Discrete-Time Conewise Linear Systems with Finitely Many Switches

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***Abstract*—We investigate discrete-time conewise linear systems (CLS) for which all the solutions exhibit a finite number of switches. By switches, we mean transitions of a solution from one cone to another. Our interest in this class of CLS comes from the optimization-based control of an insulin infusion model, for which the fact that solutions switch finitely many times appears to be key to establish the global exponential stability of the origin. The stability analysis of this class of CLS greatly simplifies compared to general CLS, as all solutions eventually exhibit linear dynamics. The main challenge is to characterize CLS satisfying this finite number of switches property. We first present general conditions in terms of set intersections for this purpose. To ease the testing of these conditions, we translate them as a non-negativity test of linear forms using Farkas lemma. As a result, the problem reduces to verify the non-negativity of a single solution to an auxiliary linear discrete-time system. Interestingly, this property differs from the classical non-negativity problem, where any solution to a system must remain non-negative (component-wise) for any non-negative initial condition, and thus requires novel tools to test it. We finally illustrate the relevance of the presented results on the optimal insulin infusion problem.**

***Index Terms*—Conewise linear systems, Lyapunov stability, optimization-based control, insulin infusion, Farkas lemma.**

I. Introduction

Conewise linear systems (CLS) are dynamical systems for which the state space is partitioned into a finite number of non-overlapping polyhedral cones [1]–[3]. The dynamics within each cone is governed by a linear time-invariant dynamical system called a mode. These systems pose significant challenges due to their piecewise linear nature. In particular, it has been shown in [4] that the stability analysis of CLS is a NP-hard problem. Hence, algorithms for deciding stability of CLS are inherently inefficient. We can mention the converse Lyapunov results in [3], which lead to necessary and sufficient stability conditions. The results in [3] state that the origin of a CLS is globally exponentially stable (GES) if and only if it admits a conewise linear Lyapunov function, whose associated conic partition does not coincide with the original system partition in general. As a consequence, the method proposed in [3] may be undecidable or computationally intractable. We can also mention that CLS are known to be equivalent to a class of linear complementarity systems [1] for which cone-copositive Lyapunov functions may be synthesized, as done recently in [5] in continuous-time. The advantage is the derivation of converse results with polynomial approximations but, again, there is no guarantee of computational tractability in general. To alleviate the computational obstruction of the CLS stability analysis, an alternative approach is to exploit additional properties for classes of CLS. This is the approach pursued in this work, where we focus on discrete-time CLS, whose solutions all exhibit a finite number of switches. By switches, we mean the transition of a solution from one cone to another.

Our motivation to study CLS with finitely many switches comes from the application of the optimization-based control approach of [6], namely quadratic control-Lyapunov policy (QCLP), for optimal insulin infusion in [7]. The primary objective is to minimize peak blood glucose level (BGL) caused by a food impulse, while adhering to the constraint that insulin flow must be positive [7]. Under the assumption that the response of the meal lasts longer than the response of an insulin impulse, as observed in specific situations (e.g., low glycemic index or high-fat/protein meals), it is proved in [7] that the optimal infusion policy is given by an open-loop policy combining an insulin bolus (applied with the meal) and a specific form of decaying insulin flow thereafter. This strategy exhibits the shortcomings of being open-loop and of requiring the perfect knowledge of the meals. This justifies the need for alternative efficient, suboptimal, closed-loop feedback policies. It appears that the problem can be formalized as a Linear Quadratic Regulator (LQR) problem with positive control inputs, as we show. This constrained optimal LQR problem arises in numerous real-world systems and leads to major methodological challenges [8]–[12]. Necessary and sufficient optimality conditions for the continuous-time case are available in [9]. Nevertheless, the associated numerical algorithm suffers from the curse of dimensionality, approximations are thus required to determine the stationary infinite horizon optimal feedback, moreover and importantly, no stability guarantees are provided. This is the reason why we opted for QCLP [11] instead, which is easy to implement and shows remarkable performances in simulations when applied to the model in [7]; see Figure 1 in Section II. We show that the obtained closed-loop system can be modelled as a CLS but ensuring its stability remains challenging. Indeed, despite our attempts to apply existing methods for stability analysis of CLS [3] or optimization-based control systems, such as [13] and its extensions [14]–[16], as well as sum-of-squares-based methods like [17] or the discrete-time counterpart of [5], we were unable to obtain a stability certificate for the CLS under consideration.

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We realized that the above-mentioned motivating example satisfies a distinctive feature: the number of switches of any solution to the closed-loop system is uniformly bounded. This property is extremely useful to investigate stability, as any solution is eventually given by a linear dynamical system. As, for this example, each mode has a Schur state matrix, the global exponential stability of the origin can then be established. Our goal in this work is to formalize and generalize these findings.

We start by assuming that a given general discrete-time CLS is such that all its solutions exhibit a finite number of switches, and we give a necessary and sufficient condition for the origin to be globally exponentially stable. This stability property relies on the assumption that solutions switch finitely many times, which is non-robust to exogenous disturbances a priori. We might thus deduce that the ensured stability property is not robust. We prove that this is not the case by establishing that, for a CLS, global exponential stability of the origin implies exponential input-to-state stability with respect to a general class of additive disturbances. Afterwards, we focus on the main challenge of this work that is to derive conditions under which all solutions to a CLS exhibit a finite number of switches. We are not aware of such results in the literature. We can mention e.g., [18], which deals with linear complementarity systems with continuous-time dynamics and not discrete-time dynamics as we do, and ensures finitely many switches for any solution on finite time intervals only, while we are seeking for results over the whole domain of the solutions. There is therefore a need for novel methodological tools that allows establishing that any solution to given discrete-time CLS switch a given maximum number of times.

In this context, we present general conditions in terms of set intersections. To ease the testing of these conditions, we derive alternative, tractable conditions, which boil down to verifying whether a specific solution to an auxiliary discrete-time system is non-negative thanks to the use of Farkas lemma. These conditions are derived for the case of a partition of the state space made of two cones, only to avoid over complicating the used notation. It is interesting to note that the required non-negativity property differs from the abundant literature on positive systems e.g., [19], which concentrate on systems for which *all* solutions take non-negative values (component-wise). In our case, a single solution has to be non-negative, we therefore present tailored conditions for this purpose, which have their own interest and which are successfully applied to the optimal insulin infusion problem.

The rest of the paper is organized as follows. The motivating example inspired by [7] and the problem statement are presented in Section II. Stability results are established in Section III assuming all the solutions of the considered CLS have a finite number of switches. Section IV provides conditions under which solutions to a CLS exhibit no more than a given maximum number of switches. Section V is dedicated to the non-negativity analysis of the auxiliary system derived from Farkas Lemma. The results are finally applied to examples, including the insulin infusion problem, in Section VI. Section VII concludes the paper. Some results are postponed to the appendix to avoid breaking the flow of exposition.

Notations. $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ the set of non-negative real numbers, $\mathbb{N}$ the set of non-negative integers, $\mathbb{N}^\star := \mathbb{N}\backslash\{0\}$, and $\mathbb{C}$ the set of complex numbers. For real matrices or vectors $(^\top)$ indicates transpose. The identity matrix of the considered set of matrices is denoted $\mathbf{I}$. For any symmetric matrix $X > 0$ ($X \geqslant 0$) means that X is positive (semi-)definite. Given $a_1,\dots,a_n \in \mathbb{R}$ with $n \in \mathbb{N}^\star$, we use $\mathrm{diag}(a_1,\dots,a_n)$ to denote the diagonal matrix, whose diagonal components are $(a_1,\dots,a_n)$. For any real square matrix M, $\sigma(M)$ denotes its spectrum. For any vectors v, we write $v \geqslant 0$ when all its entries are non-negative. The notation $\lfloor s \rfloor$ for $s \in \mathbb{R}$ stands for the integer part of s, and recall that $s-1 \leqslant \lfloor s \rfloor$. The interior of a set S is denoted $\mathrm{int}(S)$, its closure $\mathrm{cl}(S)$ and we use $\mathbb{B}(0,r)$ for the closed ball centered at the origin of radius $r > 0$ of the considered Euclidean space. Also, $\mathrm{card}(S)$ is the cardinal of the set S. Given $n \in \mathbb{N}^\star$, we say that set $\mathcal{C} \subset \mathbb{R}^n$ is a closed convex cone of $\mathbb{R}^n$ if $\mathcal{C}$ is closed and, for any $x, y \in \mathcal{C}$ and any $a, b \in \mathbb{R}_+$, $ax + by \in \mathcal{C}$. Given two sets A and B, A^B stands for the set of functions defined from B to A.

II. Motivation and problem formulation

We first focus on the LQR problem with scalar positive inputs (Section II-A), which covers the optimal insulin infusion problem presented in more details afterwards (Section II-B). After having illustrated the potential of QCLP for the near-optimal insulin infusion, we formalize the stability analysis of the obtained closed-loop system as the stability problem of a CLS (Section II-C). There, we also discuss possible, but unfortunately unsuccessful, Lyapunov-based approaches to establish stability properties for the considered example thereby motivating the problem statement (Section II-D).

A. LQR with positive inputs

The envisioned optimal insulin infusion problem is modeled as a LQR problem with positive inputs. We thus consider the deterministic discrete-time linear system

$$x_{t+1} = Ax_t + Bu_t, \tag{1}$$

where $x_t = (x_t^1,\dots,x_t^n)^\top \in \mathbb{R}^n$ is the state and $u_t \in \mathbb{R}_+$ is the non-negative scalar control input at time $t \in \mathbb{N}$, with $n \in \mathbb{N}^\star$. We denote the solution to (1) initialized at state $x \in \mathbb{R}^n$ with input sequence $\mathbf{u} \in \mathbb{R}_+^{\mathbb{N}}$ at time $t \in \mathbb{N}$ as $\phi(t,x,\mathbf{u})$ and $\phi(0,x,\cdot) = x$. The cost function is given, for any $x \in \mathbb{R}^n$ and infinite-length sequence of non-negative inputs $\mathbf{u} = (u_0,u_1,\dots) \in \mathbb{R}_+^{\mathbb{N}}$, by

$$J(x,\mathbf{u}) \quad := \quad \sum_{t=0}^{\infty} \ell(\phi(t,x,\mathbf{u}),u_t), \tag{2}$$

where $\ell(z,v) := z^\top Qz + 2z^\top Sv + Rv^2$ for any $(z,v) \in \mathbb{R}^n \times \mathbb{R}_+$, $Q \in \mathbb{R}^{n\times n}$ such that $Q = Q^\top \geqslant 0$, $R \in (0,\infty)$, $S \in \mathbb{R}^n$ and $\begin{bmatrix} Q & S \\ S^\top & R \end{bmatrix} \geqslant 0$. We know from Bellman equation that the optimal value function associated with (2), namely $V^\star(x) := \min_{\mathbf{u}\in\mathbb{R}_+^{\mathbb{N}}} J(x,\mathbf{u})$ satisfies, for any $x \in \mathbb{R}^n$,

$$V^\star(x) := \min_{u\in\mathbb{R}_+} \Big(\ell(x,u) + V^\star(Ax+Bu)\Big) \tag{3}$$

based on which the optimal feedback policy is given by

$$g^\star(x) = \min_{u\in\mathbb{R}_+} \Big(\ell(x,u) + V^\star(Ax+Bu) \Big). \tag{4}$$

Hence to construct the optimal policy $g^\star$, we need to know $V^\star$, which is very challenging in general because of the constraint that u has to be non-negative; see [9] for results in the continuous-time case. Consequently, we exploit an alternative suboptimal policy proposed in [11] known as QCLP. The idea is to construct the feedback policy using a known control Lyapunov function V_{clf} instead of $V^\star$ in (4). This leads to, for any $x \in \mathbb{R}^n$,

$$g(x) := \arg\min_{u\in\mathbb{R}_+} \Big(\ell(x,u) + V_{\text{clf}}(Ax+Bu) \Big). \tag{5}$$

In this work, and as proposed in [6], we define V_{clf} as the optimal value function associated with the *unconstrained* LQR problem in the sense that the input can take any value in $\mathbb{R}$, i.e., $V_{\text{clf}}(x) := x^\top P x$ for any $x \in \mathbb{R}^n$ where P is the unique, real, symmetric, positive definite solution to the Riccati equation $A^\top PA - P - (A^\top PB + S)(B^\top PB + R)^{-1}(A^\top PB + S)^\top + Q = 0$, which exists as long as the pair (A, B) is controllable.

In view of the expressions of ℓ and V_{clf}, (5) can be re-written as

$$\begin{aligned} g(x) &= \arg\min_{u\in\mathbb{R}_+} \left((R + B^\top PB)(u - Kx)^2 \right) \\ &= \max\{0, Kx\}, \end{aligned} \tag{6}$$

with $K = -(R + B^\top PB)^{-1}(B^\top PA + S^\top)$ the optimal gain for the unconstrained LQR problem. This equivalence follows from (5) and the equalities $K^\top(R+B^\top PB)K = A^\top PA - P + Q$ and $-(R + B^\top PB)K = B^\top PA + S^\top$.

B. Optimal insulin infusion

We apply the approach of Section II-A to the problem of optimal insulin infusion of [7] where the primary objective is to minimize the peak of the blood glucose level (BGL) resulting from a food impulse while ensuring that insulin flow remains non-negative. We reformulate the problem as a LQR problem with a control non-negativity constraint and apply the associated QCLP strategy (5). The continuous-time model of [7] was obtained from clinical trials. Here, we use a sampled-data version with a sampling period $T = 5$ min; see Appendix A for more details. As a result, the discrete-time state model is given by (1) with $n = 4$, and

$$\begin{aligned} A &= \begin{bmatrix} 0.8351 & -0.1150 & -0.0521 & 0 \\ 0.0716 & 0.9954 & -0.0021 & 0 \\ 0.0014 & 0.0390 & 1.0000 & 0 \\ -0.0082 & -0.3249 & -16.4423 & 0.9277 \end{bmatrix} \\ B &= \begin{bmatrix} 1.6702 \\ 0.1431 \\ 0.0029 \\ -0.0163 \end{bmatrix}. \end{aligned} \tag{7}$$

We have selected the weighting matrices Q, S and R as in Appendix A, which leads to

$$K = \begin{bmatrix} -0.4936 & -6.9988 & -104.7360 & 1.6626 \end{bmatrix}. \tag{8}$$

A solution to the corresponding closed-loop system with (6) is depicted in Figure 1 together with the solutions using the optimal open-loop strategy given in [7] and the zero-input strategy. As it can be seen in Figure 1, the control is effective in reducing the excursion due to an impulse of food ingested (60 g) compared to the open-loop behavior with zero input in black dash-dotted line. Our result shows a maximum BGL excursion of 2 mmol/L. The best feedback policy in [7, Section 7], for the corresponding continuous-time model (see Appendix A) has a maximum BGL excursion of 1.28 mmol/L. However, the associated controller assumes the BGL can be exactly differentiated, which is unfortunately unrealistic. To make the strategy realistic, the bandwidth of these differentiators has been restricted in [7], and the results show larger peaks (5.71 mmol/L).

