Event-triggered fixed-time stabilization of two time scales linear systems

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***Abstract*—This paper investigates the fixed-time stabilization of uncertain linear time-invariant systems exhibiting two time scales using a state-feedback event-triggered controller. We proceed by emulation and assume that we know a controller that solves the problem in the absence of sampling. We then take sampling into account and present an event-triggered strategy, which consists of two independent sampling mechanisms, associated with the slow and the fast subsystems, respectively. In this setting, the fixed-time stability property becomes practical, where the adjustable parameters are constants used to define the triggering rules. The existence of a strictly positive time between any two successive transmissions is ensured for each transmission law. A numerical example is provided to illustrate the effectiveness of the results.**

***Index Terms*—fixed-time stabilization, two time scales system, event-triggered control.**

I. Introduction

Systems that evolve on two time scales appear in many domains of applications such as, e.g., biology [1], electric power management [2], and chemical engineering [3]. One way to model such dynamical systems is to introduce a small strictly positive parameter multiplying the time derivative of the variables exhibiting fast dynamics: we talk of singularly perturbed systems [4]. As a result, standard control design techniques are typically no longer suitable due to numerical issues. Appropriate methodological control tools are therefore needed, see e.g., [4]–[8].

In this context, we aim at developing controllers, which ensure a fixed-time stability property for linear time-invariant systems exhibiting two time scales. By fixed-time stability, we mean that the origin is stable and that any solution converges to the origin in, at most, a given (finite) strictly positive time, which we are free to select and which is thus independent of the initial conditions. In that way, all solutions have reached the attractor in the desired time. Fixed-time stability [9] is relevant in applications like spacecrafts [10], mobile robots [11] and underwater vehicles [12]. This property is attracting an increasing attention and Lyapunov-based methods have been proposed as an effective tool for analysing the fixed-time stability of e.g., double integrators [13], linear systems [14] and sliding mode control systems [15]. As far as we know, the related literature concentrates on single time scale systems: no results are available for singularly perturbed systems. Moreover, most works on fixed-time stabilization focus on continuous-time systems [16]–[19]. In practice, feedback laws are commonly implemented digitally, and the induced sampling may severely affect the properties of the closed-loop system if not handled carefully.

In this context, we consider plants modeled by uncertain two time scales linear time invariant systems and our objective is to investigate the scenario where the controller is implemented digitally so that it only communicates with the plant at some sampling instants. We start by assuming the knowledge of a state-feedback controller, which ensures the fixed-time stability of the closed-loop system in the absence of sampling. Easily checkable sufficient conditions for such a design are provided in the paper. We then take sampling into account and design event-triggering conditions to approximately preserve the desired stability property; we therefore follow an emulation approach [20]. Instead of generating the transmissions between the plant and the controller periodically, or more generally using clocks as in traditional sampling control setups, event-triggered control consists in transmitting information only when this is needed according to the plant state to (possibly) save communication, and computation resources [21]. To design the triggering condition so that there exists a strictly positive minimum inter-event time is the main difficulty in this context because of the two time scales nature of the system, and the non-smoothness of the feedback law. Before we present our solution, we briefly review the existing literature on the event-triggered control of singularly perturbed systems.

While numerous event-triggering control techniques are nowadays available, see e.g., [20], [21] and the references therein, solutions for singularly perturbed systems are scarce and their application is subject to limitations. In [22], two transmission laws, inspired by the work in [23] and based on spatial- and time-regularization respectively, are proposed for a class of nonlinear systems, assuming the origin of the fast subsystem is globally asymptotically stable. The latter assumption has been relaxed in [24]. In both [22] and [24], (practical) asymptotic stability properties are ensured. The novelty in the present work is to ensure *fixed-time* stability properties, which is stronger as we can directly tune the convergence speed of the state. However, this requires the use of non-smooth feedback laws as mentioned above, which renders the analysis more challenging. Like in [25], we design two triggering laws in this paper, one for the slow dynamics and one for the fast dynamics, in agreement with the two

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time scales nature of the plant. This distributed strategy is known to potentially exhibit the Zeno phenomenon [26] if we apply "off-the-shelf" existing triggering techniques for single time-scale systems [23]. To avoid this issue, we design transmission rules based on absolute and relative threshold strategies [27], [28] as well as decreasing thresholds [20] adapted to the context of the paper. As a result, the original fixed-time stability property ensured by the continuous-time controller becomes practical when it is implemented digitally, where the adjustable parameters are the absolute thresholds. We guarantee the existence of a uniform semiglobal strictly positive times between any two transmissions generated by each triggering law, in the sense that there exists a strictly positive minimum time between any two transmissions, whose value can be taken uniform over given ball of initial conditions centered at the origin.

The rest of the paper is organized as follows. The problem formulation is stated in Section II. The main results are presented in Section III. An illustrative example is presented in Section IV. Conclusions are drawn in Section V.

Notation. $\mathbb{Z}^+$ is the set of strictly positive integers, $\mathbb{R}^{m\times n}$ denotes the set of $m \times n$ real matrices with $m, n \in \mathbb{Z}^+$. The notation I_n stands for the n-dimensional identity matrix with $n \in \mathbb{Z}^+$. For a given real symmetric matrix P, $P > 0$ means that P is a positive definite matrix, and $\lambda_{\min}(P)$ and $\lambda_{\max}(P)$ represent the minimum and the maximum eigenvalues of P, respectively. The notation $\|\cdot\|$ denotes the Euclidean norm for vectors or the induced 2-norm for matrices depending on the context. For any $z_i \in \mathbb{R}^{n_i}$ with $i = 1,\ldots,n$ and $n_i \in \mathbb{Z}^+$, $(z_1,\ldots,z_n)$ stands for $(z_1^\top,\ldots,z_n^\top)^\top$. Diagonal matrices are written as $\mathrm{diag}\,(d_1,\ldots,d_N)$ where $d_1,\ldots,d_N$ are the diagonal coefficients. Given a vector $x = (x_1,\ldots,x_n) \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}_{>0}$, we define $\mathrm{sig}(x)^\alpha := [\mathrm{sign}(x_1)|x_1|^\alpha,\ldots,\mathrm{sign}(x_n)|x_n|^\alpha]^\top$, where $\mathrm{sign}(\cdot)$ is the sign function. We say that function $f : [0,\infty)^2 \to \mathbb{R}^{m\times n}$ is a $O(\varepsilon)$ if there exist positive constants k and ε^* strictly positive such that $\|f(t,\varepsilon)\| \le k\varepsilon$, for all $t \in [0,\infty)$ and $\varepsilon \in [0,\varepsilon^*]$. We consider $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions as defined in [5, Chapter 4].

II. Problem Statement

Consider the uncertain two time scales system

$$\begin{cases} \dot{x}(t) = (A_{11}+\Xi_{11}(t))x(t)+(A_{12}+\Xi_{21}(t))z(t)+B_1u(t),\\ \varepsilon\dot{z}(t) = (A_{21}+\Xi_{21}(t))x(t)+(A_{22}+\Xi_{22}(t))z(t)+B_2u(t), \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ and $z(t) \in \mathbb{R}^{n_z}$ are the slow and the fast states at time $t \ge 0$, respectively, ε is a small strictly positive parameter inducing the time-scale separation between the slow and the fast dynamics, $u(t) \in \mathbb{R}^{n_u}$ is the control input. Matrices $A_{ij}, B_i, i,j = 1,2$, are known constant matrices of appropriate dimensions, and $\Xi(t) := \begin{pmatrix} \Xi_{11}(t) & \Xi_{12}(t) \\ \Xi_{21}(t) & \Xi_{22}(t) \end{pmatrix} = \sum_{k=1}^{m} q_k(t)\Xi_k$ represents the structured uncertainty with uncertain parameters $q_k(t)$ and known constant matrices Ξ_k. We omit the time dependency of Ξ in the sequel.

To control system (1), we make the next assumptions, which are standard in the singularly perturbed literature [4].

Assumption 1. *The matrix A_{22} is invertible.*

Assumption 1 is essential to separate the slow and fast dynamics, see Section III.

