Stability crossing curves of SISO systems controlled by delayed output feedback

Constantin-Irinel Morărescu[1] and Silviu-Iulian Niculescu[2]

[1] HeuDiaSyC (UMR CNRS 6599), Université de Technologie de Compiègne,
Centre de Recherche de Royallieu, BP 20529, 60205, Compiègne, France.
The author is also with Faculty of Mathematics, University of Bucharest
and with the Department of Mathematics, University "Politehnica" of Bucharest, Romania.

[2] HeuDiaSyC (UMR CNRS 6599), Université de Technologie de Compiègne,
Centre de Recherche de Royallieu, BP 20529, 60205, Compiègne, France.
Corresponding author; Tel: +33(0)3.44.23.44.84, Fax: +33.(0)3.44.23.44.77.

Abstract. This paper focuses on closed-loop stability analysis of a class of linear single-input single-output (SISO) systems subject to delayed output feedback. The considered approach makes use of some geometric arguments in frequency-domain, arguments that simplify the understanding of the delay stabilizing mechanism. More precisely, the geometry of stability crossing curves of the closed-loop system is explicitly characterized (classification, tangent and smoothness, direction of crossing) in the parameter space defined by the pair (gain, delay). Such stability crossing curves divide the corresponding parameter space into different regions, such that, within each region, the number of characteristic roots in the right-half plane is fixed. This naturally describes the regions of (gain, delay)-parameters where the system is stable. Various illustrative examples complete the presentation.

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1   Introduction

In general, the existence of a time delay at the actuating input in a feedback control system is frequently associated to instability phenomena and/or poor (or bad) performances for the closed-loop schemes (see, for instance, [6, 15], and the references therein). At the same time, there exists situations in which the presence of some appropriate delay in the input may have the "opposite" effect, that is to induce *stability* in closed-loop under the assumption that the same control law without delay does not lead to stable behaviors in the corresponding closed-loop systems (as discussed by [1] in the delayed output feedback control of some second-order oscillatory systems). To the best of the authors' knowledge, such a "dichotomic" character of the delay (stabilizing/destabilizing) in feedback systems was not sufficiently discussed, and exploited in the literature, and there exists only a few papers devoted to the subject.

Consider now the following class of *proper SISO* open-loop trasnfer function:

$$H_{yu}(s) := \frac{P(s)}{Q(s)} = c^T(sI_n - A)^{-1}b + d \tag{1}$$

where (A, b, c^T, d) is a state-space representation of the open-loop system, and consider the *control law*:

$$u(t) = -ky(t - \tau). \tag{2}$$

As mentioned above, we are interested in finding *all* the *pairs* (k, τ) *such that the controller (2) stabilizes the SISO system (1).*

Such a problem proves its interest in the case of controlling congestion mechanisms in high-speed networks [10, 16]. Discussions on this subject can be also found in [14], but without any attempt to treat the problem in the general setting.

The classical approach in the control literature for such systems consists in finding first a controller gain k (if any) which stabilizes the system free of delay, and next in computing the *delay margin* τ_{max} such that the stability of the closed-loop system is ensured for all the delay values $\tau \in [0, \tau_{max})$ for the corresponding gain k (if such a stabilizing gain exists). The computation of such delay margins received a lot of attention in the literature as pointed out in [15, 6]. In the case of a finite τ_{max}, such an approach simply describes the so-called *destabilizing effect* of the delay. By duality, we can also consider situations where the closed-loop system is supposed *unstable* when the delay τ is set to zero for some gain value, but it can be stabilized by increasing the delay value. Such a problem largely treated in [17] describes the so-called *stabilizing effect* of the delay.

Starting from (1) with the control law given by (2), the characteristic equation of the closed-loop system simply rewrites as:

$$Q(s) + kP(s)e^{-s\tau} = 0. \tag{3}$$

The aim of this paper is to understand the underlying *stability/instability mechanisms* in presence of delays, that is to see the way the closed-loop system behaves in the *parameter-space* defined by the pair (k, τ). We think that a *geometric approach* is helpful in understanding such mechanisms. In this sense, we will focus on the *characterization* of the *stability crossing curves* in the space defined by the controller's *parameters (gain,delay)*. Such curves simply represent the collection of all the pairs (k, τ) for which the characteristic equation above has at least one root on the imaginary axis of the complex plane. The parameters behavior in the (gain, delay) space in the neighborhood of such curves is "controlled" by standard continuity properties with respect to the parameters under consideration (for the continuity with respect to the delay value, see, for instance, [4, 5]).

The presentation proposed in this paper is inspired by the classifications proposed in [7, 13] but in a completely different setting. More precisely, [7] deals with general quasipolynomials including two discrete delays, and [13] is devoted to the geometry of the crossing curves of linear systems including a particular class of distributed delay (γ-distributed delay with a gap; see, for instance, [12] for further

comments and discussions). To the best of the authors' knowledge, there does not exist any similar result in the literature, and we believe that the analysis method considered here gives new insights on such a stabilization problem. Finally, we mention that the method proposed in the paper is easy to follow, and leads to a *simple algorithm* for checking the stability of SISO systems for a given pair (gain, delay).

Without any loss of generality, the method can be resumed as follows: first, we shall find the *stability crossing set*, that is the set of frequencies corresponding to all the points in the stability crossing curves. Next, we shall *classify* the corresponding *stability crossing curves*, including some simple geometric characterization (tangent, smoothness). Finally, we shall give a method to find the *orientation* (stability/instability) of the roots crossing in *each point* of the crossing curves. It is important to point out that the methodology considered here represents a complementary approach for the characterization of all stabilizing pairs to the *analytic* one proposed by Niculescu *et al.* in [17].

The remaining paper is organized as follows: In Section 2 we briefly present the problem formulation and some prerequisites necessary to develop our (frequency-domain) stability analysis. The main results are presented in Section 3, and illustrative examples are given in Section 4. Some concluding remarks end the paper. The notations are standard.

2   Problem formulation and Preliminaries

As mentioned in the Introduction, we are interested in finding the stability regions, in the (k, τ)-parameter space, of the system whose dynamics are described by the following characteristic equation in closed-loop:

$$H(s, k, \tau) = Q(s) + kP(s)\mathrm{e}^{-s\tau} = 0 \qquad (4)$$

with polynomials P and Q satisfying the following assumptions:

Assumption 1 *Assume* $\deg(Q) \geq \deg(P)$.