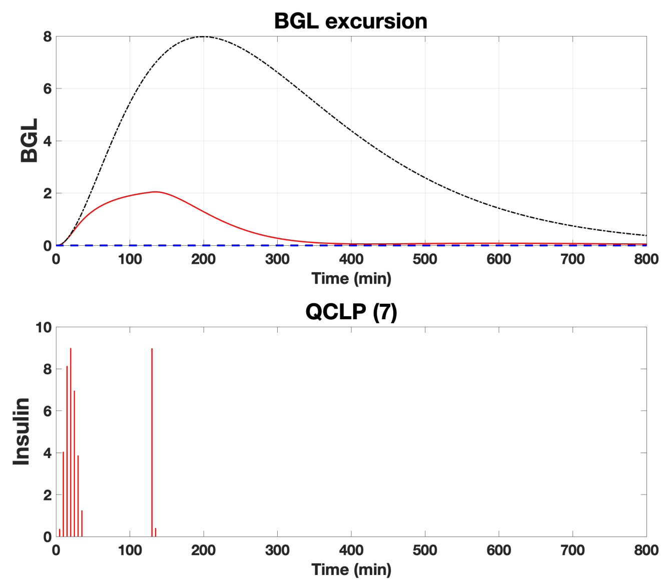


Fig. 1: Top: glucose evolution (open-loop with zero input in black dash-dotted line, closed-loop with K in red solid line, optimal open-loop of [7] in blue dashed line). Bottom: Insulin flow (closed-loop with K in (8)).

QCLP (6) is thus promising for this application. However, the stability of the corresponding closed-loop system is an open question, as we now explain.

C. Limitations of existing Lyapunov-based techniques

System (1) in closed-loop with (6) is a CLS given by

$$x_{t+1} = \begin{cases} A_1 x_t & x_t \in \mathcal{C}_1, \\ A_2 x_t & x_t \in \mathcal{C}_2, \end{cases} \tag{9}$$

with $A_1 = A + BK$ and $A_2 = A$ and

$$\begin{aligned} \mathcal{C}_1 &= \{x \in \mathbb{R}^n \ : \ Kx \geqslant 0\}, \\ \mathcal{C}_2 &= \{x \in \mathbb{R}^n \ : \ -Kx \geqslant 0\} = \text{cl}\left(\mathbb{R}^n \backslash \mathcal{C}_1\right), \end{aligned} \tag{10}$$

where $K \in \mathbb{R}^{1\times n}$ is the gain associated with the optimal unconstrained LQR problem. It appears that certifying the stability of the origin for system (9) is challenging [4]; even when A_1 and A_2 are Schur as in Section II-B.

Interestingly, this stability problem can be formulated in other frameworks such as optimization-based systems due to (5), piecewise linear systems to which system (9) belongs to, or linear complementarity systems[1] (LCS) a well-established class of nonsmooth systems [20]. This connection is significant, as it situates our problem within a broader theoretical framework. However, while the LCS theory provides a useful perspective, solving the problem at hand within this framework is nontrivial and cannot be fully addressed by existing methods in the literature as far as we know. This opens an interesting avenue for future research, requiring the development of new techniques to handle the specific challenges posed by the considered system.

To elaborate more on the fact that none of the existing Lyapunov-based stability tools of the literature certifies the stability of the system in Section II-B, we introduce below a novel comparison between three distinct classes of Lyapunov function candidates commonly encountered in the above-mentioned fields. The first one is found in the optimization-based framework [13], [21], [22], and corresponds to the value function associated with (5), i.e., for any $x \in \mathbb{R}^n$,

$$V(x) := \min_{u \in \mathbb{R}_+} \Big(x^\top Q x + 2x^\top S u + u^\top R u + (Ax+Bu)^\top P (Ax+Bu) \Big). \tag{11}$$

The second Lyapunov function candidate is the quadratic form inherited from the LCS literature [23], i.e., for any $x \in \mathbb{R}^{n_x}$ and $u = g(x)$ as in (5),

$$V_{\text{quad}}(x) := \begin{bmatrix} x^\top & u^\top \end{bmatrix} \underbrace{\begin{bmatrix} X_1 & X_2 \\ X_2^\top & X_3 \end{bmatrix}}_{\mathcal{X}} \begin{bmatrix} x \\ u \end{bmatrix} \tag{12}$$

where $X_1 = X_1^\top \in \mathbb{R}^{n\times n}$, $X_2 \in \mathbb{R}^{n\times m}$, and $X_3 = X_3^\top \in \mathbb{R}^{m\times m}$ and $\mathcal{X}$ strictly copositive [23]. The third one is the piecewise quadratic function, as in [24], that is for any $x \in \mathbb{R}^n$,

$$V_{\text{PWQ}}(x) =: \begin{cases} x^\top P_1 x, & \text{if } x \in \mathcal{C}_1, \\ x^\top P_2 x, & \text{if } x \in \mathcal{C}_2, \end{cases} \tag{13}$$

where $P_1 = P_1^\top \in \mathbb{R}^{n\times n}$ and $P_2 = P_2^\top \in \mathbb{R}^{n\times n}$. This function is also a conewise quadratic function with the same partition as (9).

The relationship between these three Lyapunov function candidates unfolds as follows. Function V is a particular instance of V_{quad}. Specifically, if we set, in (12), $X_1 = A^\top P A + Q$, $X_2 = A^\top P B + S$, and $X_3 = B^\top P B + R$, then we obtain $V_{\text{quad}} = V$. Function V_{PWQ} is the most general form, encompassing both (11) and (12), as it accommodates V_{quad} (which is more general than V) by setting $P_1 = X_1 + K^\top X_2^\top + X_2 K + K^\top X_3 K$ and $P_2 = X_1$, leading to $V_{\text{PWQ}} = V_{\text{quad}}$. Having identified the most general Lyapunov function candidate among these three options for system (9), LMI conditions under which the stability of the origin of (9) would be guaranteed using (13) are provided in Appendix B. Regrettably, these conditions are infeasible for the system discussed in Section II-B. Furthermore, all our attempts to ensure the stability of this system based on other existing tools from the literature [3], [5], [13]–[16] failed.

Now, for all the tested values of Q, R and S in (2), all the solutions of the corresponding system (9) have the distinctive feature to switch a maximum of 4 times, which, once analytically established, allows concluding on the stability of the origin for the considered system, see Section VI-B. In the following, we formalize these findings for general CLS.

[1]This can be seen by noticing that the necessary and sufficient optimality Karush-Kuhn-Tucker (KKT) conditions of the quadratic problem (5) write $(R + B^\top PB)v - (R + B^\top PB)Kx_t - \lambda = 0$, $v \geqslant 0$, $\lambda \geqslant 0$, $\lambda v = 0$, where $\lambda \in \mathbb{R}$ is a Lagrange multiplier. As a result, the closed-loop system (1)-(5) can also be written as a discrete-time LCS $x_{t+1} = Ax_t + Bu_t$, $y_t = Cx_t + Du_t$, $0 \leqslant u_t \perp y_t \geqslant 0$, where $C = -(R + B^\top PB)K$ and $D = R + B^\top PB$.

D. Problem statement

Motivated by the above developments, we consider in this paper general discrete-time CLS of the form

$$x_{t+1} = A_i x_t \qquad x_t \in \mathcal{C}_i, \quad i \in \{1, \ldots, m\}, \tag{14}$$

where $m \in \mathbb{N}^\star$ is the number of non-empty closed convex cones $\mathcal{C}_i$, $i \in \{1, \ldots, m\}$, which partition the state space $\mathbb{R}^n$, i.e.,

- $\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_m = \mathbb{R}^n$,
- $\text{int}(\mathcal{C}_i) \cap \text{int}(\mathcal{C}_j) = \varnothing$ for all $i \neq j \in \{1, \ldots, m\}$.

Assumption 1. *$A_i \xi = A_j \xi$ for any vector $\xi \in \mathcal{C}_i \cap \mathcal{C}_j$.*

As in [3], the continuity condition of Assumption 1 ensures consistent behavior across the switching surfaces (the cone boundaries) and guarantees the uniqueness of solutions for system (14) for any initial condition. This assumption is crucial because, in discrete-time systems, it is possible for trajectories to converge in finite time towards sliding surfaces, as discussed in [25]. The continuity assumption we make here prevents the occurrence of sliding modes and attractive surfaces, which would otherwise complicate the system dynamics.

Assumption 2. *Matrices $A_1, \ldots, A_m$ are are invertible.*

The full-rank condition imposed by Assumption 2 is quite classical for discrete-time systems. In our study, this assumption will allow us to avoid additional technical difficulties.

For the sake of convenience and with some slight abuse of notation, we use ϕ to denote solutions to (14), so that for initial condition $x \in \mathbb{R}^n$, the corresponding solution to (14) at time $t \in \mathbb{N}$ is denoted $\phi(t,x)$. We refer to the dynamics of (14) for a given $i \in \{1, \ldots, m\}$ as a mode. We next formalize what is meant by a switch and the maximum number of switches of a solution to (14). Intuitively, one would say that, given a solution to (14), a switching time occurs when the solution leaves a set $\mathcal{C}_i$. However, since our sets are not strictly disjoint, this simple definition is not suitable. For instance, with $m = 3$, if we have a sequence such that $x \in \mathcal{C}_1 \backslash (\mathcal{C}_2 \cup \mathcal{C}_3)$, $\phi(1,x) \in (\mathcal{C}_1 \cap \mathcal{C}_2) \backslash \mathcal{C}_3$, $\phi(2,x) \in \mathcal{C}_1 \cap \mathcal{C}_2 \cap \mathcal{C}_3$, $\phi(3,x) \in (\mathcal{C}_2 \cap \mathcal{C}_3) \backslash \mathcal{C}_1$ and $\phi(t,x) \in \mathcal{C}_3 \backslash (\mathcal{C}_1 \cup \mathcal{C}_2)$ for every $t \geqslant 4$, we would say that switching times are 3 and 4, since $\mathcal{C}_1$ (respectively $\mathcal{C}_2$) is left at time $t = 3$ (respectively $t = 4$). However, for this particular case, the *correct* switching time is $t = 3$, since $\phi(t,x) \in \mathcal{C}_1$ for $t \in \{0,1,2\}$ and $\phi(t,x) \in \mathcal{C}_3$ for $t \geqslant 3$. Thus, in order to define the switching times and the number of switches, we need to explore all possible sequences of modes. To this end, in the spitit of [3, Section 5], for every $x \in \mathbb{R}^n$, we define the possible sequences of modes

$$\mathcal{I}(x) = \left\{ (i_t)_{t\in\mathbb{N}} \in \{1,\dots,m\}^{\mathbb{N}} \;:\; \forall t \in \mathbb{N},\; \phi(t,x) \in \mathcal{C}_{i_t} \right\}.$$

Let us now define for $(i_t)_{t\in\mathbb{N}} \in \{1,\dots,m\}^{\mathbb{N}}$ the jumping times:

$$\mathcal{T}\left((i_t)_t\right) = \{t \in \mathbb{N}^* \;:\; i_t - i_{t-1} \neq 0\} \subset \mathbb{N}^*$$

and the number of jumps

$$\jmath\left((i_t)_t\right) = \mathrm{card}\left(\mathcal{T}\left(i_t\right)_t\right)\right) \in \mathbb{N} \cup \{\infty\}.$$

We are now in position to define the number of switches and switching times associated with a solution of (14).

Definition 1 (Switching times and number of switches)**.** *Given an initial condition $x \in \mathbb{R}^n$, the number of switches of the solution $\phi(\cdot,x)$ to* (14) *is given by*

$$\varsigma(x) = \min\left\{\jmath\left((i_t)_{t\in\mathbb{N}}\right) \;:\; (i_t)_{t\in\mathbb{N}} \in \mathcal{I}(x)\right\} \in \mathbb{N} \cup \{\infty\}.$$

Corresponding switching times are the elements of $\mathcal{T}\left((i_t^*)_t\right)$, where $(i_t^*)_t \in \mathcal{I}(x)$ is a sequence such that $\jmath\left((i_t^*)_t\right) = \varsigma(x)$.

Note that, in the above definition, $\varsigma(x)$ and $(i_t^*)_t$ are well-defined since we take the minimum of a non-negative function having integer values. In view of Definition 1, a solution $\phi(\cdot,x)$ for some $x \in \mathbb{R}^n$ exhibits a finite number of switches if and only if $\varsigma(x) < \infty$.

The main objective of this work is to derive conditions under which all solutions to system (14) exhibits a finite number of switches, see Section IV. This is motivated by the fact that the stability analysis simplifies for this class of CLS, as we show in the next section. In Section IV, to ensure that $\varsigma(x)$ is uniformly bounded with respect to $x \in \mathbb{R}^n \backslash \{0\}$, we will ensure that $\max\left\{\jmath\left((i_t)_{t\in\mathbb{N}}\right) \;:\; (i_t)_{t\in\mathbb{N}} \in \mathcal{I}(x)\right\}$ is uniformly bounded with respect to $x \in \mathbb{R}^n \backslash \{0\}$.

III. Stability results

We first provide a necessary and sufficient condition for $x = 0$ to be GES for system (14), again, assuming all its solutions switch a finite number of times; conditions to ensure this property are provided in Section IV. We then show that this stability property is robust, in the sense that an input-to-state stability property holds when (14) is perturbed by exogenous disturbances.

