Common quadratic Lyapunov functions for sets of second-order linear systems: a simple graphical criterion

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***Abstract*—This paper presents a simple graphical criterion to obtain all common quadratic Lyapunov functions for nonlinear systems contained in the convex-hull of a set of second-order linear time-invariant systems. It simply consists in finding the intersection between the interior of some ellipses. The proposed criterion is declined for both continuous-time and discrete-time systems and is illustrated by numerical examples.**

***Index Terms*—Quadratic Lyapunov function, Graphical criterion, Linear Differential Inclusion, Second-order system, Ellipse**

I. Introduction

A. Lyapunov stability

In his seminal 1892 work on stability, Lyapunov introduced his so-called second method to demonstrate that a system of differential equations has stable solutions [1]. His methodology, known today as the Lyapunov stability criterion or the direct method, relies on exhibiting a positive-definite function around an equilibrium point of the system, whose derivative along the system trajectories remains negative. If such a Lyapunov function exists, then the system is proven Lyapunov stable around this equilibrium. Lyapunov stability guarantees that any trajectory starting near to this equilibrium will always remain in its neighborhood. Moreover, a strictly negative derivative of a Lyapunov function also guarantees the asymptotic stability of the system, i.e. the asymptotic convergence of the trajectories towards this equilibrium [2], [3], [4]. In a nutshell, exhibiting a Lyapunov function provides a powerful way to learn about a system behavior without having to explicitly compute its trajectories. This made Lyapunov functions one of the most widely used tool in the fields of dynamical systems and control theory. The Lyapunov stability framework has been enriched with numerous new notions, such as uniform stability, exponential stability, finite-time stability, etc. [2], [3], [4].

However, Lyapunov stability criterion is only a sufficient stability condition, and converse results had to be established to complete Lyapunov's theory. Three well-known converse results are reported below.

- An autonomous continuous differential equation is asymptotically stable if and only if there exists a smooth Lyapunov function demonstrating its asymptotic stability [3], [4].
- A continuous-time Linear Differential Inclusion (LDI) is exponentially stable if and only if there exists a piecewise-quadratic Lyapunov function demonstrating its asymptotic stability [5].
- A Linear Time-Invariant (LTI) system is exponentially stable if and only if there exists a Quadratic Lyapunov Function (QLF) demonstrating its asymptotic stability.

This last result is generally known as the Lyapunov lemma, and holds both for continuous-time and discrete-time LTI systems. Moreover, since an autonomous nonlinear system can often be qualitatively approximated by linearizing its dynamics around an equilibrium point (the Hartman-Grobman theorem), the Lyapunov lemma also holds locally in some nonlinear contexts as well [3], [4].

Lemma 1 (Lyapunov lemma [2], [6])**.** *Let $A \in \mathbb{R}^{n\times n}$. The following items are equivalent:*

1) *the LTI system $\dot{x}(t) = Ax(t)$ (resp. $x_{k+1} = Ax_k$) is exponentially stable;*
2) *the real part of the eigenvalues of A are strictly negative, i.e. A is Hurwitz (resp. the spectral radius of A is strictly less than 1, i.e. A is Schur);*
3) *there exists a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that $A^\top P + PA$ (resp. $A^\top PA - P$) is negative-definite.*

In both cases, the matrix P of item 3 provides the QLF (1) defined thereafter, demonstrating the stability of the LTI system of item 1.

$$V(x) = x^\top Px \tag{1}$$

The centrality of the Lyapunov lemma made QLFs a well-investigated class of Lyapunov functions, despite their possible conservatism to demonstrate the stability of nonlinear systems. QLFs are very simply defined, they exhibit many interesting mathematical properties, and they are easily computed numerically using semidefinite programming to solve Linear Matrix Inqualities (LMIs) [7].

