Event-triggered transmission policies for nonlinear control systems over erasure channels

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***Abstract*—We investigate the scenario where a controller communicates with a nonlinear plant via a wireless erasure channel. We present an event-based control strategy to stabilize the plant while sporadically using the unreliable wireless network. In particular, control packets may be lost at any time with a certain probability. Consequently, stability is ensured in a stochastic sense. We then compare the proposed event-based policy with a baseline policy that transmits according to the age of information, i.e., the time elapsed since the last successful reception. For any given baseline policy, we show how to design an event-based policy that ensures the same guaranteed control performance while leading, on average, to a strictly smaller channel utilization. Numerical simulations suggest that the achieved channel utilization may in fact be significantly smaller.**

I. Introduction

This work aims at devising sporadic communication strategies for wireless networked control systems (WNCS). Carefully choosing the communication instants is important as it may reduce the induced energy consumption and network load. In the wireless communication literature, various studies have investigated the design of energy-efficient communication systems, see [1] for an extensive survey. In that body of work, the goal is to minimize the power consumed while maintaining a certain data rate or packet success rate. However, designing the communication system in such a manner is a priori not well suited for control systems that have their own objectives like stability and control performance.

The main directions of research for reducing communications in WNCS are: i) event-based communications, which determine transmissions according to the past and present plant states or outputs e.g., [2]–[5], ii) self-triggered communications, which determine the next transmission based on the value of the state at the last successful transmission instant e.g., [6], [7], and iii) time-based triggering communications, which rely on the elapsed time since the last successful transmission e.g., [8]–[11]. While time-based triggering rules are easier to implement as they do not require computations or repeated measurements at the transmitter, event-triggered or self-triggered control may result in a significant reduction in communications, see, e.g., [12], [13]. In this work, we concentrate on event-triggered control strategies.

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Event-triggered control strategies for WNCS subject to packet drops generally aim to reduce communications cost with a state-dependent transmission rule. Most approaches are dedicated to linear time-invariant models, see, e.g., [13]–[16]. Related results for nonlinear systems typically rely on strong assumptions on the packet losses, specifically that there exists a maximum number of consecutive packet drops, as considered in [2], [17], [18]. A notable exception is [19], in which an event-based rule is proposed for a class of continuous-time nonlinear systems with sector bounded nonlinearities. Another relevant work is [4], where an event-triggered anytime control approach is studied, which relies on zeroing hold devices, i.e., where the control input is set to zero when the packet is dropped.

The literature on the event-triggered control for nonlinear systems with stochastic packet drops is thus quite sparse, and our main contribution is to address this issue when communication occurs over binary erasure channels. In this case, each packet may be dropped with a certain independent probability, thereby relaxing the often assumed requirement that the maximum number of consecutive packet losses is bounded. In contrast, we present an event-triggered control strategy for general nonlinear WNCS modeled in discrete-time (and not continuous-time as in [19]), and for general holding functions, which cover zeroing devices (as in [4]) as a special case. We follow an emulation approach, in the sense that we assume that we know a state-feedback law, which stabilizes the origin of the closed-loop system under perfect, all-the-time communications. The event-triggering rule consists in imposing a decaying property of a given Lyapunov function along the solutions. Our approach is thus akin to that in [5], [7], [20], [21] for different, deterministic contexts. We explain how to tune the event-triggering parameters based on the (known) packet success probability so that the origin of the closed-loop system is stable in a stochastic sense. In addition, these parameters can be adjusted to enforce a desired guaranteed convergence rate on the expectation of the Lyapunov function along solutions, thereby ensuring a given control performance.

Afterwards, we compare the proposed event-based strategy with a baseline policy from [10] that transmits according to the age of information (AoI), i.e., the time elapsed since the last successful transmission, and ensures the same *guaranteed* convergence property as the event-based policy. We demonstrate that the average channel utilization (or transmission rate) of the event-based rule is upper bounded by that of the baseline policy. In other words, for the same guaranteed control performance, the event-based strategies reduce the

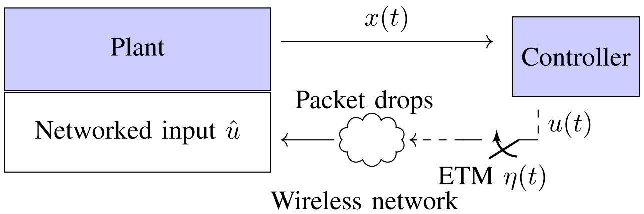

Fig. 1. Schematic of the WNCS (ETM: Event-Triggering Mechanism).

amount of communications compared to a baseline policy. Moreover, simulations demonstrate that channel utilization may be significantly reduced with the proposed event-based policy.

