A Passivity Analysis Tool for Linear Clustered Multi-Agent Systems

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***Abstract*—Passivity of a large-scale interconnected system is often broken down to the passivity of the individual subsystems that compose it. Nevertheless, there are cases in which the individual elements are not all passive, yet the overall large-scale system is. In such scenarios, we need to directly solve very large problems to conclude on the passivity. This letter proposes a methodology to analyze passivity based on the topology of the multi-agent system. In many cases, large multi-agent systems are formed by interconnected clusters, which are groups of agents densely interconnected. The clusters are sparsely interconnected with each other and this leads to a time scale-separation with a fast dynamics inside the clusters and a slow one between them. The purpose of this letter is twofold. First, we exploit the time-scale separation property inherent to such a system to provide a computationally efficient alternative to analyze its passivity. Second, we provide insight into how robust its passivity is with respect to the inter- and intra-cluster agent interactions. To achieve this, we consider the singular perturbation framework with respect to the ratio of the strength of the controls between and within the clusters, and rely on the connection between positive realness, passivity, and multi-input multi-output system phase. We consider agents with identical linear time-invariant dynamics. The method is illustrated on a numerical example.**

***Index Terms*—Agents-based systems; singular perturbation analysis; passivity; positive realness; phase shift.**

I. Introduction

MULTI-agent systems (MAS) are systems composed of interconnected autonomous dynamical agents, with research focusing on consensus, agreement, rendezvous, swarming, or other characteristics. There are many engineering applications of this framework, such as power networks [1], biological systems [2], and cyber-physical systems [3], for example. A particular case of MAS are that of *clustered network systems*, where connections inside the clusters are dense, while those between clusters are sparse.

Specific to clustered networks is the time-scale separation (TSS) property, in which one can decouple its dynamics into slow and fast ones. This property has been emphasized in [4] and has been used for analysis [5], design of synchronization laws [6], [7], aggregation [1], [3], model reduction [8]. It is noteworthy that the enumerated results, except for [8], use the Standard Singular Perturbation Form (SSPF) [9] of a system to study the previous aspects, proving it to be an effective tool for such problems. To the best of the authors' knowledge, there are no results concerning the passivity of the MAS based on its SSPF properties, as available research focuses mainly on network synchronization using passivity-based techniques.

The concept of *passivity* has been of high interest since its conception in the 1970s, because it is a natural generalization of the stability of a dynamical system [10]. It is useful both to analyze and impose system properties [11], but also for synthesis of networked systems, due to the fact that parallel and feedback connections of passive systems remain passive. Passivity has been used to impose group coordination [12], in control over networks [13] and synchronization using phase analysis [14]. Unfortunately, for large-scale MAS, the direct approach to check passivity is not tractable, due to the high dimension of the conditions to solve, which are generally further combined with other performance criteria.

Our scope is to provide sufficient conditions for the passivity of clustered MAS using its SSPF. We consider multi-input and multi-output (MIMO) linear time-invariant (LTI) agents with identical dynamics, leading to *homogeneous* clustered MAS. The ratio of the strength of the controls between and within the clusters can be seen as a singular perturbation parameter $\varepsilon > 0$. This, in turn, is considered in the ε-bound computation problem as a sensitivity measure to determine how close the overall network system is to losing its passivity if a new connection appears or is removed from the network. The contributions of this letter are: (*i*) to extend the framework in [15] for passivity analysis to include fast actuation alongside the implicit slow actuators of the SSPF; (*ii*) to directly apply said extension for the passivity analysis of homogeneous clustered MAS. We thus bypass the passivity check for the full system, which significantly reduces the computational overhead, and convert it into a passivity test for its reduced-order subsystem alongside phase conditions on a variation of the boundary layer subsystem in SSPF.

After providing a set of necessary tools in Section II, Section III provides the network setup and problem statement, Section IV applies the standard coordinate changes to enforce the time-scale separation in the MAS. Section V presents the novel results for passivity analysis of the resulting SSPF

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with slow and fast actuation. A brief numerical example is presented in Section VI, followed by remarks and extensions in Section VII.

II. Notations and Necessary Tools

The set of natural, real, complex numbers and n-dimensional vectors with components in $\{0,1\}$ are denoted by $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{Z}_2^n$. The column vectors with zero and one values of size n are represented by $\mathbf{0}_n \equiv \mathbf{0}$ and $\mathbf{1}_n \equiv \mathbf{1}$. Furthermore, O and I are the zero and identity matrices of appropriate dimensions. An orthonormal matrix $M \in \mathbb{R}^{n\times n}$ has the property $M^\top M = MM^\top = I_n$. We denote the Kronecker product by the symbol $\otimes$. $\|\cdot\|_2$ denotes the Euclidean norm. A vector-valued function $f(x,\varepsilon)$ is said to be $\mathcal{O}(\varepsilon)$ on a compact set D_x if there exist constants $k, \varepsilon^\star > 0$ such that $\|f(x,\varepsilon)\|_2 \le k\varepsilon,\ \forall \varepsilon \in [0,\varepsilon^\star],\ \forall x \in D_x$.

We use the descriptor state-space system (DSS) framework throughout the letter. We denote by Σ the continuous-time DSS, with input-state-output interface (u,x,y) and transfer matrix $\Sigma|_{y,u} \equiv \Sigma(s) = C(sE-A)^{-1}B + D$, having the equivalent signals (U,X,Y) in Laplace domain. We consider its inverse, series, and parallel connections as in [15]. We denote the derivative of a signal x as $\dot{x}$ in standard time t, $\mathrm{d}x(t_s)/\mathrm{d}t_s \equiv \mathrm{d}x/\mathrm{d}t_s$ in slow time scale t_s, $\mathrm{d}x(t_f)/\mathrm{d}t_f \equiv \mathrm{d}x/\mathrm{d}t_f$ in fast time scale t_f. System Σ is in time scale t_s, while its counterpart in time scale t_f is denoted by ${}^{t_f}\Sigma$.

For $X \in \mathbb{C}^{m\times p}$, X^H denotes its complex conjugate transpose. The concept of phase of a MIMO LTI system is well established in the literature, see, e.g., [14, Section 3], [15]:

Definition 1 ([15]): Let Ω be the frequency set for which $j\Omega$ is the set of imaginary axis poles of Σ. Given a frequency $\omega \in [-\infty,\infty]\backslash\Omega = \overline{\Omega}$, the phases of Σ at ω are the interval based on the eigenvalues of its *sectorial decomposition* [16]:

$$\varphi(\Sigma(j\omega)) = \left[\underline{\phi}(\Sigma(j\omega)), \overline{\phi}(\Sigma(j\omega))\right] = [\theta_1, \theta_{n_u}], \tag{1}$$

where $\Sigma(j\omega) = X\cdot \mathrm{diag}(e^{j\theta_1},\dots,e^{j\theta_{n_u}})\cdot X^H$, with invertible matrix X, $-\alpha \le \theta_1 \le \dots \le \theta_{n_u} \le \alpha$, with sector $\alpha \in [0,\frac{\pi}{2})$, and the phases of the system are $\varphi(\Sigma) = [\underline{\phi}(\Sigma), \overline{\phi}(\Sigma)]$, with:

$$[\underline{\phi}(\Sigma), \overline{\phi}(\Sigma)] = [\inf_{\omega\in\overline{\Omega}} \underline{\phi}(\Sigma(j\omega)), \sup_{\omega\in\overline{\Omega}} \overline{\phi}(\Sigma(j\omega))]. \tag{2}$$

We denote the vector relative degree of Σ by $\rho(\Sigma)$, as in [17].

