Robust leader-following consensus for linear multi-agent systems subjected to heterogeneous uncertainties

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Abstract

This paper presents a decentralized control strategy to ensure bounded tracking errors in a network of homogeneous linear multi-agent systems subject to heterogeneous time-varying uncertainties. A robust state–feedback decentralized protocol is designed via a convex optimization problem formulated with linear matrix inequalities (LMIs), extending the concept of quadratic boundedness to guarantee that the synchronization error between the leader and the agents converges to a positively invariant and attractive set in the presence of heterogeneous uncertainties. An integral action is incorporated into the controller to mitigate tracking errors induced by heterogeneous uncertainties and the leader's input, which can be unknown to the followers. The design procedure does not require full knowledge of the network topology, thanks to a polytopic representation of the Laplacian eigenvalues. This feature enables the design of protocols applicable to networks of arbitrary size, making the approach suitable for networks with a large number of agents. Numerical examples are provided to demonstrate the effectiveness of the proposed method.

1. Introduction

Multi-agent systems (MAS) have been the subject of many studies due to their capabilities in addressing various applications such as vehicle formation [16], mobile robot path following [5], and synchronization of power systems [37]. Consensus in MAS refers to the shared objective that all agents within the network agree upon. This idea can be viewed as the leader-follower tracking issue in MAS, where a group of followers aims to follow the leader. Leader-follower tracking is an important problem in MAS cooperative control and synchronization, as it applies to a range of real-world scenarios, including smart grids and operations of unmanned autonomous vehicles. Furthermore, in recent years, numerous efforts have been made to address various challenges in this field [41, 40, 20, 36]. These

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include event-triggered leader-follower tracking control [7], tracking control with an active leader and variable topology [10], among others.

Most attention has been devoted to the state consensus of homogeneous networks, where all agents are identical. We can mention the consensus of systems with second-order [27, 19] and high-order dynamics [17], subjected to time-delays or switching topology [31, 25]. One of the major challenges in the consensus problem of MAS is the presence of uncertainties in the agent's dynamics [39, 9], which are caused by various factors, such as parameter deviations and unmodeled dynamics [23, 42]. The majority of works assume that both the nominal dynamics and the uncertainties are identical for all agents [42, 26, 15], as this case results in a homogeneous MAS for which several tools for protocol design are available in the literature. However, this scenario is unrealistic, as the uncertainty characteristics may vary among agents within the network. Thus, from a practical standpoint, considering the class of heterogeneous uncertainties is more reasonable [1, 6, 13, 33, 34]. Assuming heterogeneous uncertainties cause the available techniques to fail in decoupling the overall dynamics of the MAS. Since the agents' models are no longer identical, any variable transformation to reduce the consensus problem to a stability one will introduce coupling terms in the new coordinate system. As a result, the consensus error will be related to the trajectories of the open-loop system (without the consensus protocol). Although some studies have examined heterogeneous uncertainties, they present certain limitations. In particular, works such as [13, 6, 33, 34] address uncertainties only in the system matrix, assuming that the input matrix is exactly known. In contrast, we address a broader class of systems, for which these methods are not directly applicable.

Considering heterogeneous uncertainties for nominal identical systems leads to the class of consensus and leader–follower problems in heterogeneous MAS. This setting is typically addressed using the internal model principle, which requires agents to embed an internal model of the consensus trajectory and to satisfy an output regulation equation [11, 12, 35, 38]. Moreover, synchronization is generally performed for the output variable, since the state dimensions and their physical interpretation may differ. In [38], output synchronization under model uncertainties is studied, but the approach imposes restrictive conditions on controller design. Specifically, the proposed dynamic (rather than static) state-feedback controller requires an acyclic communication graph and includes a leader observer acting as an internal reference model. Furthermore, the protocol is fully distributed only under certain conditions and assumes that the leader has no external inputs, a limitation that is often unrealistic.

A key approach to deal with bounded external inputs is the so-called quadratic boundedness (QB) introduced by [4]. A system is said to be quadratically bounded if the trajectories are ultimately bounded, guaranteed by a quadratic Lyapunov function. In the context of MAS, [32] proposes an observer-based protocol that ensures quadratic boundedness for the leader-following problem in which the agents follow a virtual leader under the influence of bounded disturbances. Here, as a novelty, we extend this concept to ensure that the synchronization error converges to a positively invariant and attractive set containing the origin, accounting for heterogeneous uncertainties in the agents' models.

In this work, we relax key limitations of the cooperative output regulation problem for heterogeneous MAS, including the transmission zero condition [38] and the internal model requirement [35], by extending methods originally developed for homogeneous MAS to deal with heterogeneous agents induced by the uncertainties. Compared to existing approaches for uncertain heterogeneous MAS, the proposed method avoids the need for dynamic state-feedback controllers or observers in agents that do not have access to the leader's trajectories. Moreover, to attenuate the presence of steady-state synchronization errors introduced by heterogeneous uncertainties, we incorporate an integral action into the synchronization protocol [34, 3, 8]. We can also consider a more challenging scenario of non-cooperative leaders [2], where the leader's input is unknown to all followers, which imposes extra difficulties in the controller design. The main novelties and contributions of this paper are summarized as follows:

(i) We propose a novel approach inspired by techniques for synchronizing homogeneous networks to address the leader-following problem of non-identical agents arising from the presence of time-varying heterogeneous uncertainties. This method avoids the drawbacks and limitations of existing strategies for synchronizing heterogeneous MAS [35, 12, 38], providing a more robust and effective solution based on static feedback controllers, in contrast to dynamic state-feedback controllers required by the internal model principle. Compared with techniques that do not rely on internal models to handle uncertainty-induced heterogeneity [13, 6, 33, 34], our framework accommodates a more general class of uncertainty and leader models. Moreover, we provide an estimation for the error of the emergent dynamics (i.e., the consensus manifold [22]), a characterization that, to the best of our knowledge, has not been addressed in prior works dealing with heterogeneous uncertainties.

(ii) Another novel aspect of our approach is utilizing QB to deal with heterogeneous time-varying norm-bounded uncertainties in the leader-follower problem in MAS, a concept that was initially developed to deal with exogenous disturbance with bounded amplitude.

(iii) In contrast with most works in the literature, we consider the more realistic scenario in which the leader receives external input, thereby providing a wider range of behaviors with respect to leader dynamics. Such input may be unknown to the followers or used in the control protocol for specific cases, such as a virtual leader.

(iv) In addition to the proportional controller, a proportional-integral controller is introduced to further augment the performance of the proposed solutions, as in certain scenarios, the tracking error can suffer from constant offsets. In such cases, the proportional-integral controller offers an advantage over the simpler proportional controller by eliminating offset errors.

(v) To remove the explicit dependence on all eigenvalues of the Laplacian matrix in the design conditions, which would require centralized information and full knowledge of the network structure, we consider only the convex hull defined by its extreme eigenvalues for a fixed directed graph. This approach improves robustness to topology uncertainties, accommodates non-cooperative leaders, and enhances scalability for large-scale networks.

The paper is structured as follows: Section 2 presents preliminary assumptions and problem formulation along with necessary lemmas essential for further developments, Section 3 provides sufficient conditions to design protocols guaranteeing QB for different scenarios, and in Section 4, we illustrate with numerical examples the performance of the proposed approach.

Notation: The space of real matrices with dimension $n \times m$ is denoted by $\mathbb{R}^{n\times m}$. For a matrix X, X^T and X^{-T} denote the transpose and transpose of inverse of X, respectively, $X_{(i)}$ denotes the $i-$th row; $\kappa(X) = \sigma_{\max}(X)/\sigma_{\min}(X)$, where $||X|| = \sigma_{\max}(X)$ and $\sigma_{\min}(X)$ are respectively the maximum and minimum singular values of X. If X is square, X^{-1} denotes the inverse of X; $\mathrm{He}\{X\}$ stands for $X + X^T$; and $X > 0$ ($X < 0$) indicates that matrix X is positive (negative) definite and $X \succcurlyeq 0$ ($X \preccurlyeq 0$) indicates that all the components of matrix X are nonnegative (nonpositive). For a vector $v \in \mathbb{R}^n$, $\mathrm{diag}(v) = \mathrm{diag}(v_1, \ldots, v_n)$ is a diagonal matrix composed with the elements of v, and for a matrix X, $\mathrm{diag}(X)$ is composed with the diagonal elements of X. The identity matrix of order n is denoted by I_n and the null $m \times n$ matrix is denoted by $0_{m,n}$ (or simply I and 0 if no confusion arises), $\mathbb{1}$ stands for a vector of ones of appropriate dimension. The symbol $\star$ denotes symmetric blocks in partitioned matrices and $\otimes$ denotes the Kronecker product.

2. Preliminaries and Problem statement

2.1. Graph theory

In this work, we investigate the leader-following problem of linear multi-agent systems subject to heterogeneous bounded parametric uncertainties in directed communication graphs. The network of agents is represented by the graph $\mathbb{G}(\mathcal{V}, \zeta, \mathit{\Lambda})$, with n agents where each agent corresponds to a vertex belonging to the set $\mathcal{V} = \{v_1, \ldots, v_n\}$. The set $\zeta \subseteq \mathcal{V} \times \mathcal{V}$ represents the graph connections and $\mathit{\Lambda} = [a_{ij}] \in \mathbb{R}^{n\times n}$ is the adjacency matrix containing the connection weights. A connection from agent i to agent j indicates existence of the edge (v_i, v_j) customized by the weight $a_{ij} > 0$, and $a_{ij} = 0$ otherwise. Moreover, the neighborhood of agent i is represented by $\mathcal{N}_i = \{v_j \in \mathcal{V} : (v_j, v_i) \in \zeta\}$. The corresponding Laplacian matrix is defined by

$$\mathcal{L} = [l_{ij}] \in \mathbb{R}^{n\times n} : \begin{cases} l_{ii} = \sum_{j=1}^{n} a_{ij}, & \forall i \in \{1, \ldots, n\}, \\ l_{ij} = -a_{ij}, & \text{if } i \neq j. \end{cases} \tag{1}$$

The connection between the agents (followers) and the leader is described by a diagonal pinning matrix $\Pi = \mathrm{diag}(\pi_1, \cdots, \pi_n)$, where $\pi_i > 0$ represents the connection weight between agent i and the leader, otherwise $\pi_i = 0$. A representation of the interactions in the network of interest using Laplacian and pinning matrices is

$$\mathcal{G} = \mathcal{L} + \Pi.$$

Assumption 1. *The graph $\mathbb{G}(\mathcal{V}, \zeta, \Lambda)$ is weakly connected, that is, it contains at least one directed spanning tree. On top of that, the leader is the root node of the spanning tree.*

Assumption 2. *The matrix $\mathcal{G}$ is diagonalizable, that is, there exists a matrix $T \in \mathbb{C}^{n\times n}$ such that $T\mathcal{G}T^{-1} = D = diag(\lambda_1, \ldots, \lambda_n)$, where λ_i, $i = 1, \ldots, n$, are the eigenvalues of matrix $\mathcal{G}$, and the rows of matrix T contain the eigenvectors of each respective eigenvalue. We consider T such that $\sigma_{min}(T) = 1$, without loss of generality.*

Assumption 1 is a well-known condition in the literature for agent consensus in leader-follower directed networks. Assumption 2 always holds in undirected graphs, and it is a mild assumption for directed graphs, as non-diagonalizable and diagonalizable Laplacian matrices can represent the same topology with different weights. In other words, one can readjust the weights whenever possible to make the corresponding Laplacian matrix diagonalizable.

2.2. Problem formulation and auxiliary results

The evolution of the states of the followers in time is described by

$$\dot{x}_i(t) = (A + \Delta A_i(t))x_i(t) + (B + \Delta B_i(t))u_i(t), \qquad i = 1, \ldots, n, \tag{2}$$

where $x_i(t) \in \mathbb{R}^{n_x}$ represents the agents' states and $u_i(t) \in \mathbb{R}^m$ is the external control input. The uncertainties are time-varying and bounded such that $||\Delta A_i(t)|| \leq \delta_A$ and $||\Delta B_i(t)|| \leq \delta_B$, for all $i \in \{1, \ldots, n\}$, where they can be viewed as model and actuator uncertainties, respectively. The presence of such time-varying bounded uncertainties in linear systems is well established [24]. The dynamics of the leader is expressed by

$$\dot{x}_0(t) = Ax_0(t) + Bu_0(t), \tag{3}$$

where $x_0(t) \in \mathbb{R}^{n_x}$ denotes the leader's state and $u_0(t) \in \mathbb{R}^m$ its external input. Note that when $u_0(t) = 0$ (i.e., the leader has no external input), the leader's behavior is quite limited, whereas allowing $u_0(t) \neq 0$ enables the enforcement of prescribed trajectories even when the leader's internal dynamics is unstable. It is also noteworthy that considering an active leader renders the problem formulation more challenging, since the leader's input becomes an exogenous input in the closed-loop system when it is unknown to the followers. Therefore, the analysis must explicitly account for an upper bound on the external input. The following assumption is necessary.

Assumption 3. *We assume $x_0(t)$ and $u_0(t)$ to be bounded, that is, $||x_0(t)|| \leq \delta_{x_0}$ and $||u_0(t)|| \leq \delta_{u_0}$, where δ_{x_0} and δ_{u_0} are known finite scalar values.*

Assumption 3 holds for both stable and marginally stable systems. In the case of unstable leader dynamics, the leader's input $u_0(t)$ must be characterized to ensure that the trajectories $x_0(t)$ and $u_0(t)$ remain bounded. Although the leader's states are bounded, it is still possible to consider scenarios in which the followers' dynamics are unstable.