Notation. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_{\geq 0} := [0, \infty)$, $\mathbb{Z}$ be the set of integers, $\mathbb{Z}_{>0} := \{1, 2, \ldots\}$ and $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \ldots\}$. We use $\Pr(\cdot)$ for the probability and $\mathbb{E}[\cdot]$ for the expectation taken over the relevant stochastic variables. A function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $\mathcal{K}_\infty$ ($\alpha \in \mathcal{K}_\infty$) if it is continuous, strictly increasing, $\alpha(0) = 0$ and $\lim_{s \to \infty} \alpha(s) = \infty$. For any $x_1 \in \mathbb{R}^{n_1}$ and $x_2 \in \mathbb{R}^{n_2}$ with $n_1, n_2 \in \mathbb{Z}_{>0}$, (x_1, x_2) stands for $(x_1^\top, x_2^\top)^\top \in \mathbb{R}^{n_1+n_2}$.

II. Setup

In this section, we first describe the plant and controller model, followed by the communication model. We then introduce the event-based strategy and the baseline AoI policy we will use for comparison.

A. Plant and controller model

We consider the discrete-time nonlinear plant model given by

$$x(t+1) \quad = \quad f(x(t), u(t)) \tag{1}$$

where $x(t) \in \mathbb{R}^{n_p}$ is the plant state, $u(t) \in \mathbb{R}^{n_u}$ is the control input at time $t \in \mathbb{Z}_{\geq 0}$ with $n_p, n_u \in \mathbb{Z}_{>0}$. We proceed by emulation and assume that a stabilizing state-feedback controller for system (1) is known and is of the form $u(t) = g(x(t))$. The precise assumption we make on the closed-loop system (1) with this controller is formalized in the sequel.

We are interested in the scenario where plant (1) and its controller communicate over a wireless channel as illustrated in Figure 1. Specifically, the wireless link is used to communicate information from the controller to the actuator. A practical example of such a scenario would be a remote controlled robot, with a camera providing measurements located at the controller as in [22]. Consequently, the feedback loop is no longer closed at every time instant $t \in \mathbb{Z}_{\geq 0}$, but only at certain (a priori unknown) time instants $t_k \in \mathcal{T} \subseteq \mathbb{Z}_{\geq 0}$ with $k \in \mathcal{I} \subseteq \mathbb{Z}_{\geq 0}$. Communications are attempted at these instants according to the event-triggering conditions described in the sequel, and packets are successfully received. In the absence of successful communication, the actuator uses a so-called *networked version* [11] of the control input denoted by $\hat{u} \in \mathbb{R}^{n_u}$.

We introduce the concatenated state $\chi := (x, \hat{u}) \in \mathbb{R}^{n_\chi}$ with $n_\chi := n_p + n_u$, and we write the closed-loop dynamics of the WNCS as

$$\chi(t+1) = \begin{cases} f_S(\chi(t)) & \text{for } t \in \mathcal{T} \\ f_U(\chi(t)) & \text{for } t \in \mathbb{Z}_{\geq 0} \backslash \mathcal{T}, \end{cases} \tag{2}$$

where $f_S(\chi) := (f(x, g(x)), \hat{f}(g(x)))$, $f_U(\chi) := (f(x, g(\hat{u})), \hat{f}(\hat{u}))$ for any $\chi \in \mathbb{R}^{n_\chi}$ and $\hat{f}$ is the holding function[1] used to generate $\hat{u}$, which can take various forms including the zero-order-hold strategy $\hat{f}(\hat{u}) = \hat{u}$, or the zeroing policy $\hat{f}(\hat{u}) = 0$ for any $\hat{u} \in \mathbb{R}^{n_u}$.

Remark 1: The results presented in this paper apply *mutatis mutandis* when the network is located between the plant and the controller, and not between the controller and the actuator as in Figure 1, by changing the network variable to be $\hat{x}$ instead of $\hat{u}$. However, the event-based rule that will be developed in the sequel might be hard to implement at sensor nodes that lack computational capabilities, as it requires computing the Lyapunov function (as seen in the sequel). When the network is used in both directions, the analysis becomes quite convoluted, especially if communication events occur independently. We leave this case for future work. □

The assumption we make on system (2) is stated next.