Assumption 2   *2.1) The polynomials $P(s)$ and $Q(s)$ do not have common zeros;*

 2.2) $P'(j\omega) \neq 0$ whenever $P(j\omega) = 0$.

The first assumption simply says that the open-loop system is proper. If Assumption 2.1) is violated, there exists a common factor $c(s) \neq$ constant such that $P(s) = c(s)P_1(s)$ and $Q(s) = c(s)Q_1(s)$. Choose $c(s)$ be the highest possible order polynomial, then P_1 and Q_1 do not have any common zeros and the delay-differential equation can be decomposed to an ordinary differential equation with characteristic polynomial $c(s)$ and a delay-differential equation with characteristic quasipolynomial

$$Q_1(s) + kP_1(s)\mathrm{e}^{-s\tau} = 0, \qquad (5)$$

which should necessarily satisfy Assumption 2.1). Finally, the Assumption 2.2) is made to exclude some rare singular cases in order to simplify the presentation.

As mentioned in the Introduction, our description will mainly follow the algorithm presented in [7, 13] and based on some simple geometric interpretations of the characteristic equation in the parameter-space defined by the corresponding (gain,delay)-pair. First at all, we briefly present some necessary considerations proposed by Niculescu *et al.* in [17] using a continuity principle argument for the dependence of the roots of the characteristic equation with respect to some real parameter (the gain k in our study).

Introduce now the following Hurwitz matrix associated to some polynomial $A(s) = \sum_{i=0}^{n_a} a_i s^{n_a - i}$:

$$H(A) = \begin{bmatrix} a_1 & a_3 & a_5 & \ldots & a_{2n_a-1} \\ a_0 & a_2 & a_4 & \ldots & a_{2n_a-2} \\ 0 & a_1 & a_3 & \ldots & a_{2n_a-3} \\ 0 & a_0 & a_2 & \ldots & a_{2n_a-4} \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & a_{n_a} \end{bmatrix} \in \mathbb{R}^{n_a \times n_a}, \tag{6}$$

where the coefficients a_l are assumed to be zero ($a_l = 0$), for all $l > n_a$.

We consider $H(Q)$, $H(P) \in \mathbb{R}^{n \times n}$ where $\deg(Q) = n > m = \deg(P)$, the coefficients $q_l = 0$ for all $l > n$ and the coefficients $p_l = 0$ for all $l > m$.

We consider also, the set of roots of $H(s, k, 0)$ located in the closed right half plane, denoted by $\mathcal{U}$.

Introduce now the quantity k_{max} given by:

$$k_{max} = \begin{cases} \frac{q_0}{p_0}, & \text{if } \deg(Q) = \deg(P), \\ +\infty, & \text{if } \deg(Q) > \deg(P). \end{cases} \tag{7}$$

Such a quantity will define the controller's gain domain. It is easy to see that while in the case of a strictly proper transfer ($\deg(Q) > \deg(P)$), we do not have any restriction on the gain, the case of a proper transfer ($\deg(Q) = \deg(P)$) imposes such restrictions. The explanation can be resumed as follows: in this last case, the corresponding closed-loop system is a quasipolynomial of *neutral* type (see, for instance, [9, 15] for further discussions on the topics), and one explicitly needs further constraints on the gain, that is k should satisfy the inequality $|k| < k_{max} = 1/|d|$ (stability of the corresponding difference operator). Indeed, if this is not the case, larger gain values will induce instability even for infinitesimal small delay values ($\deg(Q) = \deg(P)$ with unstable difference operators) as pointed out by [11].

As a consequence of the remarks above, it is important to point out that for all $k \in \mathbb{R}$, such that $|k| < k_{max}$, card($\mathcal{U}$) is *finite*, where card($\cdot$) denotes the cardinality (number of elements).

The following result is a slight extension of Lemma 2 in [17] (devoted only to the case of strictly proper SISO systems) and also a slight modification, and generalization of Theorem 2.1 by Chen [3]:

Lemma 1 *Let $\lambda_1 < \lambda_2 < ... < \lambda_h$, with $h \le n$ be the real eigenvalues of the matrix pencil $\Sigma(\lambda) = \lambda H(P) + H(Q)$ inside the interval $(-k_{max}, k_{max})$.*

Then, $\text{card}(\mathcal{U})$ remains constant as k varies within each interval $(\lambda_i, \lambda_{i+1})$. The same holds for the intervals $(-k_{max}, \lambda_1)$ and (λ_h, k_{max}).

We also note that the lemma above gives a simple method to compute $\text{card}(\mathcal{U})$ by computing the generalized eigenvalues of the matrix pencil $\Sigma(\lambda)$. Such a quantity $\text{card}(\mathcal{U})$ is needed to derive the stability regions in the parameter space defined by the gain and delay parameters (k, τ) (see Section 3).

3 Stability crossing curves characterization

The characterization of the stability crossing curves in the (k, τ) parameter space needs the following ingredients:

(a) first, the identification of the corresponding *crossing points*, that is the set of frequencies corresponding to all the points in the stability crossing curves. Next, we define the associated *crossing set*, which will be defined by a finite number of intervals of finite length;

(b) second, the classification of the corresponding stability crossing curves, including some simple geometric characterizations (tangent, smoothness);

(c) finally, the characterization of the way the roots cross the imaginary axis.

All these steps are detailed in the next paragraphs, and the examples illustrating various case study are considered in the next section. The presentation is as simple as possible, and intuitive.

3.1 Identification of crossing points

Let $\mathcal{T}$ denote the set of all $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ such that (4) has at least one zero on imaginary axis. Any $(k, \tau) \in \mathcal{T}$ is known as a *crossing point*. The set $\mathcal{T}$, which is the collection of all crossing points, is known as the *stability crossing curves*.

We consider also the set Ω of all real number ω such that $j\omega$ satisfy (4) for at least one pair $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$. We will refer to Ω as the *crossing set*.

