \ldots, v_{|I|}\}$, with $|I| \geq 2$, be a connected component of $\mathcal{G}'$ then $P_I(\mathcal{G}')$ is an aperiodic irreducible row stochastic matrix. Then, from the Perron-Frobenius Theorem, it follows that 1 is a simple eigenvalue of $P_I(\mathcal{G}')$. Therefore, $\lambda_2(P_I(\mathcal{G}')) \neq 1$. Let $f_I(\mathcal{G}')$ be a left eigenvector of $P_I(\mathcal{G}')$ associated to eigenvalue $\lambda_2(P_I(\mathcal{G}'))$. Let us define the vector of $\mathbb{R}^n$, $c_I(\mathcal{G}')$ whose coordinates are given by $c_{I,v_k}(\mathcal{G}') = f_{I,k}(\mathcal{G}')$ if $v_k \in I$ and $c_{I,k}(\mathcal{G}') = 0$ if $k \in \mathcal{V} \setminus I$. We define the $(n-1)$-dimensional subspace of $\mathbb{R}^n$:

$$H_I(\mathcal{G}') = \left\{x \in \mathbb{R}^n | \; c_I(\mathcal{G}') \cdot x = 0\right\}.$$

Finally, let us define the subset of $\mathbb{R}^n$:

$$Y^0 = \bigcup_{t\in\mathbb{N}} \left(\bigcup_{\mathcal{G}'\in\mathcal{S}(\mathcal{G})} \left(\bigcup_{I\in\mathcal{K}(\mathcal{G}'),\, |I|\geq 2} \left(\bigcup_{Q\in\mathcal{Q}_t} QH_I(\mathcal{G}') \right)\right)\right)$$

where $\mathcal{Q}_t$ is the set of matrices defined in (2.7). $\mathcal{S}(\mathcal{G})$ is a finite set and for all $\mathcal{G}' \in \mathcal{S}(\mathcal{G})$, $\mathcal{K}(\mathcal{G}')$ is a finite set. Moreover, for all $t \in \mathbb{N}$, $\mathcal{Q}_t$ is a finite set. Then, Y^0 is a countable union of $(n-1)$-dimensional subspaces of $\mathbb{R}^n$. Therefore Y^0 has Lebesgue measure 0.

Let X^0 be given as in equation (2.8), let $x^0 \in \mathbb{R}^n \setminus X^0$ be a vector of initial opinions. Let us assume there is a community $C \in \mathscr{C}$ with $|C| \geq 2$, such that $|\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))| \geq \rho$. Let us show that necessarily, x^0 belongs to the set Y^0. First, since $x^0 \notin X^0$, we have from the proof of Theorem 7 that C is a connected component of $\mathcal{G}^\infty = G_{\mathscr{C}}$. Therefore, from Lemma 1, there exists $T' \in \mathbb{N}$, such that for all $t \geq T'$, $x_C(t+1) = P_C(\mathcal{G}_{\mathscr{C}})x_C(t)$ and $P_C(\mathcal{G}_{\mathscr{C}})$ is an aperiodic irreducible row stochastic matrix. From the Perron-Frobenius Theorem, it follows that 1 is a simple eigenvalue of $P_C(\mathcal{G}_{\mathscr{C}})$ and all other eigenvalues of $P_C(\mathcal{G}_{\mathscr{C}})$ have modulus strictly smaller than 1. Let $e_C(\mathcal{G}_{\mathscr{C}})$ be the left Perron eigenvector of $P_C(\mathcal{G}_{\mathscr{C}})$:

$$e_C(\mathcal{G}_{\mathscr{C}})^\top P_C(\mathcal{G}_{\mathscr{C}}) = e_C(\mathcal{G}_{\mathscr{C}})^\top \text{ and } e_C(\mathcal{G}_{\mathscr{C}}) \cdot \mathbf{1}_{|C|} = 1$$

Then

$$\lim_{t\to+\infty} x_C(t) = x_C^* \quad \text{where} \quad x_C^* = (e_C(\mathcal{G}_{\mathscr{C}}) \cdot x_C(T'))\mathbf{1}_{|C|}.$$

Let us remark that for all $t \geq T'$,

$$x_C(t+1) - x_C^* = P_C(\mathcal{G}_{\mathscr{C}})(x_C(t) - x_C^*). \tag{2.9}$$

Let $f_C(\mathcal{G}_{\mathscr{C}})$ be a left eigenvector of $P_C(\mathcal{G}_{\mathscr{C}})$ associated to eigenvalue $\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))$:

$$f_C(\mathcal{G}_{\mathscr{C}})^\top P_C(\mathcal{G}_{\mathscr{C}}) = \lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))f_C(\mathcal{G}_{\mathscr{C}})^\top.$$

Then, it follows from equation (2.9) that for all $t \geq T'$,

$$f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(t) - x_C^*) = f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(T') - x_C^*)\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))^{(t-T')}.$$

Therefore, by the Cauchy-Schwarz inequality, we have for all $t \geq T'$

$$\|x_C(t) - x_C^*\| \geq \frac{|f_C(\mathcal{G}_{\mathscr{C}})\cdot(x_C(t)-x_C^*)|}{\|f_C(\mathcal{G}_{\mathscr{C}})\|} \geq \frac{|f_C(\mathcal{G}_{\mathscr{C}})\cdot(x_C(T')-x_C^*)|}{\|f_C(\mathcal{G}_{\mathscr{C}})\|}|\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))|^{(t-T')}.$$

Since we assumed $|\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))| \geq \rho$, we have for all $t \geq T'$

$$\|x_C(t) - x_C^*\| \geq \frac{|f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(T') - x_C^*)|}{\|f_C(\mathcal{G}_{\mathscr{C}})\|\rho^{T'}}\rho^t. \tag{2.10}$$

Now, let us remark that it follows from Assumption 9 that for all $t \in \mathbb{N}$

$$\|x_C(t) - x_C^*\| \le \sqrt{|C|} M \underline{\rho}^t. \tag{2.11}$$

Inequalities (2.10) and (2.11) give for all $t \geq T'$

$$\frac{|f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(T') - x_C^*)|}{\|f_C(\mathcal{G}_{\mathscr{C}})\| \rho^{T'}} \rho^t \le \sqrt{|C|} M \underline{\rho}^t.$$

Since $\underline{\rho} < \rho$, the previous inequality holds for all $t \geq T'$ if and only if $|f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(T') - x_C^*)| = 0$. Therefore, $f_C(\mathcal{G}_{\mathscr{C}}) \cdot x_C(T') = f_C(\mathcal{G}_{\mathscr{C}}) \cdot (x_C(T') - x_C^*) = 0$ which means that $x(T') \in H_C(\mathcal{G}_{\mathscr{C}})$. Therefore,

$$x^0 = P(\mathcal{G}(0))^{-1} P(\mathcal{G}(1))^{-1} \dots P(\mathcal{G}(T'-1))^{-1} x(T') \in \bigcup_{Q \in \mathcal{Q}_{T'}} Q H_C(\mathcal{G}_{\mathscr{C}})$$

which leads to $x^0 \in Y^0$. Therefore, we have proved that for all vectors of initial opinions $x^0 \in \mathbb{R}^n \setminus (X^0 \cup Y^0)$, for all communities $C \in \mathscr{C}$ such that $|C| \geq 2$, $|\lambda_2(P_C(\mathcal{G}_{\mathscr{C}}))| < \rho$. We conclude by remarking that $X^0 \cup Y^0$ is a set of Lebesgue measure 0.

In this section, we showed that the community structure $\mathscr{C}$ satisfies some properties related to the eigenvalues of the matrix $P_C(\mathcal{G}_{\mathscr{C}})$, for $C \in \mathscr{C}$. In the following, we use this result to address the problem of community detection in graphs.

2.4 Application : Community Detection in Graphs

In this section, we propose to use a model of opinion dynamics with decaying confidence to address the problem of community detection in graphs.

2.4.1 The Community Detection Problem

In the usual sense, communities in a graph are groups of vertices such that the concentration of edges inside one community is high and the concentration of edges between communities is comparatively low. Because of the increasing need of analysis tools for understanding complex networks in social sciences, biology, engineering or economics, the community detection problem has attracted a lot of attention in the recent years. The problem of community detection is however not rigorously defined mathematically. One reason is that community structures may appear at different scales in the graph : there can be communities inside communities. Another reason is that communities are not necessarily disjoint and can overlap. We refer the reader to the excellent survey [25] and the references therein for more details. Some formalizations of the community detection problem have been proposed in terms of optimization of quality functions such as modularity [54] or partition stability [39].

2.4.2 Quality functions

Modularity has been introduced in [54], the modularity of a partition measures how well the partition reflects the community structure of a graph. More precisely, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with $\mathcal{E}$ symmetric (i.e. $(i,j) \in \mathcal{E} \Leftrightarrow (j,i) \in \mathcal{E}$) and anti-reflexive (i.e. $(i,i) \notin \mathcal{E}$). For a vertex $i \in V$ the degree d_i of i is the number of neighbors of i in $\mathcal{G}$. Let $\mathcal{P}$ be a partition of $\mathcal{V}$. Essentially, the modularity $\mathrm{Q}(\mathcal{P})$ of the partition $\mathcal{P}$ is the proportion of edges within the classes of the partition minus the expected proportion of such edges, where the expected number of edges between vertex i and j is assumed to be $d_i d_j/|\mathcal{E}|$:

$$\mathsf{Q}(\mathcal{P}) = \frac{1}{|\mathcal{E}|}\sum_{I\in\mathcal{P}}\sum_{i,j\in I}\left(a_{ij} - \frac{d_i d_j}{|\mathcal{E}|}\right)$$

where a_{ij} are the coefficients of the adjacency matrix of $\mathcal{G}$ ($a_{ij} = 1$ if $(i,j) \in \mathcal{E}$, $a_{ij} = 0$ otherwise). The higher the modularity, the better the partition reflects the community structure of the graph. Thus, it is reasonable to formulate the community detection problem as modularity maximization. However, it has been shown that this optimization problem is NP-complete [13]. Therefore, approaches for community detection rely mostly on heuristic methods. In [53], a modularity optimization algorithm is proposed based on spectral relaxations. Using the eigenvectors of the modularity matrix, it is possible to determine a good initial guess of the community structure of the graph. Then, the obtained partition is refined using local combinatorial optimization. In [7], a hierarchical combinatorial approach for modularity optimization is presented. This algorithm which can be used for very large networks, is currently the one that obtains the partitions with highest modularity.

However, modularity has the drawback that it fails to capture communities at different scales. The notion of partition stability [39] makes it possible to overcome this limitation. Let us consider a continuous-time process associated with a random walk over the graph $\mathcal{G}$ where transitions are triggered by a homogeneous Poisson process. Assume that the initial distribution is the stationary distribution. Then, the stability at time $t \in \mathbb{R}^+$ of the partition $\mathcal{P}$ is defined as

$$\mathsf{R}(\mathcal{P}, t) = \sum_{I\in\mathcal{P}} p(I,t) - p(I,\infty)$$

where $p(I,t)$ is the probability for a walker to be in the class I initially and at time t. Stability measures the quality of a partition by giving a positive contribution to communities from which a random walker is unlikely to escape within the given time scale t. For small values of t, this gives more weights to small communities whereas for larger values of t, larger communities are favored. Thus, by searching the partitions maximizing the stability for several values of t, one can detect communities at several scales.

2.4.3 Eigenvalues of the normalized Laplacian matrix

We give an alternative formulation of the community detection problem using a measure of connectivity of graphs given by the eigenvalues of their *normalized Laplacian matrix*. Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with $\mathcal{V} = \{1, \dots, n\}$, with $n \geq 2$. For a vertex $i \in \mathcal{V}$, the degree $d_i(\mathcal{G})$ of i is the number of neighbors of i in $\mathcal{G}$. The normalized Laplacian of the graph $\mathcal{G}$ is the matrix $L(\mathcal{G})$ given by

$$L_{ij}(\mathcal{G}) = \begin{cases} 1 & \text{if } i = j \text{ and } d_i(\mathcal{G}) \neq 0, \\ \frac{-1}{\sqrt{d_i(\mathcal{G}) d_j(\mathcal{G})}} & \text{if } (i,j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Let us review some of the properties of the normalized Laplacian matrix (see e.g. [18]). $\mu_1(L(\mathcal{G})) = 0$ is always an eigenvalue of $L(\mathcal{G})$, it is simple if and only if $\mathcal{G}$ is connected. All other eigenvalues are real and belong to the interval $[0, 2]$. The second smallest eigenvalue of the normalized Laplacian matrix is denoted $\mu_2(L(\mathcal{G}))$. It can serve as an algebraic measure of the connectivity : $\mu_2(L(\mathcal{G})) = 0$ if the graph $\mathcal{G}$ has two distinct connected components, $\mu_2(L(\mathcal{G})) = n/(n-1)$ if the graph is the complete graph (for all $i, j \in \mathcal{V}$, $i \neq j$, $(i,j) \in \mathcal{E}$), in the other cases $\mu_2(L(\mathcal{G})) \in (0, 1]$.

Remark 8 *The second smallest eigenvalue of the (non-normalized) Laplacian matrix is called algebraic connectivity of a graph. In this paper, we prefer to use the eigenvalues of the normalized Laplacian matrix because it is less sensitive to the size of the graph. For instance, if $\mathcal{G}$ is the complete graph then $\mu_2(L(\mathcal{G})) = n/(n-1)$ whereas its algebraic connectivity is n.*

Let $\mathcal{P}$ be a partition of the set of vertices V. For all $I \in \mathcal{P}$, with $|I| \geq 2$, $L(\mathcal{G}_I)$ denotes the normalized Laplacian matrix of the graph $\mathcal{G}_I = (I, \mathcal{E}_I)$ consisting of the set of vertices I and of the set of edges of $\mathcal{G}$ between elements of I. Let us define the following measure associated to the partition $\mathcal{P}$

$$\underline{\mu_2}(\mathcal{P}) = \min_{I \in \mathcal{P}, |I| \geq 2} \mu_2(L(\mathcal{G}_I)).$$

Essentially, $\underline{\mu_2}(\mathcal{P})$ measures the connectivity of the less connected component of $\mathcal{G}_{\mathcal{P}}$.

We now propose a new formulation of the community detection problem :

Problem 1 *Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a real number $\delta \in (0,1]$, find a partition $\mathcal{P}$ of $\mathcal{V}$ such that for all $I \in \mathcal{P}$, such that $|I| \geq 2$, $\mu_2(L(\mathcal{G}_I)) > \delta$ (i.e. $\underline{\mu_2}(\mathcal{P}) > \delta$).*

If $\mu_2(L(\mathcal{G})) > \delta$, it is sufficient to choose the trivial partition $\mathcal{P} = \{\mathcal{V}\}$. If $\delta \geq \mu_2(L(\mathcal{G}))$, then we want to find groups of vertices that are more densely connected than the global graph. This coincides with the notion of community. The larger δ the more densely connected the communities. This makes it possible to search for communities at different scales of the graph. Let us remark that Problem 1 generally has several solutions. Actually, the trivial partition $\mathcal{P} = \{\{1\}, \ldots, \{n\}\}$ is always a solution. In the following, we show how non-trivial solutions to Problem 1 can be obtained using a model of opinion dynamics with decaying confidence. We evaluate the modularity of the partitions we obtain and compare our results to those obtained using modularity optimization algorithms presented in [53, 7].

2.4.4 Opinion Dynamics for Community Detection

Let $\alpha \in (0, 1/2)$, we consider the opinion dynamics with decaying confidence model given by :

$$x_i(t+1) = \begin{cases} x_i(t) + \dfrac{\alpha}{|N_i(t)|} \displaystyle\sum_{j \in N_i(t)} (x_j(t) - x_i(t)), & \text{if } N_i(t) \neq \emptyset \\ x_i(t) & \text{if } N_i(t) = \emptyset \end{cases} \tag{2.12}$$

where $N_i(t)$ is given by equation (2.3). It is straightforward to check that this model is a particular case of the model given by equations (2.1) and (2.2) and that Assumption 6 (Stochasticity) holds. Moreover, since $\alpha \in (0, 1/2)$ it follows that for all $i \in V$, $t \in \mathbb{N}$, $p_{ii}(t) > 1/2$. Therefore the matrix $P(t)$ is strictly diagonally dominant and hence it is invertible. Also, $P(t) = P(\mathcal{G}(t))$, where for a subgraph $\mathcal{G}'$, $P(\mathcal{G}') = Id - \alpha Q(\mathcal{G}')$ where Id is the identity matrix and

$$Q_{ij}(\mathcal{G}') = \begin{cases} 1 & \text{if } i = j \text{ and } d_i(\mathcal{G}') \neq 0, \\ \frac{-1}{d_i(\mathcal{G}')} & \text{if } (i,j) \in E', \\ 0 & \text{otherwise.} \end{cases} \tag{2.13}$$

where $d_i(\mathcal{G}')$ denotes the degree of i in the graph $\mathcal{G}'$. Therefore, Assumption 7 (Invertibility and graph to matrix mapping) holds as well. Let us remark that the matrix $P(t)$ is generally not average preserving and therefore Assumption 8 does not hold.

Before stating the main result of this section, we need to prove the following lemma :

Lemma 2 *Let $\mathcal{P}$ be a partition of $\mathcal{V}$, $I \in \mathcal{P}$ such that $|I| \geq 2$. Then, λ is an eigenvalue of $P_I(\mathcal{G}_{\mathcal{P}})$ if and only if $\mu = (1-\lambda)/\alpha$ is an eigenvalue of $L(\mathcal{G}_I)$.*

Proof : First, let us remark that $P_I(\mathcal{G}_\mathcal{P}) = Id - \alpha Q(\mathcal{G}_I)$ where $Q(\mathcal{G}_I)$ is defined as in equation (2.13). Then, let us introduce the matrices $R(\mathcal{G}_I)$ and $D(\mathcal{G}_I)$ defined by

$$R_{ij}(\mathcal{G}_I) = \begin{cases} \frac{1}{\sqrt{d_i(\mathcal{G}_I)}} & \text{if } i = j \text{ and } d_i(\mathcal{G}_I) \neq 0, \\ \frac{-1}{d_i(\mathcal{G}_I)\sqrt{d_j(\mathcal{G}_I)}} & \text{if } (i,j) \in E_I, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$D_{ij}(\mathcal{G}_I) = \begin{cases} \sqrt{d_i(\mathcal{G}_I)} & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Let us remark that $L(\mathcal{G}_I) = D(\mathcal{G}_I)R(\mathcal{G}_I)$ and $Q(\mathcal{G}_I) = R(\mathcal{G}_I)D(\mathcal{G}_I)$. It follows that $L(\mathcal{G}_I)$ and $Q(\mathcal{G}_I)$ have the same eigenvalues. The stated result is then obtained from the fact that the matrix $Q(\mathcal{G}_I) = (Id - P_I(\mathcal{G}_\mathcal{P}))/\alpha$.

We now state the main result of the section which is a direct consequence of Theorem 8 and Lemma 2 :

Corollary 1 *Let $\rho = 1 - \alpha\delta$, under Assumption 9 (Fast convergence), for almost all vectors of initial opinions x^0, the set of communities $\mathscr{C}$ obtained by the opinion dynamics model (2.12) is a solution to Problem 1.*

2.5 Examples

In this section, we propose to evaluate experimentally the validity of our approach on three benchmarks taken from [53].

2.5.1 Zachary karate club

We propose to evaluate our approach on a standard benchmark for community detection : the karate club network initially studied by Zachary in [73]. This is a social network with 34 agents shown on the top left part of Figure 2.1. The original study shows the existence of two communities represented on the figure by squares and triangles.

We propose to use our opinion dynamics model (2.12) to uncover the community structure of this network. We chose 4 different values for δ and 2 different values for parameters R and α. The parameter ρ was chosen according to Corollary 1 : $\rho = 1 - \alpha\delta$. For each combination of parameter value, the model was simulated for 1000 different vectors of initial opinions chosen randomly in $[0,1]^{34}$. Simulations were performed as long as enabled by floating point arithmetics.

The experimental results are reported in Table 2.1. For each combination of parameter value, we indicate the partitions in communities that are the most frequently obtained after running the opinion dynamics model. For each partition $\mathscr{C}$, we give the number of communities in the partition, the measure $\underline{\mu}_2(\mathscr{C})$, this value being greater than δ indicates that Problem 1 has been solved. We computed the modularity $\mathsf{Q}(\mathscr{C})$ in order to evaluate the quality of the obtained partition. We also indicate the number of times that each partition occurred over the 1000 simulations of the opinion dynamics model.

We can check in Table 2.1 that all the partitions are solutions of Problem 1. Let us remark that in general the computed partition depends on the initial vector of opinions, this is the case for $\delta = 0.3$ and $\delta = 0.4$. Also, changing the parameters R and α seems to have some effect on the probability of obtaining a given partition. For instance, for $\delta = 0.3$, the probabilities of obtaining one partition are significantly different for $R = 1$ and $R = 10$. Also, for $\delta = 0.4$, the probabilities are slightly different for $\alpha = 0.1$ and $\alpha = 0.2$.

δ	$\|\mathscr{C}\|$	$\underline{\mu}_2(\mathscr{C})$	$Q(\mathscr{C})$	Occurences $R = 1, \alpha = 0.1$	Occurences $R = 10, \alpha = 0.1$	Occurences $R = 1, \alpha = 0.2$	Occurences $R = 10, \alpha = 0.2$
0.1	1	0.132	0	1000	1000	1000	1000
0.2	2	0.250	0.360	1000	999	1000	999
0.3	3	0.334	0.399	691	105	679	63
0.3	3	0.363	0.374	283	891	298	937
0.4	4	0.566	0.417	924	994	884	897
0.4	5	0.566	0.402	15	6	54	98

TABLE 2.1 – Properties of the partitions of the karate club network obtained by the opinion dynamics model (1000 different vectors of initial opinions for each combination of parameter values).