Assumption 2. *The pairs (A_0, B_0) and (A_{22}, B_2) are controllable, where $A_0 := A_{11} - A_{12}A_{22}^{-1}A_{21}$, $B_0 := B_1 - A_{12}A_{22}^{-1}B_2$.*

Assumption 2 is instrumental for the design of asymptotically stabilizing feedback gains for the reduced order and boundary layer dynamics [4]. Like in [25], under Assumption 2, there exist positive definite, symmetric matrices $\bar{P}_1$, $\bar{P}_2$, $\bar{Q}_1$, $\bar{Q}_2$, such that

$$\bar{P}_1A_0 + A_0^\top\bar{P}_1 + 2\bar{P}_1B_0B_0^\top\bar{P}_1 + 2\bar{Q}_1 = 0, \tag{2}$$

$$\bar{P}_2A_{22} + A_{22}^\top\bar{P}_2 + 2\bar{P}_2B_2B_2^\top\bar{P}_2 + 2\bar{Q}_2 = 0. \tag{3}$$

Moreover, defining $K_0 := B_0^\top P_1$, $K_2 = B_2^\top P_2$ yields that $A_0 + B_0K_0$ and $A_{22} + B_2K_2$ are both Hurwitz.

We assume that there exists a state-feedback control law of the form

$$u = g(\xi) = K_{\text{stab}}\xi - \mu_1\mathrm{sig}(K_{\text{ft}}\xi)^\alpha - \mu_2\mathrm{sig}(K_{\text{ft}}\xi)^\beta, \tag{4}$$

such that the origin of the closed-loop system (1), (4) is T-globally fixed-time stable, as formalized in the following, where $\xi := (x,z)$ is the concatenated state, $\alpha > 1$, $\beta \in [\frac{1}{2}, 1)$, $\mu_1, \mu_2 > 0$, and $K_{\text{stab}}, K_{\text{ft}}$ are design parameters.

Definition 1. *The origin of system* (1), (4) *is* T-globally fixed-time stable *with $T > 0$ if there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0,\bar{\varepsilon}]$ there exists a class $\mathcal{KL}$ function $\beta_{T,\varepsilon}$ verifying, for any $s \ge 0$ and $t \ge T$, $\beta_{T,\varepsilon}(s,t) = 0$, such that for any solution (x,z) and any $t \ge 0$, $\|(x(t),z(t))\| \le \beta_{T,\varepsilon}(\|(x(0),z(0))\|,t)$.*

The notion of T-global fixed-time stability in Definition 1 corresponds to the global fixed-time stability as defined in [14] when the maximum convergence time is known to be T, adapted to the two time scales context.

We consider the case where the matrix gain $K_{\text{stab}} := (K_1, K_2)$ in (4) is designed using the decoupling of the slow and fast dynamics i.e., $K_1 = (1 - K_2A_{22}^{-1}B_2)K_0 + K_2A_{22}^{-1}A_{21}$ with $K_0 = B_0^\top\bar{P}_1$, and $K_2 = B_2^\top\bar{P}_2$. This design is based on the following coordinate transformation

$$\begin{pmatrix} x_s \\ z_f \end{pmatrix} := T_c^{-1}\begin{pmatrix} x \\ z \end{pmatrix}. \tag{5}$$

where $T_c := \begin{pmatrix} I_{n_x} & \varepsilon H \\ -L & I_{n_z} - \varepsilon LH \end{pmatrix}$ and L, H are the solution of the following equations

$$\begin{aligned} &\Lambda_{21} - \Lambda_{22}L + \varepsilon L\Lambda_{11} - \varepsilon L\Lambda_{12}L = 0,\\ &\Lambda_{12} - H\Lambda_{22} + \varepsilon\Lambda_{11}H - \varepsilon\Lambda_{12}LH - \varepsilon HL\Lambda_{12} = 0, \end{aligned} \tag{6}$$

and $\Lambda_{ij} := A_{ij} + B_iK_j$, for $i,j = 1,2$.

We write (4) as $u = u_{\text{stab}} + u_{\text{ft}}$ with $u_{\text{stab}} := K_{\text{stab}}\xi$ and $u_{\text{ft}} := -\mu_1\mathrm{sig}(K_{\text{ft}}\xi)^\alpha - \mu_2\mathrm{sig}(K_{\text{ft}}\xi)^\beta$ for the sake of convenience. The term u_{stab} corresponds to the standard composite controller used to asymptotically stabilize the origin of (1), see [4, Chapter 3], while u_{ft} is introduced to enforce the desired fixed-time convergence property. Let $\xi_c := (x_s, z_f)$. We write $\dot{\xi}_c = f(\xi_c)$ for the sake of convenience. Controller (4) is assumed to be designed such that the next assumption holds.

Assumption 3. *Given $T > 0$, controller* (4) *is designed such that there exist positive definite symmetric matrices $Q := \text{diag}(Q_1, Q_2)$, P_1, P_2 and $b, c > 0$, such that $\frac{1}{c(1-\beta)} + \frac{1}{b(\alpha-1)} \leq T$ and $V_c(\xi_c) := \xi_c^\top P_\varepsilon \xi_c$ verifies, for any $\xi_c \in \mathbb{R}^{n_x+n_z}$,*

$$\langle \nabla V_c(\xi_c), f(\xi_c)\rangle \leq -\frac{1}{2}\xi_c^\top Q\xi_c - b(\xi_c^\top P\xi_c)^{\frac{1+\alpha}{2}} - c(\xi_c^\top P\xi_c)^{\frac{1+\beta}{2}}, \tag{7}$$

where $P_\varepsilon := \text{diag}(P_1, \varepsilon P_2)$, $P := \text{diag}(P_1, P_2)$.

Assumption 3 guarantees the existence of a state-feedback controller (4), which renders the origin of system (1), (4) T-globally fixed-time stable in view of Lemma 1 in the appendix.

Remark 1. *When $B := [B_1^\top, B_2^\top]^\top$ is full-row rank, we can always design controller* (4) *so that Assumption 3 holds in view of Lemma 4 in the Appendix.*

To prepare the design of the triggering rules, we write system (1), (4) in the (x_s, z_f)-coordinates in (5)

$$\begin{pmatrix} \dot{x}_s \\ \dot{z}_f \end{pmatrix} = (A_D + \Xi_c)\begin{pmatrix} x_s \\ z_f \end{pmatrix} - B_D u_{\text{ft}},$$

where $\Xi_c := T_c^{-1}E^{-1}\Xi T_c$, $E = \text{diag}(I_{n_x}, \varepsilon I_{n_z})$, A_D is a block-diagonal matrix with

$$\begin{aligned}
&A_D := \begin{pmatrix} A_s + B_s K_s & 0 \\ 0 & \frac{A_f + B_f K_2}{\varepsilon} \end{pmatrix}, B_D := \begin{pmatrix} B_d \\ \frac{B_f}{\varepsilon} \end{pmatrix},\\
&A_s := A_0 - \varepsilon A_{12}A_{22}^{-1}L(A_{11} - A_{12}L),\\
&B_s := B_0 - \varepsilon A_{12}A_{22}^{-1}LB_1, K_s := K_1 - K_2 L,\\
&A_f := A_{22} + \varepsilon L A_{12}, B_f := B_2 + \varepsilon L B_1,\\
&B_d := B_1 - HB_2 - \varepsilon HLB_1.
\end{aligned} \tag{8}$$

III. Event-triggered control

We investigate the scenario where controller (4) is digitally implemented and communicates with system (1) using event-triggered transmission schemes. Our objective is to approximately preserve the fixed-time stability property of system (1), (4) ensured by Assumption 3 and to guarantee the existence of a strictly positive lower bound on the minimum inter-times for each triggering rule thereby excluding Zeno phenomenon.

A. Hybrid model

Because of sampling, the implementation of controller (4) leads to

$$u = g(\hat{\xi}) = K_{\text{stab}}\hat{\xi} - \mu_1 \text{sig}(K_{\text{ft}}\hat{\xi})^\alpha - \mu_2 \text{sig}(K_{\text{ft}}\hat{\xi})^\beta, \tag{9}$$

where $\hat{\xi} = (\hat{x}, \hat{z})$, and $\hat{x}$ and $\hat{z}$ are the sampling-induced versions of x and z, respectively. We assume that these variables are generated using zero-order-hold devices. Hence, between any two successive sampling instants $\dot{\hat{x}} = 0$ and $\dot{\hat{z}} = 0$. We design two triggering conditions in the sequel: one associated to x, and one associated to z, in agreement with the two time scales of system (1). We denote the corresponding sequence of transmissions instants as $t_{s,k}$, $k \in \mathcal{I}_s \subseteq \mathbb{Z}$, and $t_{f,k}$, $k \in \mathcal{I}_f \subseteq \mathbb{Z}$, respectively. In that way, at $t = t_{s,k}$, only x is sampled so that $(\hat{x}(t_{s,k}^+), \hat{z}(t_{s,k}^+)) = (x(t_{s,k}), \hat{z}(t_{s,k}))$ for $k \in \mathcal{I}_s$, and at $t = t_{f,k}$, $(\hat{x}(t_{f,k}^+), \hat{z}(t_{f,k}^+)) = (\hat{x}(t_{f,k}), z(t_{f,k}))$ for $k \in \mathcal{I}_f$.