Remark 1. *In this paper, we consider the leader free of uncertainties for simplicity. The presence of uncertainties in (3) can be taken into account by adjusting the upper bound of the norm of the external disturbance in the expression of the closed-loop system of the error dynamics.*

In the leader–tracking problem, if all agents share identical dynamics, they can follow the leader with asymptotically stable tracking–error dynamics. However, in the presence of heterogeneous parametric uncertainties, a constant offset in the tracking error may occur. In this work, we focus on designing a decentralized protocol that ensures the trajectories converge to a region containing the origin. Our goal is to estimate a positively invariant and attractive set, denoted by $\mathcal{S}$, as small as possible, such that the error between the followers' and leader's trajectories converges to $\mathcal{S}$. The problem addressed in this paper can thus be summarized as follows.

Problem 1. *Determine a decentralized protocol and a positively invariant and attractive region $\mathcal{S}$, as small as possible, such that the error between the states of followers in (2) and the states of the leader in (3) converge exponentially toward $\mathcal{S}$. In other words,*

$$\lim_{t\to\infty} \inf_{\varepsilon\in\mathcal{S}} ||e_i(t) - \varepsilon|| = 0, \quad i = 1, \dots, n, \tag{4}$$

where $e_i(t) \in \mathbb{R}^{n_x}$ is the tracking error defined as $e_i(t) = x_i(t) - x_0(t)$.

In the following, we state definitions and lemmas that are instrumental for further developments alongside the table of all the variables used in this paper.

Definition 1 ([4]). *A system $\dot{x}(t) = Ax(t) + B\omega(t)$, where $\omega(t) \in \Omega$, with Ω a known closed and bounded set, is said to be quadratically bounded with Lyapunov function $V(x(t)) = x(t)^T Px(t),\ P = P^T > 0$ if*

$$V(x(t)) > 1 \quad \textit{implies} \quad \frac{dV(x(t))}{dt} = 2x^T P(Ax(t) + B\omega(t)) < 0.$$

That is, the set $\mathcal{S} = \{x \in \mathbb{R}^{n_x} : x^T Px \leq 1\}$ is positively invariant and attractive.

Lemma 1 ([21]). *For a positive scalar ϵ, a symmetric positive definite matrix S and matrices X and Y with appropriate dimensions, the following inequality holds*

$$X^TY + Y^TX \leq \frac{1}{\epsilon}X^TS^{-1}X + \epsilon Y^TSY.$$

Lemma 2 ([4]). *Consider a vector x, two symmetric positive semi-definite matrices P and B, and Q as a symmetric positive definite matrix. Then*

$$x^TQx - 2(x^TBx)^{\frac{1}{2}} > 0 \quad \textit{for} \quad x^TPx > 1,$$

if and only if there exists a scalar $\alpha > 0$ such that

$$Q - \alpha P - \alpha^{-1}B \geq 0.$$

Variable	Definition
x_i	State of the agent i
u_i	Control input of the agent i
x_0	State of the leader
u_0	Control input of the leader
e_i	Tracking error of the agent i
$\tilde{e}$	Transformed tracking error ($\tilde{e} = (T \otimes I)e$)
ν	Augmented input of the error dynamics
$\tilde{\nu}$	Transformed Augmented input
$\hat{\lambda}_k$	Extreme value of the network matrix eigenvalues
$\tilde{e}_a$	Augmented state containing transformed tracking error and the leader state
ξ	Augmented state containing transformed tracking error and its integral
$\tilde{\xi}$	transformed ξ
$\tilde{\chi}$	Augmented state containing transformed tracking error, its integral and the leader states

Table 1: Summary of variables.

3. Main results

In this section, we derive conditions for designing distributed controllers to solve Problem 1. The proposed approach employs state feedback control protocols to synchronize the followers with the leader by ensuring a positively invariant and attractive set for the error trajectories.

3.1. Proportional controllers design

In this part, we assume that each agent is controlled by a local protocol with the following structure

$$u_i(t) = \pi_i K(x_i(t) - x_0(t)) + \sum_{j \in \mathcal{N}_i} a_{ij} K(x_i(t) - x_j(t)) + h u_0(t), \qquad i = 1, \ldots, n, \tag{5}$$

where $h \in \{0, 1\}$ specifies whether the agents have access to the leaders' input ($h = 1$) or not ($h = 0$), and K is the feedback gain to be designed. This protocol considers the disagreement among the agents' states and with the leader, as well as the characteristics of the network. Let us define the tracking error $e_i(t) \in \mathbb{R}^{n_x}$ as $e_i(t) = x_i(t) - x_0(t)$ and the vectors collecting the agents' errors, states, and control inputs of the n agents as $e(t) = [e_1(t)^T, \cdots, e_n(t)^T]^T$, $x(t) = [x_1(t)^T, \cdots, x_n(t)^T]^T$, and $u(t) = [u_1(t)^T, \cdots, u_n(t)^T]^T$, respectively. Then, $e(t) = x(t) - \tilde{x}_0(t)$, and (5) becomes

$$u(t) = (\mathcal{G} \otimes K)e(t) + H(\mathbb{1} \otimes u_0(t)), \tag{6}$$

where $H = I_{nm} \otimes h$. The error dynamics is expressed as[1]

$$\dot{e}(t) = ((I_n \otimes A) + \Delta A + (\mathcal{G} \otimes BK) + \Delta B(\mathcal{G} \otimes K))e(t) + \begin{bmatrix} \Delta A & \check{B} + \Delta\check{B} \end{bmatrix} \nu(t), \tag{7}$$

where $\Delta A = \mathrm{diag}(\Delta A_1, \cdots, \Delta A_n)$, $\Delta B = \mathrm{diag}(\Delta B_1, \cdots, \Delta B_n)$, $\Delta\check{B} = \Delta B H$, $\check{B} = (I_n \otimes B)(H - I)$, and $\nu(t) = [\mathbb{1}^T \otimes x_0(t)^T \quad \mathbb{1}^T \otimes u_0(t)^T]^T$,

[1]For simplicity, the time dependency is omitted from ΔA and ΔB.

Remark 2. *In a scenario where we have access to the leader's input in advance, like a virtual leader, we can consider that all agents have access to the leader's input* $u_0(t)$ *(*$h = 1$*). Moreover, for the case of a non-cooperative leader, we set* $h = 0$ *as the agents do not have access to the leader's input. We may also have the case where only a subset of agents have access to* $u_0(t)$*, which can be modeled by replacing* h *by* h_i *in (5), with* $h_i \in \{0, 1\}$*,* $i = 1, \dots, n$*, yielding* $H = diag(I_m \otimes h_1, \dots, I_m \otimes h_n)$*. However, in this case, due to the structure of matrix* H*, the design conditions require the full information of the Laplacian matrix* $\mathcal{G}$*, which is not considered in the present work. Finally, observe from (7) that in the absence of uncertainties and with all agents having access to the leader's input, the tracking error can converge asymptotically to the origin.*

From (7), note that the presence of $\mathcal{G}$ implies full knowledge of the network structure. To relax this requirement, we seek a vector transformation that yields the spectral decomposition of $\mathcal{G}$. In accordance with Assumption 2, consider the following variable transformation

$$\tilde{e}(t) = (T \otimes I_{n_x})e(t), \quad \tilde{\nu}(t) = (I_n \otimes (T \otimes I_{n_x}))\nu(t). \tag{8}$$

Then, (7) is rewritten as

$$\dot{\tilde{e}}(t) = \mathbb{A}\tilde{e}(t) + \mathbb{B}\tilde{\nu}(t), \tag{9}$$

with

$$\begin{aligned}
\mathbb{A} &= (I_n \otimes A) + (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) + D \otimes BK + (T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes K),\\
\mathbb{B} &= \begin{bmatrix}(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) & (T \otimes I_{n_x})(\check{B} + \Delta\check{B})(T^{-1} \otimes I_{n_x})\end{bmatrix}.
\end{aligned}$$

To avoid the necessity of using all eigenvalues in the design conditions, we define the following polytopic region $\mathcal{Q}$ containing the set of all eigenvalues λ_i, $i = 1, \dots, n$, of $\mathcal{G}$

$$\mathcal{Q} := \left\{ \lambda \in \mathbb{C} : \ \lambda = \sum_{k=1}^{\kappa} \alpha_k \hat{\lambda}_k, \ \alpha \in \mathcal{U} \right\}, \tag{10}$$

where $\mathcal{U} = \{\alpha = (\alpha_1, \dots, \alpha_\kappa) \in \mathbb{R}^\kappa : \ \sum_{k=1}^{\kappa} \alpha_k = 1, \ \alpha_k \geq 0\}$, and $\hat{\lambda}_k \in \mathbb{C}, \ k = 1, \dots, \kappa$, are the vertices of the convex hull of the points λ_i, $i = 1, \dots, n$.

Remark 3. *If the communication network is such that* $\mathcal{G}$ *has only real eigenvalues, as undirected networks, one can always choose* $\kappa = 2$ *and* $\hat{\lambda}_1$ *and* $\hat{\lambda}_2$ *are the extreme values of* λ_i*,* $i = 1, \dots, n$*. If the eigenvalues of* $\mathcal{G}$ *are complex, which may arise in directed graphs,* $\mathcal{Q}$ *has at least* $\kappa \geq 3$ *vertices.*

The following theorem solves Problem 1 with control law (6).

Theorem 1. *Suppose that there exist a symmetric positive definite matrix* $W \in \mathbb{R}^{n_x \times n_x}$*, a matrix* $Z \in \mathbb{R}^{m \times n_x}$*, and positive scalars* ϵ_1, ϵ_2 *and* α*, such that the following inequality holds* *for* $k = 1, \dots, \kappa$*:*

$$\Xi_1(\hat{\lambda}_k) := \begin{bmatrix} He\{AW + \hat{\lambda}_k BZ\} + (\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{n_x} + \alpha W & \delta_\nu \psi_1 & W^T & \hat{\lambda}_k Z^T \\ \star & -\alpha I_{2n_x} & 0 & 0 \\ \star & \star & -\epsilon_1 I_{n_x} & 0 \\ \star & \star & \star & -\epsilon_2 I_m \end{bmatrix} \leq 0, \tag{11}$$

with $\psi_1 = [\delta_\Delta I_{n_x} \quad (h-1)B]$ *and*

$$\delta_\nu = \sigma_{max}(T)\sqrt{n(\delta_{x0}^2 + \delta_{u0}^2)}, \quad \delta_\Delta = \sqrt{\hat{\delta}_A^2 + \check{\delta}_B^2}, \tag{12}$$

$$\hat{\delta}_B = \sigma_{max}(T)\delta_B, \quad \hat{\delta}_A = \sigma_{max}(T)\delta_A, \quad \check{\delta}_B = h\hat{\delta}_B. \tag{13}$$

Then, the controller (6) *with* $K = ZW^{-1}$ *makes the closed-loop system* (7) *quadratically bounded with respect to the set* $\mathcal{S} = \{e \in \mathbb{R}^{nn_x} : e^T \mathbb{W}^{-1} e \leq 1, \ \mathbb{W} = I_n \otimes W\}$.

Proof. First, observe that $\lambda_i \in \mathcal{Q}$, $i = 1, \dots, n$, then $\lambda_i = \sum_{k=1}^{\kappa} \alpha_{k,i} \hat{\lambda}_k$. From linearity, $\Xi_1(\hat{\lambda}_k) \leq 0$ implies $\Xi_1(\lambda_i) = \sum_{k=1}^{\kappa} \alpha_{k,i} \Xi_1(\hat{\lambda}_k) \leq 0$. Consider the Lyapunov function $V(\tilde{e}(t)) = \tilde{e}(t)^T \mathbb{P} \tilde{e}(t)$ with $\mathbb{P} = \mathbb{P}^T = I_n \otimes P > 0$. Then, the time derivative of the Lyapunov function with respect to (9) is given by

$$\dot{V}(\tilde{e}(t)) = 2\tilde{e}(t)^T \mathbb{P} \dot{\tilde{e}}(t) = 2\tilde{e}(t)^T \mathbb{P}(\mathbb{A}\tilde{e}(t) + \mathbb{B}\tilde{\nu}(t)).$$

Thus, the time derivative of the Lyapunov function is negative if

$$\max\{2\tilde{e}(t)^T \mathbb{P}(\mathbb{A}\tilde{e}(t) + \mathbb{B}\tilde{\nu}(t))\} < 0,$$

or, equivalently,

$$\tilde{e}(t)^T Q \tilde{e}(t) - 2(\tilde{e}(t)^T \mathbb{P}\mathbb{B}\tilde{\nu}(t)\tilde{\nu}(t)^T \mathbb{B}^T \mathbb{P}\tilde{e}(t))^{\frac{1}{2}} > 0, \ Q = -\mathbb{P}\mathbb{A} - \mathbb{A}^T \mathbb{P}. \tag{14}$$