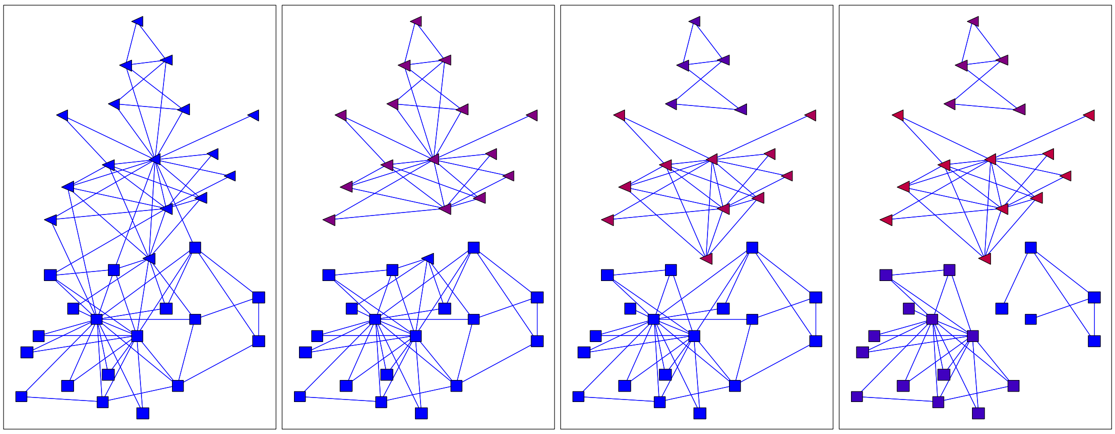

FIGURE 2.1 – Graphs $\mathcal{G}_C$ for the most frequently obtained partition of the karate club network for $\delta = 0.1$ (top left), $\delta = 0.2$ (top right), $\delta = 0.3$ (bottom left), $\delta = 0.4$ (bottom right).

However, it is interesting to note that the partitions that are obtained for the same value of parameter δ have modularities of the same order of magnitude which seems to show that these are of comparable quality. The partition with maximal modularity is obtained for $\delta = 0.4$, it is a partition in 4 communities with modularity 0.417. As a comparison, algorithms [53, 7] obtain a partition in 4 communities with modularity 0.419. This shows that our approach not only allows to solve Problem 1 but also furnishes partitions with a good modularity which might seem surprising given the fact that our approach, contrarily to [53, 7] does not try to maximize modularity.

In Figure 2.1, we represented the graphs of communities $\mathcal{G}_{\mathscr{C}}$ that are the most frequently obtained for $R = 1$, $\alpha = 0.1$ and the different values of δ. It is interesting to remark that for $\delta = 0.2$ we almost obtained the communities that were reported in the original study [73]. Only one agent has been classified differently. One may argue that this agent has originally 4 neighbors in each community so it could be classified in one or the other. It is also interesting to see that our approach allows us to search for communities at different scales of the graph. When δ increases, the communities become smaller but more densely connected. This is corroborated by computing the stability of these partitions (see Figure 2.2). We can see that the partition with maximal stability changes according to time-scale t : for small values of t the partition in 4 communities is better, for intermediate values of t the partition in 3 communities has the largest stability, for large values of t the partition in 2 communities maximizes the stability.

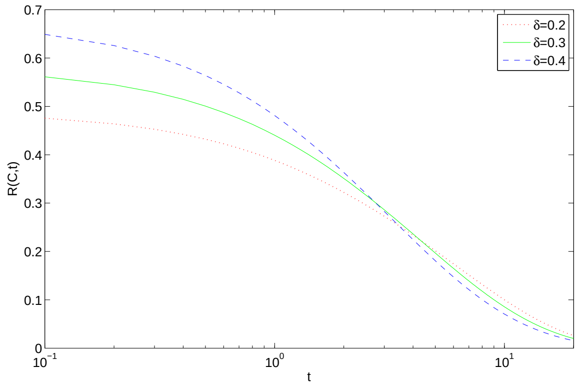

FIGURE 2.2 – Stability of the partitions presented in Figure 2.1.

2.5.2 Books on American Politics

We propose to use our approach on an example consisting of a network of 105 books on politics [53], initially compiled by V. Krebs (unpublished, see `www.orgnet.com`). In this network, each vertex represents a book on American politics bought from Amazon.com. An edge between two vertices means that these books are frequently purchased by the same buyer. The network is presented on the top left part of Figure 2.3 where the shape of the vertices represent the political alignment of the book (liberal, conservative, centrist).

We used our opinion dynamics model (2.12) to uncover the community structure of this network. We chose 3 different values for δ and 2 different values for parameters R and α. The parameter ρ was chosen according to Corollary 1 : $\rho = 1 - \alpha\delta$. For each combination of parameter value, the model was simulated for 1000 different vectors of initial opinions chosen randomly in $[0,1]^{105}$. Simulations were performed as long as enabled by floating point arithmetics. The experimental results are reported in Table 2.2.

δ	$\lvert\mathscr{C}\rvert$	$\underline{\mu_2}(\mathscr{C})$	$Q(\mathscr{C})$	Occurences $R=1, \alpha=0.1$	Occurences $R=10, \alpha=0.1$	Occurences $R=1, \alpha=0.2$	Occurences $R=10, \alpha=0.2$
0.1	2	0.134	0.457	980	1000	640	581
0.1	2	0.129	0.457	20	0	360	419
0.15	3	0.182	0.499	898	1000	905	1000
0.15	3	0.187	0.494	102	0	95	0
0.2	4	0.269	0.523	678	1000	673	1000
0.2	4	0.266	0.512	218	0	207	0
0.2	4	0.269	0.520	49	0	72	0

TABLE 2.2 – Properties of the partitions of the books network obtained by the opinion dynamics model (1000 different vectors of initial opinions for each combination of parameter values).

Let us remark that the computed partitions are solutions to the Problem 1. Also, for the same value of parameter δ, the modularity is very similar for all partitions. Actually, all the partitions obtained for the same value of δ are almost the same. As in the previous example, we can see that the choice of parameters R and α affects the probability of obtaining a given partition. The partition with maximal

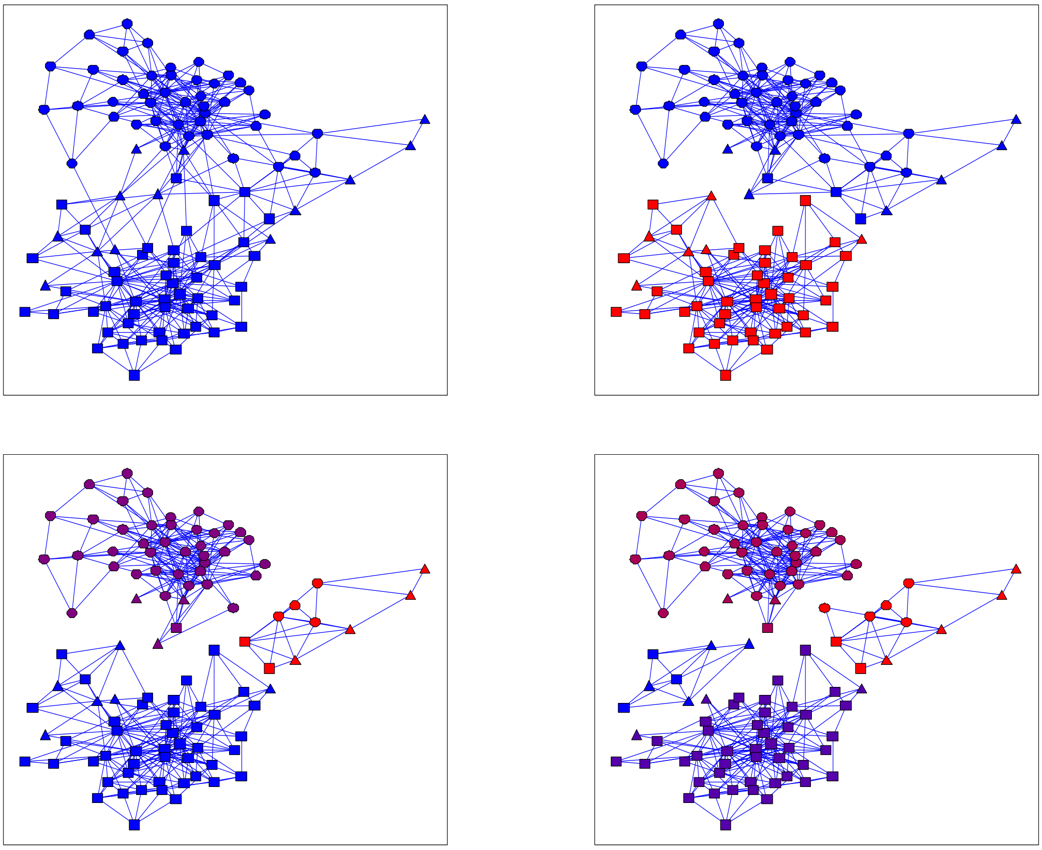

FIGURE 2.3 – Graphs $\mathcal{G}_{\mathcal{C}}$ for the most frequently obtained partition of the books network from top to bottom : initial graph, $\delta = 0.1$, $\delta = 0.15$, $\delta = 0.2$. Shapes represent political alignment of the books : circles are liberal, squares are conservative, triangles are centrist.

modularity is obtained for $\delta = 0.2$, it is a partition in 4 communities with modularity 0.523. As a comparison, algorithms [53] and [7] obtain partitions in 4 communities with modularity 0.526 and 0.527, respectively. As we can see, our partition has a modularity that is quite close from those obtained by these algorithms.

In Figure 2.3, we represented the graphs of communities $\mathcal{G}_{\mathscr{C}}$ that are the most frequently obtained for the different values of δ. Let us remark that even though the information on the political alignment of the books is not used by the algorithm, our approach allows to uncover this information. Indeed, for $\delta = 0.1$, we obtain 2 communities that are essentially liberal and conservative. For $\delta = 0.2$, we then obtain 4 communities : liberal, conservative, centrist-liberal, centrist-conservative.

In Figure 2.4, we represented the stability of the partitions shown in Figure 2.3. As in the previous example, we can see that the partition with maximal stability changes according to time-scale t which shows that our approach makes it possible to detect community at several scales using different values of parameter δ.

2.5.3 Political blogs

The last example we consider consists of a significantly larger network of 1222 political blogs [1]. In this network, an edge between two vertices means that one of the corresponding blogs contained a

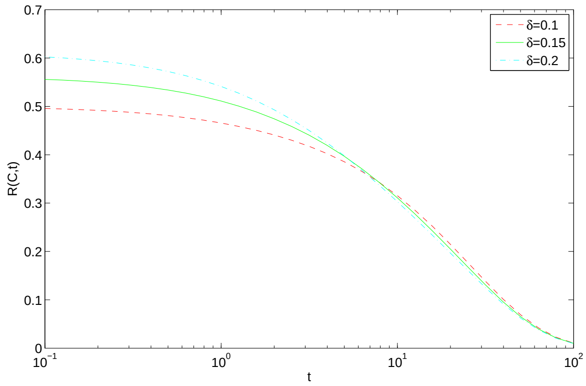

FIGURE 2.4 – Stability of the partitions presented in Figure 2.3.

hyperlink to the other on its front page. We also have the information about the political alignment of each blog based on content : 636 are conservative, 586 are liberal.

The two previous examples show that the modularity of the obtained partitions does not depend much on the parameters R and α or on the vector of initial opinions. For this reason, we decided to apply our opinion dynamics model with parameters $R = 1$ and $\alpha = 0.1$. We used 17 values of δ between 0.05 and 0.75. The parameter ρ was chosen according to Corollary 1 : $\rho = 1 - \alpha\delta$. For each value of δ, the model was simulated only once for a vector of initial opinion chosen randomly in $[0, 1]^{1222}$. Simulations were performed as long as enabled by floating point arithmetics.

The partition with maximal modularity was obtained for $\delta = 0.4$. It is a partition in 12 communities with modularity 0.426. There are 2 main communities : one with 653 blogs, from which 94% are conservative, and one with 541 blogs, from which 98% are liberal. The 28 remaining blogs are distributed in 10 tiny communities. When we progressively increase δ, we can see that the size of the two large communities reduces moderately but progressively until $\delta = 0.65$ where the conservative community splits into several smaller communities, the largest one containing 40 blogs. The liberal community remains until $\delta = 0.725$ where it splits into smaller communities, the largest one containing 54 blogs.

As a comparison, algorithm [53] obtains a partition in 2 communities with modularity 0.426 whereas algorithm [7] obtains a partition in 9 communities with modularity 0.427. As we can see, the partition we obtain is very acceptable in terms of modularity.

In Table 2.3, we give a comparative summary of the modularity of the partition obtained for the three examples by our approach and by the algorithms presented in [53, 7]. Though slightly smaller, the modularity of the partition we obtain is comparable to that of other partitions which is actually surprising since our approach, contrarily to [53, 7] does not try to maximize modularity.

Network	Karate	Books	Blogs
Number of nodes	34	105	1222
This article	0.417	0.523	0.426
[53]	0.419	0.526	0.426
[7]	0.419	0.527	0.427

TABLE 2.3 – Modularity of the partitions obtained by the approach presented in this paper and by the algorithms presented in [53, 7] for the three examples considered in this paper.

Chapitre 3

Coordination in networks of linear impulsive agents

This chapter presents the main results from [12, 51]. Many of existing works on consensus consider the networks formed by identical subsystems [59, 48, 55]. Our point of view is that real networks are formed by several clusters inside which the interactions take place often and can be seen as continuous while, due to communication constraints (harsh environment, energy optimization or opinion preferences for instance), the inter-clusters interactions are rare, thus discrete. This leads us to a network dynamics that is expressed in term of reset systems. For this reason we addressed the consensus problem in heterogeneous networks containing both linear and linear impulsive dynamics. Most agents can only update their state in a continuous way using inner-cluster agent states. On top of this, few agents also have the peculiarity to update their states in a discrete way by reseting it using information from outside their cluster.

We first consider that the clusters are represented by fixed, directed and strongly connected graphs. To every agent we associate a scalar real value representing its state. The states continuously evolve following a linear consensus protocol and approach local agreements specific to each cluster. In order to enforce a global agreement over the whole network, we consider that each cluster contains an agent that can be exogenously controlled. The state of this agent, called leader, will be quasi-periodically reseted by a local master controller that receives information from some neighboring leaders. In order to control the consensus value we first characterized it. Precisely we show that it depends only on the initial condition and the interaction topologies. Secondly, we provide sufficient Linear Matrix Inequality (LMI) conditions for the global uniform exponential stability of the consensus in presence of a quasi-periodic reset rule. The study of the network behavior is completed by a decay rate analysis. Finally we design the interaction network of the leaders which allows to reach a prescribed consensus value.

The results presented above are extended to the case where asynchronous activation of the inter-clusters links. Moreover, each cluster can contain several agents that are externally influenced and we consider more general graph topologies that does not ensure the strong connectivity property. Finally, we have adapted our results to take into account stochastic activation of inter-clusters links. The case we are treating is when each inter-clusters link is independently activated following a Poisson renewal process. The inconvenient of the asynchronous activation of inter-clusters links is that we cannot compute a priori the consensus value or impose a prescribed one.

3.1 Problem formulation

Throughout this section, we consider that the vertex set $\mathcal{V}$ is partitioned in m clusters $\mathcal{C}_1, \dots, \mathcal{C}_m$. We denote by n_i the cardinality of each cluster $\mathcal{C}_i$. For the sake of simplicity we reorder the nodes to obtain, for $i \in \{1, \dots, n\}$,

$$\mathcal{C}_i = \{m_{i-1}+1, \dots, m_i\}, \tag{3.1}$$

where $m_0 = 0,\ m_i \geq m_{i-1}+1,\ m_m = n$, and thus,

$$n_i = m_i - m_{i-1}.$$

Let us also introduce the intra-cluster graph $\mathcal{G}_L = (\mathcal{V}, \mathcal{E}_L)$ containing only the edges of $\mathcal{G}$ that connect agents belonging to the same cluster. That is

$$\mathcal{E}_L = \{(i,j) \in \mathcal{E} \mid \exists k \in \{1, \dots, m\} \text{ such that } i, j \in \mathcal{C}_k\}.$$

It is noteworthy that $\mathcal{E}_L$ is time invariant but the updating weights associated with each link in the consensus protocol can vary in time (i.e. the components of the associated Laplacian matrix L are time-varying). The state of each agent evolves continuously by taking into account the states of other agents belonging to their cluster. Doing so, the agents approach local agreements which can be different from one cluster to another. In order to reach the consensus in the entire network every inter-cluster connection is activated at some discrete instants. When the inter-cluster link $(j,i) \in \mathcal{E} \setminus \mathcal{E}_L$ is activated, the state of agent i is reset to a weighted average of the states of i and j. If several links arriving at i are activated simultaneously, all the source states of these edges are considered in the weighted average.
The previous discussion is formally described by the linear reset system defining the overall network dynamics :

$$\begin{cases} \dot{x}(t) = -L(t)x(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\ x(t_k) = P(t_k)x(t_k^-) & \forall k \in \mathbb{N} \\ x(0) = x_0 \end{cases} \tag{3.2}$$

where $x_0 \in \mathbb{R}^n$, $\mathcal{T}$ is the countable set of reset instants which are described by the diverging and increasing sequence $(t_k)_k$, $L(t) \in \mathbb{R}^{n\times n}$ is a weighted time-varying Laplacian matrix associated to the intra-cluster graph $\mathcal{G}_L$ and $P(t_k) \in \mathbb{R}^{n\times n}$ is a stochastic matrix associated to the inter-cluster graph $\mathcal{G}_P(t_k) = (\mathcal{V}, \mathcal{E}_P(t_k))$ where $\mathcal{E}_P(t_k) \neq \emptyset$ is the set of inter-cluster links activated at time t_k, so that $\mathcal{E}_P(t_k) \subseteq \mathcal{E} \setminus \mathcal{E}_L$.
It is worth noting that $L(t)$ has the following block diagonal structure

$$L(t) = \begin{pmatrix} L_1(t) & & \\ & \ddots & \\ & & L_m(t) \end{pmatrix},\ L_i(t) \in \mathbb{R}^{n_i} \tag{3.3}$$

with $L_i(t)\mathbb{1}_{n_i} = \mathbf{0}_{n_i}$ and $P(t_k)\mathbb{1}_n = \mathbb{1}_n$.

The problems that are posed in this framework are the following :

- Can we characterize the consensus value in term of initial condition and network topology ? Can we design the matrix P in order to reach an a priori given consensus value ?
- If the consensus is guaranteed, can we estimate the convergence speed of (3.2) ?
- What kind of reset sequence $\mathcal{T}$ can we consider ?

The first item can be treated only under the assumption that the interaction weights $L_{i,j}$ are fixed and the inter-cluster interactions are synchronous and fixed - $P_{i,j}$ is time-invariant. The last two items will be considered in the general framework of time-variant interactions with asynchronous activation of inter-cluster links.

3.2 Time invariant interactions with synchronous activation of inter-cluster links

Throughout this subsection we assume that each cluster/community possesses one particular agent called leader and denoted in the following by $l_i \in \mathcal{C}_i,\ \forall i \in \{1,\dots,m\}$. The set of leaders will be referred to as $\mathcal{L} = \{l_1,\dots,l_m\}$. At specific time instants $t_k,\ k \geq 1$, called reset times, the leaders interact between them following a predefined interaction map $\mathcal{E}_l \subset \mathcal{L} \times \mathcal{L}$. We also suppose that $\mathcal{G}_l = (\mathcal{L}, \mathcal{E}_l)$ is strongly connected. The rest of the agents will be called followers and denoted by f_j. For the sake of clarity we consider that the leader is the first element of its community :

$$\mathcal{C}_i = \{l_i, f_{m_{i-1}+2}, \dots, f_{m_i}\},\ \forall i \in \{1,\dots,m\} \tag{3.4}$$

where $m_0 = 0,\ m_m = n$ and the cardinality of $\mathcal{C}_i$ is given by

$$|\mathcal{C}_i| \triangleq n_i = m_i - m_{i-1}, \forall i \geq 1.$$

Example 1 *To illustrate the notation* (3.4) *we consider a simple network of 6 agents partitioned in 2 clusters having 3 elements. Then* $\mathcal{C}_1 = \{l_1, f_2, f_3\}$ *and* $\mathcal{C}_2 = \{l_2, f_5, f_6\}$.

In order to keep the presentation simple and making an abuse of notation, each agent will have a scalar state denoted also by l_i for the leader l_i and f_j for the follower f_j. We also introduce the vectors $x = (l_1, f_2, \dots, f_{m_1}, \dots, l_m, \dots, f_{m_m} = f_n)^\top \in \mathbb{R}^n$ and $x_l = (l_1, l_2, \dots, l_m)^\top \in \mathbb{R}^m$ collecting all the states of the agents and all the leaders' states, respectively.

The system (3.2) simplifies to :

$$\begin{cases} \dot{x}(t) = -Lx(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\ x_l(t_k) = P_l x_l(t_k^-) & \forall t_k \in \mathcal{T} \\ x(0) = x_0 & \end{cases}, \tag{3.5}$$

where P_l defines the interactions between leaders.
We also denote by w_i the left eigenvector of L_i associated with the eigenvalue 0 such that $w_i^\top \mathbb{1}_{n_i} = 1$. Similarly, let $v = (v_1,\dots,v_m)^\top$ be the left eigenvector of P_l associated with the eigenvalue 1 such that $v^\top \mathbb{1}_m = 1$. Due to the structure (3.4) of the communities, we emphasize that each vector w_i can be decomposed in its first component $w_{i,l}$ and the rest of its components grouped in the vector $w_{i,f}$.

Remark 9 *When dynamics* (3.5) *is considered Theorem 1 implies that between two consecutive reset instants* t_k *and* t_{k+1}*, the agents belonging to the same community try to approach a local agreement defined by* $x_i^*(k) = w_i^\top x_{\mathcal{C}_i}(t_k)$ *where* $x_{\mathcal{C}_i}(\cdot)$ *is the vector collecting the states of the agents belonging to the cluster* $\mathcal{C}_i$*. Nevertheless, at the reset times the value of the local agreement can change. Thus,*

$$\begin{aligned} &w_i^\top x_{\mathcal{C}_i}(t) = w_i^\top x_{\mathcal{C}_i}(t_k),\ \forall t \in (t_k, t_{k+1}) \text{ and possibly} \\ &w_i^\top x_{\mathcal{C}_i}(t) \neq w_i^\top x_{\mathcal{C}_i}(t_k),\ \text{ for } t \notin (t_k, t_{k+1}) \end{aligned}$$

Therefore, the agents whose collective dynamics is described by the hybrid system (3.2), may reach a consensus only if the local agreements converge one to each other.