Standing Assumption 1 (SA1): There exist $\overline{\alpha}, \underline{\alpha} \in \mathcal{K}_\infty$, $a_S \in [0, 1)$, $a_U > 1$ and $V : \mathbb{R}^{n_\chi} \to \mathbb{R}_{\geq 0}$ such that, for any $\chi \in \mathbb{R}^{n_\chi}$,

$$\underline{\alpha}(|\chi|) \leq V(\chi) \leq \overline{\alpha}(|\chi|) \tag{3a}$$

$$V(f_S(\chi)) \leq a_S V(\chi), \tag{3b}$$

$$V(f_U(\chi)) \leq a_U V(\chi). \tag{3c}$$

□

Properties (3a) and (3b) in SA1 imply that the origin of the ideal system $\chi(t+1) = f_S(\chi(t))$ is uniformly globally asymptotically stable (UGAS), which means the feedback law g has been designed to ensure that the origin of system (1) is UGAS. Moreover, a_S in (3b) is a guaranteed decay rate of the Lyapunov function along $\chi(t+1) = f_S(\chi(t))$ in the absence of the network. Property (3c) in SA1 imposes a condition on the growth rate of V along solutions to (2) when using $\hat{u}$ instead of u as control input. These assumptions can be easily verified for linear time-invariant systems and a detailed discussion of other classes of systems satisfying it is available in [10, Section V].

To conclude the description of the closed-loop system (2), we need to explain when a communication attempt is successful or not.

B. Communication setup

We now describe the sequence of successful communication instants $t_k \in \mathcal{T}$, $k \in \mathcal{I}$. We assume that the control packet can be perfectly communicated (without any additive noise) over the wireless channel with a certain probability $\pi \in (0, 1)$, the packet success rate, hence the wireless channel is an i.i.d erasure channel.

[1]Note that $\hat{u}$ is never reset to the actual value of u in (2).

The controller may decide at any time $t \in \mathbb{Z}_{>0}$, to attempt communication over the wireless channel. We use $\eta(t) \in \{0,1\}$ to denote the channel utilization at time t, with:

- $\eta(t) = 1$ implying that the channel is utilized and transmission is attempted at time t;
- $\eta(t) = 0$ implies that the transmitter was not active and the channel is not utilized at time t.

In view of random packet dropouts, we can write $\Pr(t \in \mathcal{T}) = \pi\eta(t)$.

We focus on event-based transmission policies that determine the channel utilization $\eta(t)$ at each instant $t \in \mathbb{Z}_{>0}$. Before describing this triggering rule, we first introduce the AoI variable $\tau(t) \in \mathbb{Z}_{>0}$ for all $t \in \mathbb{Z}_{>0}$, which counts the number of time instants elapsed since the last successful communication as follows

$$\tau(t+1) = \begin{cases} 1 & \text{for } t \in \mathcal{T} \\ \tau(t)+1 & \text{for } t \in \mathbb{Z}_{\geq 0}\backslash\mathcal{T}. \end{cases} \tag{4}$$

We assume that the initial time is a successful communication instant, i.e., we set $t_1 = 0$ resulting in $0 \in \mathcal{T}$ and $\tau(0) = 1$. The event-triggered policy is described in the following subsection.

C. Event-based transmission policies

Inspired by [7], [20], [21], the idea is to enforce that $V(\chi(\cdot))$, with V from SA1, decreases (in expectation) with a given decay rate along the solutions to (2). We introduce for this purpose the variable $\widehat{V}$ to keep track of the value of $V(\chi)$ at the last successful transmission instant. We set $\widehat{V}(0) := V(\chi(0))$ and note that

$$\widehat{V}(t+1) = \begin{cases} V(\chi(t)) & \text{for } t \in \mathcal{T} \\ \widehat{V}(t) & \text{for } t \in \mathbb{Z}_{\geq 0}\backslash\mathcal{T}. \end{cases} \tag{5}$$