Definition 2 ([17]): A $n_u \times n_u$ linear system $\Sigma = (A,B,C)$ has a *vector relative degree* $\rho(\Sigma) = (\rho_1,\dots,\rho_{n_u}) \in \mathbb{N}^{n_u}$ iff:

1) $\forall j \in \overline{1,n_u}$, $\forall k \in \overline{0,\rho_j - 2}$: $C_{(j,\cdot)}A^kB = \mathbf{0}_{n_u}$;
2) $\mathrm{rank}\left((C_{(1,\cdot)}A^{\rho_1-1}B)^\top \dots (C_{(n_u,\cdot)}A^{\rho_{n_u}-1}B)^\top\right)^\top = n_u$.

The passivity property is both an energy- and frequency-based characterization of the input–output behavior of a DSS, inherent to the system, and does not depend on the control law.

Definition 3 ([10]): System Σ is *input-output passive* if there exists a continuously differentiable positive semidefinite storage function $V(x)$ such that: $u^\top y \ge \dot{V} = \frac{\partial V}{\partial x}\dot{x}$, $\forall (x,u)$.

For LTI systems, passivity and positive realness are equivalent properties [19], with necessary and sufficient conditions expressed in terms of stability, MIMO phase and relative degree.

Lemma 1 ([14]): An $r\times r$ MIMO transfer matrix Σ for which $\Sigma(s) + \Sigma(-s)^\top \not\equiv 0$ is *strictly positive real* if and only if $\Sigma(s)$ is Hurwitz and $\varphi(\Sigma) \subseteq \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$. If Σ is semi-stable and $\varphi(\Sigma) \subseteq \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$, then the system is *positive real* only.

III. Problem Statement

Consider a network of n identical square (i.e. same number of inputs and outputs) MIMO agents partitioned into m non-empty clusters $\mathcal{C}_1,\dots,\mathcal{C}_m \subset \mathcal{V}$. A clustered network (in the sense of [1], [4]) refers to a network that is partitioned into distinct groups of agents having dense connection structures, whereas the connections between the clusters are sparse. We denote by $\mathcal{M} = \{1,2,\dots,m\}$ the set of clusters, while n_k represents the cardinality of cluster $\mathcal{C}_k$, with $n = \sum_{k=1}^m n_k$. Each agent in the network is identified by a couple $(k,i) \in \mathcal{C}_k$, where k refers to the cluster $\mathcal{C}_k$ and i is the index of the agent in cluster $\mathcal{C}_k$. The notation $(k,j) \in \mathcal{N}_{k,i}$ represents the neighbors of agent (k,i) in the same cluster $\mathcal{C}_k$. Each agent $(k,i) \in \mathcal{C}_k$, $k \in \mathcal{M}$, is assigned a *state* $x_{k,i} \in \mathbb{R}^{n_x}$:

$$\dot{x}_{k,i} = Ax_{k,i} + Bu_{k,i}, \quad y_{k,i} = Cx_{k,i}, \tag{3}$$

where $u_{k,i}, y_{k,i} \in \mathbb{R}^{n_u}$, $A \in \mathbb{R}^{n_x\times n_x}$, $B \in \mathbb{R}^{n_x\times n_u}$ and $C \in \mathbb{R}^{n_u\times n_x}$. For each cluster $\mathcal{C}_k$, let $x_k = (x_{k,1}^\top,\dots,x_{k,n_k}^\top)^\top \in \mathbb{R}^{n_k\cdot n_x}$ be its state, $u_k = (u_{k,1}^\top,\dots,u_{k,n_k}^\top)^\top \in \mathbb{R}^{n_k\cdot n_u}$ its control input and $y_k = (y_{k,1}^\top,\dots,y_{k,n_k}^\top)^\top \in \mathbb{R}^{n_u\cdot n_k}$ its output. Thus, the cluster dynamics takes the form, for all $k \in \mathcal{M}$:

$$\dot{x}_k = (I_{n_k}\otimes A)\,x_k + (I_{n_k}\otimes B)\,u_k;\ \ y_k = (I_{n_k}\otimes C)\,x_k. \tag{4}$$

The overall network dynamics will have the following form:

$$\dot{x} = (I_n\otimes A)\,x + (I_n\otimes B)\,u; \quad y = (I_n\otimes C)\,x, \tag{5}$$

where $x = \left(x_1^\top,\dots,x_m^\top\right)^\top \in \mathbb{R}^{n\cdot n_x}$, $u = \left(u_1^\top,\dots,u_m^\top\right)^\top \in \mathbb{R}^{n\cdot n_u}$, and $y = \left(y_1^\top,\dots,y_m^\top\right)^\top \in \mathbb{R}^{n\cdot n_u}$ are the network state, network control input, and network output, respectively.

The interactions between the agents in the network are encoded by the Laplacian matrix $\mathcal{L} \in \mathbb{R}^{n\times n}$, written as $\mathcal{L} = \mathcal{L}^{\text{int}} + \mathcal{L}^{\text{ext}}$. The internal Laplacian of the network is $\mathcal{L}^{\text{int}} = \mathrm{diag}(\mathcal{L}_1^{\text{int}},\dots,\mathcal{L}_m^{\text{int}})$, where each block $\mathcal{L}_k^{\text{int}} \in \mathbb{R}^{n_k\times n_k}$ refers to the Laplacian of the cluster $\mathcal{C}_k$ excluding the external connections. The external Laplacian $\mathcal{L}^{\text{ext}} \in \mathbb{R}^{n\times n}$ represents the connections between agents from different clusters.