3.2.1 Consensus value

Before presenting our next result, let us introduce the following vectors :

$$\begin{aligned} x^*(t) &= (x_1^*(t), x_2^*(t), ..., x_m^*(t))^\top \in \mathbb{R}^m \\ u &= (v_1/w_{1,l}, v_2/w_{2,l}, ..., v_m/w_{m,l})^\top \in \mathbb{R}^m \end{aligned} \tag{3.6}$$

where $x_i^*(\cdot)$ represents the local agreement of the cluster $\mathcal{C}_i$ and $v \in \mathbb{R}^m$ and $w_i \in \mathbb{R}^{n_i}$ are defined at the beginning of the section as left eigenvectors associated with the matrices describing the reset dynamics of the leaders and the continuous dynamics of each cluster, respectively. Let us also introduce the matrix of the left eigenvectors of the communities :

$$W = \begin{bmatrix} w_1^\top & 0 & \cdots & 0 \\ 0 & w_2^\top & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_m^\top \end{bmatrix} \in \mathbb{R}^{m\times n}. \tag{3.7}$$

It is noteworthy that $x^*(t)$ is time-varying but piecewise constant : $x^*(t) = x^*(k)\ \forall t \in (t_k, t_{k+1})$.

Proposition 3 *Consider the system (3.5) with L and P_l defined by (1.1) and (1.2), respectively. Then,*

$$u^\top x^*(t) = u^\top x^*(0),\ \forall t \in \mathbb{R}_+. \tag{3.8}$$

Proof : The following relation holds :

$$x^*(t) = Wx(t) \qquad \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \tag{3.9}$$

Since $w_i = (w_{il}, w_{if})$, we define a permutation matrix T such that $WT^\top = U = (U_1, U_2)$. The matrix U_1 is a diagonal matrix corresponding to the leaders' components $w_{i,l}$, while U_2 is a block diagonal matrix corresponding to the followers' components $w_{i,f}$. In other terms

$$U_1 = \begin{bmatrix} w_{1,l} & 0 & \cdots & 0 \\ 0 & w_{2,l} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{m,l} \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{3.10}$$

$$U_2 = \begin{bmatrix} w_{1,f} & 0 & \cdots & 0 \\ 0 & w_{2,f} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & w_{m,f} \end{bmatrix} \in \mathbb{R}^{m\times (n-m)}. \tag{3.11}$$

Finally, we can rewrite equation (3.9) as :

$$x^*(t) = WT^\top x(t) = U \cdot (x_l(t), x_f(t)). \tag{3.12}$$

Note that at the reset time t_k one has $x_f(t_k) = x_f(t_k^-)$. This yields

$$\begin{aligned} x^*(t_k) - x^*(t_k^-) &= U \cdot (x_l(t_k) - x_l(t_k^-), x_f(t_k) - x_f(t_k^-)) \\ &= U \cdot (x_l(t_k) - x_l(t_k^-), 0) = U_1 \cdot (x_l(t_k) - x_l(t_k^-)) + U_2 \cdot 0 \\ &= U_1(x_l(t_k) - x_l(t_k^-)). \end{aligned}$$

Thus,

$$x^*(t_k) = x^*(t_k^-) + U_1 \cdot (P_l - I_m)x_l(t_k^-). \tag{3.13}$$

Multiplying equation (3.13) by $u^\top$ and using $u^\top U_1 = v^\top$ as well as $v^\top P_l = v^\top$ one obtains

$$\begin{aligned} u^\top x^*(t_k) &= u^\top x^*(t_k^-) + u^\top U_1(P_l - I_m)x_l(t_k^-) \\ &= u^\top x^*(t_k^-) + v^\top (P_l - I_m)x_l(t_k^-) \\ &= u^\top x^*(t_k^-) + v^\top x_l(t_k^-) - v^\top x_l(t_k^-) = u^\top x^*(t_k^-) \end{aligned} \tag{3.14}$$

According to Remark 13, $x^*(t)$ remains constant for all $t \in (t_k, t_{k+1})$ leading to

$$u^\top x^*(t) = u^\top x^*(0) \qquad \forall t \in \mathbb{R}_+. \tag{3.15}$$

Corollary 2 *Consider the system (3.5) with L and P_l defined by (1.1) and (1.2), respectively. Assuming the agents of this system reach a consensus, the consensus value is*

$$x^* = \frac{u^\top W x(0)}{\sum_{i=1}^m u_i}. \tag{3.16}$$

Proof : Let x^* be the consensus value reached by the system (3.2). It means that $x(t) \to x^* \mathbb{1}_n$. Thus, when t goes to ∞ in (3.15) one obtains

$$u^\top x^* \mathbb{1}_n = u^\top x^*(0) = u^\top W x(0)$$

leading to (3.16).

In order to simplify the presentation and without loss of generality, in what follows, we consider that $\sum_{i=1}^m u_i = 1$.

Remark 10 *It is important to note that the consensus value depends only on the system matrices L, P_l and does not depend on the reset sequence $\mathcal{T}$.*

A trivial result which may be seen as a consequence of Corollary 2 is the following.

Corollary 3 *If the matrices L, P_l are symmetric (i.e. i^{th} agent takes into account the state of j^{th} agent as far as j^{th} takes into account the i^{th} one and they give the same importance one to another) the consensus value is the average of the initial states.*

Proof : In this case $w_i = \frac{1}{n_i}\mathbb{1}_{n_i}$ and $v = \frac{1}{m}\mathbb{1}_m$ which leads to $u = (\frac{n_1}{m}, \frac{n_2}{m}, \ldots, \frac{n_m}{m})$. The result follows from (3.16).

3.2.2 Stability analysis

We notice that convergence of (3.5) toward consensus can be guaranteed by adapting Theorem 5. However we give here an alternative version based on LMIs verification. We do that because this method can be easily adapted to enforce convergence toward a prescribed consensus value.

Since the consensus value is previously computed we can first define the disagreement vector $y = x - x^* \mathbb{1}_n$. We also introduce the stochastic matrix P_{ex} as follows :

$$P_{ex} = T^\top \begin{bmatrix} P_l & 0 \\ 0 & I_{n-m} \end{bmatrix} T \tag{3.17}$$

where T is the permutation matrix used in the proof of Proposition 3. It is noteworthy that $L\mathbb{1}_n = \mathbf{0}_n$ and $P_{ex}\mathbb{1}_n = \mathbb{1}_n$. Thus, the disagreement dynamics is exactly the same as the system one :

$$\left\{ \begin{array}{ll} \dot{y}(t) = -Ly(t), & \forall t \in \mathbb{R}_+ \setminus \mathcal{T} \\ y(t_k) = P_{ex}y(t_k^-) & \forall t_k \in \mathcal{T} \\ y(0) = y_0 & \end{array} \right. . \tag{3.18}$$

Let us first introduce the reset sequences that are considered in this paragraph. Due to uncertainties that affect the reset instant in practice, instead of considering a periodic reset sequence, we consider a nearly periodic one defined by $t_{k+1} - t_k = \delta + \delta'$ where $\delta \in \mathbb{R}_+$ is the fixed period and $\delta' \in \Delta$ is a jitter belonging to the compact set $\Delta \subset \mathbb{R}_+$. Thus the set of reset times $\mathcal{T}$ belongs to the set of all admissible reset sequences associated with Δ :

$$\Phi(\Delta) \triangleq \left\{ \{t_k\}_{k\in\mathbb{N}}, t_{k+1} - t_k = \delta + \delta'_k, \delta'_k \in \Delta, \forall k \in \mathbb{N} \right\} \tag{3.19}$$

where we always consider $t_0 = 0$.

Stability analysis of the equilibrium point $y^* = 0$ of the system (3.18) with $\mathcal{T}$ belonging to $\Phi(\Delta)$ defined in (3.19) can be numerically treated by using the results in [23]. Other classical results guarantees the consensus is reached [59]. In this manuscript we follow the presentation in [12] since the LMI criterion stated there can be adapted for the control design as well.

Remark 11 • *We note that in practice periodic events are difficult to ensure while nearly periodic is simple. In the case of social network periodic meetings can be impossible while quasi periodic ones are more realistic. Thus, quasi-periodic reset sequences increase the accuracy of the model with respect to practical applications.*
• *The case $\delta'_k = 0, \forall k \in \mathbb{N}$ recovers the purely periodic reset strategy. In this situation system (3.2) rewrites as a discrete dynamics $x(t_{k+1}) = P_{ex}e^{-L\delta}x(t_k)$. The stability issue in this case can be solved without using the LMI based criterium presented below. Indeed, in order to guarantee the consensus, we can use the strong connectivity of the clusters and of the graph of leaders in order to prove that $P_{ex}e^{-L\delta}$ is not only stochastic but also primitive (i.e. irreducible and aperiodic). This is a necessary and sufficient condition to reach consensus starting from any initial condition.*

We recall that for any $\mathcal{T} \in \Phi(\Delta)$ and any initial condition x_0 the system (3.2) has a unique solution denoted by $\varphi(t, x_0)$.

Definition 6 *We say that the equilibrium $y^* = \mathbf{0}_n$ of the system (3.18) is Globally Uniformly Exponentially Stable (GUES) with respect to the set of reset sequences $\Phi(\Delta)$ if there exist positive scalars c, λ such that for any $\mathcal{T} \in \Phi(\Delta)$, any $y_0 \in \mathbb{R}^n$, and any $t \geq 0$*

$$\|\varphi(t, y_0)\| \leq ce^{-\lambda t}\|y_0\| \tag{3.20}$$

The following theorem is instrumental :

Theorem 9 (Theorem 1 in [32]) *Consider the system (3.18) with the set of reset times $\mathcal{T} \in \Phi(\Delta)$. The equilibrium $y^* = \mathbf{0}_n$ is GUES if and only if there exists $S_{[\cdot]} : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$, $S_{[y]} = S_{[y]}^\top = S_{[ay]} > 0,\ \forall x \neq 0, a \in \mathbb{R},\ a \neq 0$ defining a positive function $V : \mathbb{R}^n \mapsto \mathbb{R}_+$ strictly convex,*

$$V(y) = y^\top S_{[y]} y,$$

homogeneous (of second order), $V(0) = 0$, such that $V(y(t_k)) > V(y(t_{k+1}))$ for all $y(t_k) \neq 0,\ k \in \mathbb{N}$ and any of the possible reset sequences $\mathcal{T} \in \Phi(\Delta)$.

In the sequel, we define a quasi-quadratic Lyapunov function satisfying Theorem 9 by means of some LMI. Therefore, the following result gives sufficient conditions for the stability of the equilibrium point $y^* = \mathbf{0}_n$ for the system (3.18) or equivalently of $x^*\mathbb{1}_n$ for the system (3.2). Even if other sufficient condition for GUES can be given, we present the following result since it will be useful in the next section.

Theorem 10 *Consider the system* (3.2) *with* $\mathcal{T}$ *in the admissible reset sequences* $\Phi(\Delta)$*. If there exist matrices* $S(\delta')$*,* $S(\cdot):\Delta \mapsto \mathbb{R}^{n\times n}$ *continuous with respect to* δ'*,* $S(\delta') = S^\top(\delta') > 0,\ \delta' \in \Delta$ *such that the LMI*

$$\begin{aligned}
&\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S(\delta_a)\Big(I_n - \mathbb{1}_n u^\top W\Big) - \\
&\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top S(\delta_b)\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) > 0, \\
&Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)}
\end{aligned} \tag{3.21}$$

is satisfied on $span\{\mathbb{1}_n\}^\perp$ *for all* $\delta_a, \delta_b \in \Delta$*, then* x^* *is GUES for* (3.2)*. Moreover, the stability is characterized by the quasi-quadratic Lyapunov function* $V(t) = V(x(t)) \triangleq \max_{\delta'\in\Delta}(x(t) - x^*\mathbb{1}_n)^\top S(\delta')(x(t) - x^*\mathbb{1}_n)$ *satisfying* $V(t_k) > V(t_{k+1})$*.*

Proof : Using the disagreement vector $y(t) = x(t) - x^*\mathbb{1}_n$ and supposing that there exist matrices $S(\delta')$ satisfying (3.21) for all $\delta_a, \delta_b \in \Delta$ we define the Lyapunov matrix

$$S_{[y]} = S(\delta^*(y)) \text{ with } \delta^*(y) = \arg\max_{\delta'\in\Delta} y^\top S(\delta') y \tag{3.22}$$

Following [32] the Lyapunov function

$$V(y) = y^\top S_{[y]} y = \max_{\delta'\in\Delta} y^\top S(\delta') y,$$

is convex and homogeneous of the second order.
Let us show now that $S(\cdot)$ solution of (3.21) ensures that $V(\cdot)$ defined above satisfies Theorem 9.
We note first that any $x(t_k) \in \mathbb{R}^n$ can be decomposed as $x(t_k) = \bar{x}(t_k) + \tilde{x}(t_k)$ with $\bar{x}(t_k) \in span\{\mathbb{1}_n\}^\perp$ and $\tilde{x}(t_k) \in span\{\mathbb{1}_n\}$. Moreover,

$$\Big(I_n - \mathbb{1}_n u^\top W\Big)\tilde{x}(t_k) = 0,\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)\tilde{x}(t_k) = 0$$

hence

$$\begin{aligned}
\Big(I_n - \mathbb{1}_n u^\top W\Big)x(t_k) &= \Big(I_n - \mathbb{1}_n u^\top W\Big)\bar{x}(t_k), \\
\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)x(t_k) &= \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)\bar{x}(t_k)
\end{aligned}$$

Thus, the LMI (3.21) yields

$$\begin{aligned}
&x(t_k)^\top\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S(\delta_a)\Big(I_n - \mathbb{1}_n u^\top W\Big)x(t_k) > \\
&x(t_k)^\top\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top S(\delta_b)\Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)x(t_k), \\
&\quad \forall \delta_a, \delta_b \in \Delta
\end{aligned}$$

Consequently, using $u^\top W x(t_k) = x^*$ one gets

$$\begin{aligned}&\Big(x(t_k) - \mathbb{1}_n x^*\Big)^\top S(\delta_a)\Big(x(t_k) - \mathbb{1}_n x^*\Big) > \\ &\Big(Y(\delta_a)x(t_k) - \mathbb{1}_n x^*\Big)^\top S(\delta_b)\Big(Y(\delta_a)x(t_k) - \mathbb{1}_n x^*\Big), \\ &\quad \forall \delta_a, \delta_b \in \Delta,\ x(t_k) \in \mathbb{R}^n \end{aligned} \tag{3.23}$$

For any $\{t_k\}_{k\in\mathbb{N}} \in \Phi(\Delta)$ we have $x(t_{k+1}) = Y(\delta'_k)x(t_k)$ with some $\delta'_k \in \Delta$. Thus, for $\delta_a = \delta'_k$, (3.23) rewrites as :

$$\begin{aligned}&\Big(x(t_k) - x^* \mathbb{1}_n\Big)^\top S(\delta'_k)\Big(x(t_k) - x^* \mathbb{1}_n\Big) > \\ &\Big(x(t_{k+1}) - x^* \mathbb{1}_n\Big)^\top S(\delta_b)\Big(x(t_{k+1}) - x^* \mathbb{1}_n\Big) \\ &\quad \forall \delta'_k,\ \delta_b \in \Delta,\ x(t_k) \in \mathbb{R}^n \end{aligned}$$

or equivalently

$$y(t_k)^\top S(\delta'_k) y(t_k) > y(t_{k+1})^\top S(\delta_b) y(t_{k+1}) \quad \forall \delta'_k, \delta_b \in \Delta$$

Taking $\delta_b = \delta^*(y(t_{k+1}))$, defined by (3.22) one obtains

$$V(y(t_k)) > y(t_k)^\top S(\delta'_k) y(t_k) > V(y(t_{k+1}))$$

for all $y(t_k)$, which ends the proof.

3.2.3 Convergence toward prescribed consensus

In what follows we assume that the value x^* is a priori fixed and at least a vector u satisfying (3.16) exists. Under this assumption we are wondering if there exists a matrix P_l that allows system (3.5) to reach the consensus value x^*. It is worth noting that the network topology is considered fixed and known for each cluster. Under these assumptions, a consensus value is imposed by a certain choice of v such that $v^\top \mathbb{1}_m = 1$ and v left eigenvector of P_l associated with the eigenvalue 1. In other words we arrive to a joint design of P_l and the Lyapunov function V guaranteeing the trajectory of (3.5) ends up on x^*.

Theorem 11 *Let us consider the system (3.5) with $\mathcal{T}$ in the admissible reset sequences $\Phi(\Delta)$ and let x^* be a priori fixed by a certain choice of v. If there exist matrices $R(\delta')$, $R(\cdot) : \Delta \mapsto \mathbb{R}^{n\times n}$ continuous with respect to δ', $R(\delta') = R^\top(\delta') > 0,\ \delta' \in \Delta$ and P_l stochastic such that the LMI*

$$\begin{aligned}&\begin{bmatrix} Z(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & R(\delta_b) \end{bmatrix} > 0, \\ &Y(\delta_a) \triangleq P_{ex} e^{-L(\delta+\delta_a)} \\ &Z(\delta_a) \triangleq \Big(I_n - \mathbb{1}_n u^\top W\Big)^\top + \Big(I_n - \mathbb{1}_n u^\top W\Big) - R(\delta_a)\end{aligned} \tag{3.24}$$

with the constraint

$$v^\top P_l = v^\top$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ for all $\delta_a, \delta_b \in \Delta$, then x^* is GUES for (3.5). Moreover, the stability is characterized by the quasi-quadratic Lyapunov function $V(t) = V(x(t)) = \max_{\delta'\in\Delta}(x(t) - x^*\mathbb{1}_n)^\top R(\delta')^{-1}(x(t) - x^*\mathbb{1}_n)$ satisfying $V(t_k) > V(t_{k+1})$.

Proof : First notice that

$$\Big((I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a) - I_n\Big) S(\delta_a)^{-1} \Big(S(\delta_a)(I_n - \mathbb{1}_n u^\top W) - I_n\Big) \geq 0$$

leads to

$$(I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a)(I_n - \mathbb{1}_n u^\top W) \geq (I_n - \mathbb{1}_n u^\top W)^\top + (I_n - \mathbb{1}_n u^\top W) - S(\delta_a)^{-1}$$

Thus, once the solution to the LMI problem (3.24) is obtained we can define $S(\delta_a) = R(\delta_a)^{-1}$ and $S(\delta_b) = R(\delta_b)^{-1}$. Then :

$$\begin{bmatrix} Z(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & S(\delta_b)^{-1} \end{bmatrix} > 0$$

where

$$Z(\delta_a) = (I_n - \mathbb{1}_n u^\top W)^\top + (I_n - \mathbb{1}_n u^\top W) - S(\delta_a)^{-1}$$

and hence

$$\begin{bmatrix} \bar{Z}(\delta_a) & \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big)^\top \\ \Big(Y(\delta_a) - \mathbb{1}_n u^\top W\Big) & S(\delta_b)^{-1} \end{bmatrix} > 0$$

where

$$\bar{Z}(\delta_a) = (I_n - \mathbb{1}_n u^\top W)^\top S(\delta_a)(I_n - \mathbb{1}_n u^\top W).$$

By Schur complement, the last LMI is nothing than (3.21) in Theorem 10. Moreover, the constraints $v^\top P_l = v^\top$, $P_l \mathbb{1}_m = \mathbb{1}_m$ and the coefficients of P_l positive ensure the matrix P_l is stochastic and the consensus value is exactly x^*.

3.2.4 Numerical treatment of parametric LMIs

In order to render this manuscript self-contained, we consider the problem of approximation of the parametric LMI (3.21) by a finite number of conditions using polytopic embeddings. The matrix exponential $e^{-L\delta_a}$ is approximated by its $h-$ order Taylor expansion $\sum_{i=0}^{h} \frac{(-L)^i}{i!}\delta_a^i$. Thus the set $\{X \in \mathbb{R}^{n\times n} \mid X = e^{-L\delta_a},\ \delta_a \in \Delta\}$ can be embedded into the polytopic set defined by the vertices $Z_1, \ldots, Z_{h+1}$ where

$$Z_1 = I_n$$
$$Z_i = \sum_{l=0}^{i-1} \frac{(-L)^l}{l!}\delta_{max}^l,\ \forall i \in \{2, \ldots, h+1\}$$

with $\delta_{max} = \max_{\delta' \in \Delta} \delta'$, $(-L)^0 = I_n$ and $0! = 1$. Then, Theorem 10 can be replaced by the following result.

Theorem 12 *Consider the system (3.5) with $\mathcal{T}$ in the admissible reset sequences $\Phi(\Delta)$. If there exist symmetric positive definite matrices S_i, $1 \leq i \leq h+1$ such that the LMI*

$$\Big(I_n - \mathbb{1}_n u^\top W\Big)^\top S_i \Big(I_n - \mathbb{1}_n u^\top W\Big) - \Big(Y(\delta)Z_i - \mathbb{1}_n u^\top W\Big)^\top S_j \Big(Y(\delta)Z_i - \mathbb{1}_n u^\top W\Big) > 0,$$
$$Y(\delta) \triangleq P_{ex} e^{-L(\delta)} \qquad (3.25)$$

is satisfied on span$\{\mathbb{1}_n\}^\perp$ *for all* $i, j \in \{1, \dots, h+1\}$*, then* x^* *is globally uniformly exponentially stable for* (3.5)*.*

Proof : Assume that the set of LMIs (3.25) is satisfied for a set of matrices $S_i,\ 1 \leq i \leq h+1$. Thus,

$$\left(I_n - \mathbb{1}_n u^\top W\right)^\top \left(\sum_{i=1}^{h+1} \mu_i S_i\right) \left(I_n - \mathbb{1}_n u^\top W\right) -$$
$$\left(Y(\delta) \sum_{i=1}^{h+1} \mu_i Z_i - \mathbb{1}_n u^\top W\right)^\top \left(\sum_{j=1}^{h+1} \mu_j S_j\right) \left(Y(\delta) \sum_{i=1}^{h+1} \mu_i Z_i - \mathbb{1}_n u^\top W\right) > 0,$$

is satisfied for all $\mu_i, \mu_j \in [0,1],\ i, j \in \{1, \dots, h+1\}$ such that $\sum_{i=1}^{h+1} \mu_i = \sum_{j=1}^{h+1} \mu_j = 1$. It is noteworthy that the polytopic embedding provided above implies that for all $\delta_a \in [0, \delta_{max}]$ there exists the set of scalars $\mu_i \in [0,1]$ such that $e^{-L\delta_a} = \sum_{i=1}^{h+1} \mu_i Z_i$ and $\sum_{i=1}^{h+1} \mu_i = 1$. In other words, Theorem 10 holds with $S(\delta') = \sum_{i=1}^{h+1} \mu_i(\delta') S_i$.