Under the proposed policy, the channel is utilized in the following manner

$$\eta(t) = \begin{cases} 1 & \text{if } V(f_U(\chi(t))) > \mu^{\tau(t)+1}\nu\widehat{V}(t) \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $\nu \in [0,1]$ and $\mu \in (a_S, 1)$ are tunable parameters, and f_U and $\tau(t)$ come from (2) and (4). The policy in (6) implies that, at every time t, the transmitter compares the value of the Lyapunov function if no transmission is attempted to the imposed upper-bound $\mu^{\tau(t)+1}\nu\widehat{V}(t)$. Parameter μ denotes the desired convergence rate of the Lyapunov function and ν must be selected carefully in order to ensure the desired control property as explained in Section III. If $\pi = 1$, then $\nu = 1$ can be chosen to ensure that the Lyapunov function is exponentially decreasing with rate μ, along the solutions to (2), for any given $\mu \in (a_S, 1)$. However, since the transmissions are stochastic, a ν strictly smaller than one must be selected to compensate for potential packet drops. Additionally, policy (6) enforces that (re)transmissions are attempted until successful as long as (6).

Remark 2: Since we consider noiseless system dynamics and measurements, and deal with state feedback, assuming that the controller knows f_S, it can easily determine if a packet was successfully transmitted at t by comparing $x(t+1)$ with the predicted value using f_S. However, when the system is noisy, packet acknowledgements (which are often implemented in wireless links) can be used in order to determine $\tau(t)$. □

Under policy (6), the WNCS in (2) becomes

$$\begin{pmatrix} \chi(t+1) \\ \widehat{V}(t+1) \\ \tau(t+1) \end{pmatrix} = \begin{cases} \begin{pmatrix} f_S(\chi(t)) \\ V(\chi(t)) \\ 1 \end{pmatrix} & \text{if } V(f_U(\chi(t))) > \mu^{\tau(t)+1}\nu\widehat{V}(t) \text{ with probability } \pi, \\ \begin{pmatrix} f_U(\chi(t)) \\ \widehat{V}(t) \\ \tau(t)+1 \end{pmatrix} & \text{otherwise.} \end{cases} \tag{7}$$

D. AoI policies

As mentioned in the introduction, we will compare the proposed event-based strategy with an AoI policy, which we now present. As before, we assume that the initial time is a successful communication instant, i.e., we set $t_1 = 0$ resulting in $0 \in \mathcal{T}$ and $\tau(0) = 1$. The AoI policy is characterized by $\bar{\tau} \in \mathbb{Z}_{>0}$, which denotes the threshold on the AoI after which transmissions are attempted [10]. Under the proposed policy, the channel utilization evolves in the following manner

$$\eta(t) = \begin{cases} 1 & \text{if } \tau(t) \geq \bar{\tau} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Implementing this policy implies that after each successful communication, the next transmission is only attempted after $\bar{\tau}$ steps have passed. The (random) closed loop dynamics can thus be described as follows

$$\begin{pmatrix} \chi(t+1) \\ \tau(t+1) \end{pmatrix} = \begin{cases} \begin{pmatrix} f_S(\chi(t)) \\ 1 \end{pmatrix} & \text{if } \tau(t) \geq n+1 \text{ with probability } \pi \\ \begin{pmatrix} f_U(\chi(t)) \\ \tau(t)+1 \end{pmatrix} & \text{otherwise.} \end{cases} \tag{9}$$

III. Design specifications

A. Control guarantees

The primary objective of this work is to preserve the stability of the WNCS and the secondary objective is to reduce a communication cost compared to an AoI policy, which ensures the same control guarantees as the proposed event-based strategy. Due to the stochastic nature of communication success, we can no longer ensure the original UGAS property guaranteed by SA1. Instead, we rely on the stochastic notion of stability defined next, which is inspired by [4].

Definition 1: The set $\{(\chi, \widehat{V}, \tau) : \chi = 0\}$ is *stochastically stable* for system (7), if there exists $\alpha \in \mathcal{K}_\infty$, such that for any solution to (7) with $\widehat{V}(0) = V(0), \tau(0) = 0$, $\sum_{t=0}^{\infty} \mathbb{E}[\alpha(|\chi(t)|)] < \infty$. □

Definition 1 implies that we are interested in stability of the set where $\chi = 0$ consistently with SA1. In addition to the stability property described above, we also want to ensure that the Lyapunov function V in SA1 converges in expectation, with a given decay rate $\mu \in (a_S, 1)$ defined previously, along any solution χ to (7) with $\tau(0) = 0$ and $\widehat{V}(0) = 0$, i.e.,

$$\mathbb{E}[V(\chi(t))] \leq \mu^t V(\chi(0)), \ \forall t \in \mathbb{Z}_{>0}. \tag{10}$$

The desired stability property (10) serves as a measure of the control performance of system (2) and satisfying it automatically ensures Definition 1 as $\mu < 1$ in view of (3a).