B. Common quadratic Lyapunov functions

QLFs have been widely studied in the context of continuous-time and discrete-time LDI, which are nonlinear systems whose trajectories are included at each instant in the convex-hull of a finite set of LTI systems, respectively defined by:

$$\dot{x}(t) \in \mathrm{conv}\{A_i x(t) : i = 1, \ldots, m\} \quad \text{(2a)}$$

$$x_{k+1} \in \mathrm{conv}\{A_i x_k : i = 1, \ldots, m\} \quad \text{(2b)}$$

If used on these systems, the QLF (1) is then called a Common Quadratic Lyapunov Function (CQLF) to the set of LTI systems defined by the matrices $\{A_i\}_{1\leq i\leq m}$, and it leads to the following result [7].

Lemma 2 (Exponential stability of LDI)**.** *Let $\{A_i\}_{1\leq i\leq m}$ be a set of $\mathbb{R}^{n\times n}$ matrices. The LDI (2a) (resp. (2b)) is exponentially stable if there exists a symmetric positive-definite matrix $P \in \mathbb{R}^{n\times n}$ such that for all i, the $A_i^\top P + PA_i$ (resp. $A_i^\top PA_i - P$) are negative-definite.*

LDI systems (2a) and (2b) encapsulate many other classes of nonlinear systems, including switched linear systems, linear time-varying systems with a bounded state matrix, polytopic (quasi-)linear parameter-varying systems, Takagi-Sugeno (fuzzy) systems, etc. Their ever-presence makes Lemma 2 a simple yet extremely common result of the modern nonlinear control literature, with many practical applications [7], [8].

As it is usually the case for Lyapunov criterion restricted to QLF, Lemma 2 only offers a sufficient condition to exponential stability, and it is well-known not to be a necessary condition. However, despite its apparent simplicity and its clear similarity to the Lyapunov lemma, this result lacks a straightforward converse. Simply put, there are no elementary criterion to know if a given set of matrices $\{A_i\}_{1\leq i\leq m}$ is going to admit a CQLF (1) or not. The problem is actually so hard, in both the continuous- and discrete-time cases, that existing results in the literature are limited to two-dimensional systems ($n = 2$) [9], [10], [11], [12], to sets made of two matrices ($m = 2$) [13], and to other restrictive conditions such as $m = 2$ and a rank one difference between the matrices [14], or both $n = 2$ and $m = 2$ [15], [16]. The necessary conditions proposed are moreover very often difficult to grasp intuitively.

This papers suggests a very simple graphical criterion to this converse problem for two-dimensional systems ($n = 2, m \in \mathbb{N}^*$). Simply put, it associates every Hurwitz (resp. Schur) matrix with the interior of an ellipse on a two-dimensional plane. If the intersection of all the ellipses associated to the set of Hurwitz (resp. Schur) matrices $\{A_i\}_{1\leq i\leq m}$ is non-empty, it can be stated without loss of generality that there exists a CQLF (1) to this set, demonstrating exponential stability for the LDI system (2a) (resp. (2b)). All existing CQLF (1) to $\{A_i\}_{1\leq i\leq m}$ can actually be retrieved from this intersection.

To the authors' knowledge, this strikingly uncomplicated result has not yet been stated in such a simple manner in the literature so far, despite its practical interest. It should be highlighted that the graphical criterion proposed in this paper is somewhat similar to the plots leading to the results found in Theorem 4.1 of [10]. However, the plots in [10] are constructed in a more convoluted fashion, and contrary to [10], every geometrical shape defined in this document is obtained through the exact same inequality, unifying the graphical criteria without requiring to check some preliminary assumptions on the matrices $\{A_i\}_{1\leq i\leq m}$. Typically, the suggested criterion also works on non-Hurwitz (resp. non-Schur) matrices, by associating them to an empty set or to an unbounded set which does not intersect any other ellipse given by a Hurwitz (resp. Schur) matrix. This suggested unification facilitates the numerical implementation of the graphical representation. Finally, it should be noted that this paper is not concerned with finding an algorithmic procedure to find a CQLF, as plotting the suggested graphical criterion is already the whole procedure. Simply put, the proposed methodology consists in "plotting and interpreting", similarly to other famous graphical criteria, such as Bode, Nyquist and Nichols plots [17]. Moreover, the plot allows to find a value to the Lyapunov matrix P in (1) when it exists.