3.3 Time varying interactions and asynchronous activation of inter-cluster links

3.3.1 Framework's assumptions

In the following we consider the consensus problem formulated at the beginning of this section in its general form. In order to prove that the reset algorithm (3.2) guarantees asymptotic consensus for every initial condition x_0 we have to impose some standard assumptions. The first one concerns a minimal connectivity property of the whole network and of each cluster.

Assumption 10 (Network structure) *The graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is such that*

1a) For each cluster C_i*, the induced graph* $\left(C_i, \mathcal{E}_L \cap \left(C_i \times C_i\right)\right)$ *contains a spanning tree,*

1b) If needed one can reorder the clusters such that : for all $i \geq 2$ *there exist* $j < i,\ l_i \in \mathcal{C}_j$ *and* r_i *a root of a spanning tree of* C_i *such that* $(l_i, r_i) \in \mathcal{E}$*. We denote by*

$$\mathcal{E}_T = \{(l_i, r_i) | i \in \{2, \dots, m\}\}$$

the set of these $m-1$ *such edges.*

The previous assumption implies that $\mathcal{G}$ contains a spanning tree having the root in $\mathcal{C}_1$ (formed by the union of the spanning trees in each cluster together with the edges in $\mathcal{E}_T$). The assumption is satisfied if the induced graph of each cluster is strongly connected and so is $\mathcal{G}$. It also holds if we replace (1b) by the requirement that the graph induced by the set of roots of all clusters contains a spanning tree. We note that Assumption 10 implies that 0 is a simple eigenvalue of each $L_i, \forall i \in \{1, \dots, m\}$ (see [61]). The first part of Assumption 10 has a direct consequence on the continuous dynamics since equation (1.1) imposes $L_{i,j} < 0$ when $(j,i) \in \mathcal{E}_L$. The second part of Assumption 10 guarantees the existence of the inter-cluster interaction structure formed by $\mathcal{E}_T$.

The next hypothesis of this work is standard in the literature (see [8]) and it ensures a minimal influence of the states implicated in the reset process of the agents.

Assumption 11 (Minimal influence) *There exists a constant $\alpha' \in (0,1)$ such that, for all reset times t_k, $P_{i,i}(t_k) \geq \alpha'$ and, if $P_{i,j}(t_k) \neq 0$ and $(i,j) \in \mathcal{E}_T$ then $P_{i,j}(t_k) \geq \alpha'$.*

Remark 12 *Assumption 11 guarantees a minimal influence of one cluster on the root of some other at the reset time.*

We also need to bound the non-zero influence occuring during the continuous dynamics.

Assumption 12 (Bounded continuous influence) *The components of the Laplacian matrix $L(t)$ satisfy the following two constraints :*

— *influence weights are uniformly upper bounded i.e., there exists $\overline{\alpha} > 0$ a finite real number such that $|L_{i,j}(t)| \leq \overline{\alpha}$, $\forall i,j \in \{1,\dots,n\}$ and $t \geq 0$,*

— *non-zero influence weights are uniformly lower bounded i.e., there exists $\underline{\alpha} > 0$ a finite real number such that $|L_{i,j}(t)| > 0 \Rightarrow |L_{i,j}(t)| \geq \underline{\alpha}$, $\forall i,j \in \{1,\dots,n\}$.*

The first item is necessary to ensure that during the continuous dynamics, the agents do not approach one to another indefinitely fast, oscillations may otherwise prevent consensus to take place. We can notice that, in practice, this assumption is very natural and is almost always satisfied. The second item makes sure that enough interaction takes place within clusters, as Assumption 11 does for interactions between clusters.

We use an extraction function ϕ_i to emphasize that an agent belonging to the cluster $\mathcal{C}_i$ resets its state at time t_k. This function selects the instants $t_k \in \mathcal{T}$ corresponding to a reset of an agent in cluster $\mathcal{C}_i$. Precisely, for any $h \in \mathbb{N}$ we denote by $t_{\phi_i(h)}$ the h-th time an agent in cluster $\mathcal{C}_i$ resets its state meaning that

$$\phi_i(h) = \min\{k > \phi_i(h-1) | \exists j \in \mathcal{C}_i, \ell \in \mathcal{V} \setminus \mathcal{C}_i, P_{j,\ell}(t_k) > 0\},$$

where for consistency, we imposed $\phi_i(-1) = -1$ and $t_{\phi_i(-1)} = 0$, for all $i \in \{1,\dots,m\}$. We do not disregard the situation in which agents from different clusters reset their state simultaneously. Therefore, we may have $\phi_i(k) = \phi_j(h)$ for $i \neq j$ and $k,h \in \mathbb{N}$.

While in discrete time, a minimal influence is guaranted by Assumption 11, in continuous time, a minimal influence can be ensured using a dwell time. This will be shown in Proposition 4 below.

Assumption 13 (Dwell time) *There exists a positive constant $\delta > 0$ such that*

$$t_{\phi_i(k+1)} - t_{\phi_i(k)} \geq \delta,\ \forall i \in \{1,\dots,m\}.$$

In other words, there exists a lower bound for the period between the consecutive reset instants on the state of agents belonging to the same cluster. Notice that according to Assumptions 10 and 14 below, all clusters in $\{2,\dots,m\}$ reset an infinite number of times, so that for these clusters, ϕ_i is well defined. Cluster $\mathcal{C}_1$ may not reset an infinite number of times. In this case, $t_{\phi_1}(k)$ is only defined for k smaller than some finite bound, and should still satisfy Assumption 13 for these k. This has no impact on the results of the paper.

Remark 13 *A simple manner to ensure Assumption 13 in a decentralized way is for each cluster $\mathcal{C}_i$, to allow only one agent to interact outside $\mathcal{C}_i$. Then, this one agent has full control of $t_{\phi_i(k)}$ and can reset respecting the dwell time condition without the need for further communication. Otherwise, since Assumption 13 concerns the resets of all agents in a cluster (unlike Assumption 14), these agents should have a way to communicate the last reset time which occured in the cluster.*

The next assumption establishes the relationship between $\mathcal{E}_T$ and the reset dynamics.

Assumption 14 (Recurrent activation of inter-cluster links) *There exists a positive constant $\delta_{max} > \delta$ satisfying the following : for all $(l,r) \in \mathcal{E}_T$,*
- *there exists $k \in \mathbb{N}$ such that $t_k \leq \delta_{max}$ and $(l,r) \in \mathcal{E}_P(t_k)$,*
- *if $(l,r) \in \mathcal{E}_P(t_k)$ there exists $\tau \in [t_k, t_k + \delta_{max}]$ such that $(l,r) \in \mathcal{E}_P(\tau)$.*

Remark 14 *Assumption 14 can be easily imposed in a decentralized way since it concerns inter-cluster links one by one in a decoupled manner. This assumption bears only on the few links in $\mathcal{E}_T$ which connect a node in a parent cluster to a root in a child cluster of the structure defined in Assumption 1. Other links may appear in $\mathcal{E}_P(t_k)$ but these are not constrained by Assumption 14. Notice that Assumption 14 ensures that edges in $\mathcal{E}_T$ reset an infinite number of times. We emphasize that a time-invariant δ_{max} is only required to ensure convergence with a geometric rate (see Theorem 13). This can be relaxed to time-varying but sufficiently slowly growing δ_{max} (see Remark 19).*

To justify Assumption 14 we provide an example where consensus is not reached when only Assumptions 1-4 hold.

Remark 15 *Notice that Assumption 14 implies that for all cluster $i \in \{2, \ldots, m\}$ and for all $k \in \mathbb{N}$,*

$$t_{\phi_i(k+1)} - t_{\phi_i(k)} \leq \delta_{max}.$$

Denote by $\Phi(t,T)$ the fundamental matrix over time interval $[t,T]$ of the global linear dynamics $\dot{x}(t) = -L(t)x(t)$, for any $T \geq t \geq 0$, which is uniquely defined [24] by $x(T) = \Phi(t,T)x(t)$. Denote $\Phi_{\mathcal{C}_i}(t,T)$ the fundamental matrix of the linear dynamics $\dot{x}_{\mathcal{C}_i}(t) = -L_i(t)x_{\mathcal{C}_i}(t)$ within cluster $\mathcal{C}_i$. It is important to mention that dynamics (3.2) leads to the collective state trajectory

$$\begin{cases} x(t) = \Phi(t_k,t)P(t_k)x(t_k^-), & \forall k \in \mathbb{N} \text{ and } \forall t \in [t_k, t_{k+1}) \\ x(0) = x_0 \end{cases} \tag{3.26}$$

but a jump occurs in $x_{\mathcal{C}_i}$ only at times $t_{\phi_i(k)}$, which involves edges with sink in cluster i. This can be formalized as

$$\begin{cases} x_{\mathcal{C}_i}(t) = \Phi_{\mathcal{C}_i}(t_{\phi_i(k)},t)P_{\mathcal{C}_i}(t_{\phi_i(k)})x(t_{\phi_i(k)}^-), \ \forall k \in \mathbb{N} \text{ and } \forall t \in [t_{\phi_i(k)}, t_{\phi_i(k+1)}) \\ x(0) = x_0 \end{cases} \tag{3.27}$$

where $P_{\mathcal{C}_i}(t_{\phi_i(k)})$ contains only the rows of $P(t_{\phi_i(k)})$ corresponding to the cluster $\mathcal{C}_i$ (i.e. the rows $m_{i-1}+1, \ldots, m_i$ of $P(t_{\phi_i(k)})$).

3.3.2 Matrix prerequisite properties

In this subsection we provide an instrumental result concerning the matrices defining the state-trajectory associated with the dynamics (3.2). With the graph $\bar{\mathcal{G}} = (\bar{\mathcal{V}}, \bar{\mathcal{E}})$ we associate a time-varying weighted adjacency matrix $A(t)$ which is a matrix with non-negative entries satisfying $A_{i,j}(t) > 0 \Leftrightarrow (i,j) \in \bar{\mathcal{E}}$ for all $t \geq 0$. The corresponding degree matrix $D(t)$ is diagonal and $D_{i,i}(t) = \sum_{j=1}^{n} A_{i,j}(t)$ where n is the size of $A(t)$ which is equal to the cardinality of $\bar{\mathcal{V}}$. The weighted Laplacian matrix associated with $A(t)$ is simply defined as $\bar{L}(t) = D(t) - A(t)$. Moreover, we suppose that $\bar{L}(t)$ satisfies Assumption 15 which implies the fact that there exist $\bar{\alpha} > 0$ and $\underline{\alpha} > 0$ such that $\underline{\alpha} \leq A_{i,j}(t) \leq \bar{\alpha}$ for all $(i,j) \in \bar{\mathcal{E}}$ and for all $t \geq 0$. Finally, denote by $\bar{\Phi}(t,T)$ the fundamental matrix over time interval $[t,T]$ of the linear dynamics $\dot{y}(t) = -\bar{L}(t)y(t)$, for any $T \geq t \geq 0$, which is uniquely defined by

$$y(T) = \bar{\Phi}(t,T)y(t).$$

Proposition 4 *Let $\bar{\mathcal{G}}$ be a directed graph without self loops with n vertices containing a spanning tree and $A(t)$ a weighted adjacency matrix associated with it. Denote r the root of a spanning tree in $\bar{\mathcal{G}}$. Let $D(t)$ and $\bar{L}(t)$ the corresponding degree and weighted Laplacian matrices. Then the fundamental matrix $\bar{\Phi}(t,T)$ is a stochastic matrix. Furthermore, if $\bar{L}$ satisfies Assumption 15, then there holds*

$$\forall\delta>0,\ \forall t\geq 0,\ \forall T\geq t+\delta,\ \forall i\in\{1,\dots,n\},\\ \begin{cases}(\bar{\Phi}(t,T))_{i,r}\geq\gamma^n,\\ (\bar{\Phi}(t,T))_{i,i}\geq\gamma^n\end{cases} \tag{3.28}$$

with

$$\gamma=(n\underline{\alpha})^{-1}\underline{\alpha}e^{-2\bar{\alpha}\delta}(1-e^{-\underline{\alpha}\delta}). \tag{3.29}$$

To prove Proposition 4, we need the following intermediate lemma.

Lemma 3 *Let $(j,i)\in\bar{\mathcal{E}}$ and $t,t'\geq 0$ such that $t'\geq t$. Then, there holds*

$$\begin{cases}(\bar{\Phi}(t,t'))_{i,j}\geq\gamma',\\ (\bar{\Phi}(t,t'))_{i,i}\geq\gamma',\end{cases}$$

with $\gamma'=(n\bar{\alpha})^{-1}\underline{\alpha}e^{-2n\bar{\alpha}(t'-t)}(1-e^{-n\bar{\alpha}(t'-t)})$.

Proof : The proof relies on ideas from [44], and is very similar to the first part of the proof of [44, Proposition 7] (although here we prove the lower bound on one element $(\bar{\Phi}(t,t'))_{i,j}$ rather than on a sum of elements, this is possible thanks to Assumption 15). For the first inequality, we set artificial states $y_j(t)=1$ and $y_k(t)=0$ for $k\neq j$. We then have

$$(\bar{\Phi}(t,t'))_{i,j}=y_i(t'),$$

as given in [44, equation (15)]. We now show that $y_i(t')$ is lower-bounded by γ'. Denote $M=\bar{\alpha}(t'-t)$. Using [44, equation (17)], we have

$$y_j(\tau)\geq e^{-nM},\forall\tau\in[t,t']. \tag{3.30}$$

As a first case, assume that $\forall\tau\in[t,t'],y_i(\tau)\leq e^{-nM}$. Then,

$$\begin{aligned}\dot{y}_i(\tau) &= A_{i,j}(\tau)(y_j(\tau)-y_i(\tau))+\sum_{k\neq i,j}A_{i,h}(\tau)(y_h(\tau)-y_i(\tau))\\ &\geq \underline{\alpha}(e^{-nM}-y_i(\tau))-(n-2)\bar{\alpha}y_i(\tau)\geq\underline{\alpha}e^{-nM}-n\bar{\alpha}y_i(\tau)\\ &\geq -n\bar{\alpha}\left(y_i(\tau)-(n\bar{\alpha})^{-1}\underline{\alpha}e^{-nM}\right),\end{aligned}$$

where we have used $y_h(\tau)\geq 0$ and Assumption 15. It follows then from Gronwall's inequality that

$$\begin{aligned}y_i(t') &\geq (n\bar{\alpha})^{-1}\underline{\alpha}e^{-nM}+e^{-nM}\left(y_i(t)-(n\bar{\alpha})^{-1}\underline{\alpha}e^{-nM}\right)\\ &\geq (n\bar{\alpha})^{-1}\underline{\alpha}e^{-n\bar{\alpha}(t'-t)}\left(1-e^{-n\bar{\alpha}(t'-t)}\right)\\ &\geq (n\bar{\alpha})^{-1}\underline{\alpha}e^{-2n\bar{\alpha}(t'-t)}\left(1-e^{-n\bar{\alpha}(t'-t)}\right).\end{aligned}$$

In the alternative case, denote τ the first time $y_i(\tau) = e^{-nM}$. Let $s \in [\tau, t']$. Using a similar reasoning as in the first case, since $y_h \geq 0, h \in \mathbb{N}$,

$$\dot{y}_i(s) = \sum_{k \neq i} A_{i,h}(s)(y_h(s) - y_i(s)) \geq -n\bar{\alpha} y_i(s),$$

which by integration over time interval $[\tau, t']$ yields

$$y_i(t') \geq e^{-2n\bar{\alpha}(t'-t)} \geq \gamma'.$$

We turn to the second inequality. A direct consequence of equation (3.30) with $j := i$ is that

$$(\bar{\Phi}(t,t'))_{i,i} \geq e^{-n\bar{\alpha}(t'-t)} \geq \gamma'.$$

Proof of Proposition 4 The stochasticity of the fundamental matrix $\bar{\Phi}(t,T)$ was proven in [44, Lemma 6]. We first prove the first item of equation (3.28) when $T = t + \delta$. Let $i \in \{1, \dots, n\}$. Since r is a root of a spanning tree in the graph, i is connected to r by a directed path $(i_0, \dots, i_d)$ with $i_0 = r$ and $i_d = i$. Denote $\tau_h = t + h\frac{\delta}{n}$ for $h \in \{0, \dots, n-1\}$. We have

$$\bar{\Phi}(t, t+\delta) = \prod_{h=n-1}^{0} \bar{\Phi}(\tau_h, \tau_{h+1}),$$

so that

$$(\bar{\Phi}(t,t+\delta))_{i,r} \geq \prod_{h=n-1}^{d} (\bar{\Phi}(\tau_h, \tau_{h+1}))_{i,i} \prod_{h=d-1}^{0} (\bar{\Phi}(\tau_h, \tau_{h+1}))_{i_{h+1},i_h} \geq \gamma^n,$$

where we used both the first and second inequalities in Lemma 3. Then, we have

$$\begin{aligned}(\bar{\Phi}(t,T))_{i,r} &= \sum_{k=1}^{n} (\bar{\Phi}(t+\delta,T))_{i,k} (\bar{\Phi}(t,t+\delta))_{k,r} \\ &\geq \sum_{k=1}^{n} (\bar{\Phi}(t+\delta,T))_{i,k} \gamma^n \geq \gamma^n,\end{aligned}$$

where we have used the stochasticity of $\bar{\Phi}(t+\delta, T)$ for the last inequality. The fact that the second item of equation (3.28) holds can be shown similarly using the second inequality of Lemma 3 applied n times.

Remark 16 *Notice that (see (3.27)) the matrix $\Phi_{\mathcal{C}_i}(t_{\phi_i(k)}, t)$ defines the state trajectory of the cluster $\mathcal{C}_i$ between two reset instants. Moreover, the graph associated with any cluster satisfies the hypothesis of Proposition 4 and the time interval between consecutive reset instants is bounded. Thus, Proposition 4 shows that Assumption 13 (Dwell time) is the corresponding of Assumption 11 (Minimal influence) for the continuous dynamics defined by L_i.*

Remark 17 *We can apply Proposition 4 to the continuous dynamics in each cluster defined in section* **??**. *For given δ and $\delta_{max} > \delta$, Proposition 4 states that for all $i \in \{1, \dots, m\}$ $\Phi_{\mathcal{C}_i}(t_{\phi_i(k)}, t)$ satisfies (3.28) for all $t \in [\delta, \delta_{max}]$. As a consequence, we can define a lower bound on both the impulsive attraction strengths and the attraction strengths resulting from the continuous dynamics as*

$$\alpha = \min(\alpha', \gamma^n),$$

where α' is defined in Assumption 11 and γ is defined in equation (3.29).

3.3.3 Convergence analysis

This part contains the main results of the paper concerning fully decentralized reset rules. The resets of clusters are not synchronized and the intervals $(t_{\phi_i(k)}, t_{\phi_i(k+1)})$ and $(t_{\phi_j(h)}, t_{\phi_j(h+1)})$ may overlap for distinct i and j. This means, $t_{k+1} - t_k$ can be arbitrarily small and the existing results in the literature are not applicable. Assumption 13 (Dwell time) only ensures a dwell time on the resets of the same cluster. In this section, we assume that Assumptions 10- 13, are satisfied. Under such assumptions, we will show that all agents eventually converge toward the same consensus state at exponential speed (Theorem 13). Prior to stating the main result we provide the necessary intermediate ingredients.

For all time $t \in \mathbb{R}_+$, we define the global diameter of the group as

$$\Delta(t) = \bar{x}(t) - \underline{x}(t)$$

with

$$\bar{x}(t) = \max_{i\in\{1,\ldots,n\}} x_i(t) \text{ and } \underline{x}(t) = \min_{i\in\{1,\ldots,n\}} x_i(t).$$

Our goal in the sequel is to show that $\Delta(t)$ approaches 0 when t increases. This requires some intermediate results presented as lemmas in the sequel. All of them are written in terms of minimum $\underline{x}(t)$ but they can be easily transformed in terms of maximum $\bar{x}(t)$.

Summary

- In Lemma 4 we prove that : if an agent resets its state by taking into account a state bigger than $\underline{x}(t)$, then its state after reset will be bigger than $\underline{x}(t)$.
- In Lemma 5 we complement Lemma 4 by proving that, if all the states in the cluster $\mathcal{C}_i$ are bigger than $\underline{x}(t)$ at some time, they will remain bigger than $\underline{x}(t)$ after a finite number of resets.
- In Lemma 6 we prove that during the continuous dynamics the root of a cluster will pull all the states of the corresponding cluster far from the minimum value. Before the next reset concerning this cluster, all its agents are at a strictly positive distance from the minimum $\underline{x}(t)$.
- In Lemma 7 we show that, the distances between the agents of an arbitrarily fixed cluster $\mathcal{C}_w$ and $\underline{x}(t)$ are uniformly lower bounded by a strictly positive value. This is done by induction on a sequence of clusters going from $\mathcal{C}_1$ to $\mathcal{C}_w$ chosen along the spanning tree in $\mathcal{G}$ (see Assumption 10). Combining Lemma 4 and Lemma 6 provides the induction step.
- Finally, in Theorem 13 we use the lemmas to prove the geometric decrease of the diameter $\Delta(t)$.

Lemma 4 (Reset) *Let $i \in \{1,\ldots,m\}$ and $t \geq 0$ fixed. Let $k \in \mathbb{N}$ such that $t_{\phi_i(k)} > t$ the first reset instant of cluster i after t. Assume that there are some $\ell \in \mathcal{V}$, some bound $X \in \mathbb{R}_+$, some $j \in \mathcal{C}_i$ and some bound $\alpha \in (0,1)$ such that*

$$x_\ell(t^-_{\phi_i(k)}) - \underline{x}(t) \geq X \textit{ and } P_{j,\ell}(t_{\phi_i(k)}) \geq \alpha.$$

Then, we have

$$x_j(t_{\phi_i(k)}) - \underline{x}(t) \geq \alpha X.$$

Proof : Using the stochasticity of $P(t_k)$, one obtains $1 = \sum_{h\in\mathcal{V},h\neq\ell} P_{j,h}(t_{\phi_i(k)}) + P_{j,\ell}(t_{\phi_i(k)})$, thus, by equation (3.2),

$$\begin{aligned} x_j(t_{\phi_i(k)})) - \underline{x}(t) = &\sum_{h\in\mathcal{V},h\neq\ell} P_{j,h}(t_{\phi_i(k)})(x_h(t^-_{\phi_i(k)}) - \underline{x}(t)) \\ &+ P_{j,\ell}(t_{\phi_i(k)})(x_\ell(t^-_{\phi_i(k)}) - \underline{x}(t)) \geq \alpha X. \end{aligned}$$

The last inequality follows from the fact that $P(t_{\phi_i(k)}) \geq \mathbf{0}_{n\times n}$, $P_{j,\ell}(t_{\phi_i(k)}) \geq \alpha$ and $x_h(t^-_{\phi_i(k)}) \geq \underline{x}(t)$ since $\underline{x}$ is non-decreasing (i.e. $\underline{x}(t) \leq \underline{x}(t^-_{\phi_i(k)})$).