A. Global exponential stability

We define next the set $\mathcal{F}$, which characterizes the region of the state space where solutions to (14) stop switching,

$$\mathcal{F} := \left\{ x \in \mathbb{R}^n \;:\; \exists i \in \{1,\dots,m\},\, \forall t \in \mathbb{N}\; A_i^t x \in \mathcal{C}_i \right\}. \quad (15)$$

In other words, $\mathcal{F} = \{x \in \mathbb{R}^n \;:\; \varsigma(x) = 0\}$ with the notation of Definition 1. Obviously, $0 \in \mathcal{F}$ and $\mathcal{F}$ is forward invariant for system (14). The next theorem gives a necessary and sufficient condition for the origin of system (14) to be GES, i.e., there exist $c_1 \geqslant 1$ and $c_2 > 0$ such that for any $x \in \mathbb{R}^n$, $|\phi(t,x)| \leqslant c_1 e^{-c_2 t}|x|$ for any $t \in \mathbb{N}$, when we know that any solution exhibits a finite number of switches. Again, conditions to ensure the latter property are provided in Section IV.

Theorem 1. *Consider system (14) and suppose that any solution exhibits a finite number of switches, i.e., $\varsigma(x) < \infty$ for any $x \in \mathbb{R}^n$ with ς as in Definition 1. Then the origin is GES if and only if the origin of the restriction of (14) to $\mathcal{F}$, namely,*

$$x_{t+1} = A_i x_t \qquad x_0 \in \mathcal{C}_i \cap \mathcal{F}, \quad i \in \{1,\dots,m\}, \qquad (16)$$

is GES.

Proof: We first suppose the origin of (14) is GES. As $\mathcal{F}$ is forward invariant, it follows that the origin of (16) is GES.

Suppose now that the origin of (16) is GES. Let ϕ be a solution to (14) initialized at $x \in \mathbb{R}^n$. Since $\phi(\cdot,x)$ exhibits a finite number of switches over $\mathbb{N}$, for $t_0 = t_0(x)$ (the last switching time), we have $\phi(t_0,x) \in \mathcal{F}$. As a result, $\phi(t,x) \in \mathcal{F}$ for any $t \geqslant t_0$ as $\mathcal{F}$ is forward invariant. This implies that $\phi(t,x) \to 0$ as $t \to \infty$ as the origin is GES for system (16). Since x has been arbitrarily chosen, we have proved that any solution to (14) asymptotically converges to the origin, i.e., the origin is globally attractive for system (14). As item (i) of Theorem 6 in Appendix C holds, it yields that the origin is GES for system (14). ■

To apply Theorem 6 we use two ingredients: that the origin of (16) is GES, and that any solution enters in $\mathcal{F}$. This last condition is ensured by assuming that any solution to (14) exhibits a finite number of switches. We note that $\mathcal{C}_i \cap \mathcal{F}$, which appears in (16), is the largest forward invariant set within cone $\mathcal{C}_i$ for the dynamics $x_{t+1} = A_i x_t$. When we know in which cone(s) the solutions eventually enter and remain for all future times, this boils down to investigating the spectrum of the state matrices associated with these cones. When all the matrices A_i, $i \in \{1,\dots,m\}$, are Schur as in the examples of Section VI, the fact that the origin of (16) is GES directly follows as solutions to (16) exhibit no switches. Now ensuring the condition that all solutions exhibit a finite number of switches, requires the development of novel tools, which are presented in Section IV.

Remark 1. *A necessary condition for the origin of system (16) to be GES is that for every $i \in \{1,\dots,m\}$, if $\lambda \in \sigma(A_i) \cap \mathbb{R}$ is such that there exists an eigenvector $v \in \mathcal{C}_i$ of A_i associated with λ, then $\lambda \in [0,1)$. This comes from the fact that, with these conditions, the solution of* (14) *initialized at v, $\phi(t,v) = \lambda^t v$, belongs to $\mathcal{C}_i$ for every $t \in \mathbb{N}$. In particular, we have $\mathbb{R}_+ v \subseteq \mathcal{F}$.*

B. Exponential input-to-state stability

The property assumed in Theorem 1 that any solution to (14) switches a finite number of times is non-robust a priori, in the sense that an arbitrarily small exogenous disturbance may destroy it. It is therefore essential to analyze the robustness of the stability result in Theorem 1. This is formalized in the next theorem, which we could not find in the literature, although similar statements have been established for discrete-time systems but with different homogeneity degrees, see [26, Theorem 10], or for general continuous-time homogeneous systems, see e.g. [27].

We consider for this purpose system (14) perturbed by exogenous disturbance as follows

$$x_{t+1} \in \{A_i x_t + w_t \,:\, i \in \{1,\ldots,m\},\ x_t \in \mathcal{C}_i,\ w_t \in E(x_t)v_t\}, \tag{17}$$

where $v_t \in \mathbb{R}^{n_v}$, with $n_v \in \mathbb{N}^\star$, is the disturbance at time t and E is a set-valued map from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n_v}$ such that: (i) $E(\lambda x) = E(x)$ for any $x \in \mathbb{R}^n$ and $\lambda > 0$; (ii) there exists $m_E \geqslant 0$ such that for any $x \in \mathbb{R}^n$ and $z \in E(x)$, $|z| \leqslant m_E$. Set-valued map E covers as a special case the situation where $E(x) = \{E_i \,:\, i \in \{1,\ldots,m\},\ x \in \mathcal{C}_i\}$ for some constant matrices E_i. On the other hand, matrices A_i and sets $\mathcal{C}_i$, $i \in \{1,\ldots,m\}$, are as in (14). System (17) is a difference inclusion, as its right-hand side is set-valued. For the sake of convenience and like in Section II-A, we denote a solution to (17) initialized at $x \in \mathbb{R}^n$ with disturbance sequence $\mathbf{v} = (v_0, v_1, \ldots) \in (\mathbb{R}^{n_v})^{\mathbb{N}}$ at time $t \in \mathbb{N}$ as $\phi(t,x,\mathbf{v})$. We have the next robustness result.

Theorem 2. *Suppose the origin is GES for system (14), then system (17) is exponentially input-to-state stable, in particular there exist $c_1 \geqslant 1$, $c_2 > 0$ and $c_3 \geqslant 0$ such that for any $x \in \mathbb{R}^n$ and any $\mathbf{v} \in (\mathbb{R}^{n_v})^{\mathbb{N}}$, any solution ϕ satisfies $|\phi(t,x,\mathbf{v})| \leqslant c_1 e^{-c_2 t}|x| + c_3 \sup_{t' \in \{1,\ldots,t\}} |v_{t'}|$.*

Proof: We first apply [28, Theorem 2] to obtain a suitable homogeneous Lyapunov function for system (14). With the notation of [28], we take $\omega(\cdot) = |\cdot|$ and $\mathcal{D} = \mathbb{R}^n$. Then, Assumption 1 in [28] holds by [29, Theorems 6.30 and 7.21], as $x = 0$ is GES for system (14) and the vector field in (14) is continuous[2]. Assumption 2 in [28] also holds, as the vector field in (14) is homogeneous of degree 0. The last condition to check is [28, Assumption 3], which is verified with $G_\lambda = \lambda\mathbf{I}$ and $d = 1$. Consequently, by [28, Theorem 2], there exist $V : \mathbb{R}^n \to \mathbb{R}_+$ continuous on $\mathbb{R}^n$, smooth on $\mathbb{R}^n\backslash\{0\}$, such that $V(\lambda x) = \lambda V(x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$, as well as $\underline{\alpha}, \overline{\alpha} > 0$ and $\mu \in (0,1)$ such that, for any $x \in \mathbb{R}^n$ and any $i \in \{1,\ldots,m\}$ such that $x \in \mathcal{C}_i$,

$$\begin{aligned} \underline{\alpha}|x| \leqslant \ \ V(x) \ \ &\leqslant \overline{\alpha}|x| \\ V(A_i x) \leqslant \mu V(x). & \end{aligned} \tag{18}$$

Let $(x,v) \in \mathbb{R}^n \times \mathbb{R}^{n_v}$, $w \in E(x)v$, and any $i \in \{1,\ldots,m\}$ such that $x \in \mathcal{C}_i$. In view of (18) and Lemma 1 stated below, there exists $L \geqslant 0$ (independent of x, v and thus w) such that

$$\begin{aligned} V(A_i x + w) &\leqslant V(A_i x) + |V(A_i x + w) - V(A_i x)| \\ &\leqslant \mu V(x) + L|w| \leqslant \mu V(x) + L m_E |v|. \end{aligned} \tag{19}$$

Let $\mathbf{v} \in (\mathbb{R}^{n_v})^{\mathbb{N}}$, we derive from (19) that for any $t \in \mathbb{N}$, any solution ϕ to (17) satisfies

$$V(\phi(t,x,\mathbf{v})) \leqslant \mu^t V(x) + \frac{L m_E}{1-\mu} \sup_{t' \in \{1,\ldots,t\}} |v_{t'}|.$$

We deduce the desired property with $c_1 = \overline{\alpha}/\underline{\alpha}$, $c_2 = -\ln(\mu)$ and $c_3 = L m_E/(\underline{\alpha}(1-\mu))$ by invoking the first line in (18). ■

We have used the next lemma in the proof of Theorem 2.

[2] Recall that $A_i x = A_j x$ for any $x \in \mathcal{C}_i \cap \mathcal{C}_j$.

Lemma 1. *Let $V : \mathbb{R}^n \to \mathbb{R}$ be such that $V(\lambda x) = \lambda V(x)$ for every $\lambda > 0$ and every $x \in \mathbb{R}^n$. V is globally Lipschitz if and only if V is Lipschitz on the unit sphere of $\mathbb{R}^n$.*

A consequence of Lemma 1 is that, if V is C^1 on $\mathbb{R}^n\backslash\{0\}$, then it is globally Lipschitz on $\mathbb{R}^n$.

Proof: We set $D = \mathbb{B}(0,1)$ and $\partial D = D\backslash \mathrm{int}(\mathbb{B}(0,1))$, the unit sphere of $\mathbb{R}^n$. Obviously, if V is Lipschitz on $\mathbb{R}^n$, then V is Lipschitz on ∂D. Reciprocally, we assume that V is Lipschitz on ∂D. It is enough to prove that V is Lipschitz on D. In fact, by homogeneity, if V is Lipschitz on D, then V is (globally) Lipschitz on $\mathbb{R}^n$. Assume by contradiction that V is not Lipschitz on D. Then, for every $k \in \mathbb{N}$, there exists $x_k, y_k \in D$ such that $|V(x_k) - V(y_k)| > k|x_k - y_k|$. Obviously, we have $x_k \neq y_k$, and without loss of generality, we can assume that $|x_k| > |y_k|$. Using the homogeneity of V, we can also assume without loss of generality that $x_k \in \partial D$. In addition, since $k|x_k - y_k| < |V(x_k) - V(y_k)| \leqslant 2\max_{\xi \in \partial D}|V(\xi)|$, we get that $\lim_{k\to\infty}|y_k| = 1$, and in particular, for k large enough, we have $|y_k| \neq 0$. We then define $z_k = y_k/|y_k| \in \partial D$, and we have

$$\begin{aligned} |V(x_k) - V(z_k)| &\geqslant |V(x_k) - V(y_k)| - |V(y_k) - V(z_k)| \\ &= |V(x_k) - V(y_k)| - (1 - |y_k|)|V(z_k)| \\ &> k|x_k - y_k| - (1 - |y_k|)\max_{\partial D}|V|. \end{aligned}$$

Since V is Lipschitz on ∂D, there exists $L \in \mathbb{R}_+$ such that $|V(x_k) - V(z_k)| \leqslant L|x_k - z_k| \leqslant L\left(|x_k - y_k| + (1 - |y_k|)\right)$. We thus have,

$$k|x_k - y_k| < L|x_k - y_k| + \left(L + \max_{\partial D}|V|\right)(1 - |y_k|).$$

But, $1 - |y_k| = |x_k| - |y_k| \leqslant |x_k - y_k|$, and finally, for every $k \in \mathbb{N}$, we should have $k < 2L + \max_{\partial D}|V|$ (recall that $x_k \neq y_k$) which is a contradiction. ■

IV. CLS WITH A FINITE NUMBER OF SWITCHES

The objective of this section is to provide conditions under which all the solutions to (14) exhibit a finite number of switches, as required by Theorem 1. We will actually focus on a stronger property, that is that there exists $p \in \mathbb{N}$ such that $\varsigma(x) \leqslant p$ for any $x \in \mathbb{R}^n$, where we recall that $\varsigma(x)$ is the number of switches exhibited by the solution of (14) initialized at x, see Definition 1. To this end, we first give a sufficient condition based on the emptiness of the intersection of a finite number of sets, see Section IV-A. To ease the testing of this condition, we translate it in Section V in terms of checking the non-negativity of linear forms, by exploiting Farkas lemma. As a result, the problem reduces to verify the non-negativity of a single solution to an auxiliary linear discrete-time system.

In Section IV-A, we consider the general case $m \geqslant 2$, while for the sake of clarity and readability only, we focus on the case $m = 2$ in Section IV-B. Note however that the methodology described in Section IV-B is easily extendable to the general case.