We start from a composite control comprised of internal and external synchronization terms (as in [6]), extended with an exogenous reference input by which the passivity of the network system (5) is to be analyzed. This leads to the form:

$$u_k = u_k^{\text{int}} + u_k^{\text{ext}} + r_k, \quad \forall k \in \mathcal{M}, \tag{6}$$

where $u_k^{\text{int}} = (u_{k,1}^{\text{int}\top},\dots,u_{k,n_k}^{\text{int}\top})^\top$, $u_k^{\text{ext}} = (u_{k,1}^{\text{ext}\top},\dots,u_{k,n_k}^{\text{ext}\top})^\top$, and $r_k = (r_{k,1}^\top,\dots,r_{k,n_k}^\top)^\top$:

$$u_{k,i}^{\text{int}} = -K_k^{\text{int}} \textstyle\sum_{(k,j)\in\mathcal{N}_{k,i}} (x_{k,i} - x_{k,j})\,; \tag{7a}$$

$$u_{k,i}^{\text{ext}} = -K_k^{\text{ext}} \textstyle\sum_{(\neg k,j)\in\mathcal{N}_{\neg k,i}} (x_{k,i} - x_{\neg k,j})\,, \tag{7b}$$

where K_k^{int}, $K_k^{\text{ext}} \in \mathbb{R}^{n_u\times n_x}$. The notation $(\neg k,j) \in \mathcal{N}_{\neg k,i}$ indicates the neighbors of an agent (k,i) belonging to a different cluster. The internal control u_k^{int} is the effort required to achieve local agreement, whereas the external control u_k^{ext} is the command necessary to synchronize the agents between the clusters. Furthermore, the reference signal $r_k = r_{k,s} + r_{k,f}$ will be further split into slow and fast references, respectively, to later exploit the system's inherent time-scale separation.

This letter provides sufficient conditions to guarantee the passivity of the MAS (5) with respect to the reference r_k, based on the reduced subsystems specific to its SSPF. The analysis involves the structural properties of the system and does not perturb the network consensus provided by the composite control law (6)–(7). The design of the state feedback matrices K_k^{int}, K_k^{ext} is assumed from reference [6]. We provide conditions under which the passivity property is maintained with respect to the ratio of the strength of the controls between and within the clusters, which in SSPF corresponds to the perturbation variable ε. To develop conditions to achieve the above objective, the following assumption is made, similarly to [6].

Assumption 1: The overall graph and the graphs of the clusters are undirected and connected. The internal graphs are very dense for all clusters, i.e. we assume that all non-zero eigenvalues for an internal Laplacian can be approximated by n_k [18]: $\lambda_i\left(\mathcal{L}_k^{\text{int}}\right) \approx n_k$, $i = \overline{2, n_k}$, $\forall k \in \mathcal{M}$. Furthermore, the agents are controllable, ensuring that the synchronization of the overall network system via control law (6)–(7) is attainable.

IV. SSPF of the Network Control System

The current section presents two coordinate changes for the SSPF of the network system (5), namely the state signal scaling and the time domain scaling. We follow the overall outline from [6], with the remark that we extend the network system to also account for outputs, and our purpose in Section V is to study the MAS passivity, rather than synchronization with a cost minimization as in [6].

Consider the coordinate transformation for cluster $\mathcal{C}_k$ using the Jordan canonical form of its symmetric Laplacian $\mathcal{L}_k^{\text{int}}$ [7]:

$$\mathcal{L}_k^{\text{int}} = T_k \begin{pmatrix} 0 & \mathbf{0}^\top \\ \mathbf{0} & \Lambda_k^{\text{int}} \end{pmatrix} T_k^\top, \ \forall k \in \mathcal{M}, \tag{8}$$

where $T_k \in \mathbb{R}^{n_k \times n_k}$ is an orthonormal matrix and $\Lambda_k^{\text{int}} = \text{diag}(\lambda_{k,2}^{\text{int}}, \ldots, \lambda_{k,n_k}^{\text{int}})$ collects the $n_k - 1$ positive eigenvalues of $\mathcal{L}_k^{\text{int}}$. Consequently, the matrix T_k can be expressed as:

$$T_k = \begin{pmatrix} v_{k,1} & V_k \end{pmatrix}, \ \forall k \in \mathcal{M}, \tag{9}$$

where $v_{k,1} = \frac{1}{\sqrt{n_k}}\mathbf{1}_{n_k}$ is the eigenvector associated with the 0 eigenvalue and the matrix $V_k \in \mathbb{R}^{n_k \times (n_k - 1)}$ contains the eigenvectors corresponding to the nonzero eigenvalues of $\mathcal{L}_k^{\text{int}}$.

We now define the coordinate transformation as:

$$\overline{x}_k = \begin{pmatrix} \zeta_k^\top & \xi_k^\top \end{pmatrix}^\top = \left(\tfrac{1}{\sqrt{n_k}} T_k^\top \otimes I_{n_x}\right) x_k, \ \forall k \in \mathcal{M}. \tag{10}$$

From (9)–(10), for all $k \in \mathcal{M}$, we obtain a new state:

$$\zeta_k = \left(\tfrac{1}{n_k}\mathbf{1}_{n_k}^\top \otimes I_{n_x}\right) x_k = H_k x_k \in \mathbb{R}^{n_x}; \tag{11a}$$

$$\xi_k = \left(\tfrac{1}{\sqrt{n_k}} V_k^\top \otimes I_{n_x}\right) x_k = Z_k x_k \in \mathbb{R}^{(n_k - 1)\cdot n_x}, \tag{11b}$$

with $H_k \in \mathbb{R}^{n_x \times n_k \cdot n_x}$ and $Z_k \in \mathbb{R}^{(n_k-1)\cdot n_x \times n_k \cdot n_x}$. The first component, ζ_k, corresponds to the aggregate state [1], i.e., the average of the agents' states in clusters $\mathcal{C}_k$. It represents the collective behavior of the clusters. The second component, ξ_k, corresponds to the local state [1], i.e., the synchronization error around each cluster's average, which evolves at a faster rate.

The inverse of transformation (10), $\forall k \in \mathcal{M}$, yields:

$$\begin{aligned} x_k &= (\sqrt{n_k} T_k \otimes I_{n_x})\, \overline{x}_k = (\mathbf{1}_{n_k} \otimes I_{n_x})\zeta_k + \\ &+ (\sqrt{n_k} V_k \times I_{n_x})\xi_k = \widetilde{H}_k \zeta_k + \widetilde{Z}_k \xi_k, \end{aligned} \tag{12}$$

where $\widetilde{H}_k \in \mathbb{R}^{n_k \cdot n_x \times n_x}$ and $\widetilde{Z}_k \in \mathbb{R}^{n_k \cdot n_x \times (n_k - 1)\cdot n_x}$.