For the product $\tilde{\nu}(t)\tilde{\nu}(t)^T$, we have

$$\tilde{\nu}(t)\tilde{\nu}(t)^T \leq ||T|| \, ||\nu(t)|| \, ||\nu(t)^T|| \, ||T^T|| \leq \delta_\nu^2, \quad \delta_\nu = \sigma_{\max}(T)\sqrt{n(\delta_{x0}^2 + \delta_{u0}^2)}, \tag{15}$$

where $T\mathcal{G}T^{-1} = D$. Afterward, since for $h \in \{0, 1\}$ we have $(T \otimes I_{n_x})\Delta\check{B}(T^{-1} \otimes I_{n_x})\check{B}^T = 0$, and by considering $\mathbb{B} = \begin{bmatrix}(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) & (T \otimes I_{n_x})(\check{B} + \Delta\check{B})(T^{-1} \otimes I_{n_x})\end{bmatrix}$ in (9), for the product $\mathbb{B}\mathbb{B}^T$ we have

$$\begin{aligned} \mathbb{B}\mathbb{B}^T = (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})(T^{-T} \otimes I_{n_x})\Delta A^T (T^T \otimes I_{n_x}) \\ + (T \otimes I_{n_x})(\Delta\check{B})(T^{-1} \otimes I_{n_x})(T^{-T} \otimes I_{n_x})(\Delta\check{B}^T)(T^T \otimes I_{n_x}) \\ + (T \otimes I_{n_x})(\check{B})(T^{-1} \otimes I_{n_x})(T^{-T} \otimes I_{n_x})(\check{B}^T)(T^T \otimes I_{n_x}). \end{aligned}$$

Consequently, we have $\hat{\delta}_A \geq ||(T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x})||$ and $\check{\delta}_B \geq ||(T\otimes I_{n_x})\Delta \check{B}(T^{-1}\otimes I_{n_x})||$, and it is straight forward to show

$$\mathbb{B}\mathbb{B}^T \leq \delta_\Delta^2 I_{nn_x} + \check{B}\check{B}^T,$$

where $\delta_\Delta = \kappa(T)\sqrt{\delta_A^2 + \delta_B^2}$ and

$$\begin{aligned}\hat{\delta}_A = \delta_A||(T\otimes I_{n_x})||\ ||(T^{-1}\otimes I_{n_x})|| &\geq ||(T\otimes I_{n_x})||\ ||\Delta A||\ ||(T^{-1}\otimes I_{n_x})|| \\ &\geq ||(T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x})||,\end{aligned}$$

$$\begin{aligned}\check{\delta}_B = h\delta_B||(T\otimes I_{n_x})||\ ||(T^{-1}\otimes I_{n_x})|| &\geq ||(T\otimes I_{n_x})||\ ||\Delta BH||\ ||(T^{-1}\otimes I_{n_x})|| \\ &\geq ||(T\otimes I_{n_x})\Delta \check{B}(T^{-1}\otimes I_{n_x})||.\end{aligned}$$

Thus, we conclude that

$$\mathbb{B}\mathbb{B}^T \leq \Psi_1\Psi_1^T, \quad \Psi_1 = \begin{bmatrix}\delta_\Delta I_{nn_x} & \check{B}\end{bmatrix}. \tag{16}$$

Moreover, with respect to Assumption 2 we can have $\sigma_{\min}(T) = 1$, without loss of generality, and consequently, we have $\kappa(T) = \sigma_{\max}(T)$ and $\delta_\Delta = \sigma_{\max}(T)\sqrt{\delta_A^2 + \delta_B^2}$. From (14), (15) and (16) we obtain

$$\tilde{e}(t)^T Q\tilde{e}(t) - 2(\tilde{e}(t)^T\delta_\nu\mathbb{P}\Psi_1\Psi_1^T\mathbb{P}\delta_\nu\tilde{e}(t))^{\frac{1}{2}} \geq \tilde{e}(t)^T Q\tilde{e}(t) - 2(\tilde{e}(t)^T\mathbb{P}\mathbb{B}\tilde{\nu}(t)\tilde{\nu}(t)^T\mathbb{B}^T\mathbb{P}\tilde{e}(t))^{\frac{1}{2}} > 0.$$

Consequently, with respect to $V(\tilde{e}(t)) > 1$ and by exploiting Lemma 2, (14) can be rewritten as

$$Q - \alpha\mathbb{P} - \alpha^{-1}(\delta_\nu\mathbb{P}\Psi_1\Psi_1^T\mathbb{P}\delta_\nu) \geq 0. \tag{17}$$

By using Schur complement lemma, we can present inequality (17) in the following form

$$\begin{bmatrix}\mathrm{He}\{\mathbb{P}\mathbb{A}\} + \alpha\mathbb{P} & \delta_\nu\mathbb{P}\Psi_1 \\ \star & -\alpha I_{2nn_x}\end{bmatrix} \leq 0, \tag{18}$$

and by pre-and-post multiplying (18) by $\mathrm{diag}(\mathbb{W}, I)$, $\mathbb{W} = \mathbb{P}^{-1}$, we obtain

$$\begin{bmatrix}\mathrm{He}\{\mathbb{A}\mathbb{W}\} + \alpha\mathbb{W} & \delta_\nu\Psi_1 \\ \star & -\alpha I_{2nn_x}\end{bmatrix} \leq 0. \tag{19}$$

Observe that the term $\mathrm{He}\{\mathbb{A}\mathbb{W}\}$ can be expressed as

$$\begin{aligned}\mathrm{He}\{\mathbb{A}\mathbb{W}\} = \mathrm{He}\{&(I_n\otimes A)\mathbb{W} + D\otimes BKW\} \\ &+ \mathrm{He}\left\{(T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x})\mathbb{W}\right\} + \mathrm{He}\left\{(T\otimes I_{n_x})\Delta B(\mathcal{G}T^{-1}\otimes KW)\right\},\end{aligned}$$

where using Lemma 1 with $X = \mathbb{W}$, $Y = ((T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x}))^T$ and $S = I$, we have

$$\mathrm{He}\left\{(T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x})\mathbb{W}\right\} \leq \epsilon_1\hat{\delta}_A^2 I_{nn_x} + \epsilon_1^{-1}(\mathbb{W}^T\mathbb{W}),$$

and using Lemma 1 with $X = (\mathcal{G}T^{-1} \otimes KW)$, $Y = ((T \otimes I_{n_x})\Delta B)^T$ and $S = T^T T$ yields

$$\mathrm{He}\left\{(T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes KW)\right\} \leq \epsilon_2 \hat{\delta}_B^2 I_{nn_x} + \epsilon_2^{-1}(T^{-T}\mathcal{G}^T T^T T\mathcal{G}T^{-1} \otimes Z^T Z),$$

where $\hat{\delta}_B$ is an upper bound for $||(T \otimes I_{n_x})\Delta B(T^{-1} \otimes I_{n_x})||$, that is,

$$\begin{aligned}\hat{\delta}_B = \delta_B ||(T \otimes I_{n_x})|| \; ||(T^{-1} \otimes I_{n_x})|| &\geq ||(T \otimes I_{n_x})|| \; ||\Delta B|| \; ||(T^{-1} \otimes I_{n_x})|| \\ &\geq ||(T \otimes I_{n_x})\Delta B(T^{-1} \otimes I_{n_x})||.\end{aligned}$$

Moreover, one can write

$$\begin{bmatrix} \mathrm{He}\left\{(I_n \otimes A)\mathbb{W} + D \otimes BKW\right\} + \epsilon_1 \hat{\delta}_A^2 I_{nn_x} + \alpha \mathbb{W} + \mathrm{He}\left\{(T \otimes I_{n_x})\Delta B(\mathcal{G}T^T \otimes KW)\right\} & \delta_\nu \Psi_1 \\ \star & -\alpha I_{2nn_x} \end{bmatrix} + \begin{bmatrix} \mathbb{W}^T \\ 0 \end{bmatrix} \epsilon_1^{-1} I_{nn_x} \begin{bmatrix} \mathbb{W} & 0 \end{bmatrix} \leq 0,$$

which is equivalent to

$$\begin{bmatrix} \mathrm{He}\left\{(I_n \otimes A)\mathbb{W} + D \otimes BKW\right\} + \epsilon_1 \hat{\delta}_A^2 I_{nn_x} + \alpha \mathbb{W} + \mathrm{He}\left\{(T \otimes I_{n_x})\Delta B(\mathcal{G}T^T \otimes KW)\right\} & \delta_\nu \Psi_1 & \mathbb{W}^T \\ \star & -\alpha I_{2nn_x} & 0 \\ \star & \star & -\epsilon_1 I_{nn_x} \end{bmatrix} \leq 0.$$

Similarly, we can write

$$\begin{bmatrix} \mathrm{He}\left\{(I_n \otimes A)\mathbb{W} + D \otimes BKW\right\} + \alpha \mathbb{W} + \epsilon_1 \hat{\delta}_A^2 I_{nn_x} + \epsilon_2 \hat{\delta}_B^2 I_{nn_x} & \delta_\nu \Psi_1 & \mathbb{W}^T \\ \star & -\alpha I_{2nn_x} & 0 \\ \star & \star & -\epsilon_1 I_{nn_x} \end{bmatrix} + \begin{bmatrix} T^{-T}\mathcal{G}^T T^T \otimes Z^T \\ 0 \\ 0 \end{bmatrix} \epsilon_2^{-1} I_{nm} \begin{bmatrix} T\mathcal{G}T^{-1} \otimes Z & 0 & 0 \end{bmatrix} \leq 0,$$

and, by using the Schur complement lemma, one recovers

$$\begin{bmatrix} \mathrm{He}\{I_n \otimes AW + D \otimes BZ\} + (\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{nn_x} + \alpha \mathbb{W} & \delta_\nu \Psi_1 & \mathbb{W}^T & D^T \otimes Z^T \\ \star & -\alpha I_{2nn_x} & 0 & 0 \\ \star & \star & -\epsilon_1 I_{nn_x} & 0 \\ \star & \star & \star & -\epsilon_2 I_{nm} \end{bmatrix} \leq 0.$$

Since all the entries of the above matrix are of diagonal structure, it is straightforward to find an appropriate congruence transformation to obtain (11) from the above inequality.

It should be noted that initially the invariant and attractive set that we are obtaining is defined by $\tilde{\mathcal{S}} = \{\tilde{e} \in \mathbb{R}^{nn_x} : \tilde{e}^T \mathbb{W}^{-1} \tilde{e} \leq 1,\ \mathbb{W} = I_n \otimes W\}$. To establish the quadratically bounded set with respect to $e(t)$, we can rewrite this set as $\tilde{\mathcal{S}} = \{e \in \mathbb{R}^{nn_x} : e^T(T^T T \otimes W^{-1})e \leq 1\}$. Due to the fact that we have $\sigma_{\min}(T) = 1$, we know $T^T T \otimes W^{-1} \geq I_n \otimes W^{-1}$ and thus, $\tilde{\mathcal{S}} \subseteq \mathcal{S} = \{e \in \mathbb{R}^{nn_x} : e^T \mathbb{W}^{-1} e \leq 1\}$. Hence, the set $\mathcal{S}$ forms an invariant and attractive set. $\square$

Remark 4. *In Theorem 1, instead of using vertices of the convex hull of $\mathcal{Q}$ ($\Xi_1(\hat{\lambda}_k) \leq 0$, $k = 1, \ldots, \kappa$), one can use all of the eigenvalues of the network ($\Xi_1(\lambda_i) \leq 0$, $i = 1, \ldots, n$), which requires a full knowledge of the network structure, but yields better results at the price of a higher computational effort. Observe also that designing a protocol such that (11) holds for all $\hat{\lambda} \in \mathcal{Q}$ implies that (5) solves Problem 1 for a larger class of typologies than $\mathcal{G}$, improving the robustness of the approach.*

Remark 5. *Note that in the case of undirected graphs, the matrix $\mathcal{G}$ is a real symmetric matrix and T is orthonormal, that is, $T^{-1} = T^T$ and $||T|| = 1$. Consequently, the bounds defined in (13) would reduce to $\hat{\delta}_B = \delta_B$, $\hat{\delta}_A = \delta_A$, and $\check{\delta}_B = h\delta_B$, as $\kappa(T) = 1$.*

Remark 6. *To obtain an approximation of the upper bound for $||x_0(t)||$, identified by δ_{x_0}, we exploit the triangle inequality considering $||B|| = \delta_B$ and $||u_0(t)|| = \delta_{u_0}$. The states of the system (3) are thus bounded by $||x_0(t)|| \leq \hat{\delta}_{x_0} + \delta_B \delta_{u_0}$, where $\hat{\delta}_{x_0} = \max\{||S e^{\Lambda t} S^{-1} x_0(0)||$, $t = [0, T]\}$ [29], with Λ being a diagonal matrix containing the eigenvalues of A, and S the matrix of eigenvectors corresponding to these eigenvalues. It is important to note that even if the system is unstable, $x_0(t)$ can remain bounded under an appropriately chosen input signal u_0. A candidate for such practice is a state feedback protocol defined by $u_0(t) = -N x_0(t)$, that transform (7) such that $\Delta \check{B} = \Delta B H \tilde{N}$ and $\check{B} = (I_n \otimes B)(H - I)\tilde{N}$, where $\tilde{N} = I_n \otimes N$. Note that the closed-loop system in this case maintains the same structure with minor adjustments in matrices $\Delta \check{B}$ and $\check{B}$, and the leader dynamics in (3) becomes*

$$\dot{x}_0(t) = (A - BN)x_0(t). \tag{20}$$

If matrix A in (2) is Hurwitz, we can define the following augmented state $\tilde{e}_a(t) = [\tilde{e}(t)^T \quad \tilde{x}_0(t)^T]^T$, where $\tilde{x}_0(t) = (T \otimes I_{n_x})(\mathbb{1} \otimes x_0(t))$, with respect to (2)–(7). By using this augmented vector, we can establish a slightly different setup in the closed-loop system to provide a less conservative solution. We can re-express the closed-loop system in the following form

$$\dot{\tilde{e}}_a(t) = \mathbb{A}_a \tilde{e}_a(t) + \mathbb{B}_a \tilde{\nu}_a(t),\ \tilde{\nu}_a(t) = (I_2 \otimes (T \otimes I_{n_x})) \left[\mathbb{1}_2^T \otimes \tilde{u}_0(t)^T\right]^T, \tag{21}$$

where

$$\mathbb{A}_a = \begin{bmatrix} \mathbb{A} & (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) \\ 0 & (I_n \otimes A) \end{bmatrix},$$

$$\mathbb{B}_a = (I_2 \otimes (T \otimes I_{n_x})) \begin{bmatrix} \Delta \check{B} + \check{B} & 0 \\ 0 & (I_n \otimes B) \end{bmatrix} (I_2 \otimes (T^{-1} \otimes I_{n_x})).$$

Following the same steps of Theorem 1, we propose conditions to design (5) with respect to the closed-loop system (21) in the following corollary.