To design the triggering conditions, we also introduce an auxiliary open-loop variable $\varphi := (\varphi_s, \varphi_f) \in \mathbb{R}^2_{\geq 0}$, whose dynamics between consecutive triggering instants is given by

$$\dot{\varphi}_i = -\omega_{1,i}\varphi_i - \omega_{2,i}\text{sig}(\varphi_i)^\alpha - \omega_{3,i}\text{sig}(\varphi_i)^\beta, \; i \in \{s, f\}, \tag{10}$$

and at any sampling instants $\varphi_i^+ = \varphi_i$, where $\omega_{1,i}$, $\omega_{2,i}$, $\omega_{3,i}$ are strictly positive constants to be designed. Note that variable φ is initialized in $\mathbb{R}^2_{\geq 0}$, we will elaborate more on the initialization of φ in the following, see Remark 3.

Given φ, the triggering rule for the slow variable is given by $\varpi_1 e_s^\top e_s + \varpi_2(e_s^\top e_s)^\alpha + \varpi_3(e_s^\top e_s)^\beta \geq \varpi_4 x^\top x + \varpi_5\varphi_s^2 + \nu_s$, and the fast variable is sampled whenever $\varpi_1 e_f^\top e_f + \varpi_2(e_f^\top e_f)^\alpha + \varpi_3(e_f^\top e_f)^\beta \geq \varpi_4 z^\top z + \varpi_5\varphi_f^2 + \nu_f$, where $e_s := \hat{x} - x$, $e_f := \hat{z} - z$, ϖ_i, for $i = 1, \ldots, 4$, ν_s, ν_f are strictly positive constants to be designed.

We are ready to model the overall system using the hybrid formalism of [29] for which a jump corresponds to a data transmission generated by one of the two triggering mechanisms. We introduce for this purpose the concatenated state vector $\chi := (x, z, \hat{x}, \hat{z}, \varphi) \in \mathbb{X} =: \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^{n_x} \times \mathbb{R}^{n_z} \times \mathbb{R}^2_{\geq 0}$. Let $\omega_i = \text{diag}(\omega_{i,s}, \omega_{i,f})$, $i = 1, 2, 3$. The hybrid model is written in the formalism of [29] and is given by

$$\dot{\chi} = F(\chi) \quad \chi \in C, \qquad \chi^+ \in G(\chi) \quad \chi \in D, \tag{11}$$

where

$$F(\chi) := \begin{pmatrix} (A_{11} + \Xi_{11})x + (A_{12} + \Xi_{21})z + B_1 g(\hat{\xi}) \\ \frac{1}{\varepsilon}((A_{21} + \Xi_{21})x + (A_{22} + \Xi_{22})z + B_2 g(\hat{\xi})) \\ 0 \\ 0 \\ -\omega_1\varphi - \omega_2\text{sig}(\varphi)^\alpha - \omega_3\text{sig}(\varphi)^\beta \end{pmatrix},$$

$$G(\chi) := \begin{cases} (x, z, x, \hat{z}, \varphi) \text{ for } \chi \in D_s \backslash D_f, \\ (x, z, \hat{x}, z, \varphi) \text{ for } \chi \in D_f \backslash D_s, \\ \{(x, z, x, \hat{z}, \varphi), (x, z, \hat{x}, z, \varphi)\} \text{ for } \chi \in D_f \cap D_s. \end{cases}$$

The sets C and D are defined in (12) according to the triggering rules. In particular, the triggering condition for the slow (fast) system is associated to D_s (to D_f), so that, when (x, z) lies in D_s (D_f) a jump, i.e., a transmission, may occur. When χ is in both D_s and D_f, two instantaneous transmissions may occur according to the definition of the jump map G, which ensures that it is outer-semicontinuous, which is one of the hybrid basic conditions [29].

The goal of this paper is to solve the next problem.

Problem 1. *Given $T > 0$ and $\nu > 0$, design the parameters in* (10) *and in the definitions of sets C and D in* (12) *so that there exists $\bar{\varepsilon} > 0$ such that for any $\varepsilon \in (0, \bar{\varepsilon}]$, there exist $\gamma \in \mathcal{K}$ independent of (T, ν, ε), and, $\beta_{T,\varepsilon} \in \mathcal{KL}$ independent of ν, which satisfies that, for any $s \in [0, \infty)$ and $t \geq T$, $\beta_{T,\varepsilon}(s, t) = 0$, such that, for any solution χ to* (11) *and any $(t, j) \in dom\chi$,*

$$\|(x(t, j), z(t, j))\| \leq \beta_{T,\varepsilon}(\|\chi(0, 0)\|, t) + \gamma(\nu), \tag{13}$$

and, if χ is maximal, it is complete. Moreover, system (11) *has to generate solutions with a uniform semiglobal average dwell-time, i.e., for any $\delta > 0$, there exist $\tau(\delta) > 0$ and $n_0(\delta) \in$*

$$
\begin{aligned}
C := \{\chi \in \mathbb{X} : & \varpi_{1,s} e_s^\top e_s + \varpi_{2,s}(e_s^\top e_s)^\alpha + \varpi_{3,s}(e_s^\top e_s)^\beta \le \varpi_{4,s} x^\top x + \varpi_{5,s}\varphi_s^2 + \nu_s \text{ and} \\
& \varpi_{1,f} e_f^\top e_f + \varpi_{2,f}(e_f^\top e_f)^\alpha + \varpi_{3,f}(e_f^\top e_f)^\beta \le \varpi_{4,f} z^\top z + \varpi_{5,f}\varphi_f^2 + \nu_f\}, \\
D_s := \{\chi \in \mathbb{X} : & \varpi_{1,s} e_s^\top e_s + \varpi_{2,s}(e_s^\top e_s)^\alpha + \varpi_{3,s}(e_s^\top e_s)^\beta \ge \varpi_{4,s} x^\top x + \varpi_{5,s}\varphi_s^2 + \nu_s\}, \\
D_f := \{\chi \in \mathbb{X} : & \varpi_{1,f} e_f^\top e_f + \varpi_{2,f}(e_f^\top e_f)^\alpha + \varpi_{3,f}(e_f^\top e_f)^\beta \ge \varpi_{4,f} z^\top z + \varpi_{5,f}\varphi_f^2 + \nu_f\},
\end{aligned}
\tag{12}
$$

$\mathbb{Z}^+$, such that for any solution χ to (11) with $\|\chi(0,0)\| \le \delta$, $k - i \le \frac{1}{\tau(\delta)}(t-s) + n_0(\delta)$, for any (s,i), $(t,k) \in$ dom χ with $s + i \le t + k$.

B. Main results

We explain how to design the parameters of the proposed triggering mechanism to solve Problem 1 in the next theorem.

Theorem 1. *Given $T, \nu > 0$, suppose Assumptions 1-3 hold. There exist $\alpha_{1,s}$, $\alpha_{1,f}$, $\beta_{1,s}$, $\beta_{1,f}$, $\varpi_{1,s}$, $\varpi_{1,f} > 0$ such that*

$$
\begin{pmatrix} -\frac{Q_1}{8} + (\frac{\mu_1}{\alpha_{1,s}} + \frac{2^{1-\beta}\mu_2}{\beta_{1,s}}) P_1 B_d B_d^\top P_1 & P_1 B_d K_{\text{stab}} \\ (P_1 B_d K_{\text{stab}})^\top & -\frac{\varpi_{1,s}}{2} I_{n_x+n_z} \end{pmatrix} \le 0,
\tag{14}
$$

$$
\begin{pmatrix} -\frac{Q_2}{8} + (\frac{\mu_1}{\alpha_{1,f}} + \frac{2^{1-\beta}\mu_2}{\beta_{1,f}}) P_2 B_f B_f^\top P_2 & P_2 B_f K_{\text{stab}} \\ (P_2 B_f K_{\text{stab}})^\top & -\frac{\varpi_{1,f}}{2} I_{n_x+n_z} \end{pmatrix} \le 0,
\tag{15}
$$