A. A sufficient condition

Given $p > 0$ time instants $t_1, \dots, t_p > 0$ and $t \geqslant 0$, and given indexes $i_1, \dots, i_{p+1} \in \{1, \dots, m\}$ with $i_{k+1} \neq i_k$ for $k \in \{1, \dots, p\}$, we define the set $\Sigma_{\substack{t_1,\dots,t_p,t\\ i_1,\dots,i_p,i_{p+1}}}$ of initial conditions x_0 such that the solution to (14) satisfies $x_s = A_{i_1}^s x_0 \in \mathcal{C}_{i_1}$ for $0 \leqslant s < t_1$, $x_{t_1+s} = A_{i_2}^s A_{i_1}^{t_1} x_0 \in \mathcal{C}_{i_2}$ for $0 \leqslant s < t_2$, etc, and finally $x_{t_1+\dots+t_p+s} = A_{i_{p+1}}^s A_{i_p}^{t_p} \dots A_{i_1}^{t_1} x_0 \in \mathcal{C}_{i_{p+1}}$ for $0 \leqslant s \leqslant t$. The first step is to give a condition under which any solution to (14) initialized in $\Sigma_{\substack{t_1,\dots,t_p,t\\ i_1,\dots,i_p,i_{p+1}}}$ admits no more than p switches. In other words, using notations of Section II-D, we will give conditions ensuring that if $(k_t)_{t\in\mathbb{N}} \in \mathcal{I}(x_0)$ is such that $\{t_1, t_1+t_2, \dots, t_1+\dots+t_p\} \subset \mathcal{T}((k_t)_t) \cap \{1, \dots, t_1+\dots+t_p\}$ and $k_{t_1-1} = i_1, \dots, k_{t_1+\dots+t_p-1} = i_p$ and $k_{t_1+\dots+t_p+s} = i_{p+1}$ for every $s \in \{0, \dots, t\}$, then we have $\jmath((k_t)_t) = p$. This in particular ensures that $\sigma(x_0) \leqslant p$, according to Definition 1.

Writing down the definition of $\Sigma_{\substack{t_1,\dots,t_p,t\\ i_1,\dots,i_p,i_{p+1}}}$, we obtain that this set is given by the following intersections

$$\Sigma_{\substack{t_1,\dots,t_p,t\\ i_1,\dots,i_p,i_{p+1}}} = \bigcap_{\tau=0}^{t} \left(S_{\substack{t_1,\dots,t_p,\tau\\ i_1,\dots,i_p,i_{p+1}}} \right) \cap \left(\bigcap_{k=1}^{p} \bigcap_{\tau=0}^{t_k-1} S_{\substack{t_1,\dots,t_{k-1},\tau\\ i_1,\dots,i_{k-1},i_k}} \right), \tag{20}$$

where, for every $k \in \mathbb{N}$, $t_1, \cdots, t_k \in \mathbb{N}^*$, $\tau \in \mathbb{N}$ and $i_1, \cdots, i_k, i \in \{1, \dots, m\}$ we have set

$$S_{\substack{t_1,\dots,t_k,\tau\\ i_1,\dots,i_k,i}} := \left\{ x_0 \in \mathbb{R}^n \ : \ A_i^\tau A_{i_k}^{t_k} \dots A_{i_1}^{t_1} x_0 \in \mathcal{C}_i \right\}. \tag{21}$$

For notation convenience, in (20), for $k = 1$ (resp. $k = 2$), $S_{\substack{t_1,\dots,t_{k-1},\tau\\ i_1,\dots,i_{k-1},i_k}}$ is identified with $S_{\substack{\tau\\ i_1}} := \left\{ x_0 \in \mathbb{R}^n \ : \ A_{i_1}^\tau x_0 \in \mathcal{C}_{i_1} \right\}$ (resp. $S_{\substack{t_1,\tau\\ i_1,i_2}} = \left\{ x_0 \in \mathbb{R}^n \ : \ A_{i_2}^\tau A_{i_1}^{t_1} x_0 \in \mathcal{C}_{i_2} \right\}$).

To give a better intuition on the meaning of the sets $\Sigma_{\substack{t_1,\dots,t_p,t\\ i_1,\dots,i_p,i_{p+1}}}$ and $S_{\substack{t_1,\dots,t_k,\tau\\ i_1,\dots,i_k,i}}$, we consider a solution of (14) that is initialized at some $x_0 \in \mathcal{C}_1$, stays in $\mathcal{C}_1$ for times $t \in \{0, 1\}$, enters in $\mathcal{C}_2$ in time $t = 2$ and stays in $\mathcal{C}_2$ at time $t = 3$. We then have:

- $x_0 \in \mathcal{C}_1 = S_{\substack{0\\ 1}}$;
- $x_1 = A_1 x_0 \in \mathcal{C}_1$, i.e., $x_0 \in S_{\substack{1\\ 1}} = \{x_0 \in \mathbb{R}^n \ : \ A_1 x_0 \in \mathcal{C}_1\}$;
- $x_2 = A_1 x_1 = A_1^2 x_0 \in \mathcal{C}_2$, i.e., $x_0 \in S_{\substack{2,0\\ 1,2}} = \left\{ x_0 \in \mathbb{R}^n \ : \ A_1^2 x_0 \in \mathcal{C}_2 \right\}$;
- $x_3 = A_2 x_2 = A_2 A_1^2 x_0 \in \mathcal{C}_2$, i.e., $x_0 \in S_{\substack{2,1\\ 1,2}} = \left\{ x_0 \in \mathbb{R}^n \ : \ A_2 A_1^2 x_0 \in \mathcal{C}_2 \right\}$.

Gathering these constraints, we obtain $x_0 \in \left(\bigcap_{\tau=0}^{1} S_{\substack{2,\tau\\ 1,2}} \right) \cap \left(\bigcap_{\tau=0}^{1} S_{\substack{\tau\\ 1}} \right) = \Sigma_{\substack{2,1\\ 1,2}}$. Reciprocally, if $x_0 \in \Sigma_{\substack{2,1\\ 1,2}}$ the solution initialized from this x_0 satisfies:

- $x_0 \in S_{\substack{0\\ 1}} = \mathcal{C}_1$, thus $x_1 = A_1 x_0$;
- $x_0 \in S_{\substack{1\\ 1}}$, thus $x_1 = A_1 x_0 \in \mathcal{C}_1$ and $x_2 = A_1 x_1 = A_1^2 x_0$;
- $x_0 \in S_{\substack{2,0\\ 1,2}}$, thus $x_2 = A_1^2 x_0 \in \mathcal{C}_2$ and $x_3 = A_2 x_2 = A_2 A_1^2 x_0$;
- $x_0 \in S_{\substack{2,1\\ 1,2}}$, thus $x_3 = A_2 A_1^2 x_0 \in \mathcal{C}_2$.

We also emphasize that, without additional information, $x_0 \in S_{\substack{t\\ i}}$ does not imply $x_0 \in S_{\substack{t-1\\ i}}$. In addition, if $x_0 \in S_{\substack{t_1,\dots,t_k,\tau\\ i_1,\dots,i_k,i}}$, without additional conditions (which in fact, leads to $x_0 \in \Sigma_{\substack{t_1,\dots,t_k,\tau\\ i_1,\dots,i_k,i}}$), nothing ensures that the sequence defined by

$$x_1 = A_{i_1} x_0, \ \dots, \ x_{t_1} = A_{i_1}^{t_1} x_0, \ x_{t_1+1} = A_{i_2} A_{i_1} x_0, \ \dots$$

is solution of (14). In fact, nothing ensures that x_t is in the correct cone (for instance, nothing ensures that $x_0 \in \mathcal{C}_{i_1}$).

We aim to give conditions ensuring that any nontrivial solution to (14) exhibits no more than p switches. This is ensured by enforcing that, if the solution associated with an initial condition has already *changed* p times of cone, then it cannot switch anymore. That is to say that $\Sigma_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}} = \{0\}$ for every $t \geqslant 1$, and every $t_1, \dots, t_p \in \mathbb{N}^*$ and $i_1, \dots, i_{p+1}, i \in \{1, \dots, m\}$, with $i_{k+1} \neq i_k$ and $i \neq i_{p+1}$. The condition $\Sigma_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}} = \{0\}$ for every $t_1, \dots, t_{p+1} \in \mathbb{N}^*$ and every $t \in \mathbb{N}$ explicitly means that the only initial condition for the system (14) such that the corresponding solution of (14) has visited successively the cones $\mathcal{C}_{i_1}, \dots, \mathcal{C}_{i_{p+1}}$, and will visit the cone $\mathcal{C}_i$, is 0. We recall that $0 \in \bigcap_{i=1}^m \mathcal{C}_i$, and hence, $\mathcal{I}(0) = \{1, \dots, m\}^{\mathbb{N}}$. A sufficient condition for this is as follows.

Proposition 1. *Given $p \in \mathbb{N}^\star$, $i_1, \dots, i_p, i_{p+1} \in \{1, \dots, m\}$ with $i_{k+1} \neq i_k$ for every $k \in \{1, \dots, p\}$, and given $t_1, \dots, t_p \in \mathbb{N}^\star$, we also set $t_{p+1} = 1$. If there exist $\kappa \in \{1, \dots, p+1\}$, $1 \leqslant j_1 \leqslant \dots \leqslant j_\kappa \leqslant p+1$ and for every $k \in \{1, \dots, \kappa\}$, there exist $r_k \in \mathbb{N}$ and $\tau_{k,1}, \dots, \tau_{k,r_k} \in \{0, \dots, t_{j_k} - 1\}$, such that $\sum_{k=1}^{\kappa} r_k \leqslant n$ and*

$$\left(\bigcap_{k=1}^{\kappa} \bigcap_{\ell=1}^{r_k} S_{\substack{t_{j_1},\dots,t_{j_{k-1}},\tau_{k,\ell}\\ i_{j_1},\dots,i_{j_{k-1}},i_{j_k}}} \right) \cap S_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}} = \{0\}, \tag{22}$$

for every $t \in \mathbb{N}^\star$ and every $i \in \{1, \dots, m\} \backslash \{i_{p+1}\}$, then all the solutions to (14) *initialized in $\Sigma_{\substack{t_1,\dots,t_p,0\\ i_1,\dots,i_p,i_{p+1}}} \backslash \{0\}$ exhibit no more than p switches.*

We make several comments before proving Proposition 1.

- As claimed in the paragraph preceding Proposition 1, the aim of this proposition is to give conditions ensuring that $\Sigma_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}} = \{0\}$ for every $t \in \mathbb{N}^\star$.
- The proof of this result relies on the fact that given a sequence of sets $(S_i)_{i\in I}$, such that $0 \in \bigcap_{i\in I} S_i$, if there exists $J \subset I$ such that $\bigcap_{j\in J} S_J = \{0\}$ then $\bigcap_{i\in I} S_i = \{0\}$.
- Using the abstract notation of the previous point, we add two constraints on the possible choices of J:
  1) the cardinality of J is not greater than n (this constraint is $\sum_{k=1}^{\kappa} r_k \leqslant n$). This constraint is added for the numerical verification of the test (see Theorem 4), where a $n \times n$ matrix will be constructed from these sets;
  2) we do not consider sets $(S_j)_{j\in J}$ of the form $S_{\substack{t_1,\dots,t_p,\tau\\ i_1,\dots,i_p,i_{p+1}}}$ (with $\tau > 0$). This fact is expressed by the condition $t_{p+1} = 1$, together with $\tau_{k,1}, \dots, \tau_{k,r_k} \in \{0, \dots, t_{j_k} - 1\}$ (that is to say that if $j_\kappa = p+1$, we enforce $r_\kappa = 1$ and $\tau_{\kappa,r_\kappa} = 0$).

Proof of Proposition 1: For every $t \in \mathbb{N}^*$, we have

$$\begin{aligned}\Sigma_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}} &\subset \Sigma_{\substack{t_1,\dots,t_p,0\\ i_1,\dots,i_p,i_{p+1}}} \cap S_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}}\\ &= \left(\bigcap_{k=1}^{p}\bigcap_{\tau=0}^{t_k-1} S_{\substack{t_1,\dots,t_{k-1},\tau\\ i_1,\dots,i_{k-1},i_k}}\right) \cap S_{\substack{t_1,\dots,t_p,0\\ i_1,\dots,i_p,i_{p+1}}} \cap S_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}}\\ &\subset \left(\bigcap_{k=1}^{\kappa}\bigcap_{\ell=1}^{r_k} S_{\substack{t_{j_1},\dots,t_{j_{k-1}},\tau_{k,\ell}\\ i_{j_1},\dots,i_{j_{k-1}},i_{j_k}}}\right) \cap S_{\substack{t_1,\dots,t_p,t,0\\ i_1,\dots,i_p,i_{p+1},i}}.\end{aligned}$$

■

We are ready to state conditions under which *any* solution to (14), and not only those initialized in $\Sigma_{\substack{t_1,\dots,t_p,0\\ i_1,\dots,i_p,i_{p+1}}}$ as in Proposition 1, exhibits at most p switches.

Theorem 3. *Under Assumption 2, given $p \in \mathbb{N}^\star$, if for every $i_1,\dots,i_{p+1} \in \{1,\dots,m\}$ and every $t_2,\dots,t_p \in \mathbb{N}^\star$, the conditions of Proposition 1 are satisfied with $t_1 = 1$, then all the possible nontrivial solutions of* (14) *admit at most p switches.*

Proof: Assume by contradiction that there exists $x_0 \in \mathbb{R}^n\backslash\{0\}$ such that the solution initialized at x_0 switches more that p times. Then there exists $i_1,\dots,i_p,i_{p+1},i_{p+2}$ (with $i_{k+1} \neq i_k$) and $t_1,\dots,t_p,t_{p+1} \in \mathbb{N}^\star$ such that $A_{i_1}^{t_1-1}x_0$ belongs to $\Sigma_{\substack{1,t_2,\dots,t_{p+1},0\\ i_1,i_2,\dots,i_{p+1},i_{p+2}}}$. This leads to a contradiction, since by Proposition 1, this set is reduced to $\{0\}$, and by Assumption 2, A_{i_1} is invertible. ■

Remark 2. *Under the assumptions of Theorem 3, we have in fact a stronger result, which is* $\max\{\jmath((k_t)_t) \ : \ (k_t)_{t\in\mathbb{N}} \in \mathcal{I}(x)\} \leqslant p$ *for every $x \in \mathbb{R}^n\backslash\{0\}$. Obviously, this ensures the claim of Theorem 3: $\varsigma(x) \leqslant p$ for every $x \in \mathbb{R}^n\backslash\{0\}$.*

From a computational point of view, the main difficulties with the result of Theorem 3 are related to checking the intersection (22) for every $t \in \mathbb{N}^\star$ and every $i \in \{1,\dots,m\}\backslash\{i_{p+1}\}$. The problem is combinatorial by nature. However, we provide in the sequel, tractable conditions for the case $m = 2$. In Section IV-B, we propose a solution to tackle the fact that t is a priori not bounded. It consists in using Farkas lemma, which is a particular case of the S-procedure dedicated to linear forms, to transform the problem into assessing the non-negativity of a specific solution to an auxiliary discrete-time system. We propose tractable conditions to check this non-negativity in Section V. As to the combinatorial complexity related to checking the intersection (22) for every $i \in \{1,\dots,m\}\backslash\{i_{p+1}\}$, the situation is manageable as we will manipulate sets defined by linear inequalities, and checking if their intersection is reduced to $\{0\}$ can be done using linear programming.