B. Network usage

To evaluate the amount of network usage, we consider the expected average channel utilization over an infinite horizon. Specifically, the communication cost for the event-triggered policy for any given $\nu \in [0, 1]$, and any solution χ to (7) with $\tau(0) = 1$, is given by

$$J_{\text{ET}}(\nu, \chi) := \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \eta(t)\right], \tag{11}$$

where $\eta(1), \eta(2), \ldots$ is the sequence of transmission attempts applied at instances dictated by (6). Since, $t_1 = 0$ by definition, if ν and the sequence χ are given, then $V(\chi), \widehat{V}, \tau$ can be evaluated for any time and thus the sequence η is fixed. This allows us to write J_{ET} as a function of ν and χ.

We evaluate the average channel utilization for an AoI policy described in Section II-D as

$$J_{\text{AoI}}(\bar{\tau}) := \lim_{T\to\infty} \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} \tilde{\eta}(t)\right] = \frac{1}{1 + (\bar{\tau} - 1)\pi} \tag{12}$$

where $\tilde{\eta}(1), \tilde{\eta}(2), \ldots$ is the sequence of transmission attempts applied at instances dictated by the AoI policy (8). The sequence $\tilde{\eta}$ is a Markov process that is independent of the plant state as seen from (8) and this expectation becomes independent of χ, in contrast to (11). Since transmissions are attempted until a packet is successfully communicated, the expected length of attempted transmissions is given by π^{-1}. On the other hand, after every successful transmission, transmissions are stopped for a duration of $\bar{\tau} - 1$ (by definition of the AoI policy). Thus, the average channel utilization, which is the fraction of time during which transmissions are attempted, is given by $\frac{\pi^{-1}}{\pi^{-1} + (\bar{\tau}-1)} = (1 + (\bar{\tau} - 1)\pi)^{-1}$.

Minimizing J_{ET} over ν directly is challenging as the trigger times are a priori unknown and thus the communication cost for a given ν is hard to evaluate as it depends on the trajectory of χ, which is stochastic due to the random packet drops. As a consequence, our secondary objective is to find an AoI policy with the largest $\bar{\tau} \in \mathbb{Z}_{>0}$ which ensures the same guaranteed rate of convergence μ as an event-triggered policy and satisfies $J_{\text{ET}}(\nu, \chi) \leq J_{\text{AoI}}(\bar{\tau})$. This will provide us with a bound on the communication cost for the event-triggered policy.

IV. MAIN RESULTS

In this section, we first provide conditions on (π, ν) to ensure the stability property (10), when the event-triggered policy (6) is used, thereby satisfying Definition 1. Given a desired convergence rate $\mu \in (a_S, 1)$ for the expected value of V along (7) as in (10), we first identify a set of feasible ν ensuring (10).

A. Stability guarantees

We first provide conditions on ν to ensure the desired stability property in (10).

Theorem 1: Consider system (7) with $\mu \in (a_S, 1)$ and $\pi \in [0, 1]$ such that $\pi a_S < \mu - a_U(1 - \pi)$. If

$$\nu \in \left[\frac{a_S}{\mu}, \frac{\mu - a_U(1-\pi)}{\mu\pi}\right], \tag{13}$$

then (10) holds for any solution with $\tau(0) = 1$ and $\widehat{V}(0) = V(0)$, and the system is thus stochastically stable according to Definition 1. □