where B_d and B_f are defined in (8), Q_1, Q_2, P_1, P_2, K_{stab}, K_{ft}, μ_1, μ_2 are matrices and strictly positive constants defined in (4) such that $\frac{2}{c(1-\beta)} + \frac{2(4n_u)^{\frac{\alpha-1}{2}}}{b(\alpha-1)} \le T$. Problem 1 is solved by selecting, for any $i \in \{s, f\}$, $\varpi_{2,i} \ge 2^{\alpha-1}\mu_1\alpha_{1,i}\|K_{\text{ft}}\|^{2\alpha}$, $\varpi_{3,i} \ge \mu_2\beta_{1,i}(4n_u)^{\frac{1-\beta}{2}}\|K_{\text{ft}}\|^{2\beta}$, $\varpi_{4,i} \le \frac{1}{8}\lambda_{\min}(T_c^{-\top} Q T_c^{-1})$, $\varpi_{5,i} \le \omega_{1,i}$, $\omega_{1,i}$ is any given strictly positive constant, and $\omega_{2,i}$, $\omega_{3,i}$, ν_s, ν_f are strictly positive constants satisfying $\nu_s + \nu_f = \nu$ and

$$
\frac{1}{\omega_{3,i}(1-\beta)} + \frac{1}{\omega_{2,i}(\alpha-1)} \le T.
\tag{16}
$$

Since $Q_1, Q_2 > 0$, by applying Schur complement, conditions (14) and (15) can always be satisfied by selecting $\alpha_{1,i}$, $\beta_{1,i}$, $\varpi_{1,i} > 0, i \in \{s, f\}$, big enough. Moreover, for any $T > 0$, (16) can always be guaranteed by also selecting $\omega_{2,i}$, $\omega_{3,i} > 0$ big enough.

Remark 2. *The fact that ν_1 and ν_2 in (12) are strictly positive is essential, otherwise Zeno behaviour cannot be excluded. On the other hand, we need to set $\beta \in [\frac{1}{2}, 1)$ to avoid the case that the derivative of $e_s \mapsto (e_s^\top e_s)^\beta$ tends to infinity when e_s tends to 0.*

Remark 3. *The role of variables φ_s, φ_f in the event-triggered mechanism is to potentially help reducing the number of transmissions, i.e., the event-triggering frequency as shown on numerical simulation in Section IV. A similar idea is presented in e.g., [20, Section V.A], [30]. On the other hand, since the origin of system (10) is T-globally fixed time stable in view of Lemma 1 in the appendix, the initial values of φ_s and φ_f can be selected very big to (possibly) reduce the number of transmissions as illustrated on an example in Section IV.*

Remark 4. *Because we have two triggering laws, two transmissions may occur simultaneously. As a result, we cannot guarantee the existence of a dwell-time for solutions to (11), but of an average dwell-time. The latter is proved in the appendix by ensuring the existence of a a strictly positive time τ_f and τ_s between any two successive transmission of each triggering law in the appendix. Moreover, still in view of the proof of Theorem 1 in the appendix, we have $\tau_f = O(\varepsilon)\tau_s$.*

IV. Illustrative Example

To illustrate the results of Section III, we consider system (1) with $A := \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$,

$$
B_1 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix},
$$

$$
A = \begin{pmatrix} 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0.35 & 0 \\ 0 & 0.5 & -0.5 & 0.3 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \Xi = \begin{pmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \\ 0 & 0.01 & 0 & 0 \\ 0 & 0 & 0 & 0.01 \end{pmatrix},
$$

and $\varepsilon = 0.1$. Assumption 1 holds as A_{22} is invertible. Meanwhile, from the definition of A_0 and B_0 in Assumption 2, we obtain

$$
A_0 = \begin{pmatrix} 0 & 0.4 \\ 0 & 0.35 \end{pmatrix}, B_0 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0.21 & 2.7 & 0.7 & 1.21 \end{pmatrix}.
$$

The pairs (A_0, B_0) and (A_{22}, B_2) are controllable: Assumption 2 holds. Selecting $Q_1 = Q_2 = 0.25 I_2$, we have

$$
P_1 = \begin{pmatrix} 0.30 & -0.04 \\ -0.04 & 0.14 \end{pmatrix}, \; P_2 = \begin{pmatrix} 0.15 & 0.02 \\ 0.02 & 0.11 \end{pmatrix}.
$$

We design K_{stab} and K_{ft} in (4) by applying Lemma 4 given in the appendix.

We have simulated the system in closed-loop with event-triggered controller (9) with $\varphi(0,0) = (10^7, 10^7)$ and $(x_1(0,0), x_2(0,0), z_1(0,0), z_2(0,0)) = (-40000, -30000, 20000, 10000)$. We set $T = 40$, $\alpha = 1.2$, $\beta = 0.6$, and $Q_1 = Q_2 = 0.25I$, then we take $\underline{\lambda} = 0.06 \le \lambda_{\min}(P^{\frac{1}{2}} E B_D B_D^\top E P^{\frac{1}{2}})$ as in Lemma 4. From Theorem 1, we choose $\mu_1 = 8.16$, $\mu_2 = 4.8$, $\omega_{1,s} = \omega_{1,f} = 0.5$, $\omega_{2,s} = \omega_{2,f} = 0.35$, $\omega_{3,s} = \omega_{3,f} = 0.5$, thus $b = 0.345$, $c = 0.506$, $\frac{2}{c(1-\beta)} + \frac{2(4n_u)^{\frac{\alpha-1}{2}}}{b(\alpha-1)} \le T$, and (16) holds. Moreover, we take, in agreement with Theorem 1, $\varpi_{1,s} = \varpi_{1,f} = 32$, $\varpi_{2,s} = \varpi_{2,f} = 3213$, $\varpi_{3,s} = \varpi_{3,f} = 38.4$, $\varpi_{4,s} = \varpi_{4,f} = 32$, $\varpi_{5,s} = \varpi_{5,f} = 0.5$. Then, we design triggering rules by applying Theorem 1. Simulation results are presented in Fig. 1, which show that $\|x(t,j)\| \le 2$, $\|z(t,j)\| \le 2$, for all $t \ge 6$ such that $(t,j) \in \text{dom}\chi$ with $\nu_1 = \nu_2 = 10^3$. Note that the triggering instants of slow and fast states are not synchronized. In addition, for the sake of comparison, we have also run simulation for the event-triggered controller in [25] with $K = K_{\text{stab}}$, $c_0 = \frac{1}{32}$, $c_1 = \frac{5}{8}$ and $\alpha = 0.4$, which ensures an asymptotic practical stability property as opposed to a fixed-time one. The obtained evolution of the state and the associated triggering instants are given in Fig. 2, which show that the system has a faster convergence rate but more controller updates under the proposed fixed-time controller.

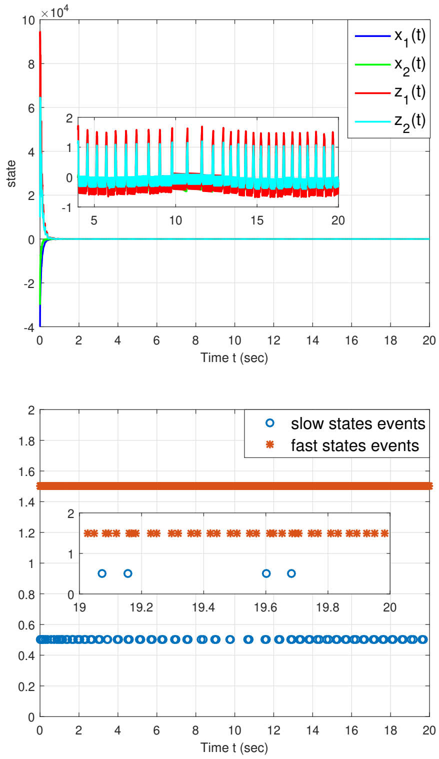

Fig. 1. States evolution of the closed-loop system (top), and the associated transmission instants (bottom)

TABLE I
CONVERGENCE TIME, ULTIMATE BOUND AND AVERAGE AMOUNT OF TRANSMISSIONS OVER 10 INITIAL CONDITIONS OF SYSTEM (1).