The paper is organised as follows: Section II states the main results of the document. Section III illustrates the graphical criterion both in a continuous- and discrete-time setting. Finally, Section IV concludes the document and offers some perspectives. The MATLAB code used for Figure 2 and 3 is provided in the Appendix.

II. Main Results

After introducing the geometry of the positive-definite cone in Lemma 3, this Section provides the graphical criterion to the existence of a CQLF to a set $\{A_i\}_{1\leq i\leq m}$ of 2×2 real matrices in Theorem 1. The ellipsoidal nature of the investigated sets is then demonstrated in Theorem 2.

Lemma 3. *Let $P = P^\top \in \mathbb{R}^{2\times 2}$ and $z_1, z_2, z_3 \in \mathbb{R}$ be such that:*

$$P = z_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z_3 I_2$$

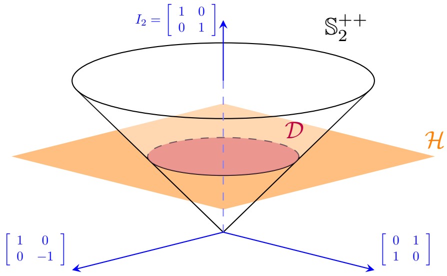

Figure 1. Cone of positive-definite matrices $\mathbb{S}_2^{++}$ in the space of symmetric 2×2 real matrices, and its intersection with the affine hyperplane $\mathcal{H}$ of symmetric 2×2 real matrices with a trace of 2.

with I_2 the 2×2 identity matrix. The matrix P is positive-semidefinite ($P \in \mathbb{S}_2^{+}$) if and only if

$$\sqrt{z_1^2 + z_2^2} - z_3 \leq 0 \tag{3}$$

Moreover, it is positive-definite ($P \in \mathbb{S}_2^{++}$) if and only if the inequality (3) is strict.

Proof. The matrix P is real-symmetric, hence both of its eigenvalues are real. Its smallest eigenvalue is:

$$\lambda_{\min}(P) = \mathrm{Tr}(P)/2 - \sqrt{\mathrm{Tr}^2(P)/4 - \det(P)}$$

It is easily verified that $\mathrm{Tr}(P) = 2z_3$ and $\det(P) = z_3^2 - z_1^2 - z_2^2$. Moreover, P is positive (semi)definite if and only if $\lambda_{\min}(P) > 0$ (resp. ≥ 0). Rewriting this condition with z_1, z_2 and z_3 directly yields (3). □

As illustrated by Figure 1, Equation (3) of Lemma 3 demonstrates that the set of positive-semidefinite matrices, denoted $\mathbb{S}_2^{+}$, is a quadratic cone (also called a Lorentz cone, or an ice-cream cone) in the space of symmetric 2×2 real matrices, oriented in the identity matrix I_2 direction [18]. Moreover, a strict inequality in (3) defines the set of positive-definite matrices $\mathbb{S}_2^{++}$ as the interior of $\mathbb{S}_2^{+}$.

Remark 1. *The negative-semidefinite ($\mathbb{S}_2^{-}$) and negative-definite ($\mathbb{S}_2^{--}$) cones are symmetric to $\mathbb{S}_2^{+}$ and $\mathbb{S}_2^{++}$ with respect to the hyperplane $\mathrm{Tr}(\cdot) = 0$. Formally, $P \in \mathbb{S}_2^{-}$ (resp. $\mathbb{S}_2^{--}$) if and only if $\sqrt{z_1^2 + z_2^2} + z_3 \leq 0$ (resp. < 0).*

Now let $\mathcal{H}$ denote the affine hyperplane of symmetric matrices with a trace of 2 (i.e. $z_3 = 1$). Formally:

$$\mathcal{H} \triangleq \{P \in \mathbb{R}^{2\times 2} : P = P^\top \text{ and } \mathrm{Tr}(P) = 2\}$$