Remark 1 *If ω is a real number and $(k, \tau) \in \mathbb{R} \times \mathbb{R}_+$ then*

$$Q(-j\omega) + kP(-j\omega)e^{j\omega\tau} = \overline{Q(j\omega) + kP(j\omega)e^{-j\omega\tau}}$$

Therefore, in the remaining paper, we only need to consider positive ω. We have the following result:

Proposition 1 *Given any $\omega > 0$, $\omega \in \Omega$ if and only if it satisfies:*

$$| P(j\omega) | \quad > \quad 0, \tag{8}$$

and all the corresponding pairs (k, τ) can be calculated by:

$$k(\omega) \quad = \quad \pm \left| \frac{Q(j\omega)}{P(j\omega)} \right|; \tag{9}$$

$$\tau_m(\omega) \quad = \quad \frac{1}{\omega} \left(\angle P(j\omega) - \angle Q(j\omega) + (2m + \epsilon_k + 1)\pi \right) \tag{10}$$
$$m = 0, \pm 1, \pm 2, \ldots$$

where $\epsilon_k = \left\{ \begin{array}{l} 0 \text{ if } k \geq 0 \\ -1 \text{ if } k < 0 \end{array} \right.$.

Proof. For the necessity of (8), let ω be a crossing frequency in Ω and apply modulus to the closed-loop equation:

$$Q(j\omega) + kP(j\omega)\mathrm{e}^{-j\omega\tau} = 0. \tag{11}$$

This implies that

$$|Q(j\omega)| = |k||P(j\omega)| \tag{12}$$

is satisfied. It becomes clear that $P(j\omega) > 0$ is necessary. Otherwise, $P(j\omega) = 0$, which implies $Q(j\omega) = 0$ for all the gains k, which contradicts the technical assumption 2 (P and Q do not have common zeros).

For the sufficiency of (8), we only need to recognize that the pair (k, τ) given by (9)-(10) makes $s = j\omega$ a solution of the corresponding characteristic equation of the closed-loop system. $\square$

Remark 2 (small gain) *Assume now that the open-loop SISO system does not include oscillatory modes, that is $Q(s)$ has no roots on the imaginary axis.*

Some simple algebraic manipulations prove that for all the gains k satisfying the following inequality:

$$| k | \quad < \quad \frac{1}{\sup_{\omega>0} \left\{ \frac{|P(j\omega)|}{|Q(j\omega)|} \right\}}, \tag{13}$$

the closed-loop system (4) is hyperbolic *(see [8, 15] for further details on such a notion), that is there does not exist any crossing roots on the imaginary axis for all positive delays τ.*

In other words, the closed-loop system is stable (unstable) for all delays value if it is stable (unstable) in the free delays case. Furthermore, the frequency-sweeping test above (13) gives a simple way to exclude some k-interval from the beginning, since in such a case crossing roots can not exist.

However, it is important to point out that such a frequency-sweeping test (13) losses all its interest if if the polynomial $Q(s)$ has roots on the imaginary axis (the corresponding upper bound becomes 0), that is in the case of linear systems including oscillatory modes (such a case will be considered in Section 4: Illustrative examples).

In these circumstances, we can assume k within some finite interval $[\alpha, \beta] \subset (-k_{max}, k_{max})$, which contains all generalized eigenvalues λ_i of the matrix pencil $\Sigma(\lambda)$, but excluding the k-interval given by (13) if the SISO system does not include oscillatory modes. Next, Lemma 1 ensures us that the choice of the interval $[\alpha, \beta]$ includes all the remaining possibilities for the system free of delay. In such a case, define $\ell_l := min\{\mid \alpha \mid, \mid \beta \mid\} \geq 0$, and $\ell_u := max\{\mid \alpha \mid, \mid \beta \mid\} < \infty$. Then, there are only a *finite number of solutions* to each of the following three equations:

$$\mid Q(j\omega) \mid \quad = \quad \ell_l \mid P(j\omega) \mid, \tag{14}$$

$$\mid Q(j\omega) \mid \quad = \quad \ell_u \mid P(j\omega) \mid, \tag{15}$$

and

$$P(j\omega) \quad = \quad 0, \tag{16}$$

because P, and Q are polynomials satisfying the Assumptions 1 and 2. Therefore, the *crossing set* Ω will be defined by all the frequencies $\omega > 0$ satisfying *simultaneously* the inequalities:

$$\begin{cases} \ell_l \mid P(j\omega) \mid \leq \mid Q(j\omega) \mid \leq \ell_u \mid P(j\omega) \mid, \\ \mid P(j\omega) \mid > 0. \end{cases} \tag{17}$$

In conclusion, due to the form of (17), and from the Assumptions 1 and 2, the corresponding crossing set Ω *consists of a finite number of intervals*. Denote these intervals as: $\Omega_1, \Omega_2, \ldots, \Omega_N$. Then:

$$\Omega \quad = \quad \bigcup_{k=1}^{N} \Omega_k.$$

Remark 3 (strictly proper SISO case) *In the case of a strictly proper SISO system $k_{max} = \infty$ (that is no any constraints on the gain k), we note that for $k \in (\beta, \infty)$ (or $k \in (-\infty, \alpha)$) we can still express Ω as a finite number of intervals, but one of them has an infinite end.*

Remark 4 (Invariance root at the origin) *If $\dfrac{Q(0)}{P(0)} \in [\alpha, \beta]$, then 0 will be a characteristic root for all τ if $k = \dfrac{Q(0)}{P(0)}$, since $e^{-s\tau} = 1$ for $s = 0$, independently of the delay value τ. The last remark allows us to eliminate the value $\dfrac{Q(0)}{P(0)}$ from Ω if it is the case.*

Remark 5 (Crossing characterization) *The frequency-sweeping test (17) above gives all the frequency intervals for which crossing roots exist for the corresponding chosen gain interval, but it does not give any information on the* crossing direction.

In other words, such a test does not make any distinction between switches *(crossing towards instability) and* reversals *(crossing towards stability). Such a problem will be considered in the next paragraphs (see, for instance, §4.3: Direction of crossing).*

In the sequel, we consider $\Omega_i = [\omega_i^l, \omega_i^r]$, for all $i = 1, 2, \ldots, N$. Without any loss of generality, we can order these intervals from left to right, i.e., for any $\omega_1 \in \Omega_{i_1}, \omega_2 \in \Omega_{i_2}, i_1 < i_2$, we have $\omega_1 < \omega_2$.