Corollary 1. *Suppose that there exist two symmetric positive definite matrices $W_1 \in \mathbb{R}^{n_x \times n_x}$ and $W_2 \in \mathbb{R}^{n_x \times n_x}$, a matrix $Z \in \mathbb{R}^{m \times n_x}$, and positive scalars ϵ_1, ϵ_2 and α, such that the following inequality holds for $k = 1, \ldots, \kappa$:*

$$\Xi_2(\hat{\lambda}_k) := \left[\begin{array}{ccc|ccc} & & & W_1 & 0 & \hat{\lambda}_k Z^T \\ & \upsilon_1 & & 0 & W_2 & 0 \\ & & & 0 & 0 & 0 \\ \hline \star & \star & \star & -\epsilon_1 I_{n_x} & 0 & 0 \\ \star & \star & \star & \star & -\epsilon_1 I_{n_x} & 0 \\ \star & \star & \star & \star & \star & -\epsilon_2 I_m \end{array}\right] \leq 0, \tag{22}$$

with

$$\upsilon_1 = \left[\begin{array}{cc|c} He\left\{AW_1 + \hat{\lambda}_k BZ\right\} + \alpha W_1 + (2\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{n_x} & 0 & \\ \star & He\{AW_2\} + \alpha W_2 & \delta_{\nu_a}\psi_2 \\ \hline \star & & -\alpha I_{3n_x} \end{array}\right]$$

and

$$\psi_2 = \begin{bmatrix} \check{\delta}_B I_{n_x} & (h-1)B & 0 \\ 0 & 0 & B \end{bmatrix}$$

where $\delta_{\nu_a} = \sqrt{n(2\delta_{u0}^2)}$ and $\hat{\delta}_A$, $\hat{\delta}_B$ and $\check{\delta}_B$ as defined in (13), then controller (6) with gain $K = ZW_1^{-1}$ makes the closed-loop system (21) quadratically bounded with respect to the set $\mathcal{S}_a = \{e_a \in \mathbb{R}^{2nn_x} : e_a^T \mathbb{W}_a^{-1} e_a \leq 1\}$, where $\mathbb{W}_a = diag(I_n \otimes W_1, I_n \otimes W_2)$.

Proof. Similar to the proof of Theorem 1, for the derivative of the Lyapunov function $V(\tilde{e}_a(t)) = \tilde{e}_a(t)^T \mathbb{P}_a \tilde{e}_a(t)$, with $\mathbb{P}_a = \mathrm{diag}(\mathbb{P}_1, \mathbb{P}_2) > 0$, one has

$$\max\{2\tilde{e}_a(t)^T \mathbb{P}_a(\mathbb{A}_a \tilde{e}_a(t) + \mathbb{B}_a \tilde{\nu}_a(t)\} < 0.$$

We can rewrite the above inequality in the following form

$$\tilde{e}_a(t)^T Q_a \tilde{e}_a(t) - 2(\tilde{e}_a(t)^T \mathbb{P}_a \mathbb{B}_a \tilde{\nu}_a(t) \tilde{\nu}_a(t)^T \mathbb{B}_a^T \mathbb{P}_a \tilde{e}_a(t))^{\frac{1}{2}} > 0, \quad Q_a = -\mathbb{P}_a \mathbb{A}_a - \mathbb{A}_a^T \mathbb{P}_a,$$

which is equivalent to

$$Q_a - \alpha \mathbb{P}_a - \alpha^{-1}(\delta_{\nu_a} \mathbb{P}_a \Psi_2 \Psi_2^T \mathbb{P}_a \delta_{\nu_a})^{\frac{1}{2}} \geq 0, \;\; \Psi_2 = \begin{bmatrix} \check{\delta}_B I_{nn_x} & \check{B} & 0 \\ 0 & 0 & (I_n \otimes B) \end{bmatrix}. \tag{23}$$

Thus, we have (18)–(19) replacing $\mathbb{A}$, Ψ_1 and δ_ν with $\mathbb{A}_a$, Ψ_2 and δ_{ν_a}, respectively. Then, defining $\mathbb{W}_a = \mathrm{diag}(\mathbb{W}_1, \mathbb{W}_2) = \mathbb{P}_a^{-1}$, with $\mathbb{W}_1 = I_n \otimes W_1$ and $\mathbb{W}_2 = I_n \otimes W_2$, we can describe the term $\mathrm{He}\{\mathbb{A}_a \mathbb{W}_a\}$ as

$$\begin{aligned}\mathrm{He}\{\mathbb{A}_a \mathbb{W}_a\} = &\begin{bmatrix}\mathrm{He}\{(I_n \otimes A)\mathbb{W}_1 + (D \otimes BKW_1)\} & 0 \\ \star & \mathrm{He}\{(I_n \otimes A)\mathbb{W}_2\}\end{bmatrix} \\ &+ \underbrace{\begin{bmatrix}\mathrm{He}\{(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_1\} & (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_2 \\ \star & 0\end{bmatrix}}_{\varphi_1} \\ &+ \underbrace{\begin{bmatrix}\mathrm{He}\{(T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes KW_1)\} & 0 \\ \star & 0\end{bmatrix}}_{\varphi_2}.\end{aligned}$$

Using Lemma 1 (similar to proof of Theorem 1), we obtain the following inequalities for the terms in the right side of the above equality

$$\begin{aligned}\varphi_1 = \mathrm{He}&\left\{\begin{bmatrix}(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) & (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) \\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbb{W}_1 & 0 \\ 0 & \mathbb{W}_2\end{bmatrix}\right\} \\ &\leq \underbrace{\epsilon_1^{-1}\begin{bmatrix}\mathbb{W}_1 & 0 \\ 0 & \mathbb{W}_2\end{bmatrix}\begin{bmatrix}\mathbb{W}_1 & 0 \\ 0 & \mathbb{W}_2\end{bmatrix}}_{\varphi_3} + \begin{bmatrix}\epsilon_1 2\hat{\delta}_A^2 I_{nn_x} & 0 \\ 0 & 0\end{bmatrix},\end{aligned}$$

$$\begin{aligned}\varphi_2 = \mathrm{He}&\left\{\begin{bmatrix}(T \otimes I_{n_x})\Delta B \\ 0\end{bmatrix}\begin{bmatrix}\mathcal{G}T^{-1} \otimes KW_1 & 0\end{bmatrix}\right\} \leq \\ &\underbrace{\epsilon_2^{-1}\begin{bmatrix}T^{-T}\mathcal{G}^T T^T \otimes W_1^T K^T \\ 0\end{bmatrix}\begin{bmatrix}T\mathcal{G}T^{-1} \otimes KW_1 & 0\end{bmatrix}}_{\varphi_4} + \begin{bmatrix}\epsilon_2\hat{\delta}_B^2 I_{nn_x} & 0 \\ 0 & 0\end{bmatrix},\end{aligned}$$

consequently, we have

$$\left[\begin{array}{c|c}\begin{bmatrix}\mathrm{He}\{(I_n \otimes A)\mathbb{W}_1 + (D \otimes BKW_1)\} + \alpha\mathbb{W}_1 + (2\epsilon_1\hat{\delta}_A^2 + \epsilon_2\hat{\delta}_B^2)I_{nn_x} & 0 \\ \star & \mathrm{He}\{(I_n \otimes A)\mathbb{W}_2\} + \alpha\mathbb{W}_2\end{bmatrix} + \varphi_3 + \varphi_4 & \delta_{\nu_a}\Psi_2 \\ \hline \star & -\alpha I_{3nn_x}\end{array}\right] \leq 0.$$

By applying Schur complement lemma for φ_3 and φ_4, we obtain

$$\left[\begin{array}{cccc|ccc} & & & & \mathbb{W}_1 & 0 & D^T \otimes Z^T \\ & \Upsilon_1 & & & 0 & \mathbb{W}_2 & 0 \\ & & & & 0 & 0 & 0 \\ & & & & 0 & 0 & 0 \\ \hline \star & \star & \star & \star & -\epsilon_1 I_{nn_x} & 0 & 0 \\ \star & \star & \star & \star & \star & -\epsilon_1 I_{nn_x} & 0 \\ \star & \star & \star & \star & \star & \star & -\epsilon_2 I_{nm}\end{array}\right] \leq 0, \qquad (24)$$

with

$$\Upsilon_1 = \left[\begin{array}{cc|c} \mathrm{He}\left\{(I_n \otimes A)\mathbb{W}_1 + (D \otimes BZ)\right\} + \alpha \mathbb{W}_1 + (2\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{nn_x} & 0 & \multirow{2}{*}{$\delta_{\nu_a}\Psi_2$} \\ \star & \mathrm{He}\left\{(I_n \otimes A)\mathbb{W}_2\right\} + \alpha \mathbb{W}_2 & \\ \hline \multicolumn{2}{c|}{\star} & -\alpha I_{3nn_x} \end{array}\right].$$

Analogous to the proof of Theorem 1, there exists an appropriate congruence transformation to obtain (22) from (24). It is noteworthy that similar to the proof of Theorem 1, the invariant and attractive set obtained here is specified by $\tilde{\mathcal{S}}_a = \{\tilde{e}_a \in \mathbb{R}^{2nn_x} : \tilde{e}_a^T \mathbb{W}_a^{-1} \tilde{e}_a \leq 1\}$ that can be rewritten as $\tilde{\mathcal{S}}_a = \{e_a \in \mathbb{R}^{2nn_x} : e_a^T \begin{bmatrix} T^T T \otimes W_1^{-1} & 0 \\ 0 & T^T T \otimes W_2^{-1} \end{bmatrix} e_a \leq 1\}$. Since $\sigma_{\min}(T) = 1$, we have $\begin{bmatrix} T^T T \otimes W_1^{-1} & 0 \\ 0 & T^T T \otimes W_2^{-1} \end{bmatrix} \geq \mathbb{W}_a$ and consequently, $\tilde{\mathcal{S}}_a \subseteq \mathcal{S}_a = \{e_a \in \mathbb{R}^{2nn_x} : e_a^T \mathbb{W}_a^{-1} e_a \leq 1\}$. Thus, considering the designed feedback gain K, the set $\mathcal{S}_a$ forms an invariant and attractive set for the closed-loop system (21). $\square$

Remark 7. *To further improve Corollary 1, we can relax the diagonal structure of matrices $\mathbb{W}_i = I_n \otimes W_i \in \mathbb{R}^{nn_x \times nn_x}$, $i = 1, 2$, to full symmetric structures. However, this implies that condition (24) is satisfied instead of (22), which requires using all the eigenvalues of the Laplacian matrix contained in matrix D instead of the vertices of $\mathcal{Q}$ in (10).*

Remark 8. *Note that estimating the set $\mathcal{Q}$ in (10) and evaluating $\sigma_{max}(T)$ requires certain knowledge of the network structure that can be seen as a drawback. In the case of evaluating $\sigma_{max}(T)$, there exist methods to estimate an upper bound using partial knowledge of network structure, but this research area is out of the scope of this paper. Moreover, there exist applications such as vehicle platooning [14], where the network has a standard structure, or power networks, where the full structure of the network of interest is available.*

In the next section, we consider the design of proportional-integral controllers.

3.2. Proportional-integral controllers design

In this section, we develop conditions to solve Problem 1 using the proportional-integral (PI) controllers. The integral action is incorporated in the protocol as follows

$$\begin{cases} u(t) = (\mathcal{G} \otimes K)e(t) + (\mathcal{G} \otimes K_I)e_I(t) + H(\mathbb{1} \otimes u_0(t)) \\ \dot{e}_I(t) = e(t), \end{cases} \tag{25}$$

where $e_I(t) \in \mathbb{R}^n$ is the state corresponding to the error integral. The integrator in controller (25) eliminates the offset in the closed-loop system response for constant inputs of $u_0(t)$ and $\omega_0(t)$. To obtain the collective dynamics of the system with respect to the controller (25), we define the augmented vector

$$\xi(t) = \begin{bmatrix} e(t) \\ e_I(t) \end{bmatrix}$$

which results in the following collective dynamics

$$\dot{\xi}(t) = \begin{bmatrix} (I_n \otimes A) + \mathcal{G} \otimes BK + \Delta A + \Delta B(\mathcal{G} \otimes K) & \mathcal{G} \otimes BK_I + \Delta B(\mathcal{G} \otimes K_I) \\ I_{nn_x} & 0 \end{bmatrix} \xi(t) \\ + \begin{bmatrix} \Delta A & \Delta \breve{B} + \breve{B} \\ 0 & 0 \end{bmatrix} \nu(t). \quad (26)$$

Similar to the transformation (8), we can consider the following variable transformation for the augmented vector $\xi(t)$ such that

$$\tilde{\xi}(t) = (I_2 \otimes (T \otimes I_{n_x}))\xi(t).$$

Afterwards, we can reformulate (26) as

$$\dot{\tilde{\xi}}(t) = \mathbb{A}\tilde{\xi}(t) + \mathbb{B}\tilde{\nu}(t), \quad (27)$$

where

$$\mathbb{A} = \underbrace{\begin{bmatrix} (I_n \otimes A) + (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x}) & 0 \\ I_{nn_x} & 0 \end{bmatrix}}_{\mathcal{A}} \\ + \underbrace{\begin{bmatrix} D \otimes B + (T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes I_m) \\ 0 \end{bmatrix}}_{\mathcal{B}} \underbrace{\begin{bmatrix} I_n \otimes K & I_n \otimes K_I \end{bmatrix}}_{\mathcal{K}},$$

$$\mathbb{B} = (I_2 \otimes (T \otimes I_{n_x})) \begin{bmatrix} \Delta A & \Delta \breve{B} + \breve{B} \\ 0 & 0 \end{bmatrix} (I_2 \otimes (T^{-1} \otimes I_{n_x})).$$

We propose the following theorem to solve Problem 1 using controller (25).