The intersection between $\mathbb{S}_2^{++}$ and $\mathcal{H}$ can be found using (3) to be an open disk of radius 1. In the $\mathcal{H}$ plane, this disk is denoted $\mathcal{D}$, and it is formally defined by:

$$\mathcal{D} \triangleq \left\{z \in \mathbb{R}^2 : z_1^2 + z_2^2 < 1\right\}$$

The main idea of the suggested graphical criterion consists in restricting the set of symmetric matrices $P \in \mathbb{R}^{2\times 2}$ for which $A^\top P + PA$ (resp. $A^\top PA - P$) is negative-definite to the affine hyperplane $\mathcal{H}$. It will be noticed that this restriction is made without loss of generality from $\mathbb{S}_2^{++}$ to $\mathcal{D}$, as a simple scaling of P allows to set its trace to 2 while preserving, by homogeneity, the negativeness of $A^\top P + PA$ (resp. $A^\top PA - P$). This ultimately reduces the problem of finding a CQLF to a set of matrices $\{A_i\}_{1\leq i\leq m}$ to the geometrical problem of finding this common P in the open disk $\mathcal{D}$ at the intersection of $\mathbb{S}_2^{++}$ and $\mathcal{H}$.

Definitions. *Given a matrix $A \in \mathbb{R}^{2\times 2}$ such that:*

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

given the following linear operators:

$$\mathcal{L}_A^c(P) = A^\top P + PA, \quad \mathcal{L}_A^d(P) = A^\top PA - P$$

and given $P_z \in \mathcal{H}$ linearly defined for $z \in \mathbb{R}^2$ by:

$$P_z \triangleq z_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + z_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + I_2$$

the following sets and functions are introduced:

$$\begin{aligned} \mathcal{Q}_A &\triangleq \{z \in \mathbb{R}^2 : \mathcal{L}_A^c(P_z) \in \mathbb{S}_2^{--}\} \\ &= \{z \in \mathbb{R}^2 : \lambda_{\max}(\mathcal{L}_A^c(P_z)) < 0\} \qquad \text{(4a)} \\ &= \{z \in \mathbb{R}^2 : \sqrt{f_1^2(z) + f_2^2(z)} + f_3(z) < 0\} \\ \mathcal{R}_A &\triangleq \{z \in \mathbb{R}^2 : \mathcal{L}_A^d(P_z) \in \mathbb{S}_2^{--}\} \\ &= \{z \in \mathbb{R}^2 : \lambda_{\max}(\mathcal{L}_A^d(P_z)) < 0\} \qquad \text{(4b)} \\ &= \{z \in \mathbb{R}^2 : \sqrt{g_1^2(z) + g_2^2(z)} + g_3(z) < 0\} \end{aligned}$$

$$\begin{aligned} f_1(z) &\triangleq z_1(a_{12} - a_{21}) + z_2(a_{11} + a_{22}) + a_{12} + a_{21} \\ f_2(z) &\triangleq z_1(a_{11} + a_{22)}) + z_2(a_{21} - a_{12}) + a_{11} - a_{22} \\ f_3(z) &\triangleq z_1(a_{11} - a_{22}) + z_2(a_{12} + a_{21}) + a_{11} + a_{22} \\ g_1(z) &\triangleq \frac{z_1}{2}(a_{21}^2 + a_{12}^2 - a_{11}^2 - a_{22}^2 + 2) + \ldots \\ &\quad z_2(a_{22}a_{12} - a_{11}a_{21}) + (a_{22}^2 + a_{12}^2 - a_{21}^2 - a_{11}^2)/2 \\ g_2(z) &\triangleq z_1(a_{11}a_{12} - a_{21}a_{22}) + a_{11}a_{12} + \ldots \\ &\quad a_{21}a_{22} + z_2(a_{11}a_{22} + a_{12}a_{21} - 1) \\ g_3(z) &\triangleq \frac{z_1}{2}(a_{11}^2 + a_{12}^2 - a_{21}^2 - a_{22}^2) - 1 + \ldots \\ &\quad z_2(a_{11}a_{21} + a_{12}a_{22}) + (a_{11}^2 + a_{12}^2 + a_{21}^2 + a_{22}^2)/2 \end{aligned}$$

Hereafter, $\mathcal{L}_A^{c,d}(P)$ stands for $\mathcal{L}_A^c(P)$ (resp. $\mathcal{L}_A^d(P)$).