We note that ω_1^l can be 0 and in this case Ω_1 is open to the left.

It is clear that $k(\omega_i^l), k(\omega_i^r) \in \{\alpha, \beta\}$ for all $i = 1 \ldots N$ if $\omega_1^l \neq 0$. We will not restrict $\angle Q(j\omega)$ and $\angle P(j\omega)$ to a 2π range. Rather, we allow them to vary continuously within each interval Ω_i. Thus, for each fixed m, (9) and (10) give us two continuous almost everywhere curves. We can lose the continuity of the curve in the points which correspond to the case $Q(j\omega) = 0$. For example, if $Q(j\omega^*)$ is a real polynomial and its sign is changing at ω^*, then $\angle(Q(j\omega))$ is not continuous in ω^*.

It should be noted that condition (9) and k finite, imply $P(j\omega) \neq 0, \quad \forall \omega \in \Omega$. We denote the curves defined by (9) and (10) with $\mathcal{T}_i^{m\pm}$. Therefore, corresponding to a given interval Ω_i, we have an infinite number of continuous stability crossing curves $\mathcal{T}_i^{m\pm}, m = 0, \pm 1, \pm 2, \ldots$.

Finally, it should be noted that, for some m, part or the entire curve may be outside of the range $\mathbb{R} \times \mathbb{R}_+$, and therefore, may not be physically meaningful. The collection of all the points in $\mathcal{T}$ corresponding to Ω_i may be expressed as

$$\mathcal{T}_i = \bigcup_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+) \right) \right]$$

Obviously,

$$\mathcal{T} = \bigcup_{i=1}^{N} \mathcal{T}_i.$$

Also it is easy to see that, for each Ω_i, we define two curves, one to the right of the $O\tau$ axis and the other to the left. According to the fixed limits α, β of the interval where k varies we can eliminate some of these curves.

3.2   Classification of the crossing curves

In this paragraph, we give a classification of the crossing curves with respect to their shape. In order to do this we first classify the ends of the crossing curves. It is not difficult to see that each end point ω_i^l or ω_i^r must belong to one of the following three types:

Type 1. It satisfies the equation $k(\omega) = \alpha$.

Type 2. It satisfies the equation $k(\omega) = \beta$.

Type 3. It equals 0.

Obviously, only ω_1^l can be of type 3. We note that all the crossing curves are situated in the vertical strip $\mathcal{D}$ between the lines $k = \alpha$ and $k = \beta$. Now, let ω_* be an end point of the interval Ω_i. We already said that each $\mathcal{T}_i^{m+}$ is an continuous

almost everywhere curve, so, $(k(\omega_*), \tau_m(\omega_*))$ is an end point of $\mathcal{T}_i^{m\pm}$, and it can be characterized as follows:

- If ω_* is of type 1, then $k(\omega_*) = \alpha$ and $\tau(\omega_*)$ are finite. More precisely, $\mathcal{T}_i^{m+}$ intersects the vertical line $k = \alpha$, which is the left bound of the strip $\mathcal{D}$.

- If ω_* is of type 2 then $k(\omega_*) = \beta$ and $\tau(\omega_*)$ are finite. Or, we may say that $\mathcal{T}_i^{m+}$ intersects the vertical line $k(\omega) = \beta$, which is the right bound of the strip $\mathcal{D}$.

- If ω_* is of type 2 then τ approaches ∞ and k approaches $\dfrac{Q(0)}{P(0)}$. In other words, $\mathcal{T}_i^{m+}$ has a vertical asymptote given by $k = \dfrac{Q(0)}{P(0)}$.

Remark 6 *The previous description holds also for $\mathcal{T}_i^{m-}$.*

We say an interval Ω_k is of type lr if its left end is of type l and its right end is of type r. We may divide accordingly these intervals into the following 6 types:

Type 11. In this case, $\mathcal{T}_i^{m\pm}$ starts at a point on the vertical line $k = \alpha$, and ends at another point on the vertical line $k = \alpha$.

Type 12. In this case, $\mathcal{T}_i^{m\pm}$ starts at a point on the vertical line $k = \alpha$, and ends at a point on the vertical line $k = \beta$.

Type 21. This is the reverse of type 12. $\mathcal{T}_i^{m\pm}$ starts at a point on the vertical line $k = \beta$, and ends at a point on the vertical line $k = \alpha$.

Type 22. In this case, $\mathcal{T}_i^{m\pm}$ starts at a point on the vertical line $k = \beta$, and ends at another point on the vertical line $k = \beta$.

Type 31. In this case, $\mathcal{T}_i^{m\pm}$ begins at ∞ with a vertical asymptote $k = \dfrac{Q(0)}{P(0)}$. The other end is on the vertical line $k = \alpha$.

Type 32. In this case, $\mathcal{T}_i^{m\pm}$ again begins at ∞ with a vertical asymptote $k = \dfrac{Q(0)}{P(0)}$. The other end is on the vertical line $k = \beta$.

More details and some figures for the above cases will be given in the illustrative examples section. More precisely, we will see, for example, an interval of type 31 for a scalar system, type 32 for a linear (second-order) oscillator, types 11, 22, 31 and 32 for a third-order system.