Considering $\bar{x}_{\mathcal{C}_i}(t) = \max_{j\in\mathcal{C}_i} x_j(t), \underline{x}_{\mathcal{C}_i}(t) = \min_{j\in\mathcal{C}_i} x_j(t)$, the previous lemma can be complemented as follows.

Lemma 5 (Reset) *Let $i \in \{1,\ldots,m\}$ and $t \geq 0$ fixed. Let $t_{\phi_i(k)} > t$ be some reset instant. Assume that there is some bound $X \in \mathbb{R}_+$, such that*

$$\underline{x}_{\mathcal{C}_i}(t^-_{\phi_i(k)}) - \underline{x}(t) \geq X.$$

Then, for all $h \in \mathbb{N}$, for all $\tau \in [t^-_{\phi_i(k)}, t_{\phi_i(k+h)}]$,

$$\underline{x}_{\mathcal{C}_i}(\tau) - \underline{x}(t) \geq \alpha^{h+1} X.$$

Proof : Using Assumption 11 (Minimal influence) and equation (1.2), we have $P_{j,j}(t_{\phi_i(k+h)}) \geq \alpha$ for all $h \in \mathbb{N}$ and $j \in \mathcal{C}_i$. Thus we can apply Lemma 4 with $l := j$ for all $j \in \mathcal{C}_i$. Also, $\underline{x}_{\mathcal{C}_i}$ is non-decreasing between two consecutive reset instants, thus the bound from Lemma 4 is preserved until the next reset of the cluster. This allows us to iterate on h to conclude.

Lemma 6 (Continuous dynamics) *Let $i \in \{1,\ldots,m\}$ and $t \geq 0$ fixed. Let $k \in \mathbb{N}$ such that $t_{\phi_i(k)} > t$ and denote for conciseness the matrix $R = \Phi_{\mathcal{C}_i}(t_{\phi_i(k)}, t_{\phi_i(k+1)})$. Assume that for the root r_i of one spanning tree of the cluster $\mathcal{C}_i$, there exist some bounds $Y \in \mathbb{R}_+$ and $\alpha \in [0,1]$ such that*

$$x_{r_i}(t_{\phi_i(k)}) - \underline{x}(t) \geq Y \text{ and } \forall j \in \mathcal{C}_i, R_{j,r_i} \geq \alpha.$$

Then, we have

$$\underline{x}_{\mathcal{C}_i}(t^-_{\phi_i(k+1)}) - \underline{x}(t) \geq \alpha Y.$$

Proof : Since $x_{\mathcal{C}_i}(t^-_{\phi_i(k+1)}) = R x_{\mathcal{C}_i}(t_{\phi_i(k)})$ with R stochastic, the proof is the same as the one in Lemma 4. The difference is that $\forall j \in \mathcal{C}_i, R_{j,r_i} \geq \alpha$. The proof can be applied for all $j \in \mathcal{C}_i$ and a minimum can be taken at the end.

Before giving the next result, let us introduce some notation that will simplify the presentation. Let $\mathcal{C}_w$ be some cluster. According to Assumption 10, there is a sequence of clusters $(K_1,\ldots,K_q)$ with $q \leq m$ connecting $\mathcal{C}_w$ to $\mathcal{C}_1$, meaning that $K_1 = \mathcal{C}_1$, $K_q = \mathcal{C}_w$ and for each intermediate cluster $h \in \{1,\ldots,q-1\}$, there is a node $l \in K_h$ and a root r of a spanning tree of K_{h+1} with $(l,r) \in \mathcal{E}_T$.

Let $t \geq 0$ be fixed. We define a sequence of integers

$$t \leq f_1 < s_1 < f_2 < s_2 < \ldots < f_q < s_q \tag{3.31}$$

such that

- f_1 is the first reset instant after t of a root of a spanning tree of cluster K_1, if a root reseting its state exists in K_1. This may not be the case since K_1 may not be influenced by the other clusters (according to Assumption 10), then $f_1 = t$ and $s_1 = t + \delta$.
- For all $h \in \{2,\ldots,q\}$, we define s_h the first instant after f_h when an agent of K_h resets its state.
- For all $h \in \{1,\ldots,q-1\}$ we define f_{h+1} as the first reset instant of a root of a spanning tree of cluster K_{h+1} after time s_h.

It is noteworthy that, thanks to Assumption 14,

$$f_{h+1} - s_h \leq \delta_{max} \text{ and } s_h - f_h \leq \delta_{max}. \tag{3.32}$$

This also gives

$$s_q - f_1 \leq (2q-1)\delta_{max} \leq (2m-1)\delta_{max}. \tag{3.33}$$

Let also introduce

$$\mu = \lfloor \delta_{max}/\delta \rfloor \tag{3.34}$$

where $\lfloor y \rfloor$ denotes the biggest integer smaller than y.

Remark 18 *Due to Assumptions 14 (Maximum inactivation time) and 13 (Dwell time), we have at most μ resets of a root of cluster K_h between s_h and f_{h+1}.*

In the sequel, iteratively applying Lemmas 4 and 6, we will show in Theorem 13 that $\Delta(s_q)$ geometrically decreases. For the next result we assume that a root r_1 of a spanning tree of $K_1 = \mathcal{C}_1$ satisfies

$$x_{r_1}(f_1) - \underline{x}(f_1^-) \geq \Delta(f_1^-)/2.$$

If it is not the case, we instead consider the system where all the states have been reversed : $x_i := -x_i$ and apply the same reasoning. In other words we relate the reasoning to the maximum instead of the minimum. In the sequel, we use $\underline{x}_{K_h}(t) = \min_{i \in K_h} x_i(t)$.

Lemma 7 (Path of clusters) *For all $h \in \{1, \dots, q\}$, we have*

$$\underline{x}_{K_h}(s_h^-) - \underline{x}(f_1^-) \geq \frac{\alpha^{(\mu+3)(h-1)+1}\Delta(f_1^-)}{2}. \tag{3.35}$$

where μ is given in equation (3.34).

Proof : We show the lemma by induction. Due to Assumptions 13 (Dwell time) and 14 (Maximum inactivation time), one has $\delta_{max} \geq s_1 - f_1 \geq \delta$, so that Proposition 4 applies to $R = \Phi_{\mathcal{C}_{i_1}}(f_1, s_1)$. The value α is chosen as in Remark 17. As a consequence we can apply previous lemmas with the same α. Lemma 6 yields

$$\underline{x}_{K_1}(s_1^-) - \underline{x}(f_1^-) \geq \frac{\alpha\Delta(f_1^-)}{2},$$

which shows equation (3.35) for $h = 1$. Assume the proposition is true for some $h \in \{1, \dots, p\}$ where $p \leq q-1$ and we prove the same for $h = p+1$. As mentioned in Remark 18, there will be at most μ resets of cluster K_p over (s_p, f_{p+1}). Thus, denoting ℓ such that $t_{\phi_p(\ell)} = s_p$, we have $f_{p+1} \leq t_{\phi_p(\ell+\mu)}$.

We can apply Lemma 5 so that

$$\underline{x}_{K_p}(f_{p+1}) - \underline{x}(f_1^-) \geq \alpha^{\mu+1} \cdot \frac{\alpha^{(\mu+3)(p-1)+1}\Delta(f_1^-)}{2}.$$

At time f_{p+1}, cluster K_{p+1} resets. A root r_{p+1} of K_{p+1} receives influence from at least one agent j in cluster K_p. Because of Assumption 11, $P_{r_{p+1},j}(f_{p+1}) \geq \alpha$. So, we apply Lemma 4 on K_{p+1} to get

$$x_{r_{p+1}}(f_{p+1}) - \underline{x}(f_1^-) \geq \alpha^{\mu+2} \cdot \frac{\alpha^{(\mu+3)(p-1)+1}\Delta(f_1^-)}{2}.$$

To conclude, we apply Lemma 6 on $\mathcal{C}_{p+1}$ with $R = \Phi_{\mathcal{C}_{p+1}}(f_{p+1}, s_{p+1})$ and we get

$$\underline{x}_{K_{p+1}}(s_{p+1}^-) - \underline{x}(f_1^-) \geq \frac{\alpha^{(\mu+3)p+1}\Delta(f_1^-)}{2}.$$

A corollary of Lemma 7 is the following proposition.

Proposition 5 *We have*

$$\underline{x}((2m-1)\delta_{max}+f_1)-\underline{x}(f_1^-)\geq \alpha^{\nu+1}\frac{\alpha^{(\mu+3)(m-1)+1}\Delta(f_1^-)}{2} \tag{3.36}$$

with $\nu = \lfloor (2m-1)\delta_{max}/\delta \rfloor$.

Proof : Taking $h = q$ in Lemma 7 gives a lower bound on the minimum of $\mathcal{C}_w = K_q$. This is true for any cluster $\mathcal{C}_w$. Using $h \leq m$ in equation (3.35), the bound can be replaced by

$$\frac{\alpha^{(\mu+3)(m-1)+1}\Delta(f_1^-)}{2}.$$

Then, equation (3.33) guarantees that there is no more than ν resets of cluster $\mathcal{C}_w$ over $[f_1^-, (2m-1)\delta_{max}+f_1^-]$, thus Lemma 5 gives that for all cluster $\mathcal{C}_w$,

$$\underline{x}_{\mathcal{C}_w}((2m-1)\delta_{max}+f_1)-\underline{x}(f_1^-)\geq \alpha^{\nu+1}\frac{\alpha^{(\mu+3)(m-1)+1}\Delta(f_1^-)}{2}.$$

In other words, equation (3.36) holds.

Once Proposition 5 is given, the exponential decay of the network diameter comes easily.

Theorem 13 *Denote* $\nu = \lfloor (2m-1)\delta_{max}/\delta \rfloor$ *and* $\mu = \lfloor \delta_{max}/\delta \rfloor$ *and let us define*

$$\beta = (1-\alpha^{\nu+1}\frac{\alpha^{(\mu+3)(m-1)+1}}{2}) \in [0,1)$$

with $\alpha = \min(\gamma^n, \alpha)$ *where* α' *is defined in Assumption 11 and* γ *is defined in equation (3.29). Then, for all* $t \in \mathbb{R}_+$,

$$\Delta(2(m+1)\delta_{max}+t) \leq \beta\Delta(t).$$

Remark 19 *From the definition of* β *in Theorem 13 one can see that, considering a time-varying* δ_{max} *leads to a time-varying* β. *The consensus is still guaranteed as far as* δ_{max} *is not growing too fast i.e. for all* $t \in \mathbb{R}_+$ *one has*

$$\lim_{k\to\infty}\prod_{i=1}^{k}\beta(t+2(m+1)\sum_{j=1}^{i}\delta_{max}^i)=0, \tag{3.37}$$

where $\left(\delta_{max}^i\right)_{i\geq 1}$ *denotes the sequence of upper bounds of the time intervals between consecutive activations of inter-clusters links. When* δ_{max} *grows sufficiently fast to make the limit in (3.37) strictly positive, the consensus is no longer guaranteed as proven in Example 1.*

Proof of Theorem 13 : Let $t \geq 0$ be fixed and define f_1 as in (3.31). It follows that

$$t \leq f_1 < f_1 + (2m-1)\delta_{max} \leq t+2(m+1)\delta_{max}.$$

Since $\bar{x}$ is non-increasing and $\underline{x}$ is non-decreasing, one has

$$\Delta(t+2(m+1)\delta_{max}) \leq \Delta((2m-1)\delta_{max}+f_1),\ \Delta(f_1^-)\leq\Delta(t). \tag{3.38}$$

On the other hand, using Proposition 5, we have

$$\Delta((2m-1)\delta_{max}+f_1) = \bar{x}((2m-1)\delta_{max}+f_1) - \underline{x}((2m-1)\delta_{max}+f_1)$$
$$\leq \bar{x}(f_1^-) - \underline{x}(f_1^-) - \alpha^{\nu+1}\frac{\alpha^{(\mu+3)(m-1)+1}\Delta(f_1^-)}{2} \leq (1-\alpha^{\nu+1}\frac{\alpha^{(\mu+3)(m-1)+1}}{2})\Delta(f_1^-).$$

The proof ends by combining this with (3.38).

3.4 Design of event triggered reset rule for consensus

The dynamics (3.2) can be used for consensus in fleets of robots that are partitioned in clusters. The robots that are relatively close one to another continuously interact and form a cluster. Inter-cluster interactions need supplementary energy associated to long distance communications between clusters and consequently, they have to be activated only if needed. In order to avoid unnecessary inter-cluster communications we can define the reset sequence using an event-based strategy. One example of such strategy is analyzed in this section. Precisely, we consider the asynchronous resets case and suppose Assumptions 10 (Network structure) and 11 (Minimal influence) are satisfied. We define two event triggered reset rules and show that they satisfy Assumption 13 and 14. Therefore, Theorem 13 applies and the consensus occurs.
The first rule defining the reset sequence is not fully decentralized. Each reseting element r_i needs to know the states of all the agents in its own cluster $\mathcal{C}_i$ at the reset instants $t_{\phi_i(k)}$. This information is used to compute the local agreement value $x_i^*(t_k)$. It is worth noting that this information, which is centralized at cluster level, is needed only at isolated instants corresponding to resets. Indeed, the function $x_i^*(\cdot)$ is constant on the interval $[t_{\phi_i(k)}, t_{\phi_i(k+1)})$. The second rule defining the reset sequence is fully decentralized since it requires only local/decentralized information. The reseting element r_i of cluster $\mathcal{C}_i$ needs to know only the state of its neighbors. Nevertheless, each reseting element r_i has to compute the maximal distance to its neighbors at every instant $t \in [t_{\phi_i(k)}, t_{\phi_i(k+1)})$.
In the following we use the following instrumental assumption :

Assumption 15 *The components of the Laplacian matrix L are uniformly bounded i.e. there exists* $\overline{\alpha} > 0$ *finite real number such that* $|L_{i,j}| \leq \overline{\alpha}$, $\forall i,j \in \{1,\dots,n\}$.

This is necessary to ensure that during the continuous dynamics, the agents do not approach one to another indefinitely fast. We can notice that, in practice, this assumption is very natural and is almost always satisfied.

3.4.1 Semi-decentralized event-triggered resets

First, let us introduce the diameter of the cluster $\mathcal{C}_i$ as

$$\Delta_i(t) = \max_{j\in\mathcal{C}_i} x_j(t) - \min_{j\in\mathcal{C}_i} x_j(t).$$

Definition 7 *The distance between the node* r_i *and the local agreement value of the cluster* $\mathcal{C}_i$ *at time t is denoted by* $d_i(t) =| x_{r_i}(t) - x_i^*(t) |$. *Considering* $\epsilon > 0$ *a fixed scalar, the reset sequence* $(t_k)_{k\in\mathbb{N}}$ *associated with the dynamics (3.2) is defined as follows : for all* $i \in \{1,\dots,m\}$ *and for all* $k \geq 0$,

— *if* $d_i(t_{\phi_i(k-1)}) > \epsilon$ *we define*

$$t_{\phi_i(k)} = \min_{t\geq t_{\phi_i(k-1)}} \left\{ d_i(t) \leq \frac{d_i(t_{\phi_i(k-1)})}{a_i} \right\},$$

— *otherwise,* $t_{\phi_i(k)} = t_{\phi_i(k-1)} + \delta$ *with*

$$\delta = \min_{i\in\{1,\dots,m\}} \frac{a_i - 1}{a_i}\frac{\epsilon}{2n_i\bar{\alpha}\Delta_i(0)},$$

where the $a_i > 1$ *are design parameters fixed a priori. (We recall that for consistency, we denote* $t_{\phi_i(-1)} = 0$*).*

Remark 20 *In our numerical illustrations we consider* $a_i = a_j,\ \forall i, j \in \{1, \dots, m\}$ *but these values can be used by the designer to change the reset frequency of some/all clusters.*

Proposition 6 *Let us consider the dynamics* (3.2) *under Assumptions 10, 11 and 15. Then, the associated reset sequence introduced by Definition 7 satisfies the Assumptions 14, 13.*

Proof : If $d_i(t_{\phi_i(k)}) \leq \epsilon$, the second point in Definition 7 applies and Assumptions 14, 13 hold. Otherwise, the first point applies and for a fixed cluster $\mathcal{C}_i,\ i \in \{1, \dots, m\}$ we have to show that the reset sequence satisfies Assumptions 14, 13.

• Again, we start by proving that Assumption 13 holds. This means, a dwell time δ exists between a reset time $t_{\phi_i(k)}$ and the first time t such that when $d_i(t) \leq \dfrac{d_i(t_{\phi_i(k)})}{a_i}$.

We recall that for all $t \in [t_{\phi_i(k)}, t_{\phi_i(k+1)})$ one has $\dot{x}_i^*(t) = 0$ and

$$\dot{x}_{r_i}(t) = -\sum_{j\in\mathcal{C}_i} L_{r_i,j}(x_j(t) - x_{r_i}(t))$$

Thus, using Assumption 15, one obtains that between two reset instants the following holds :

$$\dot{d}_i(t) \geq -n_i\bar{\alpha}\Delta_i(t).$$

Moreover, since $\Delta(\cdot)$ is non-increasing function one obtains that

$$\dot{d}_i(t) \geq -n_i\bar{\alpha}\Delta_i(0).$$

Integrating the last equation we finally get that

$$d_i(t) \geq d(t_{\phi_i(k)}) - n_i\bar{\alpha}\Delta_i(0)(t - t_{\phi_i(k)}).$$

Thus, in order to have $d_i(t) \leq \dfrac{d_i(t_{\phi_i(k)})}{a_i}$ one needs

$$\frac{d_i(t_{\phi_i(k)})}{a_i} \geq d(t_{\phi_i(k)}) - n_i\bar{\alpha}\Delta_i(0)(t - t_{\phi_i(k)})$$

which is equivalent with

$$(t - t_{\phi_i(k)}) \geq \frac{a_i - 1}{a_i}\frac{d(t_{\phi_i(k)})}{n_i\bar{\alpha}\Delta_i(0)} > \frac{a_i - 1}{a_i}\frac{\epsilon}{n_i\bar{\alpha}\Delta_i(0)}$$

This part of the proof finishes by choosing

$$\delta = \min_{i\in\{1,\dots,m\}} \frac{a_i - 1}{a_i}\frac{\epsilon}{n_i\bar{\alpha}\Delta_i(0)}.$$

• Let us show now that Assumption 14 also holds. Between $d_i(t_{\phi_i(k)})$ and $d_i(t_{\phi_i(k+1)})$ the overall dynamics of the cluster $\mathcal{C}_i$ is described by a fixed Laplacian matrix

$$\dot{x}_{\mathcal{C}_i}(t) = -L_i x_{\mathcal{C}_i}(t).$$

Thus, there exists $M_i > 0$ and $\rho_i > 0$ such that

$$\Delta_i(t) \leq M_i e^{-\rho_i(t-t_{\phi_i(k-1)})} \Delta_i(t_{\phi_i(k)}) \tag{3.39}$$

where ρ_i is the convergence speed (see for instance [57]) associated with the matrix L_i. Moreover, one has $d_i(t) \leq \Delta_i(t),\ \forall i \in \{1, \ldots, m\}$ and $\forall t \geq 0$. Combining this with (3.39) yields

$$d_i(t) \leq M_i e^{-\rho_i(t-t_{\phi_i(k-1)})} \Delta_i(t_{\phi_i(k)}) \tag{3.40}$$

Using (3.40), straightforward mathematical manipulation shows that $d_i(t) > \frac{d_i(t_{\phi_i(k)})}{a_i}$ implies

$$\begin{aligned} t - t_{\phi_i(k-1)} &< \frac{1}{\rho_i} \ln\left(\frac{a_i M_i \Delta_i(t_{\phi_i(k)})}{d_i(t_{\phi_i(k)})}\right) \\ &< \frac{1}{\rho_i} \ln\left(\frac{a_i M_i \Delta_i(0}{\epsilon}\right). \end{aligned}$$

The proof ends by defining

$$\delta_{max} = \max_{i \in \{1,\ldots,m\}} \frac{1}{\rho_i} \ln\left(\frac{a_i M_i \Delta_i(0}{\epsilon}\right).$$

3.4.2 Fully decentralized event-triggered resets

Definition 8 *The distance between the node r_i and its fairest neighbor in cluster $\mathcal{C}_i$ at time t is denoted by*

$$\tilde{d}_i(t) = \max_{j \text{ such that } (r_i,j)\in\mathcal{E}_L} \mid x_{r_i}(t) - x_j(t) \mid.$$

Considering $\epsilon > 0$ a fixed scalar, the reset sequence $(t_k)_{k\in\mathbb{N}}$ associated with the dynamics (3.2) is defined as follows : for all $i \in \{1, \ldots, m\}$ and for all $k \geq 0$,

— *if $\tilde{d}_i(t_{\phi_i(k-1)}) > \epsilon$ we define*

$$t_{\phi_i(k)} = \min_{t \geq t_{\phi_i(k-1)}} \left\{ \tilde{d}_i(t) \leq \frac{\tilde{d}_i(t_{\phi_i(k-1)})}{a_i} \right\},$$

— *otherwise, $t_{\phi_i(k)} = t_{\phi_i(k-1)} + \delta$ with*

$$\delta = \min_{i \in \{1,\ldots,m\}} \frac{a_i - 1}{a_i} \frac{\epsilon}{n_i \bar{\alpha} \Delta_i(0)},$$

where the $a_i > 1$ are design parameters fixed a priori. (We recall that for consistency, we denote $t_{\phi_i(-1)} = 0$).

Remark 21 *Once again we can fix $a_i = a_j,\ \forall i, j \in \{1, \ldots, m\}$ or we can use a_i to change the reset frequency of some/all clusters.*

Proposition 7 *Let us consider the dynamics (3.2) under Assumptions 10, 11 and 15. Then, the associated reset sequence introduced by Definition 8 satisfies the Assumptions 14, 13.*

Proof : The proof is similar to the one of Proposition 6. Nevertheless, some technical details have to be pointed out. If $\tilde{d}_i(t_{\phi_i(k)}) \leq \epsilon$, the second point in Definition 7 applies and Assumptions 14, 13 hold. Otherwise, the first point applies and for a fixed cluster $\mathcal{C}_i,\ i \in \{1, \ldots, m\}$ we have to show that the reset sequence satisfies Assumptions 14, 13.