Remark 2. *By linearity of $\mathcal{L}_A^c(P_z)$ and $\mathcal{L}_A^d(P_z)$ with respect to $z \in \mathbb{R}^2$, and by convexity of $\mathbb{S}_2^{--}$, $\mathcal{Q}_A$ and $\mathcal{R}_A$ are convex.*

Theorem 1 (Graphical criterion). *Given $\{A_i\}_{1\leq i\leq m}$ a set of $\mathbb{R}^{2\times 2}$ matrices, there exists a matrix $P \in \mathbb{S}_2^{++}$ such that for all i, $\mathcal{L}_{A_i}^c(P) \in \mathbb{S}_2^{--}$ if and only if*

$$\mathcal{Q}_{A_1} \cap \cdots \cap \mathcal{Q}_{A_m} \cap \mathcal{D} \neq \emptyset$$

Similarly, there exists a matrix $P \in \mathbb{S}_2^{++}$ such that for all i, $\mathcal{L}_{A_i}^d(P) \in \mathbb{S}_2^{--}$ if and only if

$$\mathcal{R}_{A_1} \cap \cdots \cap \mathcal{R}_{A_m} \cap \mathcal{D} \neq \emptyset$$

Proof. $\boxed{\Rightarrow}$ If there exists a $P \in \mathbb{S}_2^{++}$ such that $\mathcal{L}_{A_i}^{c,d}(P) \in \mathbb{S}_2^{--}$ for all i, then by homogeneity [7], $P' = 2P/\operatorname{Tr}(P) \in \mathcal{H} \cap \mathbb{S}_2^{++}$, and $\mathcal{L}_{A_i}^{c,d}(P') \in \mathbb{S}_2^{--}$ for all i as well. Moreover as $P' \in \mathcal{H}$ there exists $z \in \mathbb{R}^2$ such that $P_z = P'$. By applying Lemma 3, since $P_z \in \mathbb{S}_2^{++}$, then $z \in \mathcal{D}$, and since $\mathcal{L}_{A_i}^{c,d}(P_z) \in \mathbb{S}_2^{--}$, then $z \in \mathcal{Q}_{A_i}$ (resp. $z \in \mathcal{R}_{A_i}$) for all i.

$\boxed{\Leftarrow}$ If there exists $z \in \mathcal{Q}_{A_1} \cap \cdots \cap \mathcal{D}$ (resp. $z \in \mathcal{R}_{A_1} \cap \cdots \cap \mathcal{D}$), then $P_z \in \mathbb{S}_2^{++}$ such that $\mathcal{L}_{A_i}^{c,d}(P_z) \in \mathbb{S}_2^{--}$ for all i. □

Remark 3. *Similarly to how the trace of P can be fixed to a positive value without loss of generality in $\mathbb{S}_2^{++}$, its determinant can also be fixed to a positive value without loss of generality in $\mathbb{S}_2^{++}$. In fact, the proposed graphical criterion can also be interpreted inside of $\mathcal{D}$ as the Klein disk model of the sheet of the hyperboloid associated with $\det(\cdot) = 1$ contained in $\mathbb{S}_2^{++}$ [19].*

Theorem 2 (Geometry of the solutions)**.** *$A \in \mathbb{R}^{2\times 2}$ is Hurwitz if and only if $\mathcal{Q}_A$ is the interior of an ellipse. If so, $\mathcal{Q}_A \subseteq \mathcal{D}$. Similarly, A is Schur if and only if $\mathcal{R}_A$ is the interior of an ellipse. If so, $\mathcal{R}_A \subseteq \mathcal{D}$.*

Proof. $\boxed{\Rightarrow}$ Let $A \in \mathbb{R}^{2\times 2}$ be a Hurwitz (resp. Schur) matrix, i.e. $\mathcal{Q}_A \cap \mathcal{D} \neq \emptyset$ (resp. $\mathcal{R}_A \cap \mathcal{D} \neq \emptyset$) according to Theorem 1. It is first shown by contradiction that $\mathcal{Q}_A \subseteq \mathcal{D}$ (resp. $\mathcal{R}_A \subseteq \mathcal{D}$).