Proof: Let μ, ν, π be such that (13) holds and consider an arbitrary solution $(\chi, \tau, \widehat{V})$ to (7) with $\tau(0) = 1$ and $\widehat{V}(0) = V(0)$. Recall that we use $\mathcal{T}$ to denote the set of time instants where the communication succeeded and $t_1 = 0 \in \mathcal{T}$ by definition. Let us first denote by t_1' the first time instant where (6) triggered. If no such time instant exists, then that implies that

$$V(\chi(t)) \leq \mu^t \nu V(\chi(0))$$

for all $t \in \mathbb{Z}_{>0}$ by definition of the event-triggering rule, leading to the desired stability property without any communication as $\nu < 1$ from (13), with the upper-limit of (13) being increasing in π and taking the value 1 when $\pi = 1$. If such a t_1' exists, then we have

$$V(f_U(\chi(t_1'))) > \mu^{\tau(t_1')+1}\nu\widehat{V}(t_1') \tag{14}$$

by (6). On the other hand (3b) implies that

$$V(\chi(t_1 + 1)) \leq \mu\nu V(\chi(t_1)) \tag{15}$$

as $\nu \geq \frac{a_S}{\mu}$. By iteration and (6), we have that for any $t \in \{t_1 + 1, \ldots, t_1'\}$, we have

$$V(\chi(t)) \leq \mu^{t-t_1}\nu V(\chi(t_1)) \tag{16}$$

as (6) will first trigger when this inequality is false at the time instant $t_1' + 1$. Thus, we have

$$V(\chi(t_1')) \leq \mu^{t_1'-t_1}\nu V(\chi(t_1)), \tag{17}$$

leading to the desired stability property and concluding the proof for this case. On the other hand, given any $t_k \in \mathcal{T}$, $k \in \mathcal{I}$, the k-th successful transmission, let us denote by $t_k' < \infty$ the first time instant after t_k where (6) was satisfied. We will use $\mathcal{T}' \subseteq \mathbb{Z}_{>0}$ to collect all such time instants where the event-triggered policy (6) was first met before communication succeeded and $\widehat{V}$ was reset. We have

$$V(f_U(\chi(t_k'))) > \mu^{\tau(t_k')+1}\nu\widehat{V}(t_k') \tag{18}$$

by (6) and we can repeat the same logic as applied for t_1, t_1' to obtain that for any $t \in \{t_k + 1, \ldots, t_k'\}$

$$V(\chi(t)) \leq \mu^{t-t_k}\nu V(\chi(t_k)) \tag{19}$$

This implies that if $t_k' = \infty$ for some k, then

$$V(\chi(t)) \leq \mu^{t-t_k}\nu V(\chi(t_k)). \tag{20}$$

for all $t > t_k$. By design, we know that for any $t \in \{t_k', \ldots, t_{k+1}\}$ communication was attempted, with a success probability π. Consider $\{P_j\}_{j\in\mathbb{Z}_{>0}}$, with $P_j \in \{\nu, a_U\mu^{-1}\}$ for any $j \in \mathbb{Z}_{>0}$, a sequence of independent Bernoulli variables, with $\Pr(P_j = \nu) = \pi$ and $\Pr(P_j = a_U\mu^{-1}) = 1 - \pi$. For any $t \in \mathbb{Z}_{>0}$, we can exploit the definition of P_j, (3c) in SA1 and

(19) to derive

$$V(\chi(t)) \leq \mu^t \prod_{j=1}^{J} P_j V(\chi(0)) \tag{21}$$

where $J \leq t$ is the number of time instants before t such that transmissions were attempted. This simplification is possible because the packet loss or success events are i.i.d and occur at all $t \in \{t'_k, \ldots, t_{k+1}\}$. We thus can replace the growth or decay of the Lyapunov function relative to μ during these instants with P_j. On the other hand, (19) bounds the Lyapunov function for the remaining time instants. We have

$$\mathbb{E}[P_j] = \pi\nu + (1-\pi)a_U\mu^{-1}. \tag{22}$$

The expectation of a product of independent random variables is simply the product of their expectations. Therefore, $\mathbb{E}[\prod_{j=1}^{J} P_j] \leq 1$ for all J, as $\pi\nu + (1-\pi)a_U\mu^{-1} \leq 1$ from (13). Since $\mathbb{E}[\prod_{j=1}^{J} P_j] \leq 1$ for all J, (21) implies that

$$\mathbb{E}[V(\chi(t))] \leq \mu^t V(\chi(0)). \tag{23}$$

for all t and therefore (10) holds, and our statement is proven. ■

Theorem 1 implies that we can pick any ν according to (13) and the event-based policy (6) will result in stochastic stability and (10) is satisfied. If π is sufficiently close to 1 and $\mu > a_S$, any $\nu \in [\frac{a_S}{\mu}, 1)$ results in the desired stability property. Next, we look to compare and upper bound the communication cost of the event-triggered policy with an AoI policy.