For the overall network, we obtain:

$$\zeta = Hx; \quad \xi = Zx; \quad x = \widetilde{H}\zeta + \widetilde{Z}\xi, \tag{13}$$

with the stacked vectors $\zeta = (\zeta_1^\top, \ldots, \zeta_m^\top)^\top \in \mathbb{R}^{m \cdot n_x}$ and $\xi = (\xi_1^\top, \ldots, \xi_m^\top)^\top \in \mathbb{R}^{(n-m)\cdot n_x}$, alongside matrices $H = \text{diag}(H_1, \ldots, H_m)$, $Z = \text{diag}(Z_1, \ldots, Z_m)$, $\widetilde{H} = \text{diag}(\widetilde{H}_1, \ldots, \widetilde{H}_m)$, and $\widetilde{Z} = \text{diag}(\widetilde{Z}_1, \ldots, \widetilde{Z}_m)$. Next, we recast the overall network dynamics in terms of the new coordinate variables. The overall network system (5)–(7) is:

$$\begin{aligned} \dot{x} = \big[&(I_n \otimes A) - (I_n \otimes B)K^{\text{int}}(\mathcal{L}^{\text{int}} \otimes I_{n_x}) \\ &- (I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\big]\, x + (I_n \otimes B) r; \end{aligned} \tag{14a}$$

$$y = (I_n \otimes C)x, \tag{14b}$$

where $K^{\text{int}} = \text{diag}((I_{n_1} \otimes K_1^{\text{int}}), \ldots, (I_{n_m} \otimes K_m^{\text{int}}))$, $K^{\text{ext}} = \text{diag}((I_{n_1} \otimes K_1^{\text{ext}}), \ldots, (I_{n_m} \otimes K_m^{\text{ext}}))$ and stacked exogenous reference $r = (r_1^\top, \ldots, r_m^\top)^\top \in \mathbb{R}^{n \cdot n_u}$. Using the transformation (13), the MAS (14) is recast in new coordinates:

$$\dot{\zeta} = \widehat{A}_{11}\zeta + \widehat{A}_{12}\xi + \widehat{B}_1 r; \tag{15a}$$

$$\dot{\xi} = \widehat{A}_{21}\zeta + \left(\widehat{A}_{22}^1 + \widehat{A}_{22}^2\right)\xi + \widehat{B}_2 r; \tag{15b}$$

$$y = \widehat{C}_1\zeta + \widehat{C}_2\xi, \tag{15c}$$

where $\widehat{C}_1 = (I_n \otimes C)\widetilde{H}$; $\widehat{C}_2 = (I_n \otimes C)\widetilde{Z}$;
$\widehat{A}_{11} = [(I_m \otimes A) - H(I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\widetilde{H}]$;
$\widehat{A}_{12} = -H(I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\widetilde{Z}$; $\widehat{B}_1 = H(I_n \otimes B)$;
$\widehat{A}_{21} = -Z(I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\widetilde{H}$; $\widehat{B}_2 = Z(I_n \otimes B)$;
$\widehat{A}_{22}^1 = -Z(I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\widetilde{Z}$;
$\widehat{A}_{22}^2 = [(I_{n-m} \otimes A) - (I_{n-m} \otimes B)K_{n-m}^{\text{int}}(\Lambda^{\text{int}} \otimes I_{n_x})]$,

$K_{n-m}^{\text{int}} = \text{diag}((I_{n_1 - 1} \otimes K_1^{\text{int}}), \ldots, (I_{n_m - 1} \otimes K_m^{\text{int}}))$, and $\Lambda^{\text{int}} = \text{diag}(\Lambda_1^{\text{int}}, \ldots, \Lambda_m^{\text{int}})$.

We define the network parameters based on [4], [6], according to the density of connections inside and between the clusters of the consensus framework:

$$\mu^{\text{ext}} = \left\|(I_n \otimes B)K^{\text{ext}}(\mathcal{L}^{\text{ext}} \otimes I_{n_x})\right\|_2; \tag{17a}$$

$$\mu^{\text{int}} = \min_{k \in \mathcal{M}} \left\|\Lambda_k^{\text{int}} \otimes BK_k^{\text{int}}\right\|_2; \quad \varepsilon = \mu^{\text{ext}}/\mu^{\text{int}}. \tag{17b}$$

The ratio of the strength of the controls between and within the clusters is the perturbation parameter ε in SSPF [9].

References [1] and [4] define the time-scale separation in system (15) based on the above definition of ε and time variables: the fast time-scale $t_f = \mu^{\text{int}} \cdot t$ and the slow time-scale $t_s = \varepsilon \cdot t_f$. This allows us to represent system (15) in (ζ, ξ)-coordinates in SSPF as the system Σ^ε:

$$\mathrm{d}\zeta/\mathrm{d}t_s = A_{11}\zeta + A_{12}\xi + B_1 r; \tag{18a}$$

$$\varepsilon\, \mathrm{d}\xi/\mathrm{d}t_s = \varepsilon A_{21}\zeta + A_{22}\xi + B_2 r; \tag{18b}$$

$$y = C_1\zeta + C_2\xi, \tag{18c}$$

where $A_{11} = \widehat{A}_{11}/(\varepsilon\mu^{\text{int}})$; $A_{12} = \widehat{A}_{12}/(\varepsilon\mu^{\text{int}})$; $A_{21} = \widehat{A}_{21}/(\varepsilon\mu^{\text{int}})$; $A_{22}^1 = \widehat{A}_{22}^1/(\varepsilon\mu^{\text{int}})$; $A_{22}^2 = \widehat{A}_{22}^2/\mu^{\text{int}}$; $A_{22} = \varepsilon A_{22}^1 + A_{22}^2$; $B_1 = \widehat{B}_1/(\varepsilon\mu^{\text{int}})$; $B_2 = \widehat{B}_2/\mu^{\text{int}}$; $C_1 = \widehat{C}_1$; $C_2 = \widehat{C}_2$.

Next, we analyze the slow and fast dynamics of system (18) using the standard convention, i.e., $\zeta = \zeta_s(t_s) + \zeta_f(t_f)$, $\xi = \xi_s(t_s) + \xi_f(t_f)$, $r = r_s(t_s) + r_f(t_f)$, $y = y_s(t_s) + y_f(t_f)$. The *reduced-order* (slow) dynamics, using the standard approach for singular perturbation analysis [9], is obtained by setting $\varepsilon = 0$. As such, (18b) degenerates into $A_{22}^2\xi_s(t_s) + B_2 r_s(t_s) = 0$. This equation has the unique solution:

$$\xi_s(t_s) = -(A_{22}^2)^{-1}B_2 \cdot r_s(t_s) = h(r_s(t_s)), \tag{19}$$

Substituting (19) into (18a) leads to the system Σ_s:

$$\mathrm{d}\zeta_s/\mathrm{d}t_s = A_{11}\zeta_s + B_s r_s; \quad y_s = C_1\zeta_s + D_s r_s, \tag{20}$$

with $B_s = B_1 - A_{12}(A_{22}^2)^{-1}B_2$ and $D_s = -C_2(A_{22}^2)^{-1}B_2$. In our setting, the slow dynamics (20) represents the collective behavior of the clusters and it may or may not be stable. Moreover, writing (18b) in the fast time scale t_f, we have:

$$\begin{aligned}\mathrm{d}\xi_f/\mathrm{d}t_f &= \varepsilon A_{21}\zeta_f + A_{22}\xi_f - \varepsilon A_{22}^1(A_{22}^2)^{-1}(r_s + r_f)\\ &\quad + B_2 r_f + \varepsilon\frac{\partial h}{\partial r_s}\,\mathrm{d}r_s/\mathrm{d}t_s\,,\end{aligned} \tag{21}$$

which includes the derivative $\mathrm{d}r_s/\mathrm{d}t_s$. Setting $\varepsilon = 0$, we have $\mathrm{d}\zeta_f/\mathrm{d}t_f = 0$, leading to the *boundary layer* (fast) dynamics:

$$\mathrm{d}\xi_f|_{\varepsilon=0}/\mathrm{d}t_f = A_{22}^2\xi_f|_{\varepsilon=0} + B_2 r_f; \quad y_f = C_2\xi_f|_{\varepsilon=0}, \tag{22}$$

where ζ_f, ξ_f, r_f, y_f are the fast parts of the corresponding variables in (18). The above singular perturbation framework is valid in the following conditions, achievable by the design of the control gains.

Assumption 2: There exist an external gain K^{ext} and an internal gain K^{int} such that the slow average dynamics (20) is synchronized (i.e., A_{11} is Hurwitz, using K^{ext}), the fast synchronization error dynamics (22) is stabilized (i.e., A_{22}^2 is Hurwitz, using K^{int}), and the network parameter $\varepsilon \ll 1$ is small enough such that the time-scale separation occurs.

The above assumption can be satisfied by an adequate selection of K^{ext} and K^{int} using [6, Lemmas 1–3]. The gains further influence the TSS derived from Assumption 1, as ε, different from [4], does not depend only on the graph structure. Furthermore, ε allows a decoupling between the state matrices from (15): $\|\widehat{A}_{11}\|_2, \|\widehat{A}_{12}\|_2, \|\widehat{A}_{21}\|_2, \|\widehat{A}_{22}^1\|_2 \leq c_1\varepsilon\mu^{\text{int}}$, $\|\widehat{A}_{22}^2\|_2 \geq \mu^{\text{int}}$, $c_1 > 0$, which, in SSPF [9], makes ζ and ξ to be slow and fast variables, respectively. With Assumption 2 in mind, we can approximate the dynamics of (18) based on its reduced-order and boundary layer subsystems, according to Tikhonov's theorem, stated next for this particular setup.

Theorem 1 ([6], [9]): Let K^{ext} such that $\|A\| \leq c_1\mu^{\text{ext}}$, $c_1 > 0$. Under Assumption 2, because $\mathrm{Re}\left\{\lambda(A_{22}^2)\right\} < 0$, there exists $\varepsilon^\star > 0$ such that, for all $\varepsilon \in (0, \varepsilon^\star]$, system (18), starting from any bounded initial conditions ζ_0 and ξ_0, is approximated for all finite time intervals $t \geq t_0$ by:

$$\zeta = \zeta_s(t_s) + \mathcal{O}(\varepsilon); \quad \xi = \xi_s(t_s) + \xi_f(t_f)|_{\varepsilon=0} + \mathcal{O}(\varepsilon), \tag{23}$$

where $\zeta_s \in \mathbb{R}^{m\cdot n_x}$ and $\xi_s, \xi_f \in \mathbb{R}^{(n-m)\cdot n_x}$ are the respective variables from the slow and fast dynamics.

V. Passivity Analysis of the Network System

This section presents conditions for system (18) to be passive based on the passivity of its reduced-order system alongside adequate phase conditions on a necessary variation of its boundary layer system. Furthermore, the current section presents an extension of the mathematical apparatus described in [15] needed to develop such conditions. Specifically, the singularly perturbed system in question additionally receives fast actuation r_f alongside the previously-considered slow actuation r_s. This leads to the summation of the reference $r = r_s(t_s) + r_f(t_f)$ in (6), typical in control of SSPF [9].

As described in [15], the passivity of a system in SSPF requires conditions involving the reduced-order system alongside a *modified variant* of the boundary layer (22), as the passivity of the full singularly perturbed system cannot be directly assessed using the passivity of its reduced-order and boundary layer systems. We make the following assumption.

Assumption 3: Subsystem Σ_s from (20) is strictly input-output passive from input r_s to output y_s.

As noted in Section II, the passivity of the slow subsystem (20) is given by its structure, and its relative degree cannot be altered through state feedback. This assumption is also necessary for the ε*-bound computation* problem in which we will further compute the maximum value $\varepsilon^\star > 0$ such that system (18) is passive for all $\varepsilon \in [0, \varepsilon^\star]$, if such a value exists.

We consider the multiplicative factorization of Σ^ε from (18) based on its reduced-order model Σ_s from (20), known as the *actuator* form [9]: $\Sigma^\varepsilon = \Sigma_s \cdot \Sigma_f^\varepsilon$, as Σ_f^ε is placed at the input of Σ_s. Then, Σ_f^ε, named *quotient* system, can be written as:

$$\Sigma_f^\varepsilon = (\Sigma_s)^{-1} \cdot \Sigma^\varepsilon = \mathcal{S}(\Sigma^\varepsilon, (\Sigma_s)^{-1}). \tag{24}$$

Remark 1: By definition, we have $\Sigma_s = \Sigma^\varepsilon|_{\varepsilon=0}$. This means that the trivial expression of $\Sigma_f^\varepsilon|_{\varepsilon\to 0}$ is not isomorphic to that of the *quotient boundary layer* system $\Sigma_f^\varepsilon|_{\varepsilon=0} = I$, due to the reduction of a differential equation to an algebraic one. In time scale t_s, this cancels the possibility to study the existence of $\varepsilon^\star > 0$ such that system (18) is passive.

As in [15], we proceed to characterize both systems involved in the above multiplication, i.e., $(\Sigma_s)^{-1}$ and Σ^ε, in order to provide an analytical state-space expression for Σ_f^ε. Standard SSPF techniques allows rewritting Σ^ε as:

$$\mathrm{d}\zeta_s/\mathrm{d}t_s = A_{11}\zeta_s + B_s r_s; \tag{25a}$$

$$\mathrm{d}\zeta_f/\mathrm{d}t_s = A_{11}\zeta_f + A_{12}\xi_f + \widetilde{B}_1 r_s + B_1 r_f; \tag{25b}$$

$$\varepsilon\,\mathrm{d}\chi_f/\mathrm{d}t_s = \varepsilon A_{21}\zeta_f + A_{22}\chi_f + B_2 r_s + B_2 r_f; \tag{25c}$$

$$y = C_1\zeta_s + C_1\zeta_f + C_2\chi_f, \tag{25d}$$

with $\widetilde{B}_1 = A_{12}(A_{22}^2)^{-1}B_2$. To remove the derivative $\mathrm{d}r_s/\mathrm{d}t_s$ of the slow input from (21), we perform the change of variables $\chi_f = \xi_f - (A_{22}^2)^{-1}B_2 r_s$. The system's interface is now $((r_s^\top, r_f^\top)^\top, (\zeta_s^\top, \zeta_f^\top, \chi_f^\top)^\top, y)$.