Assuming that there exists $z' \in \mathcal{D}$ and $z'' \notin \mathcal{D}$ such that $z', z'' \in \mathcal{Q}_A$ (resp. $\mathcal{R}_A$); by convexity, for all $t \in [0,1]$, $tz' + (1-t)z'' \in \mathcal{Q}_A$ (resp. $\mathcal{R}_A$). The norm of $tz' + (1-t)z''$ being continuous with respect to t, the intermediate value theorem provides $z^* \in \mathcal{Q}_A$ (resp. $z^* \in \mathcal{R}_A$) such that $\|z^*\|_2 = 1$. Its associated P_{z^*} belongs to $\mathbb{S}_2^{+} \setminus \mathbb{S}_2^{++}$, so there exists $v \in \mathbb{R}^2 \setminus \{0\}$ a vector in the kernel of P_{z^*}. However, this yields $v^\top \mathcal{L}_A^c(P_{z^*})v = 0$ and $v^\top \mathcal{L}_A^d(P_{z^*})v = v^\top A^\top P_{z^*} A v$, where $A^\top P_{z^*} A \in \mathbb{S}_2^{+}$ by congruence [7]. In both cases this is in contradiction with $\mathcal{L}_A^{c,d}(P_{z^*}) \in \mathbb{S}_2^{--}$, i.e. with $z^* \in \mathcal{Q}_A$ (resp. $z^* \in \mathcal{R}_A$).

Since $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) is a subset of $\mathcal{D}$, the boundary of $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) is necessarily bounded as well. Moreover, (4a) (resp. (4b)) guarantees that this boundary is a quadratic curve. The only bounded quadratic curve is the ellipse [20], meaning $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) is the interior of an ellipse (contained in $\mathcal{D}$) if A is Hurwitz (resp. Schur).

$\boxed{\Leftarrow}$ It is shown that if A is not Hurwitz (resp. not Schur), then $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) is either an empty set, or is unbounded, guaranteeing that $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) cannot be the interior of an ellipse. The eigenvalues of A are denoted λ_1, λ_2. A proof by cases is performed.

<u>If $\Re(\lambda_1) = \Re(\lambda_2) = 0$ (resp. $|\lambda_1| = |\lambda_2| = 1$).</u> In the continuous-time case, either $\lambda_1 = \lambda_2 = 0$, and v in the kernel of A guarantees $v^\top \mathcal{L}_A^c(P)v = 0$ for all P, or there exists trajectories of $\dot{x} = Ax$ following a limit cycle, and no QLF can be strictly decreasing along these trajectories [3], [4]. Either way, there are no symmetric $P \in \mathbb{R}^{2\times 2}$ such that $\mathcal{L}_A^c(P) \in \mathbb{S}_2^{--}$, so $\mathcal{Q}_A = \emptyset$. In the discrete-time case, since A is a real matrix, there exists $\theta \in [0, 2\pi)$ such that $\lambda_1 = e^{\mathrm{i}\theta}$, $\lambda_2 = e^{-\mathrm{i}\theta}$, hence $\lambda_1\lambda_2 = 1$, and Lemma 3.4 of [6] guarantees that there are no symmetric $P \in \mathbb{R}^{2\times 2}$ such that $\mathcal{L}_A^d(P) \in \mathbb{S}_2^{--}$, so $\mathcal{R}_A = \emptyset$.

<u>If $\Re(\lambda_1) < \Re(\lambda_2) = 0$ (resp. $|\lambda_1| < |\lambda_2| = 1$).</u> Since A is a real matrix, in that case the two distinct eigenvalues are necessarily real. Taking v_2 the eigenvector of A associated with λ_2, it is easily noticed that for all symmetric $P \in \mathbb{R}^{2\times 2}$, $v_2^\top \mathcal{L}_A^{c,d}(P)v_2 = 0$, so there exits no P such that $\mathcal{L}_A^{c,d}(P) \in \mathbb{S}_2^{--}$, hence $\mathcal{Q}_A = \emptyset$ (resp. $\mathcal{R}_A = \emptyset$).