B. When $m = 2$

To streamline the discussion and maintain clarity, we focus on the case where $m = 2$, i.e., the partition of $\mathbb{R}^n$ is made of two cones, while keeping in mind that the results and concepts presented here can be readily extended to the general case when $m \in \mathbb{N}^\star$. Note that the case where $m = 2$ is notoriously difficult [4].

When $m = 2$, the cones $\mathcal{C}_1$ and $\mathcal{C}_2$ can be written as in (10) for some matrix $K \in \mathbb{R}^{1\times n}$, which is not the same as in Section II in general. Recall that we assumed that the cones $\mathcal{C}_i$ are convex, hence, for $m = 2$ the two cones $\mathcal{C}_1$ and $\mathcal{C}_2$ are half-spaces of $\mathbb{R}^n$.

1) General result: Since the indexes i_k used in Section IV-A, are such that $i_{k+1} \neq i_k$, the only possibilities, when $m = 2$ are the sequences $1,2,1,\dots$ or $2,1,2,\dots$ This observation together with the expression of $\mathcal{C}_1$ and $\mathcal{C}_2$ given in (10), leads to set

$$\begin{aligned}\mathcal{S}_{i_1;t_1,\dots,t_p,t} &:= \mathcal{S}_{\substack{t_1,\dots,t_p,t\\ [i_1],\dots,[i_1+p]}}\\ &= \left\{x \in \mathbb{R}^n \ : \ (-1)^{i_1+p}KA^t_{[i_1+p+1]}A^{t_p}_{[i_1+p]}\dots A^{t_1}_{[i_1]}x \geqslant 0\right\},\end{aligned}$$

where, for every $i \in \mathbb{N}$, we have defined $[2i+1] = 1$ and $[2i] = 2$. Similarly, we define $\Sigma_{i_1;t_1,\dots,t_p,t}$. Given $i_1 \in \{1,2\}$ if we start from an initial condition in $\mathcal{C}_{i_1}$, the solution stays for the remaining times in $\mathcal{C}_{i_1}$ or enters in $\mathcal{C}_{[i_1+1]}$. Hence, the maximal number of switches starting from $\mathcal{C}_{i_1}$ is bounded by one plus the maximal number of switches starting from $\mathcal{C}_{[i_1+1]}$. This observation leads to the next corollary of Theorem 3.

Corollary 1. *Under Assumption 2, consider the system (14) with $m = 2$. Given $p \in \mathbb{N}^\star$, if there exists $i_1 \in \{1,2\}$ such that for every $t_2,\dots,t_p \in \mathbb{N}^\star$, the conditions of Proposition 1 are satisfied with $t_1 = 1$, then each nontrivial solution of* (14) *admits at most $p+1$ switches.*

2) Tractable conditions: To make the paper self-contained, we recall Farkas lemma in the form in which we will use it, although other formulations are possible, see e.g. [30] and [31, Corollary 4.3] for more detail.

Lemma 2 (Farkas lemma)**.** *Let $\ell, n \in \mathbb{N}^\star$ and $M_0,\dots,M_\ell \in \mathbb{R}^n$. We have,*

$$\begin{aligned}\left\{x \in \mathbb{R}^n \ : \ x^\top M_0 \geqslant 0\right\} \subset \bigcap_{i=1}^{\ell}\left\{x \in \mathbb{R}^n \ : \ x^\top M_i \geqslant 0\right\}\\ \Leftrightarrow \quad \exists \alpha_1,\dots,\alpha_\ell \in \mathbb{R}_+ \ \textit{s.t.}\ M_0 = \sum_{i=1}^{\ell}\alpha_i M_i.\end{aligned}$$

In other words, Lemma 2 states that, given any $x \in \mathbb{R}^n$,

$$x^\top M_i \geqslant 0 \quad \forall i \in \{1,\dots,\ell\} \quad \Longrightarrow \quad x^\top M_0 \geqslant 0$$

if and only if there exist non-negative numbers $\alpha_1,\alpha_2,\dots,\alpha_\ell$ such that $M_0 = \sum_{i=1}^{\ell}\alpha_i M_i$. In the sequel, to verify condition (22), we consider without loss of generality that $\sum_{k=1}^{\kappa} r_k = n$. This is justified because, if (22) holds with $\sum_{k=1}^{\kappa} r_k = \nu$ with $\nu < n$, it also holds with $\sum_{k=1}^{\kappa} r_k = n$. This can be achieved by including any $n-\nu$ sets from the available sets, we recall that $\sum_{k=1}^{\kappa} r_k$ is the number of sets used for testing the emptiness condition (22). For $t_1,\dots,t_{n-1} \in \mathbb{N}^\star$, we introduce the following matrices, which all belong to $\mathbb{R}^{1\times n}$,

$$
\begin{aligned}
&N_{i_1;t} = (-1)^{i_1+1}KA^t_{[i_1]}, && \text{for } 0 \leqslant t < t_1,\\
&N_{i_1;t_1,t} = (-1)^{i_1+2}KA^t_{[i_1+1]}A^{t_1}_{[i_1]}, && \text{for } 0 \leqslant t < t_2,\\
&N_{i_1;t_1,t_2,t} = (-1)^{i_1+3}KA^t_{[i_1+2]}A^{t_2}_{[i_1+1]}A^{t_1}_{[i_1]}, && \text{for } 0 \leqslant t < t_3,\\
&\vdots\\
&N_{i_1;t_1,\dots,t_{n-2},t} = (-1)^{i_1+n-1}KA^t_{[i_1+n-2]}A^{t_{n-2}}_{[i_1+n-3]}\dots A^{t_1}_{[i_1]}, && \text{for } 0 \leqslant t < t_{n-1},\\
&N_{i_1;t_1,\dots,t_{n-2},t_{n-1},0} = (-1)^{i_1+n}KA^{t_{n-1}}_{[i_1+n-2]}\dots A^{t_1}_{[i_1]}.
\end{aligned}
\tag{23}
$$

Observe that these matrices verify

$$S_{i_1;t_1,\dots,t_p,t} = \left\{x \in \mathbb{R}^n \ : \ N_{i_1;t_1,\dots,t_p,t}x \geqslant 0\right\}. \tag{24}$$

To verify (22) we investigate the intersection of $S_{i_1;1,t_2,\dots,t_{n-1},t,0}$ with n-sets taken from (23). We notice that in (23), there are exactly $1+\sum_{i=1}^{n-1} t_i$ row vectors defined (recall that $t_1,\dots,t_{n-1} \in \mathbb{N}^\star$, hence, $1+\sum_{i=1}^{n-1} t_i \geqslant n$).
In addition, we also have (with $t_1 = 1$)

$$S_{i_1;1,t_2,\dots,t_{n-1},t,0} = \{x \in \mathbb{R}^n \ : \ N_{i_1;1,t_2,\dots,t_{n-1},t,0}x \geqslant 0\},$$

with

$$N_{i_1;1,t_2\dots,t_{n-1},t,0} = (-1)^{i_1+n}KA^t_{[i_1+n-1]}A^{t_{n-1}}_{[i_1+n-2]}\dots A_{[i_1]}.$$

The next result gathers Corollary 1 and Lemma 2.

Theorem 4. *Under Assumption 2, suppose there exists $i_1 \in \{1,2\}$ such that for every $t_2,\dots,t_{n-1} \in \mathbb{N}^\star$, we either have $\Sigma_{i_1;1,t_2,\dots,t_{n-1},0} = \{0\}$ or one is able to build an invertible matrix $\mathcal{N} \in \mathbb{R}^{n\times n}$ whose lines are taken from (23) such that*

(i) $\beta_0 = -(KA_{i_{[i_1+n-1]}}\mathcal{M}\mathcal{N}^{-1})^\top$ is non-negative,

(ii) the solution to

$$\beta_{t+1} = \mathcal{L}\beta_t, \quad \text{with} \quad \mathcal{L}^\top = \mathcal{N}\mathcal{M}^{-1}A_{i_{[i_1+n-1]}}\mathcal{M}\mathcal{N}^{-1} \tag{25}$$

initialized with $\beta_0 = -(KA_{i_{[i_1+n-1]}}\mathcal{M}\mathcal{N}^{-1})^\top \geqslant 0$ remains non-negative for all future time,

where we have set,

$$\mathcal{M} := (-1)^{i_1+n}A^{t_{n-1}}_{[i_1+n-2]}\dots A^{t_2}_{[i_1+1]}A_{i_1}.$$

Then any solution to (14) exhibits no more than n switches.

Proof: Using Corollary 1, any nontrivial solution to (14) exhibits no more than n switches if (22) is satisfied with $t_1 = 1$ and $p = n-1$. This means checking that the intersection of $S^{1,\dots,t_{n-1},t,0}_{i_1,\dots,i_{n-1},i_n,i}$ with the sets characterized by (23) and (24) is reduced to $\{0\}$. To check this condition, it is enough to exhibit n sets such that, for every $t \in \mathbb{N}^\star$, their intersection with $S_{i_1;1,t_2,\dots,t_{n-1},t,0}$ is reduced to $\{0\}$. This is equivalent to find n rows in (23) such that the matrix $\mathcal{N} \in \mathbb{R}^{n\times n}$ formed by these n rows is such that

$$\mathcal{N}x \geqslant 0 \tag{26}$$

implies

$$(-1)^{i_1+n}KA^t_{[i_1+n-1]}A^{t_{n-1}}_{[i_1+n-2]}\dots A_{[i_1]}x \leqslant 0, \qquad \forall t \in \mathbb{N}^\star,$$

i.e., implies

$$-KA^t_{[i_1+n-1]}\mathcal{M}x \geqslant 0, \qquad \forall t \in \mathbb{N}^\star. \tag{27}$$

Invoking Lemma 2, checking that (26) implies (27) reduces to check whether there exists $\beta_t \in \mathbb{R}^n_+$ such that

$$-KA^t_{[i_1+n-1]}\mathcal{M} = \beta_t^\top\mathcal{N}, \qquad \forall t \in \mathbb{N}^\star,$$

that is to say, with $\mathcal{N}$ invertible,

$$\beta_t^\top = -KA^t_{[i_1+n-1]}\mathcal{M}\mathcal{N}^{-1} \geqslant 0, \qquad \forall t \in \mathbb{N}^\star. \tag{28}$$

To conclude the proof, we show that (28) is equivalent to check the positivity of a solution of a discrete time dynamics. Indeed, as the invertibility of A_1 and A_2 implies that $\mathcal{M}$ is invertible, we have

$$
\begin{aligned}
\beta_{t+1}^\top &= -KA^{t+1}_{[i_1+n-1]}\mathcal{M}\mathcal{N}^{-1}\\
&= -KA^t_{[i_1+n-1]}\mathcal{M}\mathcal{N}^{-1}\mathcal{N}\mathcal{M}^{-1}A_{[i_1+n-1]}\mathcal{M}\mathcal{N}^{-1}\\
&= \beta_t^\top\mathcal{N}\mathcal{M}^{-1}A_{[i_1+n-1]}\mathcal{M}\mathcal{N}^{-1}\\
&= (\mathcal{L}\beta_t)^\top.
\end{aligned}
$$

■

This theorem states that to verify the non-negativity of a single solution to the auxiliary linear discrete-time system described in (25) is enough to ensure that any solution to (14) switches no more than n times. This property is different from the classical non-negativity problem, where any solution to a system must remain non-negative (component-wise) for any non-negative initial condition. In this classical setting, it is well known that, for a discrete-time system to be positive, a necessary and sufficient condition is that the entries of the dynamical matrix have to be positive [19]. This does not correspond to our problem, and we will see in Section VI that the entries of the dynamical matrix $\mathcal{L}$ are not necessarily non-negative, so that we cannot invoke the results from [19] in general. This means new results are needed to ensure the non-negativity of system (25) for a given initial condition: this is addressed in the next section.

V. Non-negativity analysis

We derive in this section conditions under which the solution to (25) initialized with a given $\beta_0 \geqslant 0$ is non-negative. The results are based on a function analysis and allows deriving numerically tractable conditions.

A. Non-negativity theorem

The next theorem provides easy-to-test conditions to ensure the satisfaction of item (ii) of Theorem 4. The result of Theorem 5 is based on the necessary condition given in Proposition 2, which is given below.