The inverse of Σ_s can be computed using the descriptor system framework [15]. Denote its *inverse* input-output interface $r_i \in \mathbb{R}^{n\cdot n_u} \mapsto y_i \in \mathbb{R}^{n\cdot n_u}$ and extended state $(\zeta_s^{i1\top}, \zeta_s^{i2\top})^\top \in \mathbb{R}^{m\cdot n_x + n\cdot n_u}$, with the resulting model:

$$\mathrm{d}\zeta_s^{i1}/\mathrm{d}t_s = A_{11}\zeta_s^{i1} + B_s\zeta_s^{i2}; \quad r_i = C_1\zeta_s^{i1} + D_s\zeta_s^{i2}; \quad y_i = \zeta_s^{i2}. \tag{26}$$

In Laplace transfer matrix notation, Σ^ε from (25) is split into:

$$\begin{aligned}Y(s) &= \Sigma^\varepsilon \cdot \begin{pmatrix} R_s^\top(s) & R_f^\top(s)\end{pmatrix}^\top = Y_{d1}(s) + Y_{d2}(s)\\ &= \Sigma^\varepsilon|_{y_{d1}, r_s} \cdot R_s(s) + \Sigma^\varepsilon|_{y_{d2}, r_f} \cdot R_f(s).\end{aligned} \tag{27}$$

Furthermore, in the case of Σ_s, we have $Y_i(s) = (\Sigma_s^{-1}) \cdot R_i(s)$, according to (26). Thus, the expression of Σ_f^ε can be written as a parallel connection between two standard series connections. The quotient system Σ_f^ε in time scale t_s is a system with two vector inputs (r_s, r_f) and one vector output $y_i = y_{i1} + y_{i2}$:

$$\begin{aligned} Y_i(s) &= \Sigma_f^\varepsilon \cdot \begin{pmatrix} R_s^\top(s) & R_f^\top(s) \end{pmatrix}^\top = Y_{i1}(s) + Y_{i2}(s) \\ &= \Big[(\Sigma_s)^{-1}\, \Sigma^\varepsilon\big|_{y_1,r_s}\Big] R_s(s) + \Big[(\Sigma_s)^{-1}\, \Sigma^\varepsilon\big|_{y_2,r_f}\Big] R_f(s) \\ &= \Sigma_f^\varepsilon\big|_{y_{i1},r_s} \cdot R_s(s) + \Sigma_f^\varepsilon\big|_{y_{i2},r_f} \cdot R_f(s). \end{aligned} \tag{28}$$

Due to space constraints, we present the equations only for $\Sigma_f^\varepsilon|_{y_{i1},r_s}$, with an identical procedure to be applied to $\Sigma_f^\varepsilon|_{y_{i2},r_f}$. The quotient system $\Sigma_f^\varepsilon|_{y_{i1},r_s}$ has the interface $(r_s, (\zeta_{s1}^\top, \zeta_{f1}^\top, \chi_{f1}^\top, \zeta_{s1}^{i1\top}, \zeta_{s1}^{i2\top})^\top, y_{i1}^\top)$ and state-space:

$$\mathrm{d}\zeta_{s1}/\mathrm{d}t_s = A_{11}\zeta_{s1} + B_s\zeta_{s1}^{i2}; \tag{29a}$$
$$\mathrm{d}\zeta_{f1}/\mathrm{d}t_s = A_{11}\zeta_{f1} + A_{12}\chi_{f1} + \widetilde{B}_1\zeta_{s1}^{i2}; \tag{29b}$$
$$\varepsilon\, \mathrm{d}\chi_{f1}/\mathrm{d}t_s = \varepsilon A_{21}\zeta_{f1} + A_{22}\chi_{f1} + B_2\zeta_{s1}^{i2}; \tag{29c}$$
$$\mathrm{d}\zeta_{s1}^{i1}/\mathrm{d}t_s = A_{11}\zeta_{s1}^{i1} + B_s\zeta_{s1}^{i2}; \tag{29d}$$
$$r_s = C_1\zeta_{s1}^{i1} + D_s\zeta_{s1}^{i2}; \quad y_{i1} = C_1\zeta_{s1} + C_1\zeta_{f1} + C_2\chi_{f1}. \tag{29e}$$

In an identical manner, system $\Sigma_f^\varepsilon|_{y_{i2},r_f}$ has the interface $(r_f, (\zeta_{s2}^\top, \zeta_{f2}^\top, \chi_{f2}^\top, \zeta_{s2}^{i1\top}, \zeta_{s2}^{i2\top})^\top, y_{i2}^\top)$, with two modifications compared to (29): the counterpart of (29a) lacks the B_s effect, while the counterpart of (29b) has B_1 instead of $\widetilde{B}_1$. Now, we switch to the fast time scale t_f. $\Sigma_f^\varepsilon|_{y_{i1},r_s}$ in time scale t_f will be denoted ${}^{t_f}\Sigma_f^\varepsilon|_{y_{i1},r_s}$. It has the expression:

$$\mathrm{d}\zeta_{s1}/\mathrm{d}t_f = \varepsilon A_{11}\zeta_{s1} + \varepsilon B_s\zeta_{s1}^{i2}; \tag{30a}$$
$$\mathrm{d}\zeta_{f1}/\mathrm{d}t_f = \varepsilon A_{11}\zeta_{f1} + \varepsilon A_{12}\chi_{f1} + \varepsilon\widetilde{B}_1\zeta_{s1}^{i2}; \tag{30b}$$
$$\mathrm{d}\chi_{f1}/\mathrm{d}t_f = \varepsilon A_{21}\zeta_{f1} + A_{22}\chi_{f1} + B_2\zeta_{s1}^{i2}; \tag{30c}$$
$$\mathrm{d}\zeta_{s1}^{i1}/\mathrm{d}t_f = \varepsilon A_{11}\zeta_{s1}^{i1} + \varepsilon B_s\zeta_{s1}^{i2}; \tag{30d}$$
$$r_s = C_1\zeta_{s1}^{i1} + D_s\zeta_{s1}^{i2}; \quad y_{i1} = C_1\zeta_{s1} + C_1\zeta_{f1} + C_2\chi_{f1}. \tag{30e}$$