Theorem 2. *Suppose that there exist two symmetric positive definite matrices defined by $W_i \in \mathbb{R}^{n_x \times n_x}$, $i = 1, 2$, matrices $W_3 \in \mathbb{R}^{n_x \times n_x}$, $Z \in \mathbb{R}^{m \times n_x}$ and $Z_I \in \mathbb{R}^{m \times n_x}$, and positive scalars ϵ_1, ϵ_2 and α, such that the following inequality holds for $k = 1, \ldots, \kappa$:*

$$\Xi_3(\hat{\lambda}_k) := \left[\begin{array}{ccc|cc} & & & W_1 & \hat{\lambda}_k Z^T \\ & \upsilon_2 & & W_3^T & \hat{\lambda}_k Z_I^T \\ & & & 0 & 0 \\ \hline \star & \star & \star & -\epsilon_1 I_{n_x} & 0 \\ \star & \star & \star & \star & -\epsilon_2 I_m \end{array}\right] \leq 0, \quad (28)$$

with

$$\upsilon_2 = \left[\begin{array}{cc|c} He\left\{AW_1 + \hat{\lambda}_k BZ\right\} + \alpha W_1 + (\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{n_x} & W_1 + \hat{\lambda}_k BZ_I + \alpha W_3 + AW_3 & \\ \star & He\{W_3\} + \alpha W_2 & \delta_\nu \psi_3 \\ \hline \multicolumn{2}{c|}{\star} & -\alpha I_{2n_x} \end{array}\right],$$

and

$$\psi_3 = \begin{bmatrix} \delta_\Delta I_{n_x} & (h-1)B \\ 0 & 0 \end{bmatrix},$$

with δ_ν, δ_Δ, $\hat{\delta}_A$, $\hat{\delta}_B$ *and* $\check{\delta}_B$ *as in* (12)–(13), *then the controller* (25) *with gain* $\mathcal{K} = \mathcal{Z}\mathbb{W}_I^{-1}$, $\mathcal{Z} = [I_n \otimes Z \quad I_n \otimes Z_I]$, *makes the closed-loop system* (27) *quadratically bounded with respect to the set* $\mathcal{S}_I = \{\xi \in \mathbb{R}^{2nn_x} : \xi^T \mathbb{W}_I^{-1}\xi \leq 1\}$ *and* $\mathbb{W}_I = \begin{bmatrix} \mathbb{W}_1 & \mathbb{W}_3 \\ \mathbb{W}_3^T & \mathbb{W}_2 \end{bmatrix}$, $\mathbb{W}_i = I_n \otimes W_i$, $i = 1, 2, 3,$.

Proof. With respect to the closed-loop system in (27) and by considering the Lyapunov function $V(\tilde{\xi}(t)) = \tilde{\xi}(t)^T \mathbb{P}_I \tilde{\xi}(t)$ with $\mathbb{P}_I = \mathbb{P}_I^T > 0$, we have the time derivative of the Lyapunov function as

$$\tilde{\xi}(t)^T Q \tilde{\xi}(t) - 2(\tilde{\xi}(t)^T \mathbb{P}_I \mathbb{B} \tilde{\nu}(t)\tilde{\nu}(t)^T \mathbb{B}^T \mathbb{P}_I \tilde{\xi}(t))^{\frac{1}{2}} > 0,\ Q = -\mathrm{He}\left\{\mathbb{P}_I \mathbb{A}\right\},\ \mathbb{P}_I = \begin{bmatrix} \mathbb{P}_1 & \mathbb{P}_3 \\ \mathbb{P}_3^T & \mathbb{P}_2 \end{bmatrix},$$

using Lemma 2, one has

$$Q - \alpha\mathbb{P}_I - \alpha^{-1}(\delta_\nu \mathbb{P}_I \Psi_3 \Psi_3^T \mathbb{P}_I \delta_\nu) \geq 0,\ \Psi_3 = \begin{bmatrix} \delta_\Delta I_{nn_x} & (T \otimes I_{n_x})\check{B}(T^{-1} \otimes I_{n_x}) \\ 0 & 0 \end{bmatrix},$$

where $(T \otimes I_{n_x})\check{B}(T^{-1} \otimes I_{n_x}) = \check{B}$. By exploiting Schur complement lemma and pre-and-post multiplying its product by $\mathrm{diag}\,(\mathbb{W}_I, I)$, $\mathbb{W}_I = \mathbb{P}_I^{-1} = \begin{bmatrix} \mathbb{W}_1 & \mathbb{W}_3 \\ \mathbb{W}_3^T & \mathbb{W}_2 \end{bmatrix}$ (similar to the proof of Theorem 1), we can write

$$\left[\begin{array}{c|c} \mathrm{He}\left\{\mathbb{A}\mathbb{W}_I\right\} + \alpha\mathbb{W}_I & \delta_\nu \Psi_3 \\ \hline \star & -\alpha I_{2nn_x} \end{array}\right] \leq 0.$$

Considering $\mathcal{Z} = \mathcal{K}\mathbb{W}_I$ where $\mathcal{Z} = [I_n \otimes Z \quad I_n \otimes Z_I]$, in the expression $\mathrm{He}\left\{\mathbb{A}\mathbb{W}_I\right\} + \alpha\mathbb{W}_I$, we have

$$\begin{aligned} &\mathrm{He}\left\{(\mathcal{A} + \mathcal{B}\mathcal{K})\mathbb{W}_I\right\} + \alpha\mathbb{W}_I \\ &= \begin{bmatrix} \mathrm{He}\left\{(I_n \otimes A)\mathbb{W}_1 + D \otimes BZ\right\} + \alpha\mathbb{W}_1 & \mathbb{W}_1 + D \otimes BZ_I + (I_n \otimes A)\mathbb{W}_3 + \alpha\mathbb{W}_3 \\ \star & \mathrm{He}\left\{\mathbb{W}_3\right\} + \alpha\mathbb{W}_2 \end{bmatrix} \\ &\quad + \underbrace{\begin{bmatrix} \mathrm{He}\left\{(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_1\right\} & (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_3 \\ \star & 0 \end{bmatrix}}_{\varphi_5} \\ &\quad + \underbrace{\begin{bmatrix} \mathrm{He}\left\{(T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes Z)\right\} & (T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes Z_I) \\ \star & 0 \end{bmatrix}}_{\varphi_6}. \end{aligned} \tag{29}$$

The expressions φ_5 and φ_6 in (29) can be rewritten in the following form using Lemma 1

$$\varphi_5 = \mathrm{He}\left\{\begin{bmatrix}(T\otimes I_{n_x})\Delta A(T^{-1}\otimes I_{n_x})\\ 0\end{bmatrix}\begin{bmatrix}\mathbb{W}_1 & \mathbb{W}_3\end{bmatrix}\right\} \leq \begin{bmatrix}\epsilon_1\hat{\delta}_A^2 I_{nn_x} & 0\\ 0 & 0\end{bmatrix} + \epsilon_1^{-1}\begin{bmatrix}\mathbb{W}_1\\ \mathbb{W}_3^T\end{bmatrix}\begin{bmatrix}\mathbb{W}_1 & \mathbb{W}_3\end{bmatrix},$$

$$\begin{aligned}\varphi_6 = \mathrm{He}\left\{\begin{bmatrix}(T\otimes I_{n_x})\Delta B\\ 0\end{bmatrix}\begin{bmatrix}\mathcal{G}T^{-1}\otimes Z & \mathcal{G}T^{-1}\otimes Z_I\end{bmatrix}\right\} &\leq \begin{bmatrix}\epsilon_2\hat{\delta}_B^2 I_{nn_x} & 0\\ 0 & 0\end{bmatrix}\\ &+ \epsilon_2^{-1}\begin{bmatrix}T^{-T}\mathcal{G}^T T^T\otimes Z^T\\ T^{-T}\mathcal{G}^T T^T\otimes Z_I^T\end{bmatrix}\begin{bmatrix}T\mathcal{G}T^{-1}\otimes Z & T\mathcal{G}T^{-1}\otimes Z_I\end{bmatrix}.\end{aligned}$$

Thus, by using the Schur complement lemma, one recovers

$$\left[\begin{array}{ccc|cc} & & & \mathbb{W}_1 & D^T\otimes Z^T\\ & \Upsilon_2 & & \mathbb{W}_3^T & D^T\otimes Z_I^T\\ & & & 0 & 0\\ \hline \star & \star & \star & -\epsilon_1 I_{nn_x} & 0\\ \star & \star & \star & \star & -\epsilon_2 I_{nm}\end{array}\right] \leq 0, \tag{30}$$

with

$$\Upsilon_2 = \left[\begin{array}{cc|c}\mathrm{He}\{(I_n\otimes A)\mathbb{W}_1 + D\otimes BZ\} + \alpha\mathbb{W}_1 + (\epsilon_1\hat{\delta}_A^2 + \epsilon_2\hat{\delta}_B^2)I_{nn_x} & \mathbb{W}_1 + (D\otimes BZ_I) + \alpha\mathbb{W}_3 + (I_n\otimes A)\mathbb{W}_3 & \delta_\nu\Psi_3\\ \star & \mathrm{He}\{\mathbb{W}_3\} + \alpha\mathbb{W}_2 & \\ \hline \star & & -\alpha I_{2nn_x}\end{array}\right].$$

Due to the diagonal structure of every entry of the matrix in inequality (30), with an appropriate congruence transformation of this inequality, one recovers (28). Note that, similar to the proof of Theorem 1 and Corollary 1, it is possible to show $\tilde{\mathcal{S}}_I \subseteq \mathcal{S}_I$ and hence, the set $\mathcal{S}_I$ is established as an invariant and attractive set for (27) considering the gain $\mathcal{K}$. □

As before, we develop conditions for the particular case of matrix A in (2) being Hurwitz. The closed-loop system (27) is rewritten using the augmented state $\tilde{\chi}(t) = [\tilde{e}(t)^T\ \ \tilde{e}_I(t)^T\ \ \tilde{x}_0(t)^T]^T$

$$\dot{\tilde{\chi}}(t) = \mathbb{A}_a\tilde{\chi}(t) + \mathbb{B}_a\tilde{\nu}(t),\ \ \nu(t) = [\mathbb{1}^T\otimes x_0(t)^T\ \ \mathbb{1}^T\otimes u_0(t)^T]^T = (I_2\otimes(T^{-1}\otimes I_{n_x}))\tilde{\nu}(t) \tag{31}$$

where

$$\mathbb{A}_a = \underbrace{\begin{bmatrix}\mathcal{A} & 0\\ 0 & (I_n\otimes A)\end{bmatrix}}_{\mathcal{A}_a} + \underbrace{\begin{bmatrix}\mathcal{B}\\ 0\end{bmatrix}}_{\mathcal{B}_a}\underbrace{\begin{bmatrix}\mathcal{K} & 0\end{bmatrix}}_{\mathcal{K}_a},\quad \mathbb{B}_a = (I_3\otimes(T\otimes I_{n_x}))\begin{bmatrix}\Delta A & (\Delta\breve{B} + \breve{B})\\ 0 & 0\\ 0 & (I_n\otimes B)\end{bmatrix}(I_2\otimes(T^{-1}\otimes I_{n_x})).$$

Corollary 2. *Suppose that there exist three symmetric positive definite matrices $W_i \in \mathbb{R}^{n_x \times n_x}$, $i = 1, 2, 4$, matrices $W_3 \in \mathbb{R}^{n_x \times n_x}$, $Z \in \mathbb{R}^{m \times n_x}$ and $Z_I \in \mathbb{R}^{m \times n_x}$, and the positive scalars ϵ_1, ϵ_2, ϵ_3 and α, such that the following inequality holds for $k = 1, \ldots, \kappa$:*

$$\Xi_4(\hat{\lambda}_k) := \left[\begin{array}{cccc|cc} & & & & W_1 & \hat{\lambda}_k Z^T \\ & \multicolumn{2}{c}{\upsilon_3} & & W_3^T & \hat{\lambda}_k Z_I^T \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline \star & \star & \star & \star & -\epsilon_1 I_{n_x} & 0 \\ \star & \star & \star & \star & \star & -\epsilon_2 I_m \end{array}\right] \leq 0, \tag{32}$$

with

$$\upsilon_3 = \left[\begin{array}{ccc|c} He\left\{AW_1 + \hat{\lambda}_k BZ\right\} + \alpha W_1 + (\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2) I_{n_x} & W_1 + \hat{\lambda}_k B Z_I + AW_3 + \alpha W_3 & 0 & \\ \star & He\{W_3\} + \alpha W_2 & 0 & \delta_\nu \psi_4 \\ \star & \star & He\{AW_4\} + \alpha W_4 & \\ \hline & \star & & -\alpha I_{4n_x} \end{array}\right]$$

and

$$\psi_4 = \begin{bmatrix} \delta_\Delta I_{n_x} & \epsilon_3^{-\frac{1}{2}} \check{\delta}_B I_{n_x} & (1+\epsilon_3^{-1})^{\frac{1}{2}} (h-1) B & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+\epsilon_3)^{\frac{1}{2}} B \end{bmatrix}$$

considering (12) *and* (13)*, then, the controller* (25) *with gain $\mathcal{K}_a = \mathcal{Z}_a \mathbb{W}_{I_a}^{-1}$ makes the closed-loop system* (31) *quadratically bounded with respect to the set $\mathcal{S}_{I_a} = \{\chi \in \mathbb{R}^{3nn_x} : \chi^T \mathbb{W}_{I_a}^{-1} \chi \leq 1\}$, where*

$$\mathbb{W}_{I_a} = \begin{bmatrix} \mathbb{W}_1 & \mathbb{W}_3 & 0 \\ \mathbb{W}_3^T & \mathbb{W}_2 & 0 \\ 0 & 0 & \mathbb{W}_4 \end{bmatrix}, \mathbb{W}_i = I_n \otimes W_i,\ \ i = 1, \ldots, 4.$$