Parameters	Convergence time	Ultimate bound on (x, z)	n_s	n_f
$\varphi(0,0)=(10^3,10^3)$, $\mu_1=8.16, \mu_2=4.8$ $\nu_1=\nu_2=10^3$	4s	1.5	79	212
$\varphi(0,0)=(10^3,10^3)$ $\mu_1=8.16, \mu_2=4.8$ $\nu_1=\nu_2=10$	4s	0.2	90	325
$\varphi(0,0)=(10^3,10^3)$ $\mu_1=10, \mu_2=8$ $\nu_1=\nu_2=10$	3.5s	0.2	103	479
$\varphi(0,0)=(0,0)$ $\mu_1=10, \mu_2=8$ $\nu_1=\nu_2=10$	3.5s	0.2	141	558

In order to study the impact of the initial value of φ, μ_1, μ_2, ν_1 and ν_2 on the convergence time of (x, z), the ultimate bound and the number of transmissions, different designed parameters have been considered. Table I provides the obtained simulation results where n_s, n_f are respectively the average number of transmission instants of the slow and fast rules within the first 6 units of continuous time, and the maximum ultimate bound and convergence time observed are reported, over 10 initial conditions of system (1). These results suggest that smaller ν_1 and ν_2 make the slow and fast states converging to a smaller region around origin, and that bigger μ_1 and μ_2 lead to a faster convergence, but both lead to more controller updates. Moreover, we note that when $\varphi(0,0) = (0,0)$, φ is always equal to 0 in view of (10) and more transmissions are generated, thereby justifying the use of the auxiliary variable φ. We also have the expected result that the fast variables lead to more transmissions than the slow ones.

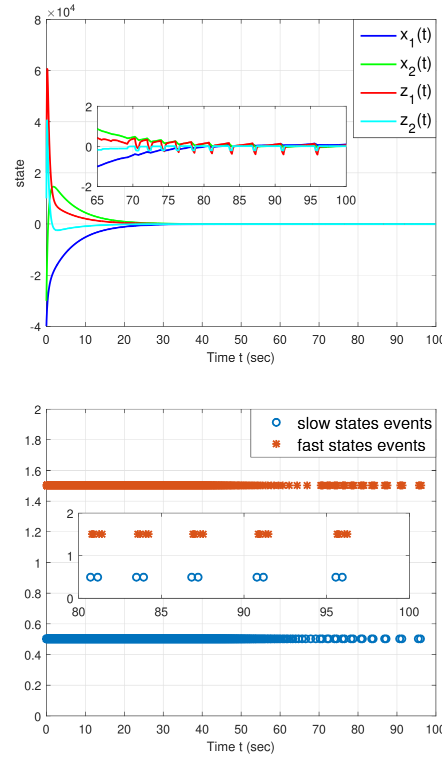

Fig. 2. States evolution of the closed-loop system (top), and the associated transmission instants (bottom)

V. CONCLUSION

The event-triggered fixed-time stabilization problem was investigated for linear two time scales system. We have supposed that we know a state-feedback controller, which solves the problem in absence of sampling and we have proposed two event-triggering conditions, one for the slow, and one for the fast subsystem, to approximately preserve the original fixed-time stability property. It would be interesting in future work to further take into consideration implementation aspects of the presented controllers, to address output feedback control, and the distributed context where several singularly controller-plant pairs have to synchronize.

VI. APPENDIX

A. Technical lemmas

We present several lemmas, which play a key role in the proof of Theorem 1. We first recall Lemma 4.1 of [19]. We provide its proof because the definition of fixed-time stability differs from [19].

Lemma 1. *Consider the following system*

$$\dot{y} = f(y), y(0) = y_0, \tag{17}$$

where $f:\mathbb{R}^n\to\mathbb{R}^n$ is continuous and $f(0)=0$. If there exists a continuously positive definite function $V:\mathbb{R}^n\to\mathbb{R}$, $b,c>0$, $\alpha>1$, $1>\beta>0$, such that, for any $y\in\mathbb{R}^n$,

$$\underline{\alpha}(|y|)\le V(y)\le\overline{\alpha}(|y|), \quad (18)$$

$$\langle\nabla V(y),f(y)\rangle+b(V(y))^{\frac{1+\alpha}{2}}+c(V(y))^{\frac{1+\beta}{2}}\le 0 \quad (19)$$

where $\underline{\alpha},\overline{\alpha}$ are two class $\mathcal{K}_\infty$ functions. Then, the origin of system (17) is T-globally fixed-time stable, and the settling time satisfies $T\le\frac{1}{c(1-\beta)}+\frac{1}{b(\alpha-1)}$.

Proof: Let $\zeta(t)$ satisfy the differential equation

$$\dot\zeta=-b\zeta^{\frac{1+\alpha}{2}}-c\zeta^{\frac{1+\beta}{2}},\zeta(0)=V(y(0))\ge 0. \quad (20)$$

Following the proof of Lemma 4.1 in [19], we have $\zeta(t)=\sigma_T(\zeta(0),t)$, where $\sigma_T\in\mathcal{KL}$ which satisfies that, for each $s\in[0,\infty)$, $\sigma_T(s,t)=0$, $t\ge\frac{1}{c(1-\beta)}+\frac{1}{b(\alpha-1)}=T$. Then, from the comparison principle, we have, for any $y(0)\in\mathbb{R}$,

$$V(y(t))\le\sigma_T(V(y(0)),t).$$

Therefore, we have $\|y(t)\|\le\underline{\alpha}^{-1}(\sigma_T(V(y(0)),t))\le\sigma_1^{-1}(\sigma_T(\overline{\alpha}(\|y(0)\|),t)):=\beta_T(\|y(0)\|),t)$ for any $y(0)\in\mathbb{R}$, where β_T is a class $\mathcal{KL}$ function, which satisfies that, for each $s\in[0,\infty)$, $\beta_T(s,t)=0$, $t\ge T$. Thus, the origin of system (17) is T-globally fixed-time stable. ■

Lemma 2. *[19] For any $x_i\in\mathbb{R}$, $i=1,\dots,n$, and $0<\beta\le 1$, $\alpha\ge 1$,*

$$\Big(\sum_{i=1}^n|x_i|\Big)^\beta\le\sum_{i=1}^n|x_i|^\beta\le n^{1-\beta}\Big(\sum_{i=1}^n|x_i|\Big)^\beta,$$
$$\Big(\sum_{i=1}^n|x_i|\Big)^\alpha\ge\sum_{i=1}^n|x_i|^\alpha\ge n^{1-\alpha}\Big(\sum_{i=1}^n|x_i|\Big)^\alpha.$$

Lemma 3. *Let $x=(x_1,x_2,\dots,x_n)$, $e=(e_s,e_f,\dots,e_n)$, where $x_i,e_i\in\mathbb{R}$, $i=1,\dots,n$,*

$$x^\top\mathrm{sig}(x+e)^\beta\ge(x^\top x)^{\frac{1+\beta}{2}}-\frac{2^{1-\beta}}{2\beta_1}x^\top x-\frac{\beta_1}{2}(4n)^{\frac{1-\beta}{2}}(e^\top e)^\beta,$$
$$x^\top\mathrm{sig}(x+e)^\alpha\ge(4n)^{\frac{1-\alpha}{2}}(x^\top x)^{\frac{1+\alpha}{2}}-\frac{1}{2\alpha_1}x^\top x-\frac{\alpha_1}{2}(e^\top e)^\alpha,$$

where $0<\beta\le 1$, $\alpha\ge 1$, $\beta_1>0$, $\alpha_1>0$.