<u>If $\Re(\lambda_1), \Re(\lambda_2) \geq 0$ (resp. $|\lambda_1|, |\lambda_2| \geq 1$).</u> Then $-A$ (resp. A^{-1}) is in the closure of the set of Hurwitz (resp. Schur) matrices. By symmetry of everything proven so far with respect to the hyperplane $\operatorname{Tr}(\cdot) = 0$, all the P (if they exist) such that $\mathcal{L}_{-A}^c(P) \in \mathbb{S}_2^{++}$, i.e. $\mathcal{L}_A^c(P) \in \mathbb{S}_2^{--}$ (resp. $\mathcal{L}_{A^{-1}}^d(P) \in \mathbb{S}_2^{++}$, i.e. $\mathcal{L}_A^d(P) \in \mathbb{S}_2^{--}$ by congruence [7]) are contained in $\mathbb{S}_2^{--}$, which does not intersect $\mathcal{H}$, and finally $\mathcal{Q}_A = \emptyset$ (resp. $\mathcal{R}_A = \emptyset$).

<u>If $\Re(\lambda_1) < 0 < \Re(\lambda_2)$ (resp. $|\lambda_1| < 1 < |\lambda_2|$).</u> Since A is a real matrix, in that case the two distinct eigenvalues are necessarily real. The eigenvalues of $A^\top$ are the same as those of A, and their associated eigenvectors are taken real, not collinear, and denoted v_1 and v_2. Clearly, $P_1 = v_1 v_1^\top \in \mathbb{S}_2^{+} \setminus (\mathbb{S}_2^{++} \cup \{0\})$ is such that $\mathcal{L}_A^{c,d}(P_1) \in \mathbb{S}_2^{-}$ and $P_2 = -v_2 v_2^\top \in \mathbb{S}_2^{-} \setminus (\mathbb{S}_2^{--} \cup \{0\})$ is such that $\mathcal{L}_A^{c,d}(P_2) \in \mathbb{S}_2^{-}$. By convexity, for all $t \in [0,1]$, $P(t) = tP_1 + (1-t)P_2$ is such that $\mathcal{L}_A^{c,d}(P(t)) \in \mathbb{S}_2^{-}$. By continuity of $\operatorname{Tr}(P(t))$ with respect to $t \in [0,1]$, the intermediate value theorem provides $t^* \in (0,1)$ such that $\operatorname{Tr}(P(t^*)) = 0$. Since v_1 and v_2 are not collinear, $P(t^*) \neq 0$. Now if $\mathcal{Q}_A = \emptyset$ (resp. $\mathcal{R}_A = \emptyset$) the proof is finished. Otherwise, there exists $z \in \mathcal{Q}_A$ (resp. $z \in \mathcal{R}_A$), and it is easily checked that for all $q > 0$, $\mathcal{L}_A^{c,d}(qP(t^*) + P_z) \in \mathbb{S}_2^{--}$. Yet, it is also easily verified that for all $q > 0$, $qP(t^*) + P_z \in \mathcal{H}$, hence $\mathcal{Q}_A$ (resp. $\mathcal{R}_A$) is unbounded. □

Remark 4. *For all $\alpha < 0$ and $\beta \in (-1,1)$, $\mathcal{D} = \mathcal{Q}_{\alpha I_2} = \mathcal{R}_{\beta I_2}$.*

III. Illustration

The graphical criterion is first applied in the continuous-time setting to the following matrices:

$$A_1 = \begin{bmatrix} -10 & 0 \\ 2 & -0.5 \end{bmatrix} \quad A_2 = \begin{bmatrix} -1 & -2 \\ 2 & -1 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0.5 \\ -1 & -1 \end{bmatrix} \quad A_4 = \begin{bmatrix} -3 & 5 \\ -7 & -2 \end{bmatrix}$$