• We start by proving that Assumption 13 holds. For any $j \in \mathcal{C}_i$ such that $(r_i, j) \in \mathcal{E}_l$ we define

$$\tilde{d}_{i,j}(t_{\phi_i(k)}) = \|x_{r_i}(t_{\phi_i(k)}) - x_j(t_{\phi_i(k)})\|.$$

Let $\tilde{j} \in \mathcal{C}_i$ be one neighbor of r_i such that $\tilde{d}_i(t_{\phi_i(k)}) = \tilde{d}_{i,\tilde{j}}(t_{\phi_i(k)})$. We emphasize that for all $t \in [t_{\phi_i(k)}, t_{\phi_i(k+1)})$ one has

$$\dot{x}_{r_i}(t) = -\sum_{j \in \mathcal{C}_i} L_{r_i,j}(x_j(t) - x_{r_i}(t))$$
$$\dot{x}_{\tilde{j}}(t) = -\sum_{j \in \mathcal{C}_i} L_{\tilde{j},j}(x_j(t) - x_{\tilde{j}}(t))$$

yielding, as in the proof of Proposition 6,

$$\dot{\tilde{d}}_{i,\tilde{j}}(t) \geq -2n_i\bar{\alpha}\Delta_i(0).$$

Integrating this equation we get

$$\tilde{d}_{i,\tilde{j}}(t) \geq \tilde{d}_{i,\tilde{j}}(t_{\phi_i(k)}) - 2n_i\bar{\alpha}\Delta_i(0)(t - t_{\phi_i(k)}).$$

But $\tilde{d}_i(t) \geq \tilde{d}_{i,\tilde{j}}(t), \forall t \geq 0$ and $\tilde{d}_i(t_{\phi_i(k)}) = \tilde{d}_{i,\tilde{j}}(t_{\phi_i(k)})$ which leads as to

$$\tilde{d}_i(t) \geq \tilde{d}_i(t_{\phi_i(k)}) - 2n_i\bar{\alpha}\Delta_i(0)(t - t_{\phi_i(k)}).$$

From this point we continue by mimicking the proof of Proposition 6. In order to have $\tilde{d}_i(t) \leq \dfrac{\tilde{d}_i(t_{\phi_i(k)})}{a_i}$ one needs

$$\frac{\tilde{d}_i(t_{\phi_i(k)})}{a_i} \geq \tilde{d}(t_{\phi_i(k)}) - 2n_i\bar{\alpha}\Delta_i(0)(t - t_{\phi_i(k)})$$

which is equivalent with

$$(t - t_{\phi_i(k)}) \geq \frac{a_i - 1}{a_i}\frac{\tilde{d}(t_{\phi_i(k)})}{2n_i\bar{\alpha}\Delta_i(0)} > \frac{a_i - 1}{a_i}\frac{\epsilon}{2n_i\bar{\alpha}\Delta_i(0)}$$

This part of the proof finishes by choosing

$$\delta = \min_{i \in \{1,\ldots,m\}} \frac{a_i - 1}{a_i}\frac{\epsilon}{2n_i\bar{\alpha}\Delta_i(0)}.$$

• To prove that Assumption 14 also holds, we simply notice that $\tilde{d}_i(t) \leq \Delta_i(t),\ \forall i \in \{1, \ldots, m\}$. Thus the arguments in the proof of Proposition 6 can be applied exactly in the same manner to define

$$\delta_{max} = \max_{i \in \{1,\ldots,m\}} \frac{1}{\rho_i} \ln\left(\frac{a_i M_i \Delta_i(0}{\epsilon}\right)$$

that upper-bounds all the intervals $[t_{\phi_i(k)}, t_{\phi_i(k+1)})$.

3.5 Numerical simulations

3.5.1 Synchronous activation of inter-cluster links

Let us first illustrate Theorems 10 and 11. The numerical treatment of the parametric LMIs will be done following the ideas in Theorem 12. In order to limit the number of LMIs to solve, we have chosen $h = 5$ and embed the set $\{X \in \mathbb{R}^{n\times n} \mid X = e^{-L\delta_a},\ \delta_a \in \Delta\}$ into the polytopic set defined by the vertices $Z_1, \ldots, Z_{h+1}$.

Small network analysis

An academic example consisting in a network of 5 agents partitioned in 2 clusters ($n_1 = 3, n_2 = 2$) is used in the sequel to illustrate the theoretical results. We consider the dynamics (3.2) with

$$L = \begin{bmatrix} 4 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & -3 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix},\ P_l = \begin{bmatrix} 0.45 & 0.55 \\ 0.25 & 0.75 \end{bmatrix} \tag{3.41}$$

and the reset sequence given by $\delta = 0.5$ and δ'_k randomly chosen in $\Delta = [0, 0.2]$. The initial condition of the system is $x(0) = (8, 7, 9, 2, 3)$ and the corresponding consensus value computed by (3.16) is $x^* = 4.6757$. The convergence of the 5 agents towards x^* is illustrated in Figure 1 emphasizing that the leaders trajectories are non-smooth while the followers trajectories are. The table below collects the

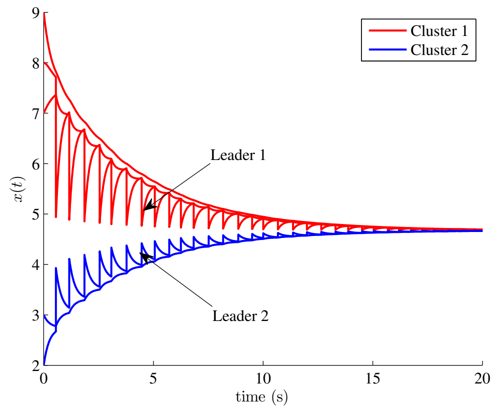

FIGURE 3.1 – The state-trajectories of the agents converging to the calculated consensus value.

first 10 time intervals between consecutive reset instants. As expected, these time intervals have random lengths within $[0.5, 0.7]$ and no monotony occurs. The jumps and decreasing of the Lyapunov function defined by Theorem 10 are pointed out in Figure 2. We emphasize that the matrices $S(\delta')$ used to define V are obtained as in the proof of Theorem 12 after solving (3.25) for $h = 5$. In order to illustrate the independence of the consensus value on the reset sequence (see Remark 10), we also considered $\delta = 5$. In this case, as can be seen in Figure 3.3, the local agreements are reached before each reset and we better emphasize their piece-wise constant behavior (see Remark 13). As expected the consensus value remains $x^* = 4.6757$.

Analyzing equation (3.15) we obtain that the consensus value is always a convex combination of the initial agreement values of the clusters. In the present case, one has two clusters and the two initial

$t_1 - t_0$	0.6189 s	$t_6 - t_5$	0.5979 s
$t_2 - t_1$	0.6131 s	$t_7 - t_6$	0.5372 s
$t_3 - t_2$	0.6433 s	$t_8 - t_7$	0.6401 s
$t_4 - t_3$	0.6023 s	$t_9 - t_8$	0.6965 s
$t_5 - t_4$	0.6553 s	$t_{10} - t_9$	0.6613 s

TABLE 3.1 – The length of the first 10 time intervals between consecutive reset instants.

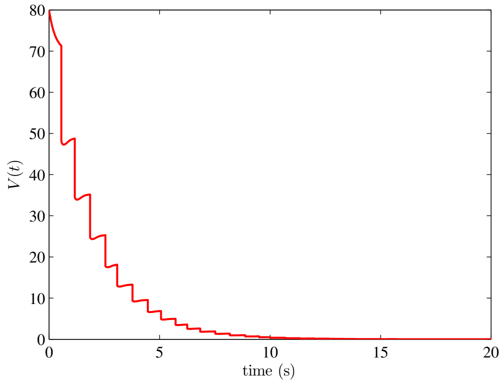

FIGURE 3.2 – The behavior of the Lyapunov function given by Theorem 10.

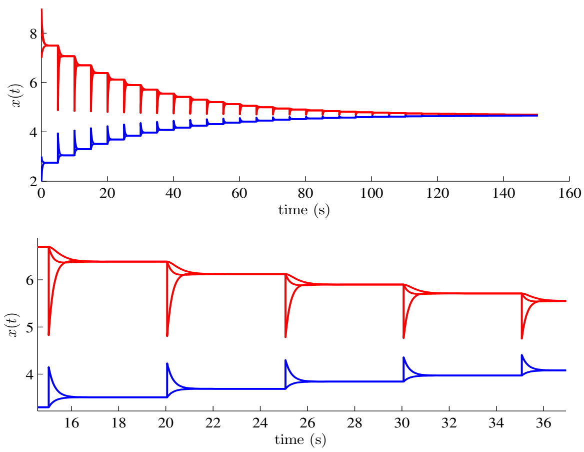

FIGURE 3.3 – Top : State trajectories of the agents converging to the piece-wise constant local agreements. The local agreements approach one of each other at the reset times. Bottom : Zoom emphasizing the state behavior. Jumps are present only in the leaders trajectories.

agreements are 2.75 and 7.5. Thus, we can try to reach only consensus values belonging to $[2.75, 7.5]$. In Figure 3.4 the consensus value is fixed at $x^* = 6.5$ and the associated P_l matrix is

$$P_l = \begin{bmatrix} 0.6870 & 0.3130 \\ 0.7825 & 0.2175 \end{bmatrix}.$$

In Figure 3.5 the consensus value is fixed as $x^* = 3.5$ and based on Theorem 11 and the polytopic

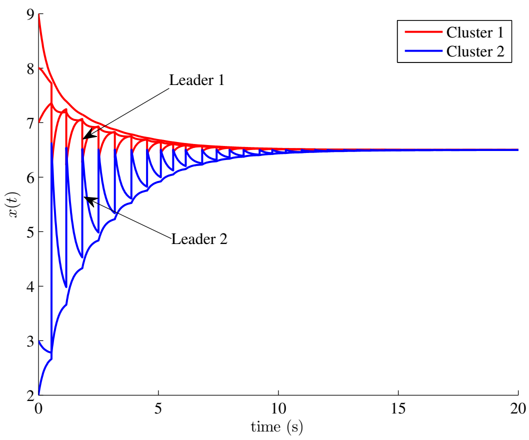

FIGURE 3.4 – The states of a system ($x^* = 6.5$).

embedding described in the previous Section we get P_l matrix is

$$P_l = \begin{bmatrix} 0.3010 & 0.6990 \\ 0.0874 & 0.9126 \end{bmatrix}. \tag{3.42}$$

Finally, in Figure 3.6 the associated Lyapunov function is plotted for both consensus values. Numerical

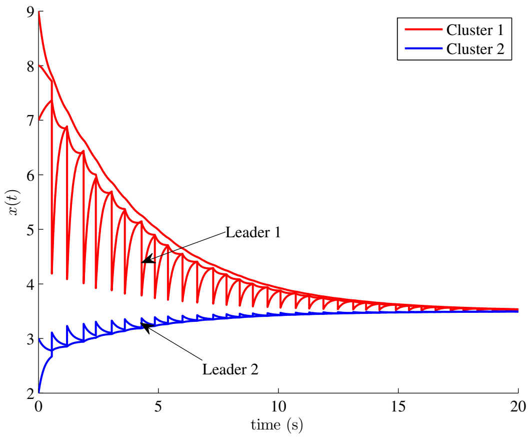

FIGURE 3.5 – The states of a system ($x^* = 3.5$).

simulations have confirmed the intuition that P_l tends to $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ when x^* approaches the initial local agreement of the second cluster 2.75 while P_l tends to $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ when x^* approaches 7.5 the initial local agreement of the first cluster .

Larger network analysis

In order to prove that the algorithms are implementable in real networks we consider in the following a larger system. Precisely, we present an example consisting of a network of 100 agents partitioned in 3

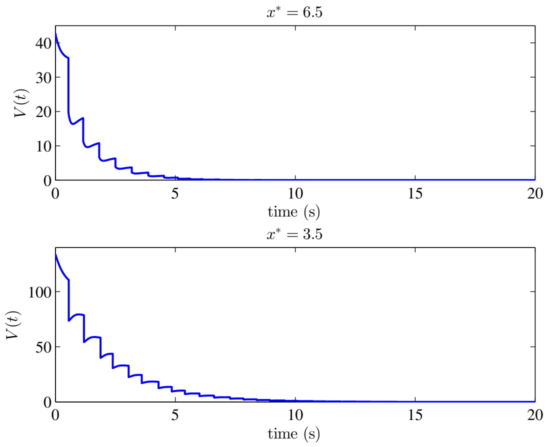

FIGURE 3.6 – The Lyapunov function of the system

clusters. The size of the clusters as well as the connections between agents are randomized, resulting in non-symmetric matrices L and P_l. The random initialization leads at $n_1 = 59, n_2 = 20, n_3 = 21$ and

$$P_l = \begin{bmatrix} 0.1538 & 0.8080 & 0.0382 \\ 0.4886 & 0.3876 & 0.1238 \\ 0.1266 & 0.2805 & 0.5929 \end{bmatrix}. \tag{3.43}$$

The initial condition is also randomized but, in order to guarantee a relatively large interval for the possible consensus value, for the first cluster the initial states of the agents are randomly chosen within $[0, 3]$, for the second one within $[3, 7]$ and for the third one within $[7, 10]$. The corresponding initial local agreement values are 1.2970, 5.2578 and 8.7556, respectively. We illustrate the theoretical results by using the dynamics (3.2) with the reset sequence given by $\delta = 0.5$. The corresponding consensus value computed by (3.16) is $x^* = 4.2562$. The convergence of the 100 agents towards x^* is shown in Figure 3.7.

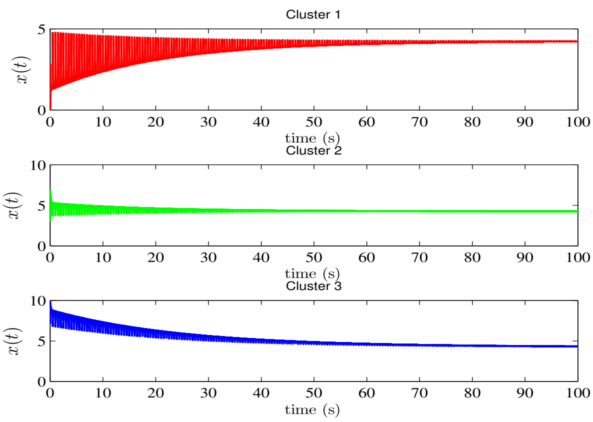

FIGURE 3.7 – The state-trajectories of the agents converging to the calculated consensus value.

As noticed before, from (3.15) we deduce that the consensus value is always a convex combination of the initial agreement values of the clusters. Therefore, for the initialization above, any consensus value can be imposed between 1.2970 and 8.7556. In Figure 3.8 the consensus value was fixed at $x^* = 6.5$ and one obtained P_l matrix is

$$P_l = \begin{bmatrix} 0.0643 & 0.3720 & 0.5637 \\ 0.3064 & 0.0358 & 0.6578 \\ 0.0360 & 0.1917 & 0.7723 \end{bmatrix}.$$

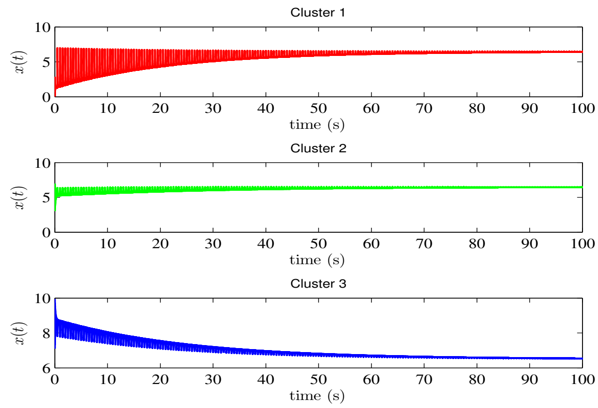

FIGURE 3.8 – The states of a system ($x^* = 6.5$).

3.5.2 Asynchronous activation of inter-cluster links

In the following we illustrate Theorem 13.

5-agent system

In the following we consider a network of five agents grouped in two clusters. The network structure satisfies Assumption 10 and is described by the following Laplacian matrix :

$$L = \begin{pmatrix} 3 & 0 & -3 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}$$

which has a block diagonal structure corresponding to the two clusters. Each cluster contains only one node able to interact with agents outside its own cluster (node 1 in the first cluster and node 4 in the second cluster). The weights of the inter-cluster interactions are chosen as follows

$$P = \begin{pmatrix} 0.7 & 0 & 0 & 0.3 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0.25 & 0 & 0 & 0.75 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

such that Assumption 11 holds. We point out that at reset times t_k either only one or both nodes 1 and 4 reset their state. Therefore, the matrices $P(t_k)$ are either equal to P or obtained by replacing the first or forth line of P by $(1,0,0,0,0)$ or $(0,0,0,1,0)$, respectively. Assumptions 13 and 14 are guaranteed by the choice of $\delta = 4$ and $\delta_{max} = 8$.

In Figure 3.9 we firstly emphasize the agreement of all five agents. A zoom-in allows to point out that each impulsive agent resets its state in its own rythme and it may happen that one of them resets twice between the reset times of the other.

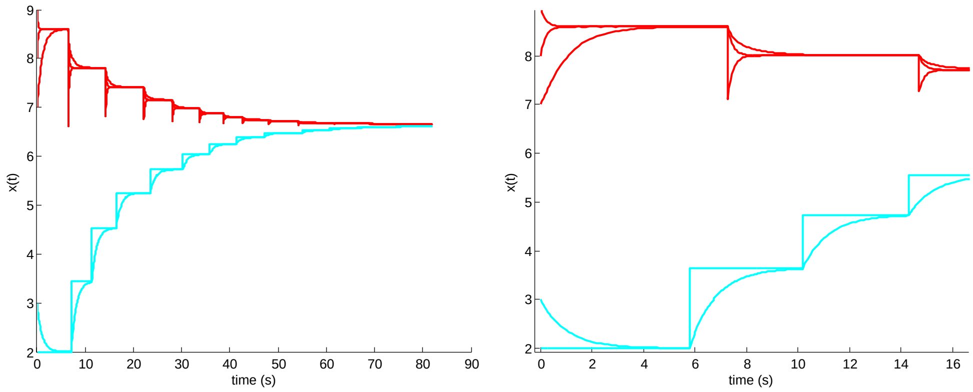

FIGURE 3.9 – Left : Consensus of the five agents grouped in 2 clusters. Right : Zoom in pointing out that the resets are not synchronized.

30-agent system

We now consider networks of 30 agents grouped in 3 clusters of similar size. We initialize the agents' state so as to enable visual distinction of the 3 clusters. The intra-cluster network is randomly constructed to ensure that each cluster contains a spanning tree (not necessarily unique). Potentially, several agents of the cluster are roots of spanning trees. In a similar way, a network of inter-cluster links between the roots of the clusters is constructed to ensure that a spanning tree connects at least one of these roots to all others. This guarantees that Assumption 10 is satisfied. For simplicity, the intra-cluster weights in L and inter-cluster weights in P are chosen constant. Also, we assume that all resets in a given cluster occur synchronously but resets in different clusters may happen asynchronously. We set the mininmum and maximum inter-activation reset threshold to $\delta = 10$ and $\delta_{max} = 20$, respectively. In Figure 3.10 and 3.11 are displayed the trajectories of the 30-agent system for two disctinct topologies. In Figure 3.10, none of the agents in the top initial cluster (in blue) is influenced by outer agents so that local consensus is quickly reached in this cluster. The top cluster influences the bottom cluster (in red) which in turn influences the middle cluster (in green). Thus, the overall interaction network is not strongly connected. The zoom-in view presented in the bottom figure shows that several agents reset their states in each cluster. The exponential decrease of the global diameter takes place, as expected thanks to Theorem 13. By contrast, Figure 3.11 presents a case where the interaction network between the 3 clusters is strongly connected : the top (blue) cluster is influenced by the bottom (red) cluster, the bottom (red) cluster is influenced by the middle (green) cluster and the middle (green) is itself influenced by the top (blue) cluster. So, the overall interaction network between clusters is a cycle. Once again, the diameter exponentially converges to 0.

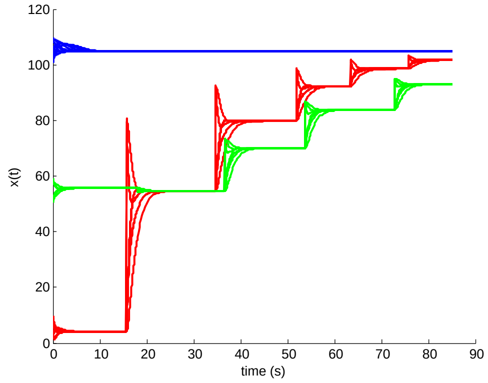

FIGURE 3.10 – Trajectory of the reset system (3.2) with 30 agents grouped in 3 clusters. The overall interaction network topology among cluster is a tree. The top (blue) cluster influences the bottom (red) clusters which influences the middle (green) cluster.

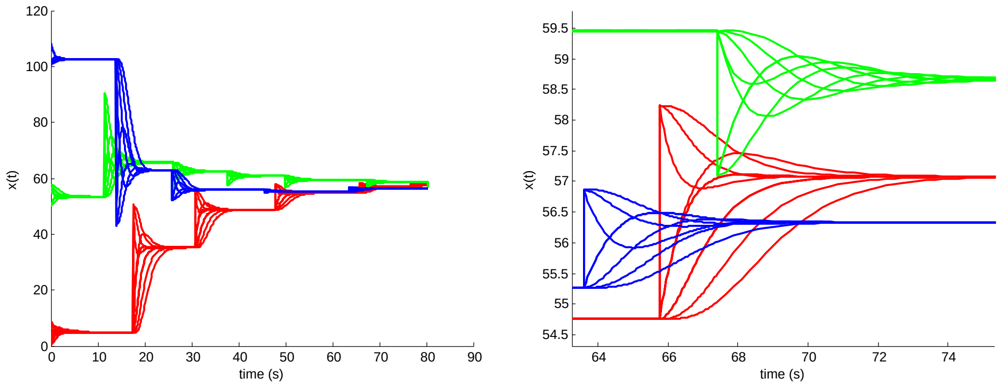

FIGURE 3.11 – Left : Trajectory of the reset system (3.2) with 30 agents grouped in 3 clusters. The overall interaction network topology among cluster is a cycle. The top (blue) cluster influences the middle (green) clusters which influences the bottom (red) cluster which influences the top (blue) cluster. Right : zoom-in of the trajectory.