Theorem 5. *Consider system (25) with a given $\beta_0 \geqslant 0$ and assume that $\sigma(A_{[i_1+n-1]}) = \{\lambda_1,\dots,\lambda_n\}$, with $\lambda_1 > \dots > \lambda_n > 0$. Then $\mathcal{L}$ is diagonalizable, and we set $v_1,\dots,v_n$ the eigenvectors of $\mathcal{L}$, and $\gamma_1,\dots,\gamma_n \in \mathbb{R}$ such that $\beta_0 = \sum_{i=1}^n \gamma_i v_i$. Let us finally set $\rho_i = \gamma_i v_i \in \mathbb{R}^n$ and $\mu_i = -\ln(\lambda_i/\lambda_1)$ with $i = 1,\dots,n$ and assume that $\rho_1 \geqslant 0$. Then, the solution β_t is non-negative for any $t \in \mathbb{N}$ if for every $\ell \in \{1,\dots,n\}$, one of the conditions of Proposition 2 is satisfied, with z_i the ℓ-th component of ρ_i.*

Proof: According to the definition of $\mathcal{L}$ in (25), the eigenvalues of $\mathcal{L}$ are also eigenvalues of $A_{[i_1+n-1]}$. They are distinct, and the n eigenvectors v_i are independent. The solution of the linear discrete-time system (25) is a weighted linear combination of modal solutions:

$$\beta_t = \sum_{i=1}^{n} \gamma_i v_i \lambda_i^t.$$

To conclude the proof of this theorem, we have to find conditions on $(\rho_1, \ldots, \rho_n)$ such that

$$\beta_t = \sum_{i=1}^{n} \gamma_i v_i \lambda_i^t = \lambda_1^t \rho_1 + \cdots + \lambda_n^t \rho_n \geqslant 0 \qquad (t \in \mathbb{N}). \tag{29}$$

We recall that $\rho_i \in \mathbb{R}^n$ and inequality (29) has to be understood component-wise, i.e., $[\beta_t]_\ell \geqslant 0$ for every $\ell \in \{1, \ldots, n\}$. According to the definition of μ_i, for every $i \in \{2, \ldots, n\}$, we have $0 < \mu_2 < \ldots < \mu_n$. Hence, the problem is equivalent with finding a condition on $(\rho_1, \ldots, \rho_n) \in \mathbb{R}^n$ such that

$$\rho_1 \geqslant -\sum_{i=2}^{n} e^{-\mu_i t} \rho_i \qquad (t \in \mathbb{N}). \tag{30}$$

Note that (30) holds if we have

$$\rho_1 \geqslant -\sum_{i=2}^{n} e^{-\mu_i s} \rho_i \qquad (s \in \mathbb{R}_+).$$

We conclude the proof using Proposition 2 ∎

Proposition 2. *Let $n \in \mathbb{N}^*$, $0 < \mu_2 < \cdots < \mu_n$ and $z_1, \ldots, z_n \in \mathbb{R}$. If $z_1 \geqslant 0$, $z_1 + \cdots + z_n \geqslant 0$ and one of the following conditions holds:*

(i) $z_2, \ldots, z_n \geqslant 0$;
(ii) $z_2, \ldots, z_n \leqslant 0$;
(iii) *there exists $j \in \{2, \ldots, n\}$ such that*

$$z_1 \geqslant -\sum_{\substack{i=2\\ i\neq j}}^{n} \left(1 - \frac{\mu_i}{\mu_j}\right) z_i e^{-\mu_i s}, \quad \forall s \in \mathbb{R}_+. \tag{31}$$

Then, the function $\varphi(s) = z_1 + z_2 e^{-\mu_2 s} + \cdots + z_n e^{-\mu_n s}$ defined for every $s \in \mathbb{R}_+$ is non-negative on $\mathbb{R}_+$.

Proof: Observe that $\varphi(0) = z_1 + \cdots + z_n$ and $\lim_{s\to\infty} \varphi(s) = z_1$. They are both non-negative in the setup of the statement. Obviously, if (i) is satisfied, $\varphi \geqslant 0$ on $\mathbb{R}$. If (ii) is satisfied, we have $\varphi' \geqslant 0$ on $\mathbb{R}$ and $\varphi(0) = z_1 + \cdots + z_n \geqslant 0$. Without the sign constraints on $z_2, \ldots, z_n$, we have either $\inf_{\mathbb{R}_+} \varphi = \varphi(0) = z_1 + \cdots + z_n$ or $\inf_{\mathbb{R}_+} \varphi = \lim_{s\to\infty} \varphi(s) = z_1$ or $\inf_{\mathbb{R}_+} \varphi = \varphi(s_0)$ for some $s_0 \in \mathbb{R}_+^*$ such that $\varphi'(s_0) = 0$. Hence, to guarantee the nonnegativity of φ on $\mathbb{R}_+$ it remains to check the positivity of $\varphi(s_0)$ in the last case. Observe that, given $j \in \{2, \ldots, n\}$, $\varphi'(s_0) = 0$ is equivalent to $\mu_j z_j e^{-\mu_j s_0} = \sum_{\substack{i=1\\ i\neq j}}^{n} -\mu_i z_i e^{-\mu_i s_0}$. Hence, we have $\varphi(s_0) = z_1 + \sum_{\substack{i=2\\ i\neq j}} \left(1 - \frac{\mu_i}{\mu_j}\right) z_i e^{-\mu_i s_0}$, and (iii) ensures that $\varphi(s_0) \geqslant 0$. ∎

Theorem 5 allows deriving numerically tractable conditions by iteratively applying (31) to any chosen value of n. As a title of example, consider the case $n = 4$. By iteratively applying (31), one can conclude that $\beta_t \geqslant 0$ for all t if

$$\begin{aligned} \rho_1 \geqslant 0, \quad \rho_1 + \left(1 - \tfrac{\mu_3}{\mu_2}\right)\rho_3 + \left(1 - \tfrac{\mu_4}{\mu_2}\right)\rho_4 \geqslant 0, \\ \text{and } \rho_1 + \left(1 - \tfrac{\mu_4}{\mu_3}\right)\left(1 - \tfrac{\mu_4}{\mu_2}\right)\rho_4 \geqslant 0. \end{aligned} \tag{32}$$

To check that this is true, first iterate (31) with $n = 4$. As $\left(1 - \frac{\mu_4}{\mu_3}\right)\left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 \geqslant 0$ or $\left(1 - \frac{\mu_4}{\mu_3}\right)\left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 \leqslant 0$, this together with $\rho_1 \geqslant 0$ and $\rho_1 + \left(1 - \frac{\mu_4}{\mu_3}\right)\left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 \geqslant 0$ ensures

$$\rho_1 \geqslant -\left(1 - \frac{\mu_4}{\mu_3}\right)\left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 e^{-\mu_4 s} \qquad (s \in \mathbb{R}_+).$$

This, together with $\rho_1 \geqslant 0$ and $\rho_1 + \left(1 - \frac{\mu_3}{\mu_2}\right)\rho_3 + \left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 \geqslant 0$ ensures

$$\rho_1 \geqslant -\left(1 - \frac{\mu_3}{\mu_2}\right)\rho_3 e^{-\mu_3 s} - \left(1 - \frac{\mu_4}{\mu_2}\right)\rho_4 e^{-\mu_4 s} \qquad (s \in \mathbb{R}_+).$$

Finally, this last fact, together with $\rho_1 \geqslant 0$ and $\beta_0 = \rho_1 + \rho_2 + \rho_3 + \rho_4 \geqslant 0$ ensures

$$\rho_1 \geqslant -\rho_2 e^{-\mu_2 s} - \rho_3 e^{-\mu_3 s} - \rho_4 e^{-\mu_4 s} \qquad (s \in \mathbb{R}_+).$$

We will exploit (32) for the insulin infusion example in Section VI-B.

B. Tailored results for small values of n

For small values of n, the function analysis in the proof of Proposition 2 can be tailored to derive tighter conditions. In this paragraph, we only present the cases $n \in \{2, 3, 4\}$. But similar results could have been obtained for larget values of n.

When $n = 2$, it is easy to see that $\varphi(s) = z_1 + z_2 e^{-\mu_2 s} \geqslant 0$ for every $s \in \mathbb{R}_+$ (with $\mu_2 \geqslant 0$) if and only if $z_1 \geqslant 0$ and $z_1 + z_2 \geqslant 0$. The next propositions provide tailored results for $n = 3$ and $n = 4$.

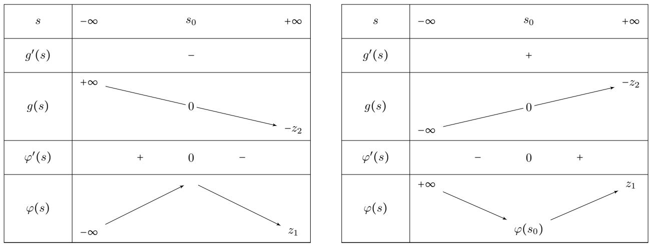
Proposition 3 (Case $n = 3$)**.** *With the notation of Proposition 2 with $n = 3$, we have $\varphi(s) \geqslant 0$ for all $s \in \mathbb{R}_+$ if and only if $z_1 \geqslant 0$, $z_1 + z_2 + z_3 \geqslant 0$ and one of the following conditions is satisfied:*

(i) $z_2 \geqslant 0$.
(ii) $z_2 < 0$, $z_3 \leqslant 0$.
(iii) $z_2 < 0$, $z_3 > 0$, $\mu_2 z_2 + \mu_3 z_3 \leqslant 0$.
(iv) $z_2 < 0$, $z_3 > 0$, $\mu_2 z_2 + \mu_3 z_3 > 0$ *and*

$$z_1 + z_2 \left(\frac{-\mu_2 z_2}{\mu_3 z_3}\right)^{\frac{\mu_2}{\mu_3 - \mu_2}} + z_3 \left(\frac{-\mu_2 z_2}{\mu_3 z_3}\right)^{\frac{\mu_3}{\mu_3 - \mu_2}} \geqslant 0.$$

Proof: Note that the cases where z_2 and z_3 have the same signs are treated in items (i) and (ii) of Proposition 2. Hence, it remains to prove the result when z_2 and z_3 do not have the same sign. Recall that $\varphi'(s) = -\mu_2 z_2 e^{-\mu_2 s} - \mu_3 z_3 e^{-\mu_3 s}$, and $\operatorname{sign} \varphi' = \operatorname{sign} g$, with $g = -z_2 - \frac{\mu_3}{\mu_2} z_3 e^{-(\mu_3 - \mu_2)s}$. We also have $g'(s) = \frac{\mu_3}{\mu_2} z_3 (\mu_3 - \mu_2) e^{-(\mu_3 - \mu_2)}$, thus $\operatorname{sign} g' = \operatorname{sign} z_3$. When z_2 and z_3 do not have the same sign, there exists one and only one $s_0 \in \mathbb{R}$ such that $g(s_0) = 0$ (see fig. 2). We thus have the following situations.

- If $z_2 > 0$ and $z_3 < 0$, we have the situation described by fig. 2a. We observe that $\varphi(s) \geqslant \min\{\varphi(0), z_1\} \geqslant 0$ for every $s \in \mathbb{R}_+$.
- If $z_2 < 0$ and $z_3 > 0$, we have the situation described by fig. 2b. We observe that $\inf_{\mathbb{R}_+} \varphi = \begin{cases} \varphi(0) & \text{if } s_0 \leqslant 0, \\ \varphi(s_0) & \text{otherwise.} \end{cases}$ But s_0 is such that $g(s_0) = 0$, i.e., $s_0 = \frac{-1}{\mu_3 - \mu_2} \ln \frac{-\mu_2 z_2}{\mu_3 z_3}$.



(a) $z_2 > 0$ and $z_3 < 0$ (b) $z_2 < 0$ and $z_3 > 0$

Fig. 2: Different situations in the proof of Proposition 3.

We have $s_0 \leqslant 0$ if and only if $\mu_2 z_2 + \mu_3 z_3 \leqslant 0$, and $\varphi(s_0) = z_1 + z_2 \left(\frac{-\mu_2 z_2}{\mu_3 z_3}\right)^{\frac{\mu_2}{\mu_3 - \mu_2}} + z_3 \left(\frac{-\mu_2 z_2}{\mu_3 z_3}\right)^{\frac{\mu_3}{\mu_3 - \mu_2}}$. ■

Proposition 4 (Case $n = 4$)**.** *With the notations used in Proposition 2 with $n = 3$, we have $\varphi(s) \geqslant 0$ for all $s \in \mathbb{R}_+$ if $z_1 \geqslant 0$, $z_1 + z_2 + z_3 + z_4 \geqslant 0$ and one of the following conditions is satisfied:*

(i) $z_4 \leqslant 0$, $z_3 \leqslant 0$.

(ii) $z_4 \leqslant 0$, $z_3 > 0$, $z_2 \geqslant 0$.

(iii) $z_4 \leqslant 0$, $z_3 > 0$, $z_2 < 0$ *and* $\frac{\mu_4}{\mu_2}\left(\frac{\mu_4 - \mu_2}{\mu_3 - \mu_2} - 1\right) e^{\frac{\mu_4 - \mu_2}{\mu_4 - \mu_3} \ln\left(\frac{z_3}{|z_4|} \frac{\mu_3}{\mu_4} \frac{\mu_3 - \mu_2}{\mu_4 - \mu_2}\right)} z_4 \geqslant z_2$.

(iv) $z_4 > 0$, $z_3 \geqslant 0$, $z_2 \geqslant 0$.

(v) $z_4 > 0$, $z_3 < 0$, $z_2 \leqslant 0$ *and* $|z_3| \mu_3 (\mu_3 - \mu_2) \geqslant z_4 \mu_4 (\mu_4 - \mu_2)$.

(vi) $z_4 > 0$, $z_3 < 0$, $z_2 > 0$ *and* $\frac{\mu_4}{\mu_2}\left(\frac{\mu_4 - \mu_2}{\mu_3 - \mu_2} - 1\right) e^{\frac{\mu_4 - \mu_2}{\mu_4 - \mu_3} \ln\left(\frac{|z_3|}{z_4} \frac{\mu_3}{\mu_4} \frac{\mu_3 - \mu_2}{\mu_4 - \mu_2}\right)} z_4 \leqslant z_2$. □

The proof of this proposition follows the lines of the one of Proposition 3. For the sake of brevity, we do not detail it here.

VI. Applications

Before applying the results to the insulin infusion example in Section VI-B, we provide in Section VI-A a second order example to illustrate the fact that the application of Corollary 1 and Theorem 4 is easy in $\mathbb{R}^2$. It reduces to check the invertibility of a second order matrix and the non-negativity of a second order dynamics.

A. Second order example

Consider system (14) with $n = 2$, $m = 2$,

$$A_1 = \begin{bmatrix} 13 & 12.5 \\ -12.5 & -12 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.93 & 0 \\ 0 & 0.95 \end{bmatrix} \qquad (33)$$
$$K = \begin{bmatrix} -8.7 & -9.2 \end{bmatrix},$$

and $\mathcal{C}_1$ and $\mathcal{C}_2$ as in (10). The solutions associated with various initial conditions selected from the unit circle are represented in Figure 3 as well as the corresponding number of switches they exhibit. These solutions predominantly reside within $\mathcal{C}_2$ and only sporadically venture into $\mathcal{C}_1$. We observe on this example that $\mathbb{R}^2$ is partitioned in three cones, which are respectively defined by the set of initial conditions that result in exactly 0, 1 and 2 switches. These cones are formed by the line $Kx = 0$ and the half-line $\mathbb{R}_+ [1 \ 0]^\top$. Observe also that $[1 \ 0]^\top \in \mathcal{C}_2$ is an eigenvector of A_2. Moreover, all points initialized in $\mathcal{C}_1$ reach the invariant region of $\mathcal{C}_2$ after a single iteration as highlighted in Figure 4 for $x_0 = [0 \ -1]^\top$.