3.3 Tangents and smoothness

For a given i, we will discuss the smoothness of the curves in $\mathcal{T}_i^{m\pm}$ and thus
$\mathcal{T} = \bigcup\limits_{m=-\infty}^{+\infty} \left[\left(\mathcal{T}_i^{m+} \cap (\mathbb{R} \times \mathbb{R}_+)\right) \cup \left(\mathcal{T}_i^{m-} \cap (\mathbb{R} \times \mathbb{R}_+)\right)\right]$. In this part we use an
approach based on the implicit function theorem. For this purpose, we consider
k and τ as implicit functions of $s = j\omega$ defined by (4). For a given m and i, as
s moves along the imaginary axis within Ω_i, $(k, \tau) = (k(\omega), \tau(\omega))$ moves along
$\mathcal{T}_i^{m\pm}$. For a given $\omega \in \Omega_i$, let

$$R_0 = Re\left(\frac{\partial H(s,k,\tau)}{\partial s}\right)_{s=j\omega}$$
$$= Re\left\{Q'(j\omega) + e^{-j\omega\tau}\left[kP'(j\omega) - k\tau P(j\omega)\right]\right\},$$
$$I_0 = Im\left(\frac{\partial H(s,k,\tau)}{\partial s}\right)_{s=j\omega}$$
$$= Im\left\{Q'(j\omega) + e^{-j\omega\tau}\left[kP'(j\omega) - k\tau P(j\omega)\right]\right\},$$

$$R_1 = Re\left(\frac{\partial H(s,k,\tau)}{\partial k}\right)_{s=j\omega} = Re\left[P(j\omega)e^{-j\omega\tau}\right],$$
$$I_1 = Im\left(\frac{\partial H(s,k,\tau)}{\partial k}\right)_{s=j\omega} = Im\left[P(j\omega)e^{-j\omega\tau}\right],$$

$$R_2 = Re\left(\frac{\partial H(s,k,\tau)}{\partial \tau}\right)_{s=j\omega} = Im\left(k\omega P(j\omega)e^{-j\omega\tau}\right),$$
$$I_2 = Im\left(\frac{\partial H(s,k,\tau)}{\partial \tau}\right)_{s=j\omega} = -Re\left(k\omega P(j\omega)e^{-j\omega\tau}\right).$$

Then, since $H(s, k, \tau)$ is an analytic function of s, k and τ, the implicit function
theorem indicates that the tangent of $\mathcal{T}_i^{m\pm}$ can be expressed as

$$\begin{pmatrix} \dfrac{\mathrm{d}k}{\mathrm{d}\omega} \\ \dfrac{\mathrm{d}\tau}{\mathrm{d}\omega} \end{pmatrix} = \frac{1}{R_1 I_2 - R_2 I_1}\begin{pmatrix} R_0 I_2 - I_0 R_2 \\ I_0 R_1 - R_0 I_1 \end{pmatrix}, \tag{18}$$

provided that
$$R_1 I_2 - R_2 I_1 \neq 0. \tag{19}$$

It follows that $\mathcal{T}_i^{m\pm}$ is smooth everywhere except possibly at the points where either
(19) is not satisfied, or when
$$\frac{\mathrm{d}k}{\mathrm{d}\omega} = \frac{\mathrm{d}\tau}{\mathrm{d}\omega} = 0. \tag{20}$$

From the above discussions, we can conclude with the following Proposition.

Proposition 2 *The curve $\mathcal{T}_i^{m\pm}$ is smooth everywhere except possibly at the point corresponding to $s = j\omega$ in any one of the following cases:*

1) *$s = j\omega$ is a multiple solution of (4), and*

2) *ω is a solution of $Q(j\omega) = 0 \Leftrightarrow k = 0$.*

Proof. If (20) is satisfied then $s = j\omega$ is a multiple solution of (4).
On the other hand, $R_1 I_2 - R_2 I_1 = -k\omega|P(j\omega)|^2$. If $P(j\omega) = 0$ we get $Q(j\omega) = 0$ so, assumption 2 is not satisfied. Therefore, (19) is violated if and only if $k = 0$. Obviously, $k = 0$ implies that $Q(j\omega) = 0$. So, we can conclude that (19) is violated if and only if ω is a solution of $Q(j\omega) = 0$. $\qquad\square$

3.4 Direction of crossing

Next we will discuss the direction in which the solutions of (4) cross the imaginary axis as (k, τ) deviates from the curve $\mathcal{T}_i^{m\pm}$. We will call the direction of the curve that corresponds to increasing ω the *positive direction*. We will also call the region on the left hand side as we head in the positive direction of the curve *the region on the left*.

To establish the direction of crossing we need to consider k and τ as functions of $s = \sigma + j\omega$, i.e., functions of two real variables σ and ω, and partial derivative notation needs to be adopted. Since the tangent of $\mathcal{T}_i^{m\pm}$ along the positive direction is $\left(\dfrac{\partial k}{\partial \omega}, \dfrac{\partial \tau}{\partial \omega} \right)$, the normal to $\mathcal{T}_i^{m\pm}$ pointing to the left hand side of positive direction is $\left(-\dfrac{\partial \tau}{\partial \omega}, \dfrac{\partial k}{\partial \omega} \right)$. Corresponding to a pair of complex conjugate solutions of (4) crossing the imaginary axis along the horizontal direction, (k, τ) moves along the direction $\left(\dfrac{\partial k}{\partial \sigma}, \dfrac{\partial \tau}{\partial \sigma} \right)$. So, if a pair of complex conjugate solutions of (4) cross the imaginary axis to the right half plane, then,

$$\left(\frac{\partial k}{\partial \omega} \frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega} \frac{\partial k}{\partial \sigma} \right)_{s=j\omega} > 0, \tag{21}$$

i.e. the region on the left of $\mathcal{T}_i^{m\pm}$ gains two solutions on the right half plane. If the inequality (21) is reversed then the region on the left of $\mathcal{T}_i^{m\pm}$ loses has two right half plane solutions. Similar to (18) we can express

$$\left(\begin{array}{c} \dfrac{\partial k}{\partial \sigma} \\ \dfrac{\partial \tau}{\partial \sigma} \end{array} \right)_{s=j\omega} = \frac{1}{R_1 I_2 - R_2 I_1} \left(\begin{array}{c} R_0 R_2 + I_0 I_2 \\ -R_0 R_1 - I_0 I_1 \end{array} \right). \tag{22}$$

Using this, we arrive at the following proposition:

Proposition 3 *Let $\omega \in (\omega_i^l, \omega_i^r)$ and $(k, \tau) \in \mathcal{T}_i$ such that $j\omega$ is a simple solution of (4) and $H(j\omega', k, \tau) \neq 0$, $\forall \omega' > 0$, $\omega' \neq \omega$ (i.e. (k, τ) is not an intersection point of two curves or different sections of a single curve of $\mathcal{T}$). Then a pair of solutions of (4) will cross the imaginary axis to the right, through $s = \pm j\omega$ if $R_2 I_1 - R_1 I_2 > 0$. The crossing is to the left if the inequality is reversed.*