Setting $\varepsilon = 0$, we overcome the problem from Remark 1, as the resulting system has an invariant number of differential and algebraic constraints in the transition from $\varepsilon \to 0_+$ to $\varepsilon = 0$. The quotient boundary layer subsystems are ${}^{t_f}\Sigma_f^0|_{y_{i1},r_s}$ and ${}^{t_f}\Sigma_f^0|_{y_{i2},r_f}$, with the former having the model:

$$\mathrm{d}\zeta_{s1}/\mathrm{d}t_f = \mathbf{0}; \quad \mathrm{d}\zeta_{f1}/\mathrm{d}t_f = \mathbf{0}; \quad \mathrm{d}\zeta_{s1}^{i1}/\mathrm{d}t_f = \mathbf{0}; \tag{31a}$$
$$\mathrm{d}\chi_{f1}/\mathrm{d}t_f = A_{22}^2\chi_{f1} + B_2\zeta_{s1}^{i2}; \quad r_s = C_1\zeta_{s1}^{i1} + D_s\zeta_{s1}^{i2}; \tag{31b}$$
$$y_{i1} = C_1\zeta_{s1} + C_1\zeta_{f1} + C_2\chi_{f1}. \tag{31c}$$

We base the forthcoming passivity results on Lemma 1. As time-rescaling is a geometric invariant of a dynamical system, it does not alter its input-output behavior, which means that passivity is invariant from time scale t_s to t_f. The MAS (27) is the parallel connection (28) premultiplied by Σ_s from (20):

$$\Sigma^\varepsilon = \Sigma_s \cdot \Sigma_f^\varepsilon|_{y_{i1},r_s} + \Sigma_s \cdot \Sigma_f^\varepsilon|_{y_{i2},r_f} = \Sigma^\varepsilon|_{y_1,r_s} + \Sigma^\varepsilon|_{y_2,r_f}. \tag{32}$$

A sufficient condition [10] for the passivity of Σ^ε is that each of its components from (32) must also be passive. We can now state the conditions for the existence of a ratio $\varepsilon > 0$ such that the MAS is strictly passive, based on its reduced-order and quotient boundary layer subsystems.

Theorem 2: If Assumptions 1–3 hold, Σ_f^{0+} is *Hurwitz* and one of the following conditions apply:

A) ${}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}$ is strictly passive, if $\rho(\Sigma_s) = (0, \ldots, 0) \equiv \overline{0}$;

B) $\varphi({}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}) \subseteq (0, \pi)$, if $\rho(\Sigma_s) = (1, \ldots, 1) \equiv \overline{1}$;

C) $\varphi({}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}) \subseteq (0, \frac{\pi}{2})$, if $\rho(\Sigma_s) \in \mathbb{Z}_2^{m\cdot n_x} \setminus \{\overline{0}, \overline{1}\}$,

then there exists $\varepsilon^\star > 0$ such that system $\Sigma^{\varepsilon^\star}$ is *strictly passive*.

Proof: In order for $\lim_{\varepsilon\to 0_+} \Sigma^\varepsilon$ to be strictly passive, we need to ensure that its symmetric part is not identically zero, it is Hurwitz, and $\varphi(\Sigma^\varepsilon) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$, according to Lemma 1. First, the precondition from Lemma 1 always holds for this class of systems described in (14) ⇔ (18), due to the existence, by Assumption 2, of matrices K^{int} and K^{ext}. It follows that the conditions from Lemma 1 can be enforced.

To ensure the strict passivity of all subsystems in (32), we have, for $\Sigma^\varepsilon|_{y_{d1},r_s}$, using Definition 1 and the phase property of a series connection [14, Lemma 2]:

$$\varphi({}^{t_f}\Sigma^\varepsilon|_{y_{d1},r_s}) \subseteq \varphi({}^{t_f}\Sigma_s) + \varphi({}^{t_f}\Sigma_f^\varepsilon|_{y_{i1},r_s}), \tag{33}$$

and similarly for ${}^{t_f}\Sigma^\varepsilon|_{y_{d2},r_f}$. Due to Assumption 3, Σ_s is also Hurwitz and $\varphi(\Sigma_s) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$ with its vector relative degree $\rho(\Sigma_s) \in \mathbb{Z}_2^{m\cdot n_x}$. It then follows that Σ_f^ε must also be Hurwitz.

Furthermore, for the strict passivity of $\Sigma^\varepsilon|_{y_1,r_s}$, due to the TSS property, ensured through Assumptions 1–2, in time scale t_f, the slow input is only perceptible at the initial moment, i.e., $r_s(0)$, which leads to the frequency equivalent of $R_s(\infty)$, according to the Initial Value Theorem [20]. This leads for the slow subsystem in (27) to $Y_{d1}(\infty) = {}^{t_f}\Sigma_s(\infty)\, {}^{t_f}\Sigma_f^{0+}|_{y_{i1},r_s}(\infty) \cdot R_s(\infty)$, which reduces to a static feedthrough term. Its passivity is ensured if and only if it is positive semidefinite. Specifically for the MAS setup, (31) does not have a feedthrough term, so ${}^{t_f}\Sigma^{0+}|_{y_{d1},r_s}(\infty) = O \geq 0$. This means that the first subsystem is irrelevant for the existence of $\varepsilon > 0$ to guarantee the passivity of the full MAS. It will only impact the ε-bound computation problem.

We now turn our attention to the fast subsystem $\Sigma_f^{0+}|_{y_{i2},r_f}$ and focus on the adapted phase condition (33):

$$\lim_{\varepsilon\to 0_+} \varphi({}^{t_f}\Sigma^\varepsilon|_{y_{d2},r_f}) \subseteq \varphi({}^{t_f}\Sigma_s) + \lim_{\varepsilon\to 0_+} \varphi({}^{t_f}\Sigma_f^\varepsilon|_{y_{i2},r_f}). \tag{34}$$

Because $\varphi(\Sigma_s) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$, due to the TSS, $\varphi({}^{t_f}\Sigma_s)$, $\varepsilon \to 0_+$, will be perceived in one of three possible ways: **A)** $\varphi({}^{t_f}\Sigma_s) = [0, 0]$, if $\rho(\Sigma_s) = \overline{0}$, the implication being $\varphi({}^{t_f}\Sigma_f^\varepsilon|_{y_{i2},r_f}) \subseteq (-\frac{\pi}{2}, \frac{\pi}{2})$, i.e., $\Sigma_f^\varepsilon|_{y_{i2},r_f}$ is strictly passive; **B)** $\varphi({}^{t_f}\Sigma_s) = [-\frac{\pi}{2}, -\frac{\pi}{2}]$, if $\rho(\Sigma_s) = \overline{1}$, so the implication becomes $\varphi({}^{t_f}\Sigma_f^\varepsilon|_{y_{i2},r_f}) \subseteq (0, \pi)$; **C)** $\varphi({}^{t_f}\Sigma_s) = [-\frac{\pi}{2}, 0]$, if $\rho(\Sigma_s) \in \mathbb{Z}_2^{m\cdot n_x}$, leading to $\varphi({}^{t_f}\Sigma_f^\varepsilon|_{y_{i2},r_f}) \subseteq (0, \frac{\pi}{2})$. The proof ends, as, thanks to Assumption 3, there are no other possible relative degree configurations $\rho(\Sigma_s) \notin \mathbb{Z}_2^{m\cdot n_x}$. ■

A similar result can be easily adapted for non-strict passivity. In this case, Assumption 3 can be relaxed to having Σ_s passive instead of strictly passive.