3.5.3 Time versus event triggering reset rules

In the following we consider a network of 5 agents partitioned in two clusters. Let us fix the initial condition of the system at $x_0 = [8, 7, 9, 2, 3]^\top$. We also introduce the following parameters $\delta_{min} = 0.1s$, $\delta_{max} = 1s$, $\delta_{ave} = \frac{\delta_{max}+\delta_{min}}{2}$ and the jitter δ' taking values within $[0, 0.001]$. Throughout this section we are using the following notation :

$$\begin{aligned}
\mathcal{T}_1 &= (t_k)_{k\geq 1} \text{ where } t_0 = 0,\ t_{k+1} = t_k + \delta_{min} + \delta',\ \forall k \geq 0,\\
\mathcal{T}_2 &= (t_k)_{k\geq 1} \text{ where } t_0 = 0,\ t_{k+1} = t_k + \delta_{max} + \delta',\ \forall k \geq 0,\\
\mathcal{T}_3 &= (t_k)_{k\geq 1} \text{ where } t_0 = 0,\ t_{k+1} = t_k + \delta_{ave} + \delta',\ \forall k \geq 0,\\
\mathcal{T}_4(a_1, a_2) &= (t_k)_{k\geq 1} \text{ given by Definition 7 with the parameters } a_1 \text{ and } a_2,\\
\mathcal{T}_5(a_1, a_2) &= (t_k)_{k\geq 1} \text{ given by Definition 8 with the parameters } a_1 \text{ and } a_2.
\end{aligned} \tag{3.44}$$

Remark 22 — *The reset sequences $\mathcal{T}_1$, $\mathcal{T}_2$, $\mathcal{T}_3$ defines nearly-periodic reset instants. Moreover the resets are synchronized i.e. all the agents that interact outside their cluster reset their state at the* *same instants. In other words, there exists only one reset matrix $P(t_k) = P$, $\forall k$. As shown in [11], in this case the consensus value does not depend on the chosen reset sequence.*

— *The reset sequences $\mathcal{T}_4(a_1, a_2)$, $\mathcal{T}_5(a_1, a_2)$ are defined by event-triggering strategies. In this case each agent resets independently its state leading to asynchrony. One agent can reset several times before another resets once. Thus, two subsequences are considered : $(t_{\phi_1(k)})_k$ contains the reset instants t_k associated with agents belonging to the first cluster and $(t_{\phi_2(k)})_k$ contains the reset instants t_k associated with agents belonging to the second cluster.*

The convergence speed in the first cluster is very high with respect to the convergence speed of the second. This situation occurs when the second cluster represents a model reduction of a large cluster. We can also consider that the interactions in the second cluster are unreliable and the agents give a small weight/importance to these informations.

Let us consider the continuous dynamics in (3.2) is given by

$$L = \begin{pmatrix} 4 & -2 & -2 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0.1 & -0.1 \\ 0 & 0 & 0 & -0.05 & 0.05 \end{pmatrix}. \quad (3.45)$$

We assume that only agents 1 and 4 can interact outside their clusters and they reset their states by using the following rules :

$$\begin{aligned} x_1(t_{\phi_1(k)}) &= 0.45 x_1(t^-_{\phi_1(k)}) + 0.55 x_4(t^-_{\phi_1(k)}) \\ x_4(t_{\phi_2(k)}) &= 0.25 x_1(t^-_{\phi_2(k)}) + 0.75 x_4(t^-_{\phi_2(k)}). \end{aligned} \quad (3.46)$$

In this case the event-triggering strategy largely outperforms the nearly periodic ones. Indeed, the consensus is reached by reseting mainly the state of agent 1 which belongs to the cluster with a rapid convergence. Unlike the event-triggering strategy, the nearly periodic ones are unnecessarily reset the state of agent 4 consuming more energy and giving finally a worst dynamic behavior. It is interesting to highlight that using $\mathcal{T}_3$ as reset sequence each cluster resets 55 times in 30 seconds and it is far from consensus. Meanwhile, using $\mathcal{T}_4(20, 40)$ the system performs a total number of 55 resets (54 for the first cluster and 1 for the second one) and the consensus is practically reached. Using $\mathcal{T}_5(20, 40)$, the leaders make 39 resets in 30 seconds to get the agents closer to the overall agreement. This shows that, in some situations, useless resets not only consume energy but also have a negative impact on the convergence speed.

In order to emphasize that event-triggering strategies are particularly suitable in the case of different time scales related to the convergence in different clusters, we are considering again the continuous time dynamics given by (3.45), but we change the reset dynamics as follows :

$$\begin{aligned} x_1(t_{\phi_1(k)}) &= 0.35 x_1(t^-_{\phi_1(k)}) + 0.65 x_5(t^-_{\phi_1(k)}) \\ x_5(t_{\phi_2(k)}) &= 0.45 x_1(t^-_{\phi_2(k)}) + 0.55 x_5(t^-_{\phi_2(k)}). \end{aligned} \quad (3.47)$$

As can be shown in Figure 3.13 the event-triggering reset strategy outperforms again the nearly periodic reset strategy.

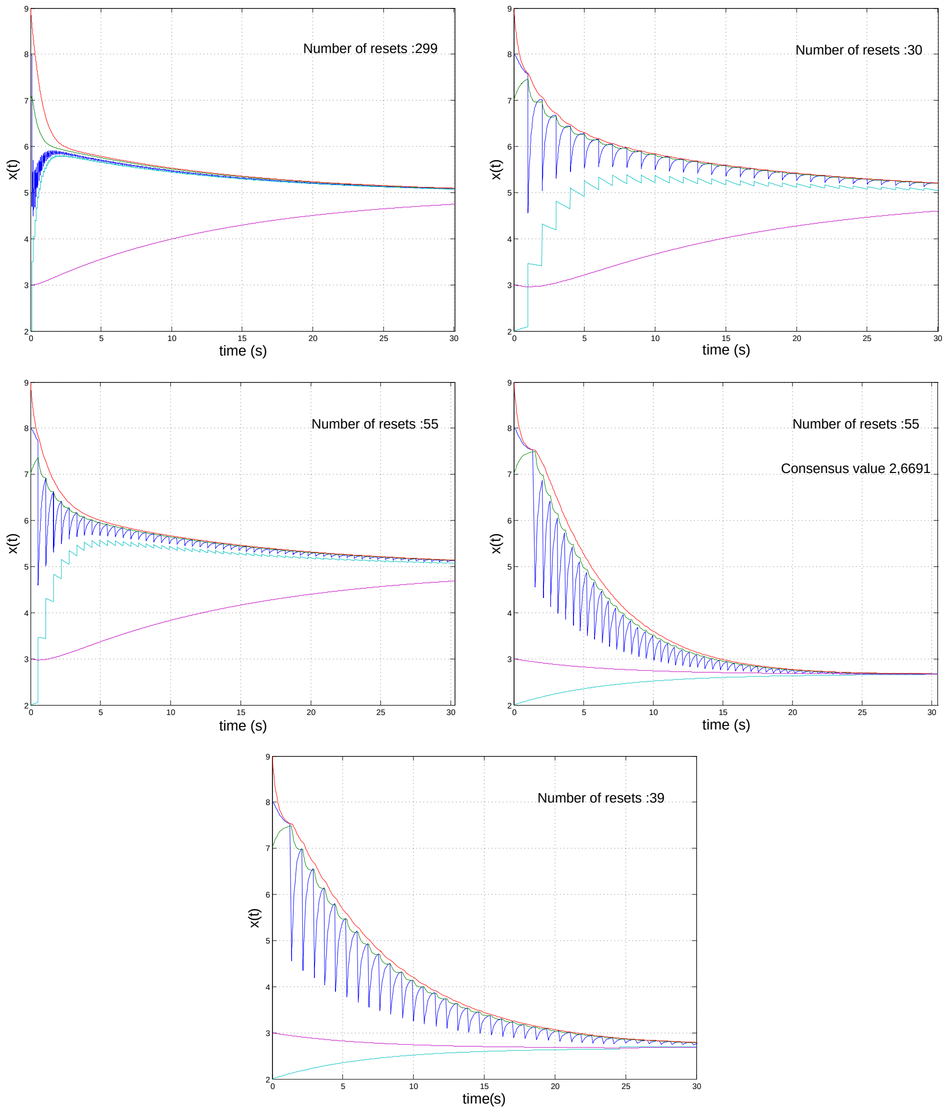

FIGURE 3.12 – Consensus of the five agents grouped in 2 clusters when L is defined by (3.45), the reset dynamics is defined by (3.46). Top left : $t_{\phi_1(k)} = t_{\phi_2(k)} = t_k$ are elements of $\mathcal{T}_1$, Top right : $t_{\phi_1(k)} = t_{\phi_2(k)} = t_k$ are elements of $\mathcal{T}_2$, Bottom left : $t_{\phi_1(k)} = t_{\phi_2(k)} = t_k$ are elements of $\mathcal{T}_3$, Bottom right : $t_{\phi_1(k)}$ and $t_{\phi_2(k)}$ are elements of $\mathcal{T}_4(20, 40)$, At the bottom : $t_{\phi_1(k)}$ and $t_{\phi_2(k)}$ are elements of $\mathcal{T}_5(20, 40)$.

3.6 Conclusion

In this chapter we have studied the consensus in heterogeneous networks containing both linear and linear impulsive dynamics. Under appropriate assumptions, we have proven that all subsystems agree and we have bounded above the convergence speed. Time-invariant linear impulsive dynamics allows at characterizing the consensus value in function of initial conditions and interconnection topologies. This permits to design the inter-clusters topology in order to achieve an a priori fixed consensus. In the time-varying dynamics case one requirement is related to a minimal dwell-time separating two consecutive

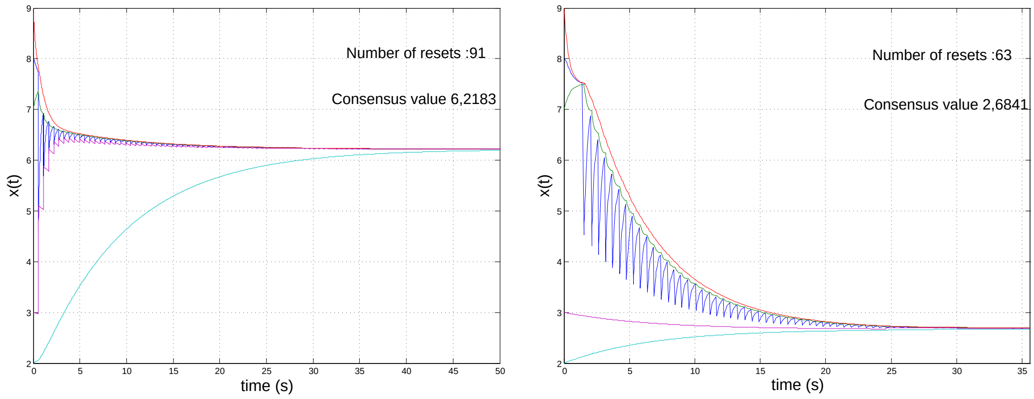

FIGURE 3.13 – Consensus of the five agents grouped in 2 clusters when L is defined by (3.45), the reset dynamics is defined by (3.47). Left : $t_{\phi_1(k)} = t_{\phi_2(k)} = t_k$ are elements of $\mathcal{T}_3$, Right : $t_{\phi_1(k)}$ and $t_{\phi_2(k)}$ are elements of $\mathcal{T}_4(20, 40)$.

reset instants of the same cluster. It is noteworthy that, in this case, the reset instants of different clusters are not synchronized, meaning that no global dwell-time is imposed between two consecutive reset instants in the network. The consensus problem has been solved under different strategies defining the reset sequence. Firstly we considered a time-triggering strategy which imposes sufficient assumptions for consensus. Secondly, we designed event-triggering reset rules and we proved that the proposed sufficient assumptions for consensus are satisfied. We have also emphasized the improvements that we get by using event-triggering reset rules.

Troisième partie

Research perspectives

Chapitre 1

Short-term perspectives

An important research direction of my near-future works is at the intersection of singular perturbation theory and multi-agent systems' analysis. Two different aspects will be addressed. The first one is directly related to the results presented in Part 2. Precisely, the presence of clusters in the network generates, as pointed out before, a time-scale separation. In other words, the agents will agree faster inside a cluster than in the whole network. The second aspect considers the consensus problem in networks of singularly perturbed agents *e.g.* whose state components evolve on different time-scales.

Another short-term perspective is the extension/generalization of some of my recent results on the stability of linear impulsive systems. I also point out in this chapter that I recently reflected on a new opinion dynamics model and in the near future we want to provide its analysis in the deterministic framework. Finally, I present here the research perspectives related to two Ph.D. application oriented theses that I co-promote in this moment or I will soon start to co-advise. The first of them is in collaboration with Arcelor Mittal an treat the cold rolling problem while the second is in collaboration with CEA List and focusses on cooperative control with real communication and computation constraints.

1.1 Singular perturbation modeling of multi-agent systems

This research will be conducted in cooperation with D. Nešić (prof. at U. Melbourne. Australia) and S. Martin (prof. at U. Lorraine, France). The multi-agent framework is widely used to model the dynamics of large number of interconnected systems. The convergence toward consensus is typically characterized by conditions that depend on the communication graph between agents. One of the most general condition (see Theorem 3) is the cut-balance communication which is a general form of communication reciprocity among the agents. Under the cut-balance assumption, convergence is ensured, and consensus may occur in groups or globally. One drawback of the cut-balance assumption is that it is a global assumption which may be hard to verify when not ensured by design. One motivation of this research direction is to provide new assumption on reciprocity of communication which can be verified locally.

It is reported in the literature that large scale networks often consist of sparsely interconnected clusters of densely coupled agents [17, 4, 50]. In this context, one option to search for a local assumption is to split the agents into clusters. In other words, we check the cut-balance assumption juste inside clusters and between entire clusters. This results in a consistent reduction of the number of subsets $\mathcal{S}$ considered in Assumption 2. Precisely, Assumption 2 should be replaced by the following three hypothesis.

Assumption 16 (Intra-cluster reciprocity) *There exists a constant K_I such that for any cluster $k \in$*

$\{1, \ldots, m\}$ *and for all non trivial subsets* $S \subset \mathcal{C}_k$,

$$w_{S \leftarrow (\mathcal{C}_k \setminus S)}(t) \leq K_I \cdot w_{(\mathcal{C}_k \setminus S) \leftarrow S}(t), \ \forall t \geq 0.$$

Assumption 17 (Inter-cluster reciprocity) *There exists a constant* K_E *such that for all non trivial subset* $S \subset \{1, \ldots, m\}$,

$$\sum_{k \in S} w_{\mathcal{C}_k \leftarrow (\mathcal{V} \setminus \mathcal{C}_k)}(t) \leq K_E \cdot \sum_{k \in S} w_{(\mathcal{V} \setminus \mathcal{C}_k) \leftarrow \mathcal{C}_k}(t), \ \forall t \geq 0.$$

Assumption 18 (Clustered communication) *There exists a constant* $\rho \geq 0$ *such that for each cluster* $k \in \{1, \ldots, m\}$ *and for all non trivial subsets* $\mathcal{S} \subset \mathcal{C}_k$,

$$w_{\mathcal{C}_k \leftarrow (\mathcal{V} \setminus \mathcal{C}_k)}(t) \leq \rho \cdot w_{S \leftarrow (\mathcal{C}_k \setminus S)}(t), \ \forall t \geq 0.$$

Assumptions 16 and 17 correspond to a cut-balance assumption [31] within each cluster and to a cut-balance assumption between clusters, respectively. In Assumption 17 the equivalent cut-balance assumption is formulated in the case where each cluster is considered as a node and the communication between clusters is weighted by the sum of agent-wise communication weights. The purpose of Assumption 18 is to ensure that the partition in clusters corresponds to the distribution of communication weights. It prevents cases where two subsets of a cluster are more connected to the outside than with each other.

Remark 23 *Note that Assumption 18 is not implicitly satisfied when Assumptions 16 and 17 hold. For instance, choose* $\mathcal{V} = \{1, 2, 3, 4\}$, $a_{12}(t) = a_{21}(t) = a_{34}(t) = a_{43}(t) = t$ *and* $a_{23}(t) = a_{32}(t) = a_{41}(t) = a_{14(t)=1}$. *If* $\mathcal{C}_1 = \{1, 2\}$ *and* $\mathcal{C}_2 = \{3, 4\}$, *Assumptions 16, 17 and 18 hold. On the other hand if* $\mathcal{C}_1 = \{1, 4\}$ *and* $\mathcal{C}_2 = \{2, 3\}$, *Assumptions 16 and 17 hold while Assumption 18 does not (see Figure 23 for an illustration). In other words, the partition in clusters has to be properly chosen.*

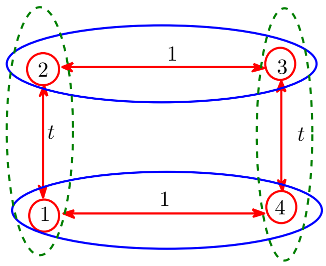

FIGURE 1.1 – Dashed green lines represents the clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponding to the first choice in Remark 23. In blue the clusters $\mathcal{C}_1$ and $\mathcal{C}_2$ corresponding to the second choice in Remark 23

Therefore, the objective is to provide assumptions ensuring convergence to consensus based on the sparse clustered structure of communication. Considering the problem in the context of interactions with a clustered structure allows to obtain assumptions that are not only less computationally intensive to check but also more natural in this context.

An important feature of these clustered networks is that the agents first converge fast toward a local agreement inside the clusters and then they slowly converge to a global consensus. This time scale separation has been emphasized and used for the analysis of power grids with fixed and undirected interconnections [17, 4, 62]. The same feature has been used in [49] for clusters detection in large scale networks. However, up to now, only symmetric and time-invariant communication networks have been dealt with. We want to use the assumptions developed in the previous step to extend the results in [17] to networks with directed and time-varying interconnections. We can still exhibit a time-scale separation in the network but the resulting singularly perturbed system is no longer in a standard form as in [17, 4, 62]. Consequently, the analysis is done by using the more natural averaging technique for two time-scales systems [6, 52, 70]. This makes the approach applicable to more realistic multi-agent systems.

Two different issues are of high interest in this framework. The first one is related to model reduction since the dynamics of a network with very large number of agents is approximated by the dynamics of the slow model. This is a network with small number of agents consisting of super-nodes, each of them aggregating the vertices of one cluster. The second issue concerns the convergence speed inside one cluster. We should be able to characterize it in order to define the time instant starting with which the model reduction becomes very effective.

1.2 Synchronization in networks of singularly perturbed systems

This topic will be dealt in cooperation with A. Girard (DR CNRS at LSS, France) and J. Daafouz (prof. at U. Lorraine, France). This research direction is motivated by the fact that many real systems are characterized by two features. The first one is that they are obtained by interconnecting a bunch of simpler subsystems that have to synchronize in order to reach a global goal. The second one is that each subsystem presents dynamics that evolves on different time-scales. Taking into account the two features leads to the problem of synchronization in networks of singularly perturbed systems. When several orders of magnitude differentiate the various time scale the analysis of the overall system becomes more difficult. In this case, standard control techniques lead to ill-conditioned problems. To overcome this, singular perturbation theory [37, 36] propose to approximate the dynamics by decoupling the slow dynamical process of the faster ones. Consequently, the control design is also decoupled with respect to each time scale and then is proven that the joint actions performs well when applied to the overall system.

The analysis and control design for multiple time-scales systems attracted a lot of interest due to their various applications going from biological systems such as gene expression systems [16], neurons behavior [33] to engineering problems [40]. General stability and stabilization of linear and nonlinear singularly perturbed systems can be found in [37, 36]. For linear singularly perturbed systems a linear quadratic optimal control design is proposed in [27]. Stabilization and exponential stability of singularly perturbed linear switched systems is considered in [2, 41]. Various biological singularly perturbed systems are analyzed from a geometric perspective in [30]. The particularity of existing studies presented above is that they consider singularly perturbed systems as being stand alone systems. Motivated by the fact that biological as technological systems are often composed of several coupled singularly perturbed systems, we propose to treat the problem of synchronization of singularly perturbed systems. In other words, the systems are no longer stand alone but coupled with other similar ones. Moreover, real multi-agent systems are often characterized by switching between different dynamics and/or state jumps. The objective of this research direction is to provide a general framework that takes into account the difficulties related to switchings, jumps as well as the presence of different time-scales.

Before tackling the problem in its generality, we analyze the synchronization in multi-agent systems with continuous singularly perturbed dynamics without switchings. Therefore, we consider a network of n identical singularly perturbed linear systems. For any $i = 1, \dots, n$, the i^{th} system at time t is characterized by the state $(x_i(t), z_i(t)) \in \mathbb{R}^{n_x+n_z}$ and there exists a small $\epsilon > 0$ such that its dynamics is given by :

$$\begin{cases} \dot{x}_i(t) = A_{11}x_i(t) + A_{12}z_i(t) + B_1u_i(t) \\ \epsilon\dot{z}_i(t) = A_{21}x_i(t) + A_{22}z_i(t) + B_2u_i(t), \quad i = 1, \ldots, n \end{cases} \tag{1.1}$$

where $u_i \in \mathbb{R}^m$ is the control input and

$$\begin{array}{ccc} A_{11} \in \mathbb{R}^{n_x \times n_x}, & A_{12} \in \mathbb{R}^{n_x \times n_z} & B_1 \in \mathbb{R}^{n_x \times m}, \\ A_{21} \in \mathbb{R}^{n_z \times n_x} & A_{22} \in \mathbb{R}^{n_z \times n_z}, & B_2 \in \mathbb{R}^{n_z \times m} \end{array}$$

such that $rank(B_1) = rank(B_2) = m$.

A standard assumption, in singular perturbation theory, which aims at ensuring the well posedness of (1.1) is the following.

Assumption 19 *The matrix A_{22} is invertible.*

In a preliminary study we considered that the interaction topology is time-invariant and undirected. We consider a weighted adjacency matrix associated with this interaction topology i.e. $G = [g_{i,j}] \in \mathbb{R}^{n \times n}$ such that

$$\begin{cases} g_{ij} > 0 \text{ if } (i,j) \in \mathcal{E} \\ g_{ij} = 0 \text{ otherwise} \end{cases}.$$

Definition 9 *The n singularly perturbed systems defined by (1.1) achieve asymptotic synchronization using local information if there exists a state feedback controller of the form*

$$u_i(t) = K_1 \sum_{j=1}^{n} g_{i,j}(x_i(t) - x_j(t)) + K_2 \sum_{j=1}^{n} g_{i,j}(z_i(t) - z_j(t)) \tag{1.2}$$
$$K_1 \in \mathbb{R}^{m \times n_x}, \ K_2 \in \mathbb{R}^{m \times n_z}$$

such that

$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0 \textit{ and } \lim_{t\to\infty} \|z_i(t) - z_j(t)\| = 0.$$

The main goal here is to characterize the feedback controllers that use local information and asymptotically synchronize the singularly perturbed systems defined by (1.1).