Proof. Similar to the development of the proof of Corollary 1, we define the Lyapunov function $V(\tilde{\chi}(t)) = \tilde{\chi}(t)^T \mathbb{P}_{I_a} \tilde{\chi}(t)$ with $\mathbb{P}_{I_a} = \mathbb{P}_{I_a}^T > 0$. The time-derivative of $V(\tilde{\chi}(t))$ with closed-loop system (31) yields

$$Q - \alpha \mathbb{P}_{I_a} - \alpha^{-1}(\delta_\nu \mathbb{P}_{I_a} \mathbb{B}_a \mathbb{B}_a^T \mathbb{P}_{I_a} \delta_\nu) \geq 0,$$

where $Q = -\text{He}\{\mathbb{P}_{I_a} \mathbb{A}_a\}$ and

$$\mathbb{P}_{I_a} = \begin{bmatrix} \mathbb{P}_1 & \mathbb{P}_3 & 0 \\ \mathbb{P}_3^T & \mathbb{P}_2 & 0 \\ 0 & 0 & \mathbb{P}_4 \end{bmatrix}, \ \mathbb{P}_i = I_n \otimes P_i,\ \ i = 1, \ldots, 4.$$

Using Lemma 1, one has

$$\mathbb{B}_a \mathbb{B}_a^T \leq \Psi_4 \Psi_4^T, \quad \Psi_4 = \begin{bmatrix} \delta_\Delta I & \epsilon_3^{-\frac{1}{2}} \check{\delta}_B I & (1+\epsilon_3^{-1})^{\frac{1}{2}} \check{B} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (1+\epsilon_3)^{\frac{1}{2}} (I_n \otimes B) \end{bmatrix},$$

and consequently,

$$Q - \alpha \mathbb{P}_{I_a} - \alpha^{-1}(\delta_\nu \mathbb{P}_{I_a} \Psi_4 \Psi_4^T \mathbb{P}_{I_a} \delta_\nu) \geq 0.$$

Applying the Schur complement lemma and pre- and post-multiplying the resulting inequality by $\mathrm{diag}\,(\mathbb{W}_{I_a}, I)$, $\mathbb{W}_{I_a} = \mathbb{P}_{I_a}^{-1}$, we obtain

$$\left[\begin{array}{c|c} \mathrm{He}\,\{\mathbb{A}_a \mathbb{W}_{I_a}\} + \alpha \mathbb{W}_{I_a} & \delta_\nu \Psi_4 \\ \hline \star & -\alpha I_{4nn_x} \end{array}\right] \leq 0.$$

For the expression $\mathrm{He}\,\{\mathbb{A}_a \mathbb{W}_{I_a}\} + \alpha \mathbb{W}_{I_a}$, we have

$$\begin{aligned} &\mathrm{He}\,\{\mathbb{A}_a \mathbb{W}_{I_a}\} + \alpha \mathbb{W}_{I_a} = \mathcal{A}_a \mathbb{W}_{I_a} + \mathcal{B}_a \mathcal{Z}_a + \alpha \mathbb{W}_{I_a} = \\ &\begin{bmatrix} \mathrm{He}\,\{(I_n \otimes A)\mathbb{W}_1 + D \otimes BZ\} + \alpha \mathbb{W}_1 & \mathbb{W}_1 + D \otimes BZ_I + (I_n \otimes A)\mathbb{W}_3 + \alpha \mathbb{W}_3 & 0 \\ \star & \mathrm{He}\,\{\mathbb{W}_3\} + \alpha \mathbb{W}_2 & 0 \\ \star & \star & \mathrm{He}\,\{(I_n \otimes A)\mathbb{W}_4\} + \alpha \mathbb{W}_4 \end{bmatrix} + \\ &\begin{bmatrix} \mathrm{He}\,\{(T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_1 + (T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes Z)\} & (T \otimes I_{n_x})\Delta A(T^{-1} \otimes I_{n_x})\mathbb{W}_3 + (T \otimes I_{n_x})\Delta B(\mathcal{G}T^{-1} \otimes Z_I) & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{bmatrix}, \end{aligned}$$

where

$$\mathcal{Z}_a = \mathcal{K}_a \mathbb{W}_{I_a} = \begin{bmatrix} I \otimes Z & I \otimes Z_I & 0 \end{bmatrix}.$$

Using the same steps similar to the proof of Theorem 2, we recover

$$\left[\begin{array}{cccc|cc} & & & & \mathbb{W}_1 & D^T \otimes Z^T \\ & \Upsilon_3 & & & \mathbb{W}_3^T & D^T \otimes Z_I^T \\ & & & & 0 & 0 \\ & & & & 0 & 0 \\ \hline \star & \star & \star & \star & -\epsilon_1 I_{nn_x} & 0 \\ \star & \star & \star & \star & \star & -\epsilon_2 I_{nm} \end{array}\right] \leq 0, \tag{33}$$

with

$$\Upsilon_3 = \left[\begin{array}{ccc|c} \mathrm{He}\,\{(I_n \otimes A)\mathbb{W}_1 + D \otimes BZ\} + \alpha \mathbb{W}_1 + (\epsilon_1 \hat{\delta}_A^2 + \epsilon_2 \hat{\delta}_B^2)I & \mathbb{W}_1 + D \otimes BZ_I + (I_n \otimes A)\mathbb{W}_3 + \alpha \mathbb{W}_3 & 0 & \\ \star & \mathrm{He}\,\{\mathbb{W}_3\} + \alpha \mathbb{W}_2 & 0 & \delta_\nu \Psi_4 \\ \star & \star & \mathrm{He}\,\{(I_n \otimes A)\mathbb{W}_4\} + \alpha \mathbb{W}_4 & \\ \hline & \star & & -\alpha I_{4nn_x} \end{array}\right].$$

With an appropriate congruence transformation of (33), one recovers (32). Moreover, since $\tilde{\mathcal{S}}_{I_a} \subseteq \mathcal{S}_{I_a}$, the set $\mathcal{S}_{I_a}$ is attractive and invariant. $\square$

Remark 9. *Similar to Remark 7, we can relax the diagonal structure of* $\mathbb{W}_4 = I_n \otimes W_4 \in \mathbb{R}^{nn_x \times nn_x}$ *to a full symmetric structure by solving* (33) *instead of* (32) *and using all the eigenvalues of the Laplacian matrix.*

Remark 10. *To indirectly minimize the sets* $\mathcal{S}$*, one can minimize the trace of* W*, that is, the minimization of the positively invariant and attractive set in each presented approach is achieved through solving the following optimization problem*

$$\begin{aligned} &\min \ \ \mathit{Trace}(W) \\ &\text{subject to } W \geq \mu I, \end{aligned}$$

where μ *is a positive scalar chosen by the designer. Note that, since* W *is directly related to the size of the aforementioned sets, a small* μ *would require an excessively large control action to stabilize the system, which may be interpreted as impractical. To address this issue, an appropriate trade-off between the stabilizing gain amplitude and the size of the invariant and attractive set (or, equivalently, the tracking error) can be achieved by selecting a suitable value of* μ*. The same procedure can be performed to minimize the sets* $\mathcal{S}_a$*,* $\mathcal{S}_I$ *and* $\mathcal{S}_{I_a}$*.*

4. Numerical examples

In this section, we present the numerical examples to illustrate the performance of the proposed approaches in a variety of different scenarios. The algorithms are implemented employing YALMIP [18] and SeDuMi [30][2].

Example 1. *Consider the leader-follower problem of three agents in an undirected network specified by the following Laplacian and pining matrices, respectively.*

$$L = \begin{bmatrix} 3 & -1 & -2 \\ -1 & 3 & -2 \\ -2 & -2 & 4 \end{bmatrix}, \quad \Pi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

with topology illustrated in Figure 1. The agents' dynamics are adapted from [28] in the presence of bounded uncertainty,

$$\begin{bmatrix} \dot{p}_i(t) \\ \ddot{p}_i(t) \end{bmatrix} = \left(\begin{bmatrix} 0 & 1 \\ -1 & -b \end{bmatrix} + \Delta A_i \right) \begin{bmatrix} p_i(t) \\ \dot{p}_i(t) \end{bmatrix} + \left(\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} + \Delta B_i \right) u_i(t), \quad i = 1, \dots, n, \tag{34}$$

with $\delta_A = 0.1$ *and* $\delta_B = 0.1$ *and* b *a parameter defined later. We consider the leader input as* $u_0 = 4$ *for* $t \in [20,\ 25]s$ *and* $u_0 = 1$*, otherwise. We aim to design controllers* (5) *and* (25) *to solve Problem 1 for the network described in Fig 1. We consider the following cases with respect to the stability of the open-loop system.*

[2]The codes related to the simulations and the details about required packages to run them can be found in this link.

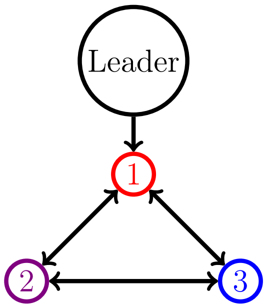

Figure 1: Undirected graph representing the connections of the agents and the leader.

1) Marginally stable system: *Consider a non-cooperative leader, where the agents do not have access to the leader's input ($h = 0$), system* (34) *with $b = 0$, and $\delta_{x_0} = 25$. Theorem 1 with $\epsilon_1 = 20$, $\epsilon_2 = 900$ and $\alpha = 100$ provided the following state feedback gain matrix for the control law* (5),

$$K = \begin{bmatrix} -241.5139 & -41.1084 \\ 100.1028 & -181.9332 \end{bmatrix}.$$

Additionally, the controller (25) *is designed using Theorem 2 with $\epsilon_2 = 5000$,$\alpha = 10$ and keeping other parameters the same, yielding*

$$K = \begin{bmatrix} -60.5861 & -3.4678 \\ 25.0983 & -32.0338 \end{bmatrix}, \; K_I = \begin{bmatrix} -339.7262 & -13.6163 \\ 102.9867 & -209.3448 \end{bmatrix}.$$

The time simulation of the leader state and errors of the followers with controllers (5) *and* (25) *are shown in Figure 2 for $x_0(0) = (-1, 1), x_1(0) = (-0.8095, -2.9443), x_2(0) = (1.4384, -0.7549)$ and $x_3(0) = (1.3703, -1.7115)$. We notice that the controller* (25) *removes the offset and reduces the magnitude of the error trajectories at the expense of a slower response than the proportional controller. Moreover, we observe that both controllers successfully eliminate the error caused by the pulse in the leader input u_0 applied between $t = 20$ and $t = 25$, which illustrates the robustness of the controllers in dealing with unexpected changes to track the leader.*

2) Stable system: *Now, consider a cooperative leader ($h = 1$) and system* (34) *with $b = 1$, which makes matrix A Hurwitz and $\delta_{x_0} = 19$. We applied Corollary 1* and *Remark 7 with $\epsilon_1 = 20$, $\epsilon_2 = 500$, $\alpha = 10$ to obtain the following state feedback gain in* (5)

$$K = \begin{bmatrix} -37.7559 & -0.2517 \\ 18.5971 & -18.0862 \end{bmatrix}.$$

Additionally, Corollary 2 with $\epsilon_1 = 20$, $\epsilon_2 = 5000$, $\epsilon_3 = 10$, and $\alpha = 0.9$ designed the following state feedback gains in (25),

$$K = \begin{bmatrix} -77.5936 & -70.9846 \\ 3.5562 & -146.1243 \end{bmatrix}, \; K_I = \begin{bmatrix} -41.9392 & -32.1421 \\ 3.6758 & -72.0933 \end{bmatrix}.$$