Proof: We firstly prove that for any $a,b\in\mathbb{R}$,

$$a\mathrm{sig}(a+b)^\beta\ge|a|\,(|a|^\beta-2^{1-\beta}\,|b|^\beta), \quad (21)$$
$$a\mathrm{sig}(a+b)^\alpha\ge|a|\,(2^{1-\alpha}\,|a|^\alpha-|b|^\alpha). \quad (22)$$

Let $a,b\in\mathbb{R}$, when $\mathrm{sign}(a)=\mathrm{sign}(b)$,

$$\begin{aligned}a\mathrm{sig}(a+b)^\beta&=|a|\,(|a|+|b|)^\beta\\&\ge|a|\,(|a|^\beta-2^{1-\beta}\,|b|^\beta),\\a\mathrm{sig}(a+b)^\alpha&=|a|\,(|a|+|b|)^\alpha\\&\ge|a|\,(2^{1-\alpha}\,|a|^\alpha-|b|^\alpha).\end{aligned}$$

From Lemma 2,

$$|a|^\beta-|b|^\beta\le(|a|-|b|)^\beta\le 2^{1-\beta}\,|a|^\beta-|b|^\beta,$$
$$|a|^\alpha-|b|^\alpha\ge(|a|-|b|)^\alpha\ge 2^{1-\alpha}\,|a|^\alpha-|b|^\alpha.$$

Thus, when $\mathrm{sign}(a)\ne\mathrm{sign}(b)$ and $|a|\ge|b|$, it has

$$\begin{aligned}a\mathrm{sig}(a+b)^\beta&=|a|\,(|a|-|b|)^\beta\\&\ge|a|\,(|a|^\beta-2^{1-\beta}\,|b|^\beta),\\a\mathrm{sig}(a+b)^\alpha&=|a|\,(|a|-|b|)^\alpha\\&\ge|a|\,(2^{1-\alpha}\,|a|^\alpha-|b|^\alpha).\end{aligned}$$

Similarly, when $\mathrm{sign}(a)\ne\mathrm{sign}(b)$ and $|a|\le|b|$, it has

$$\begin{aligned}a\mathrm{sig}(a+b)^\beta&=-|a|\,(|b|-|a|)^\beta\\&\ge-|a|\,(2^{1-\beta}\,|b|^\beta-|a|^\beta),\\a\mathrm{sig}(a+b)^\alpha&=-|a|\,(|b|-|a|)^\alpha\\&\ge-|a|\,(|b|^\alpha-2^{1-\alpha}\,|a|^\alpha).\end{aligned}$$

Thus, (21) and (22) are obtained. Let $x,e\in\mathbb{R}^n$, from (21), we have, for $0<\beta\le 1$,

$$\begin{aligned}x^\top\mathrm{sig}(x+e)^\beta&\ge\sum_{i=1}^n|x_i|\,(|x_i|^\beta-2^{1-\beta}\,|e_i|^\beta)\\&\ge(x^\top x)^{\frac{1+\beta}{2}}-\frac{2^{1-\beta}}{2\beta_1}x^\top x-\frac{\beta_1}{2}(4n)^{\frac{1-\beta}{2}}(e^\top e)^\beta\end{aligned}$$

Similarly, from (22), it can be obtained that, for $\alpha\ge 1$,

$$\begin{aligned}x^\top\mathrm{sig}(x+e)^\alpha&\ge\sum_{i=1}^n|x_i|\,(2^{1-\alpha}\,|x_i|^\alpha-|e_i|^\alpha)\\&\ge(4n)^{\frac{1-\alpha}{2}}(x^\top x)^{\frac{1+\alpha}{2}}-\frac{1}{2\alpha_1}x^\top x-\frac{\alpha_1}{2}(e^\top e)^\alpha.\end{aligned}$$

■

B. Proof of Theorem 1

Let $T,\nu>0$ and $\varepsilon\in(0,\bar\varepsilon)$ where $\bar\epsilon>0$ is specified in the following. We define $U(\chi):=\xi_c^\top P_\varepsilon\xi_c+\varphi^\top\varphi$ for any $\chi\in\mathbb{X}$, where we recall that $\chi:=(x,z,\hat x,\hat z,\varphi)$, $\xi_c:=(x_s,z_f)$, and P is defined before (30). There exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}_U$ and $\overline{\alpha}_U$ such that, for any $\chi\in\mathbb{X}$,

$$\underline{\alpha}_U(\|(\xi,\varphi)\|)\le U(\chi)\le\overline{\alpha}_U(\|(\xi,\varphi)\|). \quad (23)$$

Let $\chi\in C$. Since $\hat\xi=\xi+e$, from (7) and Lemma 3, we have

$$\begin{aligned}&\langle\nabla U(\chi),F(\chi)\rangle\\&\le-\frac{1}{2}\xi_c^\top Q\xi_c-b(\xi_c^\top P\xi_c)^{\frac{1+\alpha}{2}}-(4n_u)^{\frac{1-\beta}{2}}c(\xi_c^\top P\xi_c)^{\frac{1+\beta}{2}}\\&\quad+2\xi_c^\top P_\varepsilon B_DK_{\mathrm{stab}}e+\mu_1\alpha_1(e^\top K_{\mathrm{ft}}^\top K_{\mathrm{ft}}e)^\alpha\\&\quad+\frac{\mu_1}{\alpha_1}\xi_c^\top P_\varepsilon B_DB_D^\top P_\varepsilon\xi_c+\mu_2\beta_1(4n)^{\frac{1-\beta}{2}}(e^\top K_{\mathrm{ft}}^\top K_{\mathrm{ft}}e)^\beta\\&\quad+\frac{2^{1-\beta}\mu_2}{\beta_1}\xi_c^\top P_\varepsilon B_DB_D^\top P_\varepsilon\xi_c+2\varphi^\top\dot\varphi. \quad (24)\end{aligned}$$

where $\dot\varphi=-\omega_1\varphi-\omega_2\mathrm{sig}(\varphi)^\alpha-\omega_3\mathrm{sig}(\varphi)^\beta$. Denote $\underline{\omega}_i:=\min\{\omega_{i,s},\omega_{i,f}\}$, for $i=1,2,3$. From (14), (15), we derive

$$\begin{aligned}&\langle\nabla U(\chi),F(\chi)\rangle\\&\le-\frac{3}{8}\xi_c^\top Q\xi_c-b(\xi_c^\top P\xi_c)^{\frac{1+\alpha}{2}}-(4n_u)^{\frac{1-\beta}{2}}c(\xi_c^\top P\xi_c)^{\frac{1+\beta}{2}}\\&\quad-2(\underline{\omega}_1\|\varphi\|^2+\underline{\omega}_2\|\varphi\|^{1+\alpha}+\underline{\omega}_3\|\varphi\|^{1+\beta})\\&\quad+\varpi_1e^\top e+2^{\alpha-1}\mu_1\alpha_1\|K_{\mathrm{ft}}^\top K_{\mathrm{ft}}\|^\alpha((e_s^\top e_s)^\alpha+(e_f^\top e_f)^\alpha)\\&\quad+\mu_2\beta_1(4n)^{\frac{1-\beta}{2}}\|K_{\mathrm{ft}}^\top K_{\mathrm{ft}}\|^\beta((e_s^\top e_s)^\beta+(e_f^\top e_f)^\beta)). \quad (25)\end{aligned}$$

Then, from (25), we have

$$\begin{aligned}\langle\nabla U(\chi),F(\chi)\rangle\leq&-\frac{1}{4}\xi_c^\top Q\xi_c-\bar{b}(\xi_c^\top P_\varepsilon\xi_c)^{\frac{1+\alpha}{2}}-\bar{c}(\xi_c^\top P_\varepsilon\xi_c)^{\frac{1+\beta}{2}}\\&-2(\underline{\omega}_1\|\varphi\|^2+\underline{\omega}_2\|\varphi\|^{1+\alpha}+\underline{\omega}_3\|\varphi\|^{1+\beta})+\nu\\\leq&-\bar{a}U-\bar{b}U^{\frac{1+\alpha}{2}}-\bar{c}U^{\frac{1+\beta}{2}}+\nu,\end{aligned}\tag{26}$$

where $\bar{a}=\min\{\frac{\lambda_{\min}(Q)}{4\|P_\varepsilon\|},\underline{\omega}_1\}$, $\bar{b}=(4n_u)^{\frac{1-\beta}{2}}\min\{b,2\underline{\omega}_2\}$, $\bar{c}=\min\{c,2\underline{\omega}_3\}$. From (26), we obtain that, when $U(\chi)\geq\frac{\nu}{\bar{a}}$

$$\langle\nabla U(\chi),F(\chi)\rangle+\bar{b}U^{\frac{1+\alpha}{2}}+\bar{c}U^{\frac{1+\beta}{2}}\leq 0.\tag{27}$$

Moreover, for $\chi\in D$ and any $g\in G(\chi)$, $U(g)=\xi_c^\top P_\varepsilon\xi_c+\varphi^\top\varphi$. Using similar arguments as in the proof of Lemma 1, there exist a class $\mathcal{KL}$ function $\sigma_{T_1,\varepsilon}$ which satisfies that, for each $s\in[0,\infty)$, $\sigma_{T_1,\varepsilon}(s,t)=0$, $t\geq T_1:=\frac{1}{\bar{c}(1-\beta)}+\frac{1}{\bar{b}(\alpha-1)}$, such that for any solution χ and any $(t,j)\in\mathrm{dom}\chi$

$$U(\chi(t,j))\leq\max\left\{\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),t),\frac{\nu}{\bar{a}}\right\}.\tag{28}$$

We now prove that system (11) generates solution with a uniform semiglobal average dwell-time. In view of the definitions of the set D_f, the time between two continuous successive jumps due to D_f is lower-bounded by the time it takes for $e_s^\top e_s+\varpi_1(e_s^\top e_s)^\alpha+\varpi_2(e_s^\top e_s)^\beta$ to grow from 0 to ν_1. Let $\Delta>0$ and χ be a solution to (11) such that $||\chi(0,0)||\leq\Delta$. From (28), $\alpha>1$ and $1>\beta>\frac{1}{2}$, there exists a positive constant $\delta_1(\Delta)$, such that

$$\begin{aligned}&D^+(e_s^\top e_s+\varpi_1(e_s^\top e_s)^\alpha+\varpi_2(e_s^\top e_s)^\beta)\\&\leq(\|e_s\|+\varpi_1\|e_s\|^{2\alpha-1}+\varpi_2\|e_s\|^{2\beta-1})\|A_{11}x+A_{12}z+B_1u\|\\&\leq\delta_1(\Delta),\end{aligned}$$

where $D^+(\cdot)$ denotes the upper right-hand Dini derivative.