B. Comparison with AoI policies

We now compare control-communications tradeoff of the event-triggered policy (6) with parameters designed in a specific manner with that of the AoI policy (8), while both policies ensure the same guaranteed rate of convergence μ. For any given $\mu \in (a_S, 1)$, $\pi \in [0,1]$ and $\bar{\tau} \in \mathbb{Z}_{>0}$ (see Section II-D), we define the next quantity, which we use in the proposition below to determine if (10) holds along solutions to (9),

$$\beta_{\text{AoI}}(\bar{\tau}) := \frac{\pi a_S a_U^{\bar{\tau}-1}}{\mu^{\bar{\tau}}} + \frac{a_U(1-\pi)}{\mu}. \tag{24}$$

The next proposition gives conditions for the AoI policy to be stochastically stable, inspired by [10].

Proposition 1: Consider (9) with $\mu \in (a_S, 1)$ and $\pi \in [0,1]$. If $\beta_{\text{AoI}}(\bar{\tau}) \leq 1$, then (10) holds with V from SA1, for any solution with $\tau(0) = 0$. □

Sketch of proof: The desired result is obtained by following the proof of Theorem 1 and replacing the event-based counter with a constant $\bar{\tau}$, i.e., we can set $t'_k = t_k + \bar{\tau}$ for all $t_k \in \mathcal{T}, k \in \mathcal{I}$. We also replace $\nu := a_S a_U^{\bar{\tau}-1}\mu^{-\bar{\tau}}$ to get the condition on $\beta_{\text{AoI}}(\bar{\tau})$. ■

Proposition 1 provides conditions under which, for any given $\mu \in (a_S, 1)$, system (9) satisfies the property (10) with decay rate μ. Next, for any AoI policy such that $\beta_{\text{AoI}}(\bar{\tau}) \leq 1$, we find an event-triggered policy ensuring the same convergence rate and a reduced communication cost.

Theorem 2: Given $\mu \in (a_S, 1)$ and $\bar{\tau} \in \mathbb{Z}_{>0}$ such that $\beta_{\text{AoI}}(\bar{\tau}) \leq 1$, any solution $(\chi, \widehat{V}, \tau)$ to (7) with $\nu = a_S a_U^{\bar{\tau}-1}\mu^{-\bar{\tau}}$, $\tau(0) = 1$ and $\widehat{V}(0) = V(0)$ satisfies (10) and $J_{\text{ET}}(\nu, \chi) \leq J_{\text{AoI}}(\bar{\tau})$.

Proof: As a first step, we prove that the event-triggered policy with $\nu := a_S a_U^{\bar{\tau}-1}\mu^{-\bar{\tau}}$ has convergence rate μ, i.e., we need to show this ν respects (13). We know that $\beta_{\text{AoI}}(\bar{\tau}) \leq 1$, i.e.,

$$\frac{\pi a_S a_U^{\bar{\tau}-1}}{\mu^{\bar{\tau}}} + \frac{a_U(1-\pi)}{\mu} \leq 1 \tag{25}$$

replacing $a_S a_U^{\bar{\tau}-1}\mu^{-\bar{\tau}}$ with ν we have $\nu \leq \frac{\mu - a_U(1-\pi)}{\mu\pi}$. Furthermore, as $a_U > 1$ and $\mu < 1$, we have that $\nu \geq \frac{a_S}{\mu}$ for any $\bar{\tau} \in \mathbb{Z}_{>0}$. Thus, we have that ν respects (13) ensuring the desired control property in (10) by using Theorem 1.

Consider an arbitrary solution $(\chi, \widehat{V}, \tau)$ to (7), due to SA1, we know that the $V(\chi(t_k + \bar{\tau})) \leq a_S a_U^{\bar{\tau}-1} V(\chi(t_k))$, where $V(\chi(t_k))$ was the value at the successful transmission instant. Since $\nu = a_S a_U^{\bar{\tau}-1}\mu^{-\bar{\tau}}$, we have that $t'_k \geq t_k + \bar{\tau}$ for all $t_k \in \mathcal{T}$. Due to the erasure channel, the expected time instants spent transmitting per attempt is π^{-1}. Therefore, we can evaluate

$$J_{\text{ET}}(\nu, \chi) \leq \frac{\pi^{-1}}{\pi^{-1} + \bar{\tau} - 1} = J_{\text{AoI}}(\bar{\tau}). \tag{26}$$

concluding our proof. ■

Theorem 2 provides the design of an event-triggering policy that ensures the same guaranteed convergence rate as a given AoI policy, but has reduced communication cost. However, it is important to note that we have merely established a bound for the convergence rate of the expectation of the Lyapunov function along the solutions to (9), i.e., for the AoI policy. The simulations in the next section demonstrate that the communication rate may in fact be much smaller for the event-triggered policy even when the convergence rate is similar.