Proof. Easy computation shows that

$$\left(\frac{\partial k}{\partial \omega} \frac{\partial \tau}{\partial \sigma} - \frac{\partial \tau}{\partial \omega} \frac{\partial k}{\partial \sigma} \right)_{s=j\omega} = \frac{(R_0^2 + I_0^2)(R_2 I_1 - R_1 I_2)}{(R_1 I_2 - R_2 I_1)^2}$$

Therefore (21) can be written as $R_2 I_1 - R_1 I_2 > 0$. $\square$

Under standard regularity assumptions, Proposition 3 above gives the explicit *crossing direction* towards stability (reversals) or instability (switches) function of some quantity evaluated at the corresponding crossing point. Such a result together with Lemma 1, and Propositions 1–2 lead to the following simplified procedure: first, compute $\mathrm{card}(\mathcal{U})$ (using Lemma 1); second, derive the crossing set Ω, and related dependence $(k(\omega), \tau(\omega))$ (using Proposition 1, and its corresponding remarks and derivations); next, determine the crossing direction using Proposition 3 above.

Finally, in order to find explicitly the corresponding (closed-loop) *stability regions*, we simply count the number of crossing roots (towards stability/instability) corresponding to each region whose boundaries are given by the crossing curves (and/or eventually by the corresponding axis and/or eventually by some lines parallel to the axis). Since for $\tau = 0$, $\mathrm{card}(\mathcal{U})$ gives the complete (stability/instability) information in terms of k (closed-loop free of delays), we can apply the procedure briefly outlined above in the sense of increasing the delay value τ from 0 to some positive value.

Such an approach was explicitly applied to some of the examples considered in the next section (second-order and sixth-order example, respectively).

Remark 7 *As mentioned in the Introduction, such a geometric method represents a complementary approach to the analytical characterization proposed in [17].*

If the results are similar, however the geometric approach considered here gives more insights on the stability regions in the sense that the corresponding boundaries are appropriately classified *(stability crossing curves, excepting some singular cases).*

4   Illustrative examples

This section is devoted to show some applications of our approach. In this sense, we consider some classical examples in the literature (first-order, second-order oscillatory systems) and we compare our conclusions with the existing ones.

Example 1 (Scalar delay case) *Consider the system given by the transfer function*

$$H_{y,u}(s) = \frac{1}{s + a} \tag{23}$$

subject to the control law $u(t) = -ky(t - \tau)$. The corresponding characteristic equation of the corresponding closed-loop system can be written as:

$$s + a + ke^{-s\tau} = 0. \tag{24}$$

For $a > 0$ it is obvious that either for $k = 0$ or $\tau = 0$, $a + k > 0$, we obtain a stable equation.

Using Proposition 3, we conclude that all the crossings are towards instability. In a completely different framework, Boese [2] considered $k > 0$ and he proved that for $k \leq a$ we get a delay independent stable system and for $k > a$ we have only one stability interval $[0, \tau_0)$, where τ_0 is a decreasing function of k.

Using our method for $a = 3$ we can draw the crossing curves and establish the stability region as in figure 1. In this case, we have:
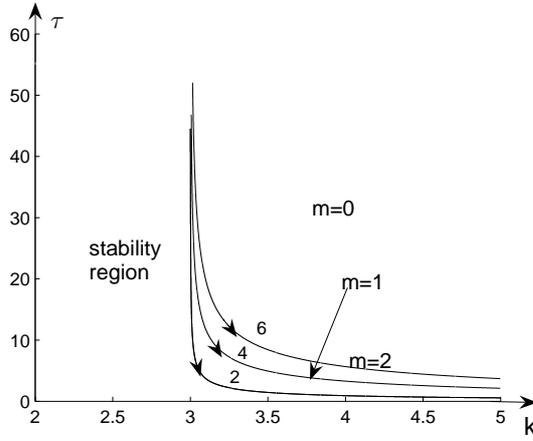
Figure 1: $\mathcal{T}_1^{m+}$, $m \in \{0, 1, 2, \}$ for the system (24)

$$\text{card}\,(\mathcal{U}) \quad = \quad \begin{cases} 0 \text{ if } k > -3, \\ 1 \text{ if } k \leq -3, \end{cases}$$

$$\tag{25}$$

and for $k \in [-5, 5]$ the crossing set Ω consists in one interval $(0, 4]$ of type 31. Therefore, we obtain only one *stability interval for $k > 3$, and this interval is $[0, \tau_0)$, where τ_0 is given by:*

$$\tau_0 = \frac{1}{\omega} \left(\pi - \arctan \frac{\omega}{3} \right) = \frac{1}{\sqrt{k^2 - 9}} \left(\pi - \arctan \frac{\sqrt{k^2 - 9}}{3} \right),$$

which is nothing else that the formula given by Boese for the corresponding upper bound of the (closed-loop) stability interval.
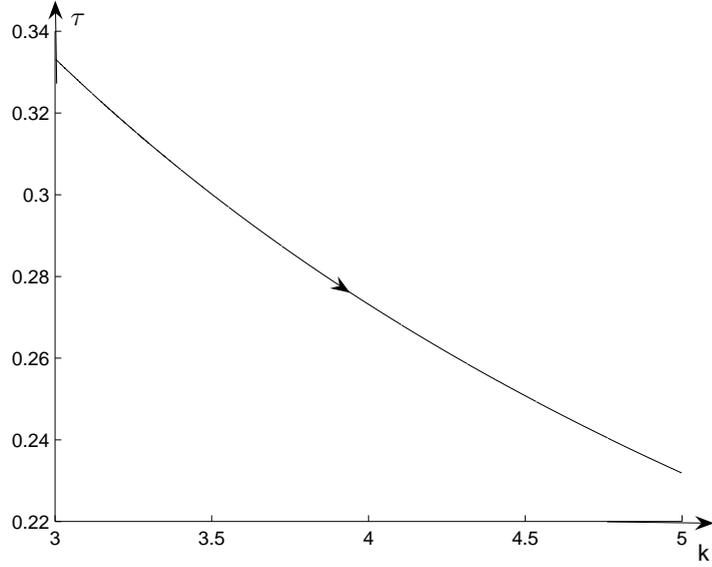
Figure 2: $\mathcal{T}_1^{0+}$ for the system (24) with $a = -3$