Thus, the continuous time between two successive jumps due to the slow triggering rule is lower bounded by $\tau_s(\Delta)=\frac{\nu_1}{\delta_1(\Delta)}>0$. Similarly, there exist a positive constant $\delta_2(\Delta)$, such that the continuous time between two successive jumps due to the slow triggering rule is lower bounded by $\tau_f(\Delta)=\frac{\varepsilon\nu_2}{\delta_2(\Delta)}>0$. Hence, for any (s,i), $(t,k)\in\mathrm{dom}\chi$ with $s+i\leq t+k$, we have $k-i\leq\frac{(t-s)}{\tau_s(\Delta)}+\frac{(t-s)}{\tau_f(\Delta)}+2$. Thus, system (11) generates solution with a uniform semiglobal average dwell-time. In view of the definition of system (11), (23), (27) and the non-increasing of U at jumps, we derive from [29, Proposition 6.1] that maximal solutions are complete.

From (28), for any solution χ to (11) and $(t,j)\in\mathrm{dom}\chi$,

$$\|(x(t,j),z(t,j))\|^2\leq\frac{\max\{\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),t),\frac{\nu}{\bar{a}}\}}{\lambda_{\min}(P_\varepsilon)}.$$

However, since $\lambda_{\min}(P_\varepsilon)$ depend on ε, we do not have yet that (13) holds with a $\gamma\in\mathcal{K}$ independent of ε. This issue is overcome below.

Let $V_2(\chi):=\varphi^\top\varphi+\varepsilon z_f^\top P_2z_f$ for any $\chi\in\mathbb{X}$, there exists a class $\mathcal{K}$ functions $\tilde{\alpha}_U$, such that $V_2(\chi)\leq\tilde{\alpha}_U(\|\chi\|)$. Using similar arguments as those invoked to obtain (26), there exists $0<\bar{\varepsilon}<\bar{\varepsilon}_1$ such that for any $0<\varepsilon\leq\bar{\varepsilon}$, $\chi\in C$,

$$\begin{aligned}&\langle\nabla V_2(\chi),F(\chi)\rangle\\&\leq-\frac{1}{4}z_f^\top Q_2z_f-b(z_f^\top P_2z_f)^{\frac{1+\alpha}{2}}-c(z_f^\top P_2z_f)^{\frac{1+\beta}{2}}\\&\quad+\frac{1}{4}x_s^\top Q_1x_s+b(x_s^\top P_1x_s)^{\frac{1+\alpha}{2}}+c(x_s^\top P_1x_s)^{\frac{1+\beta}{2}}\\&\quad-(\underline{\omega}_1\|\varphi\|^2+2\underline{\omega}_2\|\varphi\|^{1+\alpha}+2\underline{\omega}_3\|\varphi\|^{1+\beta})+\nu.\end{aligned}$$

Moreover, for $\chi\in D$, and $g\in G(\chi)$, $V_2(g)=\varepsilon z_f^\top P_2z_f+l\varphi^\top\varphi=V_2(\chi)$. Let χ be a solution to (11). Since $\|x_s(t,j)\|^2\leq\frac{1}{\lambda_{\min}(P_1)}\max\{\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),0),\frac{\nu}{\bar{a}}$, there exist two class $\mathcal{K}$ functions γ_1,γ_2, for any $(t,j)\in\mathrm{dom}\chi$,

$$\begin{aligned}&\langle\nabla V_2(\chi),F(\chi)\rangle\\&\leq-\frac{1}{4}z_f^\top Q_2z_f-b(z_f^\top P_2z_f)^{\frac{1+\alpha}{2}}-c(z_f^\top P_2z_f)^{\frac{1+\beta}{2}}\\&\quad-\underline{\omega}_1(\varphi)^2-2\underline{\omega}_2(\varphi)^{1+\alpha}-2\underline{\omega}_3(\varphi)^{1+\beta}\\&\quad+\gamma_1(\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),0))+\gamma_2(\nu)\\&\leq-\bar{a}V_2(\chi)-\bar{b}V_2^{\frac{1+\alpha}{2}}(\chi)-\bar{c}V_2^{\frac{1+\beta}{2}}(\chi)\\&\quad+\gamma_1(\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),0))+\gamma_2(\nu),\end{aligned}$$

where we omit (t,j) dependency for the state of convenience. Then, with a similar proof as above, it can be obtained that, there exist a class $\mathcal{KL}$ function $\bar{\sigma}_{T_1,\varepsilon}$, which satisfies that, for each $s\in[0,\infty)$, $\bar{\sigma}_{T_1,\varepsilon}(s,t)=0$, $t\geq T_1$, and there exist two class $\mathcal{K}$ functions γ_3 and γ_4 independent of T_1,ε, such that for $(t,j)\in\mathrm{dom}\chi$ with $t\leq T_1$,

$$\begin{aligned}\|z_f(t,j)\|^2\leq\max\{&\bar{\sigma}_{T_1,\varepsilon}(\tilde{\alpha}_U^{-1}(\|\chi(0,0)\|),t),\\&\gamma_3(\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),0))+\gamma_4(\nu)\}.\end{aligned}$$

For $t\geq T_1$, we have $\|x_s(t,j)\|^2\leq\frac{\nu}{\bar{a}\lambda_{\min}(P_1)}\leq\bar{\nu}$. Thus, for $(t,j)\in\mathrm{dom}\chi$ with $t\geq T_1$,

$$\langle\nabla V_2(\chi),F(\chi)\rangle\leq-\bar{a}V_2-\bar{b}V_2^{\frac{1+\alpha}{2}}-\bar{c}V_2^{\frac{1+\beta}{2}}+\gamma_2(\nu).$$

Then, following the same reasoning as above, for $(t,j)\in\mathrm{dom}\chi$ with $t\geq T_1$,

$$\|z_f(t,j)\|^2\leq\max\{\bar{\sigma}_{T_1,\varepsilon}(V_2(\chi(T_1,j_{T_1})),t-T_1),\gamma_4(\nu)\}.$$

Thus, there exist a class $\mathcal{KL}$ function $\hat{\sigma}_{2T_1,\varepsilon}$ which satisfies that, for each $s\in[0,\infty)$, $\hat{\sigma}_{2T_1,\varepsilon}(s,t)=0$, $t\geq 2T_1$, such that, for $(t,j)\in\mathrm{dom}\chi$,

$$\|z_f(t,j)\|^2\leq\hat{\sigma}_{2T_1,\varepsilon}(\|\chi(0,0)\|,t)+2\gamma_4(\nu).$$

Furthermore, there exist a class $\mathcal{K}$ function γ_5 such that for all $0<\varepsilon\leq\bar{\varepsilon}$, we have, for $(t,j)\in\mathrm{dom}\chi$,

$$\begin{aligned}\|\xi_c(t,j)\|^2\leq&\frac{\sigma_{T_1,\varepsilon}(\overline{\alpha}_U^{-1}(\|\chi(0,0)\|),t)}{\lambda_{\min}(P_1)}\\&+\hat{\sigma}_{2T_1,\varepsilon}(\|\chi(0,0)\|,t)+\gamma_5(\nu).\end{aligned}$$

Since $\frac{2}{c(1-\beta)}+\frac{2(4n_u)^{\frac{1-\beta}{2}}}{b(\alpha-1)}\leq T$ and $\frac{1}{\underline{\omega}_3(1-\beta)}+\frac{1}{\underline{\omega}_2(\alpha-1)}\leq T$, we have $2T_1<T$. Thus, there exists a class $\mathcal{KL}$ function $\beta_{T,\varepsilon}$ which satisfies that, for each $s\in[0,\infty)$, $\beta_{T,\varepsilon}(s,t)=0$, $t\geq T$, and a class $\mathcal{K}$ function γ independent of T,ε, such that, for $t\geq 0$, (13) is satisfied.