V. Numerical example

We illustrate our results on a single-link robot arm model, obtained by discretizing the continuous-time system using an Euler method with a sampling period of 10^{-3} seconds. System (1) with state $x = (x_1, x_2) \in \mathbb{R}^2$ is given by

$$\begin{pmatrix} x_1(t+1) \\ x_2(t+1) \end{pmatrix} = \begin{pmatrix} x_1(t) + 10^{-3}x_2(t) \\ x_2(t) + 10^{-3}(\sin(x_1(t)) + u(t)) \end{pmatrix}. \tag{27}$$

The emulated feedback law is given by $u = -\sin(x_1) - 25x_1 - 10x_2$ and we use zero-order-hold when the communication packet is dropped. We consider that the packet success rate is given by $\pi = 0.8$. SA1 is verified with $V(\chi) \mapsto \chi^\top P \chi$, $a_S = 0.98$, $a_U = 1.0009$ and where

$$P = \begin{pmatrix} 0.0384 & -0.0019 & -0.0336 & 0.0031 \\ -0.0019 & 0.0015 & 0.0033 & -0.0008 \\ -0.0336 & 0.0033 & 0.0341 & -0.0032 \\ 0.0031 & -0.0008 & -0.0032 & 0.0009 \end{pmatrix}.$$

We take a range of $\mu \in \{0.992, 0.993, \ldots, 0.996\}$ and apply Theorem 1 with $\nu = \frac{\mu - a_U(1-\pi)}{\mu\pi}$, which ensures the desired stability property for the event-triggered control after verifying that $a_S\mu^{-1} < \nu$. We draw inspiration from works like [13] that study the control-communication trade-off of event-triggered policies and compare it with simpler policies, namely AoI as described in Section IV-B. In Fig. 2, we plot the control-communication trade-off incurred by the proposed event-triggered (for various μ) and AoI policies from [10] that ensure (10) for the considered μ. Specifically, we look at the time it takes (on average over 1000 simulations with random initial conditions) for the Lyapunov function to decrease by a factor of 10^4 as a measure of the control performance and the average transmissions per time instant as an estimate of J_{ET}. We plot the same trade-off for AoI policies with $\bar{\tau} \in \{14, 16, \ldots, 22\}$. We observe that the event-triggered policies significantly outperform the AoI policies in both control and communication performance, and this figure offers insights on how to design μ based on desired performance.

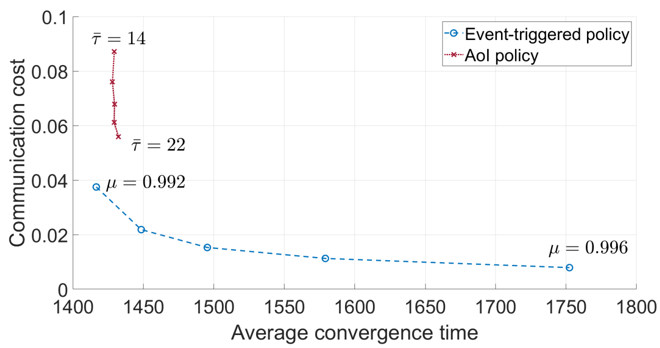

Fig. 2. The average time for the Lyapunov function to decrease by a factor of 10^4 along the solutions to the systems (7) and (9) respectively and the corresponding average channel utilization.

VI. Conclusions

We have proposed an approach to design event-triggered transmission policies for nonlinear systems communicating over a lossy channel. We have compared its control and communication performance with a AoI policy ensuring the same convergence rate guarantees on the expectation of the Lyapunov function along the respective solutions. Numerical simulations demonstrate the proposed event-triggered policy may have a smaller channel utilization even when both classes of policies have the same convergence rate. Some of the limitations of the current approach include assuming perfect measurements and noiseless dynamics, lack of communication delays, and considering that the wireless channel is only present between the plant and the actuator. Future works will focus on resolving these issues.

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