Now consider the case $a = -3$ (open-loop system unstable) and $k \in [-5, 5]$, once again we derive $\Omega = (0, 4]$ and

$$\text{card}\,(\mathcal{U}) = \left\{ \begin{array}{l} 0 \; \textit{if } k > 3 \\ 1 \; \textit{if } k \leq 3 \end{array} \right. .$$

Since all the crossing direction are towards instability, it is sufficient to plot only the first stability crossing curve. As expected (figure 2), the system becomes unstable as τ increases.   □

Example 2 (Linear (second-order) oscillators) *Consider the transfer function*

$$H_{y,u}(s) = \frac{1}{s^2 + 2} \tag{26}$$

subject to the control law $u(t) = -ky(t - \tau)$. The corresponding characteristic equation is given by:

$$s^2 + 2 + ke^{-s\tau} = 0. \tag{27}$$

For $k \in (-2, 0)$ the results regarding stability intervals of the systems can be found in [14, 17] and they say that for $\tau \in \left(0, \dfrac{\pi}{\sqrt{2 + |k|}} \right)$ the system is stable (see also [1] for a different stability argument). The number of stabilizing delay interval is a decreasing function of $|k|$.

Our computation in this case point out that for $k \in (-2, 0)$ the crossing set Ω consists in one interval $(0, 2]$ of type 32.

We note that according to Proposition 2 all the crossing curves are not continuous in the points which correspond to $k = 0$.

It is easy to see that:

$$\text{card} \, (\mathcal{U}) = \left\{ \begin{array}{l} 1 \; \text{if} \; k < -2, \\ 2 \; \text{if} \; k > -2. \end{array} \right.$$

Proposition 3 simply says that for $k < 0$ the region on the right hand side of each crossing curve has two more unstable roots.

Remark 8 *If $\omega \in (0, \sqrt{2})$ then $\tau_0(\omega) = 0$ as we can deduce from the computation below:*

$$\tau_0 = \frac{1}{\omega}(\angle(1) - \angle(2 - \omega^2) + (\epsilon_k + 1)\pi) = 0, \quad \forall \omega \in (0, \sqrt{2}) \qquad (28)$$

More precisely (see figure 2), we recover the result proposed in [14, 17]. □
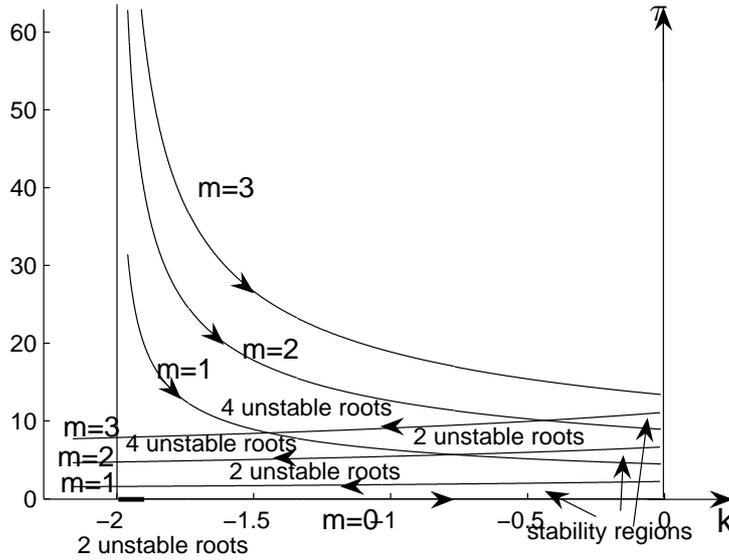
Figure 3: τ_i, $m \in \{0, 1, 2, 3\}$ versus k for the system (27)

Example 3 (Third-order unstable system) *This example is only to illustrate that it is possible to have most types of the curves enumerated in the classification section. In the sequel we present a dynamical system with crossing curves of type 11, 22, 31 and 32.*

Consider the transfer function

$$H_{y,u}(s) = \frac{1}{s^3 - 2s^2 + 9s - 8} \tag{29}$$

subject to the control law $u(t) = -ky(t - \tau)$. The corresponding characteristic equation of the closed-loop system is given by:

$$s^3 - 2s^2 + 9s - 8 + ke^{-s\tau} = 0. \tag{30}$$

We note that this system can not be stabilized by any static output feedback. Indeed, straightforward computations show us that:

$$\text{card}\,(\mathcal{U}) = \begin{cases} 1 & \text{if} & k < -10, \\ 3 & \text{if} & k \in (-10, 8), \\ 2 & \text{if} & k > 8. \end{cases}$$