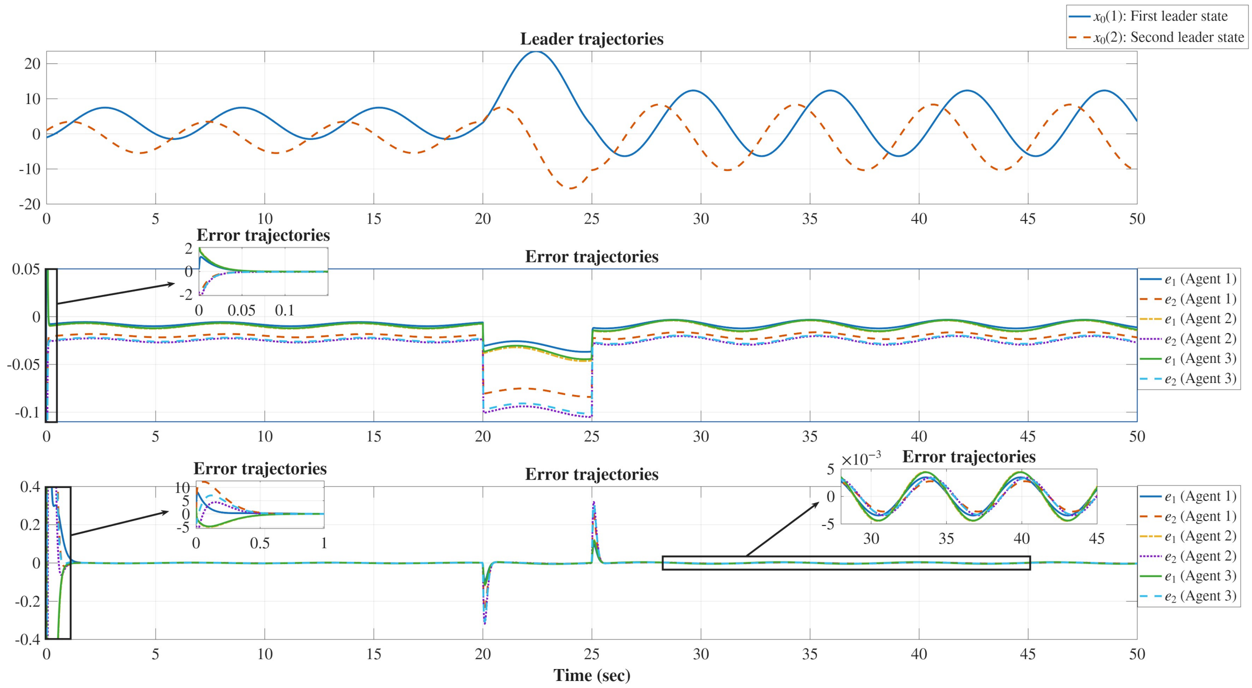

Figure 2: States trajectories of the leader (top figure) and the followers errors for marginally stable open-loop system using controllers (5) and (25) (middle and bottom figures, respectively) with $u_0 = 4$ for $t \in [20,\ 25]s$ and $u_0 = 1$ otherwise, for Example 1.

The time simulation of the leader state and error trajectories of the followers are shown in Figure 3. We can observe in Fig. 3 that controller (5) *provides a faster convergence, despite maintaining a certain level of offset in the error trajectories, which is eliminated using the controller* (25)*.*

We note that for parameters $\epsilon_1 = 20$ and $\epsilon_2 \in [13, 20]$, Corollary 1 combined with Remark 7 yields feasible results while Theorem 1 fails to synthesize the gain K. However, it remains unclear whether alternative parameter values exist for which Theorem 1 can provide a feasible solution.

Example 2. *Consider the undirected network presented in Fig. 4 and the following unstable dynamics for the agents*

$$\dot{x}_i(t) = (\begin{bmatrix} 0.25 & -2 \\ 1 & 0 \end{bmatrix} + \Delta A_i)x_i(t) + (\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} + \Delta B_i)u_i(t), \quad i = 1, \dots, 4. \tag{35}$$

Since the dynamics of the leader defined by (3) *is unstable, we choose the control law $u_0 = -Nx_0$, with*

$$N = \begin{bmatrix} 1.0674 & -0.5913 \\ -0.4087 & 0 \end{bmatrix},$$

such that the leader closed-loop system is marginally stable and consequently, $||x_0(t)||$ is bounded by $\delta_{x_0} = 2$. By utilizing Theorem 1 with $\epsilon_1 = 160$, $\epsilon_2 = 1200$, and $\alpha = 10$ with

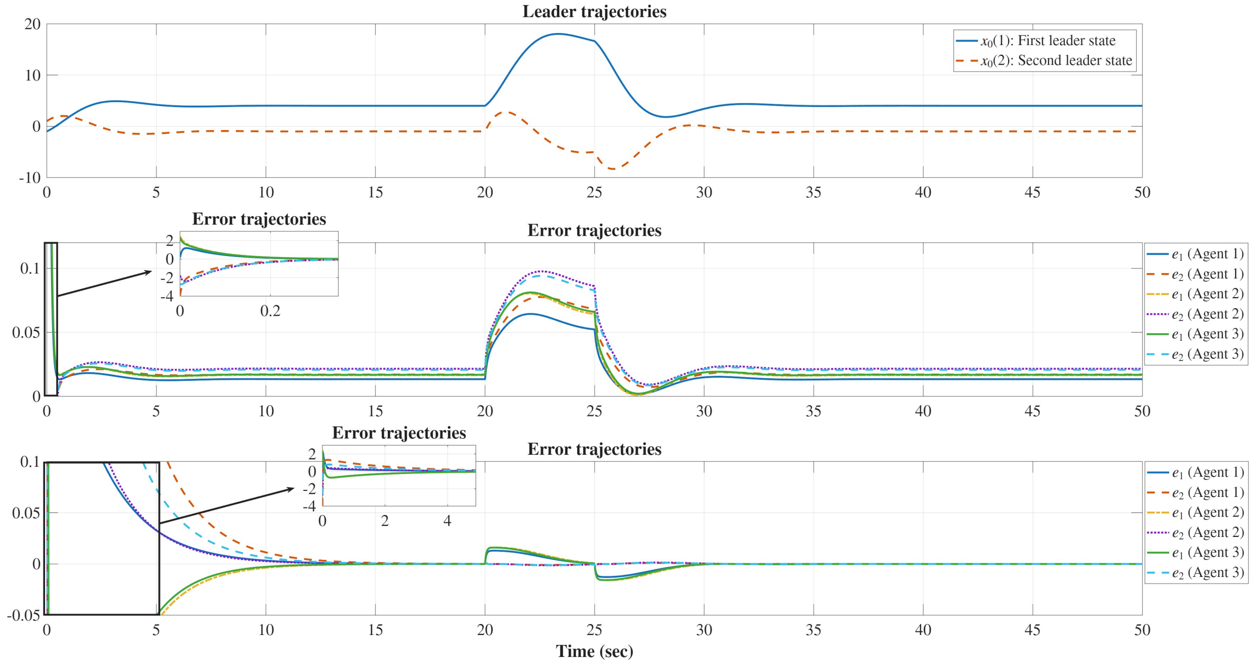

Figure 3: States trajectories of the leader (top figure) and the followers errors for stable open-loop system using controllers (5) and (25) (middle and bottom figures, respectively) with $u_0 = 4$ for $t \in [20,\ 25]s$ and $u_0 = 1$ otherwise, for Example 1.

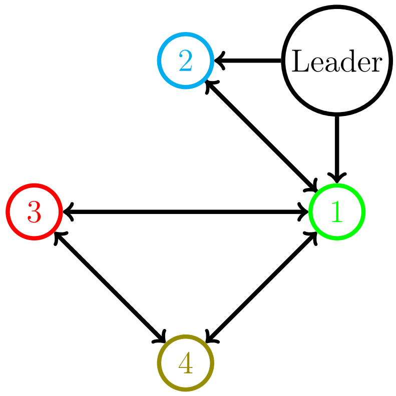

Figure 4: Undirected graph considered in Example 2.

$\delta_A = 0.1$ and $\delta_B = 0.1$, we obtain the following gain for controller (5)

$$K = \begin{bmatrix} -110.3243 & 182.5004 \\ -19.5465 & -100.8547 \end{bmatrix}.$$

The time simulation of the leader states along with the error trajectories of the follower agents with initial conditions $x_0(0) = (-1, 1), x_1(0) = (-0.7342, -2.3652), x_2(0) = (2.5127, -0.8369)$, $x_3(0) = (1.5364, -1.9102)$ and $x_4(0) = (-2.0220, 1.5894)$ is shown in

Figure [5](#). We can see that the controller successfully minimized the steady-state tracking error to a value below 0.007.

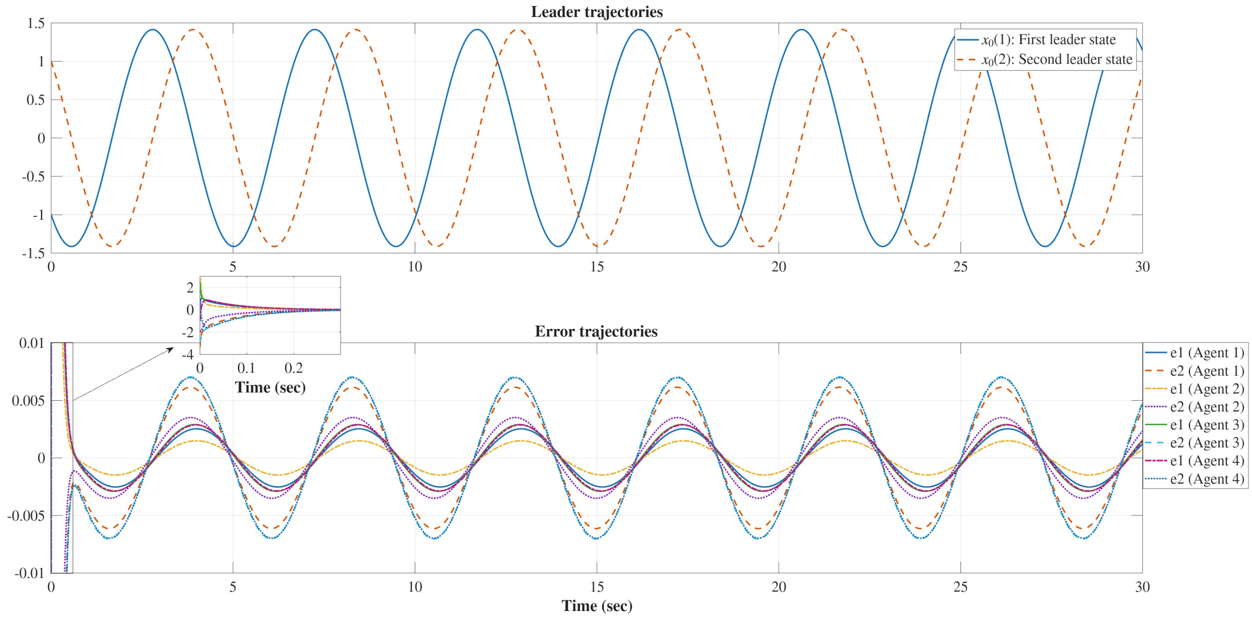

Figure 5: Trajectories of the leader states and the followers' errors for system (35).

Example 3. *Consider the directed network of 8 agents illustrated in Figure 6, where the connections between the followers have a weight of 2 and the connection weight from the leader is equal to 1. Consider system* (34) *with* $b = 0$, $\delta_A = 0.1$ *and* $\delta_B = 0.1$. *As Example 1,*

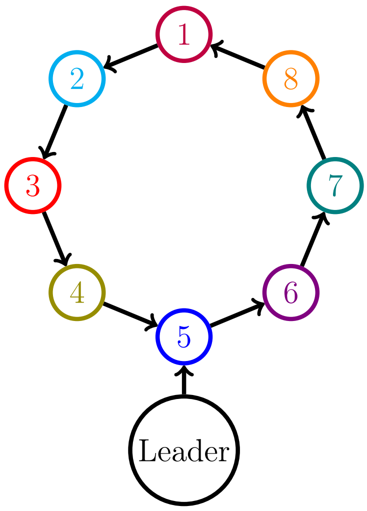

Figure 6: Directed graph considered in Example 3.

we consider the leader input as $u_0 = 4$ *for* $t \in [20,\ 25]s$ *and* $u_0 = 1$, *otherwise, and* $\delta_{x_0} = 25$.