Corollary 1: If Assumptions 1–3 hold, Σ_f^{0+} is *semi-stable*, Σ^ε from (32) has poles with multiplicity one on the imaginary axis, and one of the following conditions apply:

A) ${}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}$ is passive, if $\rho(\Sigma_s) = (0, \ldots, 0) \equiv \overline{0}$;

B) $\varphi({}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}) \subseteq [0, \pi]$, if $\rho(\Sigma_s) = (1, \ldots, 1) \equiv \overline{1}$;

C) $\varphi({}^{t_f}\Sigma_f^{0+}|_{y_{i2},r_f}) \subseteq [0, \frac{\pi}{2}]$, if $\rho(\Sigma_s) \in \mathbb{Z}_2^{m\cdot n_x} \setminus \{\overline{0}, \overline{1}\}$, then there exists $\varepsilon^\star > 0$ such that system $\Sigma^{\varepsilon^\star}$ is *passive*.

Given that system Σ^ε from (18) is simply a reformulation of system (14) through the invertible coordinate transformation (10) and different time scalings t_s, t_f, it follows that the passivity of Σ^ε is equivalent to the passivity of the overall network system with respect to the exogenous reference r_k.

The conditions of Theorem 2 and Corollary 1 can be easily verified numerically using any environment which supports linear algebra manipulations, such as `LAPACK`, `BLAS`, `SciPy`, `Eigen`, etc., as they imply only eigenvalue problems and sectorial matrix decompositions. Additionally, once the existence of $\varepsilon > 0$ is confirmed, we are left with the ε-bound computation problem, i.e., to find the largest value $\varepsilon^\star$ such that the MAS (18) is passive $\forall \varepsilon \in [0, \varepsilon^\star]$. This can be performed in a convex manner (see [15], [21]) by employing two decoupled optimizations to each subsystem from (32): one for $\Sigma^\varepsilon|_{y_1, r_s}$, leading to a solution $\varepsilon_s^\star > 0$, and one for $\Sigma^\varepsilon|_{y_2, r_f}$, leading to $\varepsilon_f^\star > 0$. Then, $\varepsilon^\star = \min(\varepsilon_s^\star, \varepsilon_f^\star)$. Furthermore, given that ε from (17b) is a parameter of the network structure, this allows us not only to infer if the MAS is passive using the proposed model, but to also have a measure of robustness as to how far from the upper bound $\varepsilon^\star$ it is.

VI. Academic Example

We briefly illustrate the proposed method via a simple numerical example, representing, e.g., a power network [1]. Consider a network consisting of $m = 3$ clusters $\mathcal{C}_k$, each with $n_k = 10$ agents with dynamics $A = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$, $B = (1\ 0)^\top$, $C = (1\ 1)$. As a baseline for robustness analysis, each cluster is initially fully connected, i.e., $\mathcal{L}_k^{\text{int}} = 10 I_{10} - \mathbf{1}_{10\times 10}$, $k = \overline{1,3}$, with additional external connections between pairs of agents $(1,1)$–$(2,1)$, $(1,1)$–$(3,1)$, $(2,1)$–$(3,1)$, $(1,1)$–$(3,5)$. Consider the state feedback gains $K_i^{\text{int}} = (39\ \ 399)$, corresponding to the closed-loop poles of the agents $\lambda_{\text{cl}_{1,2}}^{\text{int}} = -20$, and $K_j^{\text{ext}} = (3\ \ 3)$, leading to $\lambda_{\text{cl}_{1,2}}^{\text{ext}} = -2$.

Using (15)–(17), the network parameters are $\mu^{\text{ext}} = 16.97$, $\mu^{\text{int}} \approx 4009$, which gives the MAS ratio $\varepsilon = 0.00423 \ll 1$. Subsystem Σ_s from (20) is passive, with $\varphi(\Sigma_s) \subseteq [-\frac{\pi}{2}, 0]$ and $\varphi(\Sigma_s(j\infty)) = [0, 0]$, i.e., $\rho(\Sigma_s) = \overline{0}$, meaning that Assumptions 1–3 hold and condition **A**) of Corollary 1, $\varphi({}^{t_f}\Sigma_f^{0+}|_{y_{i2}, r_f}) \subseteq [-\frac{\pi}{2}, \frac{\pi}{2}]$, also holds. For the ε-bound on (18), we determine $\varepsilon^\star = 0.05539$, which means that the network SSPF model is passive $\forall \varepsilon \in [0, \varepsilon^\star]$, including the default ratio. From this value, we conclude that the inter-cluster links are very weak compared to the dense intra-cluster links, which means that the network's passivity is very robust to adding and removing links. To demonstrate this, we consider three sets of 50 Monte Carlo experiments, in which we drop at most $\mathbf{n} \in \{5, 10, 15\}$ arbitrary connections between nodes from each cluster $\mathcal{C}_k$, $k = \overline{1,3}$, which are still subject to Assumption 1. The computed $\varepsilon^\star$ bounds degrade with means and standard deviations ranging from $(0.0533, 0.0019)|_{\mathbf{n}=5}$, to $(0.0521, 0.0026)|_{\mathbf{n}=10}$, and $(0.0507, 0.0039)|_{\mathbf{n}=15}$.

In contrast, a slightly different output matrix $C = (1\ 10)$ leads to a non-passive network irrespective of ε, which justifies that the current theory provides: (*i*) a reduced-dimensionality method to ascertain if such a network system is passive and (*ii*) a precursory means to analyze its sensitivity with respect to adding/removing connections in the network graph.

VII. Conclusion and Further Extensions

The proposed passivity analysis tailored to systems in SSPF bypasses the computationally-inefficient passivity checks for large-scale systems (14), while also providing insight on its sensitivity to adding/removing links between the agents, i.e., changes to $\mathcal{L}^{\text{int}}$ and $\mathcal{L}^{\text{ext}}$. The proof of Theorem 2 can be easily adapted to more general SSPF structures. The current case (extended from [15]) considers both slow and fast actuators.

This tool will be further used as a starting point for control synthesis to design a composite law such that the network ratio ε from (17) is less than the guaranteed passivity bound $\varepsilon^\star$ with an imposed robustness margin, without affecting the synchronization. Other research directions can be to extend the proposed framework to heterogeneous networks and nonlinear agent dynamics.

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