C. Example of a controller ensuring Assumption 3

Lemma 4. *Under Assumptions 1-2 and assuming $B:=[B_1^\top,B_2^\top]^\top$ is of full-row rank, for any given $T>0$, the origin of the closed-loop system (1), (4) is T-globally fixed-time stable when*

- $K_{\text{stab}}=(K_1,K_2)$ *with* $K_1:=(1-K_2A_{22}^{-1}B_2)K_0+K_2A_{22}^{-1}A_{21}$, $K_0=B_0^\top\bar{P}_1$, $K_2:=B_2^\top\bar{P}_2$ *and* $\bar{P}_1$, $\bar{P}_2$ *as defined in* (2) *and* (3), *respectively,*
- $K_{\text{ft}}:=B^\top\begin{pmatrix}\bar{P}_1&0\\\bar{P}_2\Lambda_{22}^{-1}\Lambda_{21}-\Lambda_{22}^{-T}\Lambda_{12}^\top\bar{P}_1&\bar{P}_2\end{pmatrix}$,
- $\hat{\Xi}_k:=\bar{P}_\varepsilon T_c^{-1}E^{-1}\Xi_kT_c+T_c^\top\Xi_k^\top E^{-1}T_c^{-T}\bar{P}_\varepsilon$ *satisfies,*

$$\sum_{k=1}^m q_k^2(t)\leq\sigma_{\max}^{-1}(\sum_{k=1}^m(\bar{Q}^{-\frac{1}{2}}\hat{\Xi}_k\bar{Q}^{-\frac{1}{2}})^2),\ \forall t\geq 0\tag{29}$$

where $\bar{P}_\varepsilon:=\mathrm{diag}(\bar{P}_1,\varepsilon\bar{P}_2)$, $\bar{Q}:=\mathrm{diag}(\bar{Q}_1,\varepsilon\bar{Q}_2)$

- $0 < \underline{\lambda} \leq \lambda_{\min}(\bar{P}^{\frac{1}{2}} E B_D B_D^\top E \bar{P}^{\frac{1}{2}})$, $\bar{P} := \text{diag}(\bar{P}_1, \bar{P}_2)$, $b = \mu_1 \underline{\lambda}^{\frac{1+\alpha}{2}}$, $c = \mu_2 \underline{\lambda}^{\frac{1+\beta}{2}}$, $\mu_1, \mu_2 > 0$ *verify* $\frac{1}{c(1-\beta)} + \frac{1}{b(\alpha-1)} \leq T$.

Proof: Let $T > 0$ and $V_c(\xi_c) = \xi_c^\top \bar{P}_\varepsilon \xi_c$ for any $\xi_c \in \mathbb{R}^{n_x+n_z}$, where $\xi_c := (x_s, z_f)$. Then, there exist two class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\|\xi_c\|) = \lambda_{\min}(\bar{P}_\varepsilon)\|\xi_c\|^2$, $\overline{\alpha}(\|\xi_c\|) = \lambda_{\max}(\bar{P}_\varepsilon)\|\xi_c\|^2$, such that for any $\xi_c \in \mathbb{R}^{n_x+n_z}$,

$$\underline{\alpha}(\|\xi_c\|) \leq V_c(\xi_c) \leq \overline{\alpha}(\|\xi_c\|). \tag{30}$$

We write $\dot{\xi}_c = f(\xi_c)$ for the sake of convenience. Let $\xi_c \in \mathbb{R}^{n_x+n_z}$, we have

$$\begin{aligned}\langle \nabla V_c(\xi_c), f(\xi_c)\rangle =& x_s^\top((A_s+B_sK_s)^\top \bar{P}_1 + \bar{P}_1(A_s+B_sK_s))x_s \\ &+ z_f^\top((A_f+B_fK_2)^\top \bar{P}_2 + \bar{P}_2(A_f+B_fK_2))z_f \\ &- 2\xi_c^\top \bar{P} B_D(\mu_1 \text{sig}(K_{\text{ft}}\xi)^\alpha + \mu_2 \text{sig}(K_{\text{ft}}\xi)^\beta) \\ &+ \xi_c^\top(\bar{P}_\epsilon \Xi_c + \Xi_c^\top \bar{P}_\epsilon)\xi_c,\end{aligned}$$

From the definition of L, H and K_{ft}, we have

$$T_c^{-T}\bar{P}_\varepsilon B_D = K_{\text{ft}}^\top + O(\varepsilon), A_s + B_sK_s = A_0 + B_0K_0 + O(\varepsilon),$$
$$A_f + B_fK_2 = A_{22} + B_2K_2 + O(\varepsilon).$$

From (29), we have $\bar{Q} - \bar{P}_\epsilon \Xi_c - \Xi_c^\top \bar{P}_\epsilon \leq 0$. Thus

$$\begin{aligned}\langle \nabla V_c(\xi_c), f(\xi_c)\rangle \leq& -x_s^\top(\bar{Q}_1 + O(\varepsilon))x_s - z_f^\top(\bar{Q}_2 + O(\varepsilon))z_f \\ &- 2\xi^\top(K_{\text{ft}}^\top + O(\varepsilon))(\mu_1 \text{sig}(K_{\text{ft}}\xi)^\alpha + \mu_2 \text{sig}(K_{\text{ft}}\xi)^\beta).\end{aligned} \tag{31}$$

Since $B := [B_1^\top, B_2^\top]^\top$ is of full-row rank, we derive that $K_{\text{ft}}^\top K_{\text{ft}} > 0$ and the definition of K_{ft} in Theorem 1. Consequently, there exists $\bar{\varepsilon} > 0$ such that for all $\varepsilon \in (0, \bar{\varepsilon}]$,

$$\begin{aligned}\langle \nabla V_c(\xi_c), f(\xi_c)\rangle \leq& -\frac{1}{2}x_s^\top \bar{Q}_1 x_s - \frac{1}{2}z_f^\top \bar{Q}_2 z_f \\ &- \mu_1(\xi^\top K_{\text{ft}}^\top K_{\text{ft}}\xi)^{\frac{1+\alpha}{2}} - \mu_2(\xi^\top K_{\text{ft}}^\top K_{\text{ft}}\xi)^{\frac{1+\beta}{2}}.\end{aligned}$$

From the definition of ξ_c, we have

$$\xi^\top K_{\text{ft}}^\top K_{\text{ft}}\xi \geq \lambda_{\min}(\bar{P}^{\frac{1}{2}} E B_D B_D^\top E \bar{P}^{\frac{1}{2}})\xi_c^\top P \xi_c.$$

Thus

$$\begin{aligned}\langle \nabla V_c(\xi_c), f(\xi_c)\rangle \leq& -\frac{1}{2}x_s^\top \bar{Q}_1 x_s - \frac{1}{2}z_f^\top \bar{Q}_2 z_f \\ &- \mu_1(\xi^\top K_{\text{ft}}^\top K_{\text{ft}}\xi)^{\frac{1+\alpha}{2}} - \mu_2(\xi^\top K_{\text{ft}}^\top K_{\text{ft}}\xi)^{\frac{1+\beta}{2}} \\ \leq& -\frac{1}{2}\xi_c^\top \bar{Q}\xi_c - b(\xi_c^\top \bar{P}\xi_c)^{\frac{1+\alpha}{2}} - c(\xi_c^\top \bar{P}\xi_c)^{\frac{1+\beta}{2}} \\ \leq& -\frac{1}{2}\xi_c^\top \bar{Q}\xi_c - b(V_c(\xi_c))^{\frac{1+\alpha}{2}} - c(V_c(\xi_c))^{\frac{1+\beta}{2}}.\end{aligned}$$

According to Lemma 1, the origin of system (1)-(4) is $\frac{1}{c(1-\beta)} + \frac{1}{b(\alpha-1)}$-globally fixed-time stable. Since μ_1 and μ_2 are chosen to verify $T \geq \frac{1}{c(1-\beta)} + \frac{1}{b(\alpha-1)}$, thus the origin of system (1)-(4) is T-globally fixed-time stable. ■

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