Taking $\alpha = -10$ and $\beta = 10$, we get $\Omega = (0, 1] \cup [2, 3]$ and $\mathcal{T}_1^{m+}$ is of type 32, $\mathcal{T}_1^{m-}$ is of type 31, $\mathcal{T}_2^{m-}$ is of type 11, $\mathcal{T}_2^{m+}$ is of type 22. We present the last three curves in the figures 3-3. □
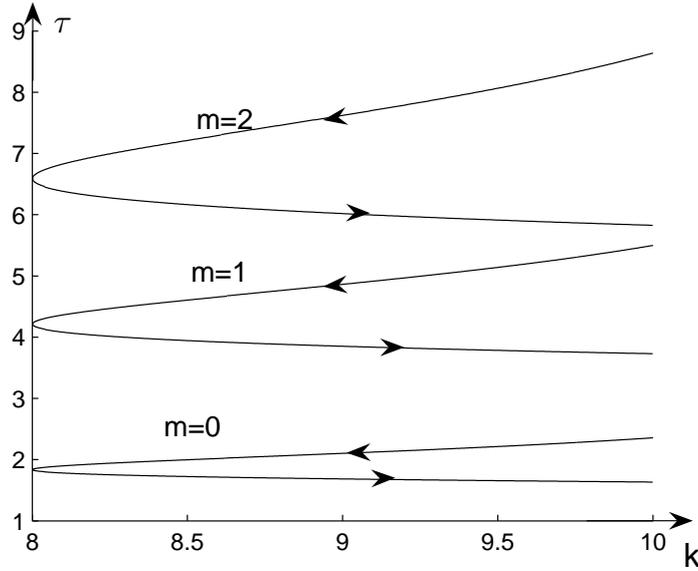
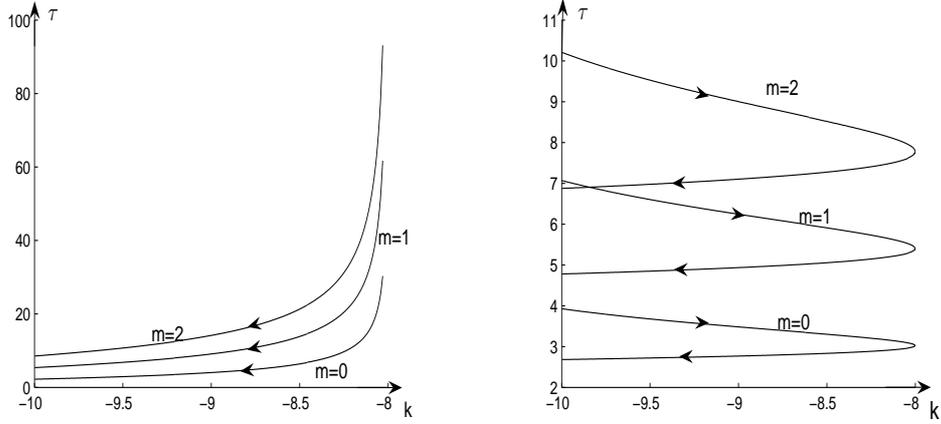Figure 4: $\mathcal{T}_2^{m+}$, $m \in \{0, 1, 2\}$ for the system (30)

Figure 5: $\mathcal{T}_i^{m-}$, $m \in \{0, 1, 2\}, i \in 1, 2$ for the system (30)

Example 4 (Sixth-order unstable system) *In this example, we consider a system that can not be stabilized by a static output feedback, but it can be stabilized by a delayed output feedback. This example is borrowed from [17].*

Consider the system:

$$H_{y,u}(s) = \frac{1}{s^6 + p_1 s^5 + p_2 s^4 + p_3 s^3 + p_4 s^2 + p_5 s + p_6} \qquad (31)$$

where

$$p_1 = -6.0000000e - 04, \quad p_2 = 1.4081634e + 00, \quad p_3 = -5.6326533e - 04,$$
$$p_4 = 4.3481891e - 01, \quad p_5 = -8,6963771e - 05, \quad p_6 = 2.6655565e - 02.$$

Using Lemma 1, we obtain:

$$\mathrm{card}\,(\mathcal{U}) = \begin{cases} 3 & if & k < -0.0707886, \\ 5 & if & k \in (-0.0707886; -0.0266556), \\ 6 & if & (-0.0266556; 0.0120036), \\ 4 & if & k > 0.0120036. \end{cases}$$

The stability crossing curves and the first two stability region for $k \in (0, 0.16)$ are plotted in figure 7 (see also the graphics of $k = k(\omega)$).

5   Concluding remarks

This paper addressed the problem of controlling a SISO system by using a delayed output feedback. More precisely, we have given a geometric characterization of the stability crossing curves in the parameter space defined by the gain, and the corresponding delay. Several examples have been presented in order to illustrate the interest of the approach.
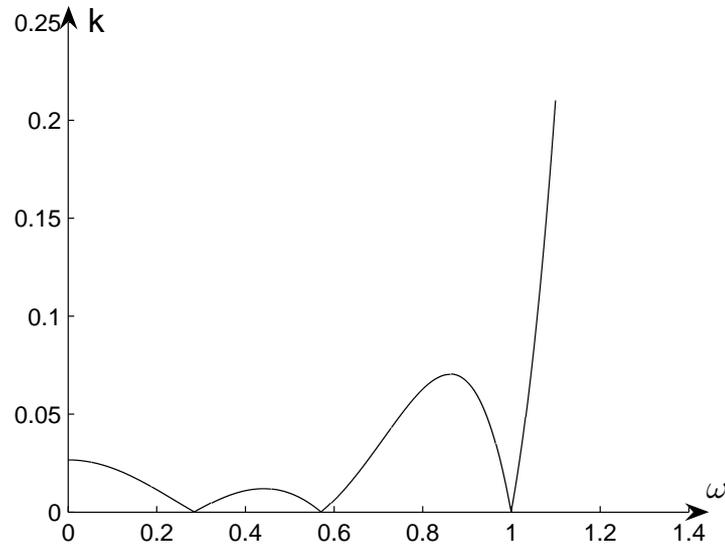
Figure 6: The dependence of the gain k as a function of ω for some positive frequencies

Acknowledgements

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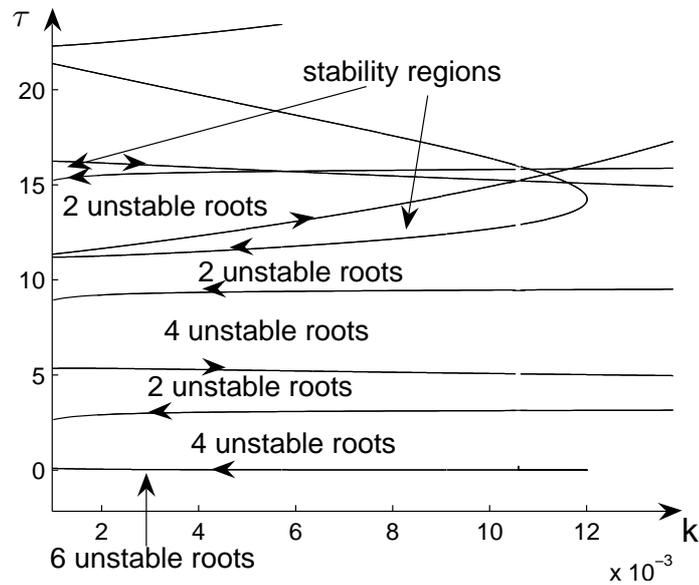
Figure 7: Stability crossing curves for the system given by (31)

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