We apply Theorem 1 with $\epsilon_1 = 20$, $\epsilon_2 = 2200$ and $\alpha = 100$, to design controller (5), yielding

$$K = \begin{bmatrix} -1075.8 & -0.5 \\ 536.9 & -540.2 \end{bmatrix}.$$

The trajectories of the agents errors and the leader states are illustrated in Figure 9 with $x_0(0) = (-1, 1)$, $x_1(0) = (-0.8095, \ -2.9443)$, $x_2(0) = (1.4384, \ -0.7549)$, $x_3(0) = (1.3703, \ -1.7115)$, $x_4(0) = (-0.9095, \ -2.5443)$, $x_5(0) = (1.7384, \ -0.2549)$, $x_6(0) = (1.9703, \ -1.5115)$, $x_7(0) = (-0.6095, \ -2.4443)$ and $x_8(0) = (1.8384, \ -0.8549)$. We can

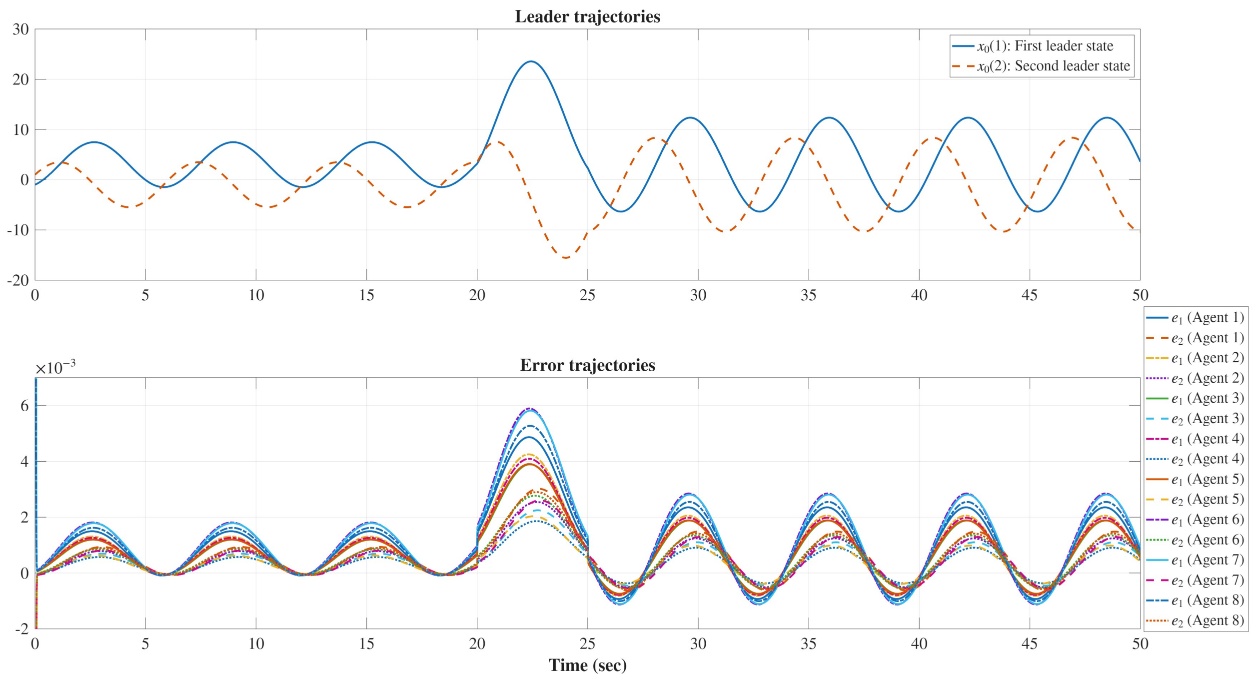

Figure 7: Trajectories of the followers' errors (bottom figure) and the leader states (top figure) with $u_0 = 4$ for $t \in [20, \ 25]s$ and $u_0 = 1$ otherwise, for Example 3.

observe that the controller successfully minimized the tracking error in this network.

Example 4. *In this example, we aim to demonstrate the effectiveness of the proposed results for networks with a large number of agents while maintaining the computational cost of the protocol design at the same level as that of small networks. For this, consider the undirected network of 20 agents illustrated in Figure 8 where the connections between the followers have a weight of 4 and the connection weight from the leader is equal to 5, and as it is evident, only one of the agents (agent 5) has access to the leader's states. Consider system (34) with $b = 0$, $h = 1$, $\delta_A = 0.1$ and $\delta_B = 0.1$. Let the leader input as $u_0 = 2$ for $t \in [20, \ 25]s$ and $u_0 = 1$ otherwise, and $\delta_{x_0} = 13$. We apply Theorem 1 with $\epsilon_1 = 496.93$, $\epsilon_2 = 9059.6$ and $\alpha = 10$, to design controller (5), yielding*

$$K = \begin{bmatrix} -781.3687 & -16.3487 \\ 382.5118 & -415.1799 \end{bmatrix}.$$

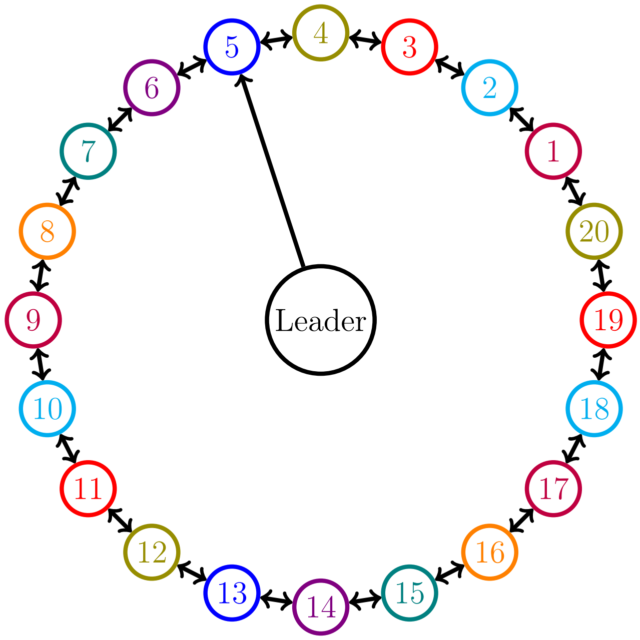

Figure 8: The graph representing the network considered in Example 4.

The trajectories of the agents' errors and the leader states are illustrated in Figure 9.

We can observe that the controller successfully minimized the tracking error in this network. Note that in this case, Theorem 1 was implemented only for the two vertices of polytopic region $\mathcal{Q}$, $\hat{\lambda}_1 = 0.0736$ *and* $\hat{\lambda}_2 = 17.434$ *(minimum and maximum eigenvalues of* $\mathcal{G}$*). Therefore, we relax the necessity of knowing all the eigenvalues of the network and keep the computational burden of finding a feasible solution low, regardless of the number of agents in the network.*

Example 5. *In this example, we adapt Example 2 of [9], which models the linearised longitudinal dynamic equation of an F-18 aircraft, by adding heterogeneous uncertain terms to the dynamics, yielding*

$$\dot{x}_i(t) = (\begin{bmatrix} -1.175 & 0.9871 \\ -8.458 & -0.8776 \end{bmatrix} + \Delta A_i)x_i(t) + (\begin{bmatrix} -0.194 & -0.03593 \\ -19.29 & -3.803 \end{bmatrix} + \Delta B_i)u_i(t), \quad i = 1, \ldots, 6, \tag{36}$$

In (36)*, the first and the second state of the system represent the angle of attack and the pitch rate, respectively. The network topology is the same as [9, Example 2] and the leader has no external input* ($u_0 = 0$)*. By considering* $\delta_A = 0.1$ *and* $\delta_B = 0.1$*, we apply Theorem 1 with* $\epsilon_1 = 6.3759$, $\epsilon_2 = 4.6840$*, and* $\alpha = 1$ *to design the controller* (5) *with gain*

$$K = \begin{bmatrix} 13.1250 & 69.6058 \\ 2.8459 & 15.0704 \end{bmatrix}.$$

The trajectories of the leader states and agents' errors are illustrated in Figure 10 considering $x_0(0) = (-1, 1)$, $x_1(0) = (-30, \ -25)$, $x_2(0) = (23, \ -11)$, $x_3(0) = (9, \ 5)$,

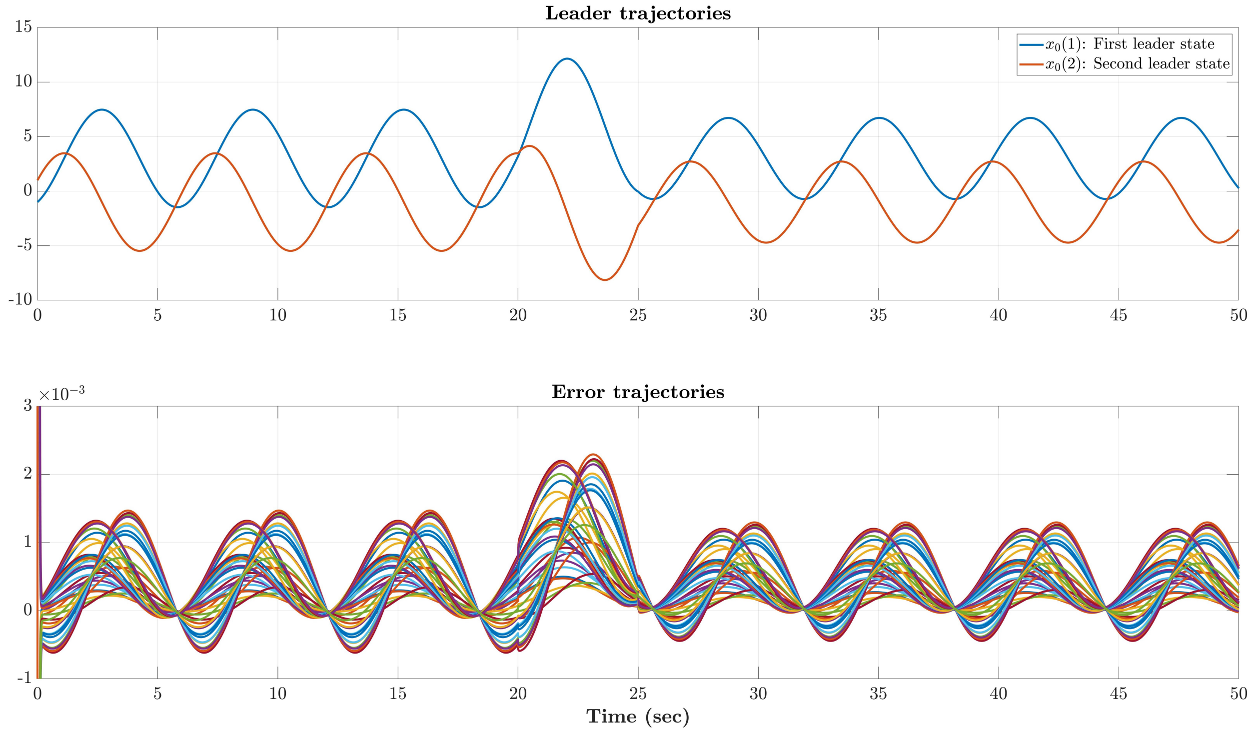

Figure 9: Trajectories of the followers' errors (bottom figure) and the leader states (top figure) with $u_0 = 2$ for $t \in [20,\ 25]s$ and $u_0 = 1$ otherwise, for Example 3.

$x_4(0) = (14,\ -3)$, $x_5(0) = (1,\ -2)$, $x_6(0) = (26,\ 29)$. *We observe that the closed-loop system is asymptotically stable, as the error trajectories converge to the origin over time. This example illustrates that the proposed method handles particular cases of the considered framework without introducing additional conservativeness when compared with existing techniques in the literature.*

5. Conclusion

This paper has presented a decentralized control strategy for the leader-follower problem in multi-agent systems subject to heterogeneous uncertainties. The main challenge that has been addressed here is mitigating offset errors caused by heterogeneous dynamics induced by such uncertainties, while avoiding the use of centralized information about the network interconnection structure. In addition, the heterogeneous uncertainties have caused the leader's input to generate tracking errors, which are attenuated by incorporating integral action in the controller. To address this issue, the offset term in the tracking error dynamics has been reformulated within the framework of quadratic boundedness, providing a novel solution. Sufficient conditions, expressed in terms of linear matrix inequalities, have been derived to guarantee that the followers' tracking errors have remained quadratically bounded. The proposed design is scalable, with conditions depending only on the extreme eigenvalues

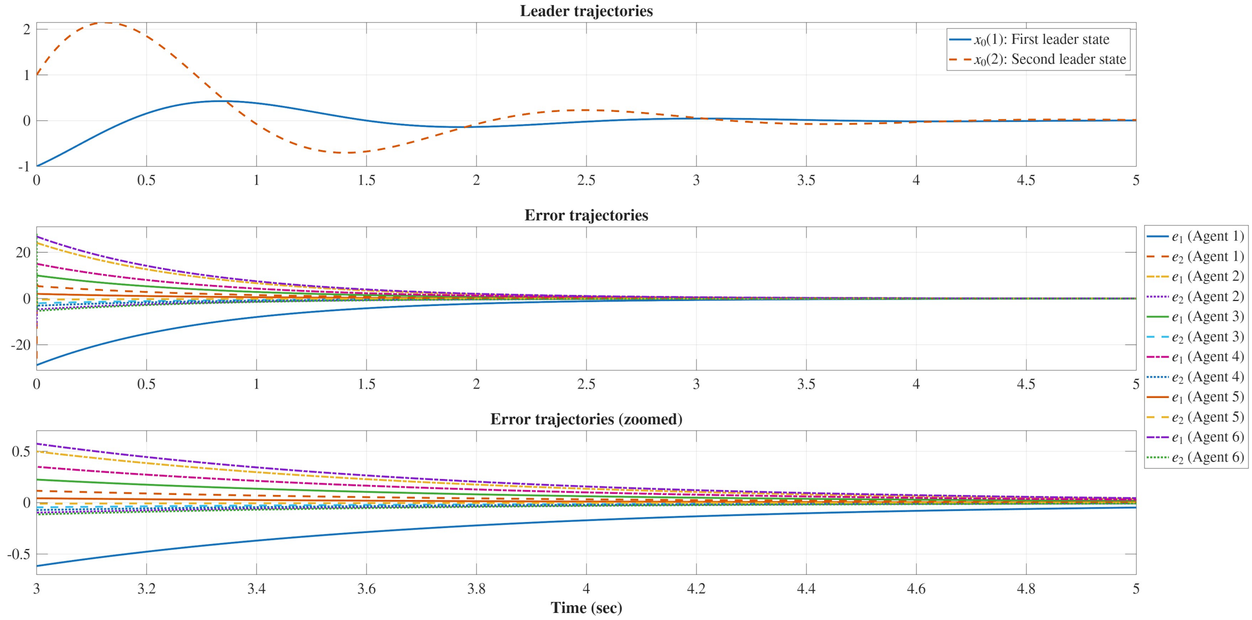

Figure 10: Trajectories of the leader states (top figure) and the followers' errors (bottom figures) for Example 5.

of the network Laplacian matrix, regardless of the number of agents. Numerical examples have been included to demonstrate the effectiveness of the method.

Declaration of Competing Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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