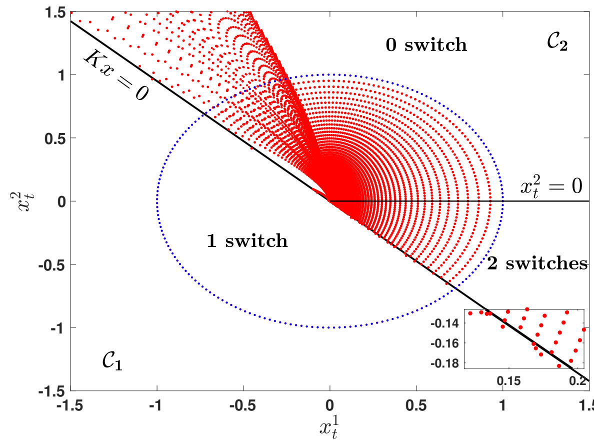


Fig. 3: Phase portrait in red with x_0 on the unit circle.

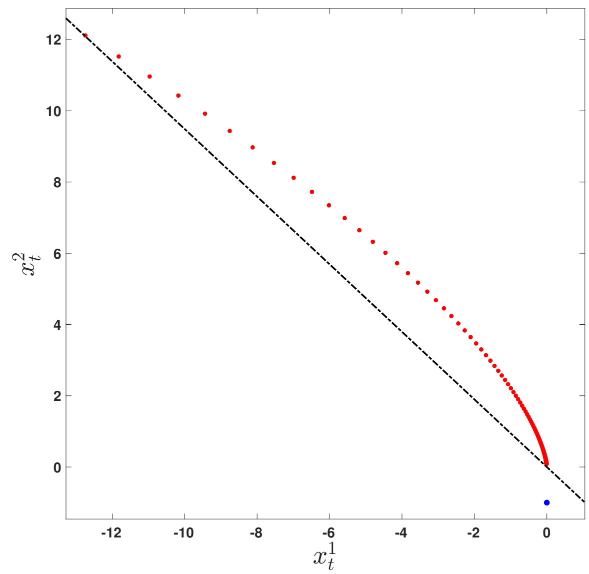


Fig. 4: Phase portrait in red with $x_0 = [0, -1]^\top$ (x_0 in $\mathcal{C}_1$).

Let us illustrate the theoretical results of Section IV-B. The application of Theorem 4 reduces to check that the following implication holds true for $x \in \mathbb{R}^n$

$$Kx \geqslant 0 \text{ and } KA_1 x \leqslant 0 \quad \Longrightarrow \quad \forall t \in \mathbb{N}^*, \quad KA_2^t A_1 x \leqslant 0$$

For $t = 1$, using Lemma 2, this is equivalent to the existence of $\alpha_1^1, \alpha_2^1 \geqslant 0$ such that

$$-KA_2 A_1 = \alpha_1^1 K - \alpha_2^1 K A_1.$$

The pair $(-A_1, K)$ is observable. Hence,

$$\begin{bmatrix} \alpha_1^1 & \alpha_2^1 \end{bmatrix} = -KA_2 A_1 \begin{bmatrix} K \\ -KA_1 \end{bmatrix}^{-1} = \begin{bmatrix} 0.1281 & 1.5541 \end{bmatrix}.$$

Now, we have to study the system

$$\beta_{t+1} = \mathcal{L}\beta_t, \quad \beta_0 = \begin{bmatrix} \alpha_1^1 & \alpha_2^1 \end{bmatrix}^\top \geqslant 0$$

$$\text{with } \mathcal{L} = \begin{bmatrix} 0.3259 & 0.1281 \\ -2.9443 & 1.5541 \end{bmatrix}, \quad \beta_0 = \begin{bmatrix} 0.1281 \\ 1.5541 \end{bmatrix}.$$

The eigenvalues of $\mathcal{L}$, which are also eigenvalues of A_2, and the associated eigenvectors are given by $\lambda_1 = 0.95$, $\lambda_2 = 0.93$, $v_1 = \begin{bmatrix} -0.2010 & -0.9796 \end{bmatrix}^\top$. $v_2 = \begin{bmatrix} -0.2074 & -0.9783 \end{bmatrix}^\top$, We have $\beta_t = \sum_{i=1}^n \gamma_i v_i \lambda_i^t$, with $\beta_0 = \sum_{i=1}^n \gamma_i v_i \geqslant 0$, where γ_1 and γ_2 are given by

$$\begin{bmatrix} \gamma_1 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1} \beta_0 = \begin{bmatrix} 28.7099 \\ -30.2574 \end{bmatrix}.$$

$$\text{and } \gamma_1 v_1 = \begin{bmatrix} 6.0818 \\ 29.6399 \end{bmatrix}, \quad \gamma_2 v_2 = \begin{bmatrix} -5.9537 \\ -28.0858 \end{bmatrix},$$

with $\lambda_1 > \lambda_2 > 0$, we conclude from Theorem 5 that

$$\beta_t = \sum_{i=1}^{2} \gamma_i v_i \lambda_i^t > 0, \quad \forall t \in \mathbb{N}.$$

As a conclusion, any solution starting in the set $\mathcal{C}_1$ switches not more than once. We can conclude that, for this example, any solution to the considered system switches not more than twice as some can be initialized in $\mathcal{C}_2$. The fact that the origin of the considered system is GES follows by Theorem 1 as A_1 and A_2 are Schur.

Remark 3. *For this example, it turns out that a piecewise quadratic Lyapunov function, given by (13), exists. Indeed, the LMIs conditions (35) given in Appendix B are feasible with* $P_1 = \begin{bmatrix} 2.5662 & 2.6008 \\ 2.6008 & 2.6427 \end{bmatrix}$, $P_2 = \begin{bmatrix} 0.1744 & 0.1570 \\ 0.1570 & 0.1482 \end{bmatrix}$, $Y_1 = 0.0091$, $Y_2 = 9.2206 \times 10^{-6}$, $Y_3 = \begin{bmatrix} 0.0017 & 0.0232 \\ 0.0232 & 0.0038 \end{bmatrix}$, $Y_4 = \begin{bmatrix} 0.0043 & 0.0058 \\ 0.0058 & 0.0297 \end{bmatrix}$.

B. Insulin infusion problem

We consider system (14) with $n = 4$, $m = 2$ and matrices A, B and K given by (7) and (8). Without the results presented in this paper, stability analysis of this 4^{th}-order example is an open question as far as we know. Let $A_1 = A + BK$ and $A_2 = A$ as in Section II-C. The eigenvalues of A_2 are $\lambda_1 = 0.9592$, $\lambda_2 = 0.9512$, $\lambda_3 = 0.9277$ and $\lambda_4 = 0.9200$. We next show that the conditions of Theorem 4 hold with $i_1 = 2$. To this end, we use Theorem 5 along with conditions given in (32).

We start by considering (23) with $i_1 = 1$, $n = 4$ and $t_1 = 1$. We have to determine the values of t_2 and t_3 that satisfy the conditions of Theorem 4, namely for every $t_2, t_3 \in \mathbb{N}^\star$, we either have $\Sigma_{i_1;1,t_2,t_3,0} = \{0\}$ or there exists an invertible matrix $\mathcal{N} \in \mathbb{R}^{4\times4}$, whose lines are taken from (23) such that items (i) and (ii) of Theorem 4 are satisfied. By employing linear programming techniques, we have identified 58 pairs of values for t_2 and t_3, as displayed in Table I.

t_2	1	3	4	5	6	7	8	9	10	11	12	13
t_3	1	1	1	1	1	1	1	1	1	1	1	1
t_2	14	15	16	17	18	19	20	21	22	23	24	1
t_3	1	1	1	1	1	1	1	1	1	1	1	2
t_2	3	4	5	6	7	8	9	10	11	12	13	14
t_3	2	2	2	2	2	2	2	2	2	2	2	2
t_2	15	16	17	1	2	3	4	5	6	7	8	9
t_3	2	2	2	3	3	3	3	3	3	3	3	3
t_2	10	1	2	3	4	1	2	3	1	2		
t_3	3	4	4	4	4	5	5	5	6	6		

TABLE I: Values of t_2 and t_3 in section IV-B.

For all these 58 pairs, we checked that Theorem 4 holds, that is, we have found for each of them an invertible matrix $\mathcal{N} \in \mathbb{R}^{4\times4}$ whose lines are taken from (23) such that items (i) and (ii) of Theorem 4 are satisfied. For item (ii), we use Theorem 5 and conditions (32). Detailed computations for two of these pairs are provided in Appendix D. The conclusion is that any solution to the considered system exhibits no more than 4 switches. As the eigenvalues of A_1 and A_2 lie inside the unit circle, the origin is GES by Theorem 1.

VII. Conclusion

This paper focuses on the stability analysis of CLS, whose solutions all exhibit a finite number of switches. This property is very useful when investigating stability, as we showed, and somehow move the problem to analytically establish that all solutions indeed switch a finite number of times. We have first presented general, sufficient conditions in terms of sets intersections. To illustrate how these conditions can be exploited, we have then concentrated on the case where two cones partitions the state space, for which we have developed numerically tractable sufficient conditions. Interestingly, these contributions required the development of novel results on the non-negativity of solutions to discrete-time dynamical systems for given initial conditions. The theoretical developments have been applied successfully to the optimization-based control of a model of insulin infusion, for which all the existing stability results we are aware of failed.

It would be interesting in future work to develop similar tools to prove that solutions exhibit a finite number of switches and exploit this property for stability analysis for other classes of hybrid dynamical systems. Another potential direction would be proving the existence of a finite number of switches using the LCS approach. This would first require expressing the CLS as a LCS, followed by the derivation of stability conditions for discrete-time LCS. This path is very challenging. Indeed, stability analysis for discrete-time LCS remains underdeveloped, with limited attention in the literature.

Appendix

A. Sampled-data modeling for optimal insulin infusion

Let y be the BGL response, u the insulin input and $f = F\delta$ an impulse of food at time $t = 0$. In [7] impulses are used as a mathematical abstraction of insulin applied, and meals consumed and the optimal insulin infusion model is given by linear transfer functions as, for any $s \in \mathbb{C}$, $y(s) = T_F(s)f(s) - T_I(s)u(s)$ where

$$T_I(s) := \frac{K_I}{(a_1 s+1)(a_2 s+1)(a_3 s+1)}, \tag{34}$$

with $K_I = 600$ (mmol/L)/(U/min), $a_1 = 60$ min, $a_2 = 100$ min, $a_3 = 120$ min, and

$$T_F(s) := \frac{K_F}{(b_1 s+1)(b_2 s+1)(b_3 s+1)}$$

with $K_F = 50$ (mmol/L)/(g/min), $b_1 = 70$ min, $b_2 = 110$ min and $b_3 = 125$ min. To enhance disturbance rejection, we add an integral action, however instead of implementing a pure integral action, we opt for a first-order filter, for any $s \in \mathbb{C}$, $\tilde{y}(s) = y(s)/(s+\tau)$ with $\tau = 0.015$. To design the state feedback control, we use a state space model with four state variables, three of them correspond to the transfer function T_I in (34). It is given by $\dot{x} = A_c x + B_c u$, $y = C_c x$ with

$$A_c := \begin{bmatrix} -0.0350 & -0.0249 & -0.0114 & 0 \\ 0.0156 & 0 & 0 & 0 \\ 0 & 0.0078 & 0 & 0 \\ 0 & 0 & -3.4133 & -0.0150 \end{bmatrix},$$

$$B_c := \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C_c := \begin{bmatrix} 0 & 0 & -3.4133 & 0 \end{bmatrix}.$$

We consider the quadratic cost

$$J_c(x_0, u) := \int_0^\infty \Big(x^\top(\tau) Q_c x(\tau) + u^\top(\tau) R_c u(\tau) \Big) d\tau,$$

with $Q_c := \mathrm{diag}(1,1,1,70)$ and $R_c = 1$. The design is performed using a sampled-data version, with a sampling period of $T = 5$ min, that is (1) with $A = e^{A_c T}$, $B = e^{A_c T} B_c$, and cost J as in (2) with $Q = \int_0^T (e^{A_c \nu})^\top Q_c e^{A_c \nu} d\nu$, $S = \int_0^T (e^{A_c \nu})^\top Q_c e^{A_c \nu} B_c d\nu$, $R = \int_0^T B_c^\top (e^{A_c \nu})^\top Q_c e^{A_c \nu} B_c d\nu + R_c$. This leads to the matrices in (7), and $R = 17.8719$,

$$Q = 10^4 \begin{bmatrix} 0.0004 & 0.0000 & 0.0009 & -0.0001 \\ 0.0000 & 0.0012 & 0.0474 & -0.0036 \\ 0.0009 & 0.0474 & 3.2140 & -0.2772 \\ -0.0001 & -0.0036 & -0.2772 & 0.0325 \end{bmatrix}$$ and

$$S = \begin{bmatrix} 8.4359 \\ 0.1558 \\ 18.8537 \\ -1.3612 \end{bmatrix}.$$

B. LMI conditions for conewise quadratic Lyapunov function (13) with the same partition as (9)

The next result reduces the conservatism of the conditions in [24] by taking into account the switches from one cone to another, and obviously applies to general CLS with $m = 2$.