A natural extension that will render the problem more realistic for real application is related to directed and time-variant interaction topologies. The passage from undirected to directed interaction should be realizable without many difficulties. It practically requires the use of triangular form instead of the diagonal one and a careful treatment of possible complex eigenvalues of the Laplacian matrix. The extension to time-varying interactions will be more cumbersome requiring a complete re-design based on the tools used in [51]. A first analysis can consider that all the topologies are represented by connected graphs. Due to topology switchings the problem will be formulated as simultaneous stabilization of a linear switched systems with impulses.

1.3 Stability conditions of linear multi-impulsive systems with asynchronous impulsion times

This work continues my collaboration with M. Fiacchini (CR CNRS at Gypsa-lab) on the stability of linear impulsive systems. This framework is useful to model controlled systems that transmit the measurements to the controller and receive the updated control input values through a network at discrete instants.

The system is assumed to have several measured outputs and several inputs, and we assume they have an independent functioning with asynchronous transmission times. Thus, every sensor transmits and every actuator updates its value at different time intervals, which are supposed to be in general independent. This system can be modelled, as illustrated in the following, as an impulsive (or reset) system, with different and asynchronous jump instants and several jump maps, one associated to each impulse sequence. Every sequence of impulses is assumed, at first to have fixed dwell time, i.e. the activation interval of every jump map is fixed. Our aim is to characterize asymptotic stability in presence of the multiple time sequences of state jumps.

For sake of simplicity, we describe the problem by a system with two different jump maps and then two sequences of impulses. This system can be used to model systems with one input and one output that transmits through the network at fixed but asynchronous time instants. Consider the following linear impulsive system :

$$\begin{cases} \dot{x}(t) = A_c x(t), & \forall t \in \mathbb{R}_+ \setminus (\mathcal{T}_1 \cup \mathcal{T}_2) \\ x(t) = A_1 x(t^-) & \forall t \in \mathcal{T}_1 \\ x(t) = A_2 x(t^-) & \forall t \in \mathcal{T}_2 \\ x(0) = x_0 \in \mathbb{R}^n \end{cases} \tag{1.3}$$

where $x \in \mathbb{R}^n$ is the state of the system, $x(t^-) = \lim_{\tau \to 0, \tau < 0} x(t+\tau)$. We will refer to jump (or impulse) of type 1 (resp. 2) if the discrete-time dynamics with transition matrix A_1 (resp. A_2) is activated. The sets of admissible impulsive sequences are

$$\Theta_1 = \Big\{ \{t_k\}_{k\in\mathbb{N}} : \ t_{k+1} = t_k + T_1, \ \forall k \in \mathbb{N} \Big\}, \qquad \Theta_2 = \Big\{ \{t_k\}_{k\in\mathbb{N}} : \ t_{k+1} = t_k + T_2, \ \forall k \in \mathbb{N} \Big\}. \tag{1.4}$$

where $T_1, T_2 \in \mathbb{R}_+$ characterize the jump intervals between the impulses/jumps of type 1 and type 2, respectively. In the following we assume that $\mathcal{T}_1 \in \Theta_1$ and $\mathcal{T}_2 \in \Theta_2$.
A hybrid model of the systems (1.3) can be directly given by defining two additional variables t_1 and t_2 taking into account the time passed from the last jump of type 1 or 2, respectively, flow and jump maps and sets. That is, denoting $z = (x, t_1, t_2) \in \mathbb{R}^{n+2}$ define

$$\begin{cases} \dot{x}(t) = A_c x(t), \\ \dot{t}_1(t) = 1, \\ \dot{t}_2(t) = 1, \end{cases} \quad \text{if } z \in C, \qquad \big\{ \ z^+ \in G(z), \quad \text{if } z \in D \tag{1.5}$$

where $C = \mathbb{R}^n \times [0, T_1] \times [0, T_2]$, $D = \mathbb{R}^n \times T_1 \times T_2$ and

$$G(z) = G((x, t_1, t_2)) = \begin{cases} G_1((x, t_1, t_2)) = (A_1 x, 0, t_2), & \text{if } t_1 = T_1, \\ G_2((x, t_1, t_2)) = (A_2 x, t_1, 0), & \text{if } t_2 = T_2. \end{cases} \tag{1.6}$$

To illustrate that the stability analysis of this system is not trivial we provide here one example emphasizing that the answer may depend on the initial condition on z.

Example 2 *Consider the system (1.3) with the following matrices*

$$A_c = 0_{2,2}, \quad A_1 = \begin{bmatrix} 0 & a \\ a^{-1} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & a^{-1} \\ a & 0 \end{bmatrix}$$

with $a > 1$ and take $T_1 = T_2 = 1$ and $x(0) = (1,\ 0)$. Consider first $(t_1(0), t_2(0)) = (0.5,\ 0)$ then the discrete dynamics is given by the transition matrix

$$A_D = A_2 e^{0 \cdot 0.5} A_1 e^{0 \cdot 0.5} = A_2 A_1 = \begin{bmatrix} a^{-2} & 0 \\ 0 & a^2 \end{bmatrix}.$$

Therefore, the trajectory with initial conditions $(x(0),\ t_1(0),\ t_2(0)) = ((1,\ 0),\ 0.5,\ 0)$ is $x(k) = a^{-2k}(1,\ 0)$ and converges exponentially fast to the origin.

Consider now the same initial condition for $x(0)$ but $(t_1(0), t_2(0)) = (0,\ 0.5)$. From reasonings analogous to the previous case, it can be proved that

$$A_D = A_1 A_2 = \begin{bmatrix} a^2 & 0 \\ 0 & a^{-2} \end{bmatrix},$$

which means that $x(k) = a^{2k}(1,\ 0)$, thus the trajectory diverges.

1.4 Continuous opinion dynamics with quantized information

This research direction will be done in cooperation with S. Martin (prof. at U. Lorraine, France) and S. Srikant (prof. at IIT Bombay). The main motivation for introducing new consensus-like opinion dynamics model comes from the fact that existing models focus mostly on consensus achievement in social networks. It is clear that in sufficiently large social networks this behavior never occurs. Thus, our goal is to propose a simple model that has no parameter to tune but provides a wide number of opinions equilibria. This will be even more complex than the polarization obtained by some existing models that are listed in the next section. We also consider a scalar opinion belonging to a finite interval, but a variety of equilibria will be reached inside this interval. One important feature of this model should emphasize that extremist people can never be convinced (or the process is long) to change their polarization. On the other hand opinions that are close of the center of the interval can oscillate around the center changing regularly their polarization.

The main opinion dynamics model proposed in the literature are : the Ising model [34], the voter model [19] the Sznajd model [68], the Deffuant model [20] and Hegselmann-Krause model [29]. It is worth mentioning that we proposed in [49] a decaying confidence model for the opinion dynamics. Unlike other studies we do not search conditions guaranteeing the emergence of consensus in the network. On the contrary, we use the apparition of different local agreements to characterize different communities in the network.

Whatever is the model employed to describe the opinion dynamics, many studies focus on the emergence of consensus in social networks [64, 5, 20, 26]. Nevertheless, this behavior is neither natural nor often in real large social networks. In order to more accurately describe the opinion dynamics and to recover more realistic behaviors, a mix of continuous opinion with discrete actions (CODA) was proposed in [45]. This model reflects the fact that even if we often face to a binary decision, the opinion is never binary and it evolves in a continuous space of values. The main drawback of many of the previous descriptions is that they assume that an agent has a perfect knowledge of the opinion of the others.
One objective of my future works is to propose and analyze more accurate consensus-like opinion dynamics models that are based on CODA. My goal is to propose models in which, instead consensus,

different equilibria can be reached by different opinions (see Figure 1.2). Moreover, these model should be able to recover oscillatory behaviors in the opinion dynamics.
We propose a model in which the opinion of each agent evolves within $[0, 1]$ (which is a normalized version of $\mathbb{R}$) by taking into account the actions of their neighbors which takes only the values 0 or 1. Two versions of this model will be considered : encounters in random pairs (like in Defffuant) and encounters with all the neighbors at every time-step (like in Hegselmann-Krause). In order to give more details we consider a network of n individuals/agents denoted by $\{1, \ldots, n\}$. Let the interaction topology be described by a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ that can be directed or not. Let p_i be the opinion of the agent $i \in \{1, \ldots, n\}$ and q_i a quantized version of p_i. Formally, $q_i = 0$ as far as $p_i < 0.5$ and $q_i = 1$ otherwise. Let us also denote by N_i the set of agents that influence i according to the graph $\mathcal{G}$. The following assumption is imposed throughout this research.

Assumption 20 *There exists* $\epsilon \in \left(0, \frac{1}{2}\right)$ *such that* $\min_{i\in\mathcal{V}} p_i(0) \geq \epsilon$ *and* $\max_{i\in\mathcal{V}} p_i(0) \leq 1 - \epsilon$.

As we can see below the opinion of the agents having p_i equals 0 or 1, does not evolve. Moreover, Assumption 20 can be used to bound the initial interaction weights.

With the above notation at hand we introduce the following opinion dynamics model :

$$p_i(k+1) = p_i(k) + p_i(k)\big(1 - p_i(k)\big)\frac{1}{|N_i|}\sum_{j\in N_i}(q_j(k) - p_i(k))$$

It is worth emphasizing that the model above contains quantized information and state-dependent interaction weights. These characteristics render the analysis of these models particularly difficult. Moreover, my ambition is to study the behavior under general interaction topologies, meaning that we cannot take advantage of some properties related to a precise network. Of course, in this case we can accomplish only a qualitative analysis describing the tendency under some local conditions. Nevertheless, for some special interaction topologies such as all-to-all, ring graph, bipartite graph, we expect to completely characterize the behavior of the opinions in function of the initial conditions.

A particular feature of the proposed model, is that "extremists" (i.e. individuals having the opinion equals either 0 or 1) never change their opinion. Moreover, these models incorporate the fact that opinions which are closer to 0 or 1 are harder to change than the opinions that are close to $\frac{1}{2}$. These agents can also be interpreted as very stubborn or self-confident individuals. Other interesting issues related to these models are :

— Consensus is not guaranteed and different equilibrium points within $[0, 1]$ can be reached by the opinion of the agents (see Figure 1.2 and Figure 1.3 Down).

— Some or all of the agents may oscillate between two values (see Figure 1.3 Up).

It is noteworthy that, under Assumption 20 the consensus is guaranteed if we replace $q_j(k)$ with $p_j(k)$. In other words, having access to perfect knowledge on the opinion of the neighbors instead of quantized one, favors global agreement in the network.

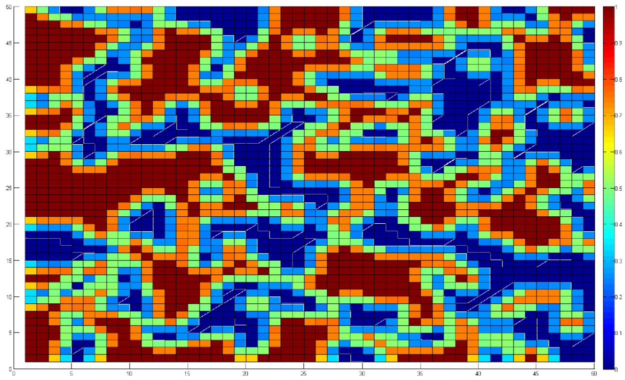

FIGURE 1.2 – End of runs for the evolution of 2500 agents having random initial opinion within $(0, 1)$ and random interconnection topology.

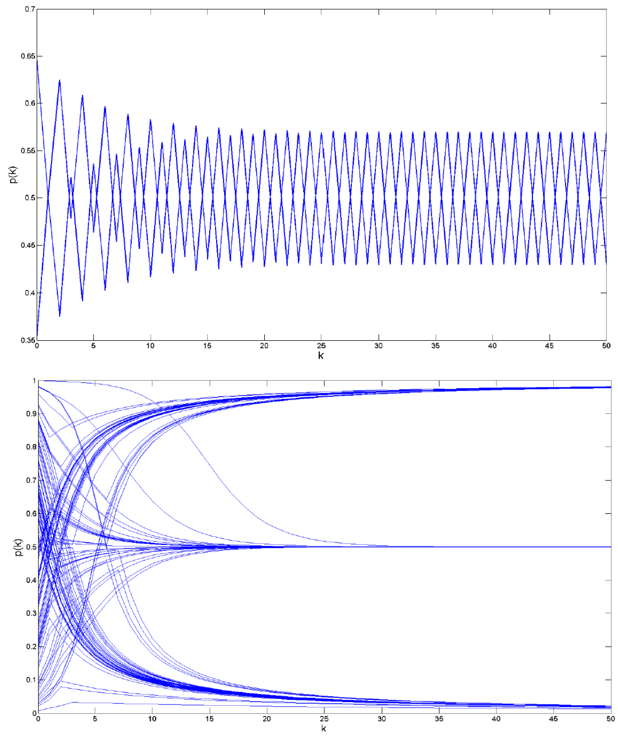

FIGURE 1.3 – Opinion dynamics in a 2-regular bipartite graph of 100 agents. Up : 50 agents with initial condition 0.64 and 50 agents with initial condition 0.36. Down : random initial opinions within (0,1).

1.5 Linear-Quadratic controllers for systems with time-varying delays

This research direction is related to the Ph.D thesis that I am co-promoting with M. Jungers (CR CNRS at CRAN) on "Advanced Multivariable Control of Cold Rolling Process : Application to Tandem Mill". In this context we have to deal with a complex physical systems described by more than 20 state interconnected variables that appear into dynamics delayed by a delay that is state-dependent. As a result, after linearization we get a huge linear delay systems of retarded-type including time-varying and state-dependent delays.

The objective of this research is to propose a Linear-Quadratic (LQ) controller that satisfies the industrial constraints being able to control the cold rolling process in presence of uncertainties on the still mechanical properties, measurement uncertainties and state-dependent delays. From practical point of view, as a first step of the study we will verify if the robustness properties of the LQ controller will be enough to guaranty a satisfactory behavior in presence of all uncertainties and with an assumption of fixed average delay. However, from theoretical point of view we consider that the problem of LQ design in presence of state-dependent delays is challenging and deserves the attention of the scientific community. Our ambition is to provide a multi-variable controller that can be implemented by ArcelorMittal in order to improve the performances of the cold rolling process.

1.6 Decentralized control with communication and computation constraints

This research direction is also related to a Ph.D. thesis that I will co-promote with M. Jungers (CR CNRS at CRAN). This subject will be considered in cooperation with C. Janneteau and M. Boc from **CEA-List/Communicating Systems Laboratory**.

The main objective of this thesis is to design decentralized controllers that integrate the communication and computation constraints. In other words, we specify the computation budget and the communication bandwidth and range and we want to design a controller that can be executed under these constraints. In order to satisfy the communication constraints we will impose a limited number of simultaneous communications per agent. The computation constraints will be taken into account by designing simple control laws that require small computation loads. This will be basically done by decoupling the controller in two parts. The first will compute reference trajectories based on standard consensus algorithms while the second will design tracking controllers for each agent independently from the others.

Chapitre 2

Mid-term research perspective

In this chapter I present some future research directions that will not be treated immediately because they need more reflection and/or investigation. A first perspective is represented by the distributed control of strings of nonlinear agents. A second perspective is related to the control of dual-switched systems. This work continues our collaboration with L. Buşoniu and the main difficulties rely on the fact that we control just a part of the control inputs while the rest act randomly and they can be considered as opponent actions. Therefore, we have to solve a minmax optimal control problem. In order to completely solve it, one has to also investigate stability guaranties which are far from being obvious when using the planning approach proposed in this work. A third mid-term perspective is represented by the analysis of the stochastic version of the opinion dynamics model presented in section 1.4.

2.1 Distributed control for strings of nonlinear agents

This subject will be talked in collaboration with Şerban Sabău (prof. at Stevens Institute of Technology, New Jersey, U.S.A). We consider a *homogeneous* group of n agents moving along the same (positive) direction of a roadway. The dynamical model for the agents, relating the control signal $u_k(t)$ of the k–th vehicle to its position $y_k(t)$ on the roadway, is given by

$$\dot{y}_k = v_k, \quad \dot{v}_k = f(v_k) + u_k \; ; \tag{2.1a}$$

$$y_k(0) = -k\,\Delta, \quad v_k(0) = 0. \tag{2.1b}$$

Let us further define

$$z_k \stackrel{def}{=} y_k - y_{k-1} - \Delta, \quad z_k^v \stackrel{def}{=} v_k - v_{k-1} \quad \text{for} \quad 1 \leq k \leq n, \tag{2.2}$$

to be the interspacing and relative velocity error signals respectively (with respect to the predecessor in the string), with Δ being the *desired* constant inter-spacing policy. By differentiating (2.2) it follows that $\dot{z}_k(t) = z_k^v(t)$, therefore implying that constant interspacing errors (in steady state) are equivalent with zero relative velocity errors and also allowing to write the following time evolution for the relative velocity error of the k–th vehicle

$$\dot{z}_k^v = f(v_k) - f(v_{k-1}) + u_k - u_{k-1}. \tag{2.3}$$

Our objective is to propose a control scheme in the platooning problem that achieves the synchronization of the trajectories of all agents in the string with the trajectory of the leader vehicle. Such trajectory tracking must be achieved while ensuring zero (steady–state) errors of the regulated measures z_k^v (*velocity matching*) and while *avoiding collisions*, *i.e.* performing the needed longitudinal steering (brake / throttle) maneuvers that guarantee the avoidance of collision with the preceding vehicle.

The inherent difficulty in platooning control is rooted in the nature of the interdependencies between the regulated measures. More specifically by the fact that the regulated measurements for the k–th agent depend on the regulated measurements of its predecessor (the $(k-1)$–st agent) and so on and so forth, by a recursive argument – going through all k–th predecessors, they ultimately depend on the trajectory of the leader vehicle, which represents the reference for the entire formation. The novel control architecture we introduce next, features certain beneficial "decoupling" properties of the closed–loop dynamics. The proposed distributed control policies, employing only information locally available to each vehicle, are built on so-called Artificial Potential Functions (**APF**). Specifically, we will look at control laws of the type

$$u_1 = -\beta_1(v_1 - v_0) - \nabla_{y_1} V_{1,0}(\|y_1 - y_0\|_\sigma) \quad \text{and} \tag{2.4a}$$

$$u_k = u_{k-1} - \beta_k(v_k - v_{k-1}) - \nabla_{y_k} V_{k,k-1}(\|y_k - y_{k-1}\|_\sigma) \quad \text{for} \quad k \geq 2, \tag{2.4b}$$

where for all $k \geq 1$, $V_{k,k-1}(\cdot)$ is an Artificial Potential Function while β_k is a proportional gain to be designed for supplemental performance requirements.

Several issues have to be clarified in this framework :

— Can we apply this methodology to general nonlinear systems or we have to reduce the analysis to some specific classes satisfying Lipshitz-like conditions.

— Study the effects of communication delays on this control strategy.

— An important step would be to extend the results to non-homogeneous strings of agents.

2.2 Artificial Inteligence-based minimax-optimal control of dual switched systems

This research will be developed in cooperation with L. Buşoniu (prof. at U.T. Cluj-Napoca, Romania) and J. Daafouz (prof. at U. Lorraine, France). In the main direction of the research, we will consider switched systems where some of the switches are under the control of the control algorithm, while others are not (either because they are influenced by disturbances from the external environment, or by opponent entities). Such systems are called dual switched in the very recent literature that started to consider them in the last couple of years. We will handle this in a minimax framework, where control decisions are chosen so as to maximize the performance under the worst possible disturbance or opponent decisions.

To this end, we will overhaul the AI-based, optimistic planning algorithm from our CDC 2015 paper to the case where minimax solutions are sought. We will focus here on the challenge of ensuring that after switching, the system remains in the same mode for a certain number of steps - the dwell time. This is because for some systems fundamental properties (stability, performance, etc.) can be guaranteed only under dwell time constraints. Another reason is that in practice, it may be unsuitable or impossible to switch arbitrarily fast, so the designer must guarantee by construction a minimum dwell time.

Therefore, we will start by deriving an algorithm that can handle dwell time constraints on the action of the controller (maximizer), as well as on that of the disturbance (minimizer). This will be done by enforcing the dwell time constraint on each newly examined solution. Initial tests in simulated experiments will be performed to check the performance of the algorithm, and its design may be revisited if performance is unsatisfactory.

The main thrust of this research line will be the analysis of the algorithm, where near-optimality will be studied as a function of invested computation. Near-optimality is characterized in terms of difference between the long-term cost accumulated by the solution returned by the algorithm to the minimax-optimal value. To this end, it will be necessary to derive a novel measure of the quantity of near-optimal solutions in a given problem. This is a challenging step, but has been performed for a number of algorithms in the group at Cluj before so, we are confident we will succeed.

Finally, we will identify relevant benchmark dual switched problems in the area of networked systems, and we will evaluate the algorithm comparing to existing approaches when it is possible (i.e. when mode dynamics are linear). We will examine the real performance as a function of computation, confirming that the bounds derived by the analysis are satisfied.

2.3 Stochastic model for continuous opinion dynamics with quantized information

Basically here we want to analyze the behavior of a stochastic version of the model introduced in Section 1.4. More precisely, the model in Section 1.4 is an averaged version of a more accurate stochastic model presented below. In this model we assume that at each time-step k, agent i encounters a randomly chosen neighbor $j \in N_i$ and updates its opinion using the following rule :

$$p_i(k+1) = p_i(k) + p_i(k)\big(1 - p_i(k)\big)(q_j(k) - p_i(k))$$

Basically the opinion dynamics is defined by a Markov process that is expressed in the form of a consensus protocol with state-dependent interaction weights. Beside an increased accuracy this model is numerically more tractable than the first CODA model. Indeed the deterministic version requires communication with all the neighbors at each update. When large social network (millions of agents) are considered, intensive computation are required in order to simulate the opinion dynamics when using our first CODA model. The stochastic version of the CODA model presents of course some challenges as the modeling of the probability kernel defining the encounters and the characterization of equilibria of such Markov process.

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