Proposition 5. *The origin of (9)-(10) is GES if there exist symmetric matrices P_1, P_2, and symmetric matrices Y_i, $i \in \{1,\ldots,4\}$ with non-negative entries, satisfying*

$$\begin{aligned}
&P_1 - \begin{bmatrix} K \\ KA_1 \end{bmatrix}^\top Y_1 \begin{bmatrix} K \\ KA_1 \end{bmatrix} > 0 \\
&A_1^\top P_1 A_1 - P_1 + \begin{bmatrix} K \\ KA_1 \end{bmatrix}^\top Y_1 \begin{bmatrix} K \\ KA_1 \end{bmatrix} < 0 \\
&P_2 - \begin{bmatrix} K \\ KA_2 \end{bmatrix}^\top Y_2 \begin{bmatrix} K \\ KA_2 \end{bmatrix} > 0 \\
&A_2^\top P_2 A_2 - P_2 + \begin{bmatrix} K \\ KA_2 \end{bmatrix}^\top Y_2 \begin{bmatrix} K \\ KA_2 \end{bmatrix} < 0 \\
&A_2^\top P_1 A_2 - P_2 + \begin{bmatrix} -K \\ KA_2 \end{bmatrix}^\top Y_3 \begin{bmatrix} -K \\ KA_2 \end{bmatrix} < 0 \\
&A_1^\top P_2 A_1 - P_1 + \begin{bmatrix} K \\ -KA_1 \end{bmatrix}^\top Y_4 \begin{bmatrix} K \\ -KA_1 \end{bmatrix} < 0.
\end{aligned} \tag{35}$$

Proof: The proof uses (13) as a Lyapunov function candidate. The next conditions ensure that $x \mapsto x^\top P_i x$, $i \in \{1,2\}$, strictly decreases along solutions to (9), for $x \neq 0$,

$$\begin{aligned}
&\begin{cases} x^\top P_1 x > \mathbf{0}, \\ x^\top (A_1^\top P_1 A_1 - P_1) x < \mathbf{0}, \end{cases} && x \in \mathcal{C}_1 \text{ and } A_1 x \in \mathcal{C}_1, \\
&\begin{cases} x^\top P_2 x > \mathbf{0}, \\ x^\top (A_2^\top P_2 A_2 - P_2) x < \mathbf{0}, \end{cases} && x \in \mathcal{C}_2 \text{ and } A_2 x \in \mathcal{C}_2, \\
&x^\top (A_2^\top P_1 A_2 - P_2) x < \mathbf{0}, && x \in \mathcal{C}_2 \text{ and } A_2 x \in \mathcal{C}_1, \\
&x^\top (A_1^\top P_2 A_1 - P_1) x < \mathbf{0}, && x \in \mathcal{C}_1 \text{ and } A_1 x \in \mathcal{C}_2.
\end{aligned}$$

The desired result is obtained by using the S-procedure, which leads to the LMIs conditions in (35). ∎

C. Discrete-time version of the results in [32]

The aim of this appendix is to prove that global attractivity of the origin is equivalent to the origin to be GES for homogeneous discrete-time systems of degree 1, which includes system (14). We consider for this purpose the system

$$x_{t+1} = f(x_t), \tag{36}$$

where $x_t \in \mathbb{R}^n$ is the state at time $t \in \mathbb{N}$, $n \in \mathbb{N}^\star$, and $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies the next assumption.

Assumption 3. *Vector field f in* (36) *verifies the properties:*

(i) *f is continuous on $\mathbb{R}^n$.*

(ii) *f is homogeneous of degree 1, i.e., for any $x \in \mathbb{R}^n$ and any $\lambda \in \mathbb{R}_+$, $f(\lambda x) = \lambda f(x)$.*

We denote the solution to (36) initialized at $x \in \mathbb{R}^n$ at time $t \in \mathbb{N}$ as $\phi(t,x)$. The goal is to prove the next result, which is stated in [32] for continuous-time systems. Note that the result below is invoked in [33], [34] with no proofs.

Theorem 6. *Consider system* (36) *and suppose that Assumption 3 holds. The following statements are equivalent.*

(i) *$x = 0$ is globally attractive, i.e., any solution ϕ asymptotically converges to 0 as time grows.*

(ii) *$x = 0$ is GES.*

Before proving Theorem 6, we need to state the next lemmas. Define for $x \in \mathbb{R}^n$ and Ω an open set of $\mathbb{R}^n$,

$$\tau(x,\Omega) = \inf\{t \in \mathbb{N} \ : \ \phi(t,x) \in \Omega\},$$

with the convention that $\inf \varnothing = \infty$. The next lemma is a discrete-time special case of [35, Corollary III.3].

Lemma 3. *Consider* (36) *and suppose the following holds.*

(i) Item (i) of Assumption 3 holds.

(ii) There exist $\mathcal{D} \subset \mathbb{R}^n$ compact, $\Omega \subseteq \mathbb{R}^n$ open, $\mathcal{S}$ compact with $\mathcal{S} \subset \Omega$ and

$$\forall x \in \mathcal{D},\ \exists t \in \mathbb{N} \text{ s.t. } \phi(t,x) \in \mathcal{S}. \tag{37}$$

Then there exists $t_0 \in \mathbb{N}$ such that $\tau(x,\Omega) \leqslant t_0$ for every $x \in \mathcal{D}$.

Proof: Assume by contradiction that there exist $(x_m)_{m\in\mathbb{N}} \in \mathcal{D}^{\mathbb{N}}$ such that $\tau(x_m,\Omega) \geqslant m$. Since $\mathcal{D}$ is compact, up to an extraction, we can assume that $(x_m)_m$ converges to some $x^* \in \mathcal{D}$. But according to (37), there exist $t^* \in \mathbb{N}$ such that $\phi(t^*,x^*) \in \mathcal{S}$. In particular, there exists $\varepsilon > 0$ such that $\mathbb{B}(x^*,\varepsilon) \subset \Omega$. But for every $m > t^*$, we have $\phi(t^*,x_m) \notin \Omega$, hence, $|\phi(t^*,x_m) - \phi(t^*,x^*)| \geqslant \varepsilon$. This leads to a contradiction with $x \mapsto \phi(t^*,x)$ continuous on $\mathbb{R}^n$. ■

The second and final lemma is stated without proof, as it directly follows from the continuity assumption made on f.

Lemma 4. *Consider system* (36) *and suppose that item (i) of Assumption 3 holds. Let Ω be a bounded set of $\mathbb{R}^n$ and let $t_0 \in \mathbb{N}$. There exist $c > 0$ depending on Ω and t_0 such that $|\phi(t,x)| \leqslant c$, for every $x \in \Omega$ and every $t \in \{0,\dots,t_0\}$,*

We are ready to prove Theorem 6.

Proof of Theorem 6: The implication (ii)⇒(i) directly follows from the corresponding statements. We therefore focus on proving (i)⇒(ii). By homogeneity, it is enough to prove the implication for $|x| = 1$. Lemma 3 together with item (i) of Theorem 6 ensure the existence of $t_0 \in \mathbb{N}$ and of $t_{x,0} \in \{0,\dots,t_0\}$ such that $\phi(t_{x,0},x) \in \mathbb{B}(0,1/2)$. Now, for every $t \in \mathbb{N}$, $\phi(t_{x,0}+t,x) = \phi(t,\phi(t_{x,0},x)) = |\phi(t_{x,0},x)|\phi(t,y)$ for $y \in \mathbb{R}^n$, such that $|y| = 1$ and $|\phi(t_{x,0},x)|y = \phi(t_{x,0},x)$. Hence, there exists $t_{y,0} \in \{0,\dots,t_0\}$ such that $|\phi(t_{y,0},y)| \leqslant 1/2$, i.e., $|\phi(t_{x,0}+t_{y,0},x)| \leqslant 1/4$. Thus, using the homogeneity relation, and repeating the above argument, we conclude to the existence of $(t_{x,\ell})_{\ell\in\mathbb{N}} \in \mathbb{N}^{\mathbb{N}}$ such that $t_{x,\ell} \leqslant t_{x,\ell+1} \leqslant t_{x,\ell} + t_0$ and $\phi(t_{x,\ell},x) \in \mathbb{B}(0,2^{-\ell})$ for every $\ell \in \mathbb{N}$. Finally, using the homogeneity, together with Lemma 4, we conclude to the existence of a constant $c = c(t_0)$ such that for every $\ell \in \mathbb{N}$ and every $t \in \{t_{x,\ell},\dots,t_{x,\ell+1}\}$, $|\phi(t,x)| \leqslant c2^{-\ell}$. That is to say that $|\phi(t,x)| \leqslant c2^{-\nu_x(t)}$, where $\nu_x(t) = \operatorname{card}\{\ell \in \mathbb{N} : t_{x,\ell} \leqslant t\}$. Observe that $\nu_x(t) \geqslant \lfloor t/t_0 \rfloor \geqslant (t/t_0) - 1$. We conclude the proof by setting $c_1 = 2c$ and $c_2 = \frac{\ln 2}{t_0}$. ■

D. Details for the example of Section VI-B

To illustrate how the conditions of Theorem 4 are checked, we treat two pairs of (t_2,t_3) among the 58 pairs: $(t_2,t_3) = (1,1)$ and $(t_2,t_3) = (3,2)$.

For $t_2 = 1$ and $t_3 = 1$, we have from (23)[3]

$$\begin{aligned} N_{1;t} &= K, \\ N_{1;t_1,t} &= -KA_1, \\ N_{1;t_1,t_2,t} &= KA_2A_1, \\ N_{1;t_1,t_2,t_3,0} &= -KA_1A_2A_1. \end{aligned} \tag{38}$$

This leads to

$$\mathcal{N} = \begin{bmatrix} -0.4936 & -6.9988 & -104.7360 & 1.6626 \\ 0.0139 & -3.5448 & -93.6330 & 2.0398 \\ -0.0248 & 9.9744 & 246.4813 & -4.9790 \\ -0.0140 & 1.8974 & 69.7493 & -1.8635 \end{bmatrix}.$$

To check that the intersection (22) is reduced to $\{0\}$ for every $t \in \mathbb{N}^\star$, we have to check that $\mathcal{N}x \geqslant 0$ implies $-KA_2^t\mathcal{M}x \geqslant 0$, for all $t \in \mathbb{N}^\star$ or equivalently $-KA_2^t\mathcal{M} = \beta_t^\top \mathcal{N}$, for all $t \in \mathbb{N}^\star$. We obtain

$$\beta_0 = \begin{bmatrix} 0.0100 & 3.3166 & 1.0539 & 3.7582 \end{bmatrix}^\top,$$

$$\mathcal{L} = \begin{bmatrix} 0.0000 & 0.0000 & -0.0000 & 0.0100 \\ -77.6570 & -0.0000 & 0.0000 & 3.3166 \\ -22.1708 & -0.0129 & 0.0000 & 1.0539 \\ 300.8926 & 22.1708 & -77.6570 & 3.7582 \end{bmatrix},$$

$$\gamma_1 v_1 = 10^5 \begin{bmatrix} 0.0090 \\ 2.2552 \\ 0.7100 \\ 0.8637 \end{bmatrix}, \quad \gamma_2 v_2 = 10^5 \begin{bmatrix} -0.0148 \\ -3.6872 \\ -1.1608 \\ -1.4041 \end{bmatrix},$$

$$\gamma_3 v_3 = 10^5 \begin{bmatrix} 0.0141 \\ 3.4783 \\ 1.0951 \\ 1.3027 \end{bmatrix}, \quad \gamma_4 v_4 = 10^5 \begin{bmatrix} -0.0083 \\ -2.0462 \\ -0.6442 \\ -0.7622 \end{bmatrix}.$$

As $\beta_0 = \sum_{i=1}^4 \gamma_i v_i \lambda_i^0 > 0$, $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, and conditions (32) hold, we conclude using Theorem 5 that

$$\beta_t = \sum_{i=1}^4 \gamma_i v_i \lambda_i^t > 0, \quad \forall t \in \mathbb{N}.$$

Now, we consider the case $t_2 = 3$ and $t_3 = 2$. The number of possible matrices having 4 lines that can be generated from (23) is 35. We have to check if there is an invertible matrix $\mathcal{N}$ that allows to conclude that the intersection (22) is reduced to $\{0\}$ for every $t \in \mathbb{N}^\star$. We found,

$$\mathcal{N} = \begin{bmatrix} -0.4936 & -6.9988 & -104.7360 & 1.6626 \\ 0.0139 & -3.5448 & -93.6330 & 2.0398 \\ 0.0248 & -9.9744 & -246.4813 & 4.9790 \\ -0.0210 & 11.0313 & 262.8501 & -4.9946 \end{bmatrix}.$$

As previously, we compute

$$\beta_0 = \begin{bmatrix} 0.0017 & 0.0607 & 0.3860 & 2.7102 \end{bmatrix}^\top,$$

$$\mathcal{L} = \begin{bmatrix} 1.1997 & -0.0049 & -0.00076 & 0.00166 \\ 210.63 & -1.1799 & -0.42343 & 0.0607 \\ -239.25 & 1.9792 & 1.0282 & 0.386 \\ 32044 & -286.92 & -98.232 & 2.7102 \end{bmatrix},$$

$$\gamma_1 v_1 = 10^4 \begin{bmatrix} 0.0122 \\ 1.0926 \\ 0.8372 \\ 1.8402 \end{bmatrix}, \quad \gamma_2 v_2 = 10^4 \begin{bmatrix} -0.0201 \\ -1.7998 \\ -1.3368 \\ -2.9338 \end{bmatrix},$$

[3] Recall Theorem 4 is used with $t_1 = 1$, and recall that in (23), $N_{i_1;t_1,\dots,t_p,t}$ is defined for $0 \leqslant t < t_{p+1}$. Hence, in (38), the variable t is 0 in any cases.

$$\gamma_3 v_3 = 10^4 \begin{bmatrix} 0.0190 \\ 1.7366 \\ 1.1679 \\ 2.5604 \end{bmatrix}, \quad \gamma_4 v_4 = 10^4 \begin{bmatrix} -0.0112 \\ -1.0295 \\ -0.6683 \\ -1.4666 \end{bmatrix}.$$

We have $\beta_0 = \sum_{i=1}^4 \gamma_i v_i \lambda_i^0 > 0$, $\lambda_1 > \lambda_2 > \lambda_3 > \lambda_4 > 0$, and we verified that conditions (32) are satisfied. We conclude that

$$\beta_t = \sum_{i=1}^4 \gamma_i v_i \lambda_i^t > 0, \quad \forall t \in \mathbb{N}.$$

The same procedure has been repeated for all the 58 possible values of t_2 and t_3 meaning that we were able to exhibit a matrix $\mathcal{N}$ such that, for every $t \in \mathbb{N}^\star$, the intersection (22) is reduced to $\{0\}$ in all these cases.

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