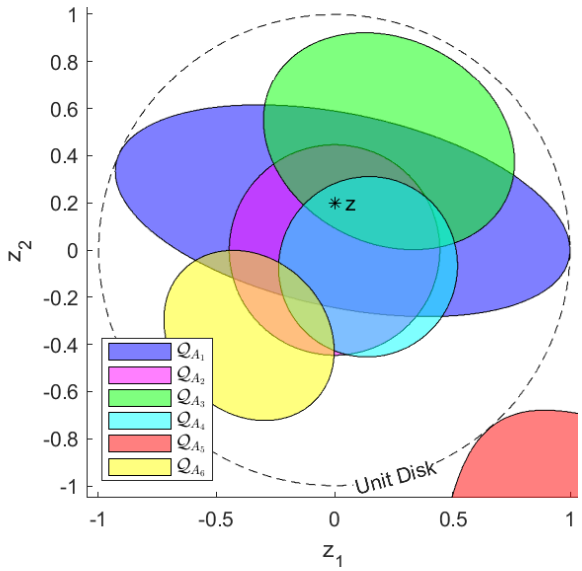

Figure 2. Plot of the graphical criterion applied to the set of matrices (5) in the continuous-time setting.

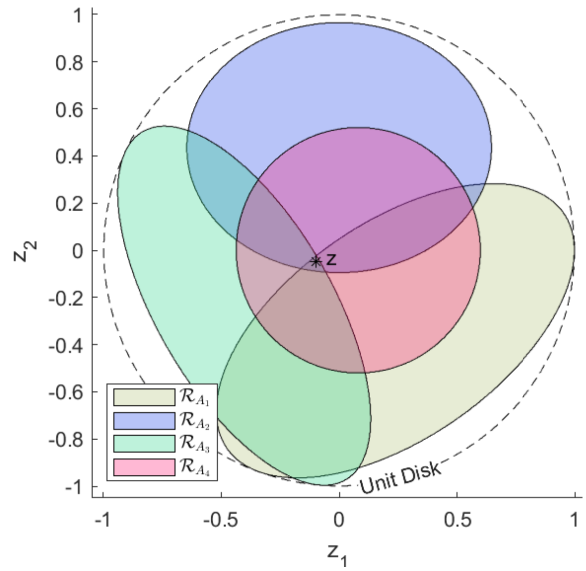

Figure 3. Plot of the graphical criterion applied to the set of matrices (6) in the discrete-time setting.

$$A_5 = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad A_6 = \begin{bmatrix} 0 & -0.5 \\ 0.2 & -0.3 \end{bmatrix} \tag{5}$$

The resulting plot is presented in Figure 2. It can be noticed that the set $\mathcal{Q}_{A_5}$ is outside $\mathcal{D}$, hence no $P \in S_2^{++}$ such that $\mathcal{L}^c_{A_5}(P) \in \mathbb{S}_2^{--}$ can be found, and A_5 is not Hurwitz. From there, it is clear that no CQLF can be found to the set of matrices (5). However, since $\mathcal{Q}_{A_1} \cap \mathcal{Q}_{A_2} \cap \mathcal{Q}_{A_3} \cap \mathcal{Q}_{A_4} \cap \mathcal{D}$ and $\mathcal{Q}_{A_1} \cap \mathcal{Q}_{A_2} \cap \mathcal{Q}_{A_4} \cap \mathcal{Q}_{A_6} \cap \mathcal{D}$ are not empty, Theorem 1 guarantees that a CQLF can be found to $\{A_1, A_2, A_3, A_4\}$ and $\{A_1, A_2, A_4, A_6\}$. Moreover, as $\mathcal{Q}_{A_3}$ and $\mathcal{Q}_{A_6}$ are non-intersecting ellipses, there are no CQLF to the sets of matrices containing $\{A_3, A_6\}$.

Graphically, taking $z = (0; 0.2)$ provides $z \in \mathcal{Q}_{A_1} \cap \mathcal{Q}_{A_2} \cap \mathcal{Q}_{A_3} \cap \mathcal{Q}_{A_4} \cap \mathcal{D}$, hence the positive-definite matrix P_z given by

$$P_z = \begin{bmatrix} 1 & 0.2 \\ 0.2 & 1 \end{bmatrix}$$

is such that $\mathcal{L}^c_{A_i}(P_z) \in \mathbb{S}_2^{--}$ for $i = 1, \ldots, 4$.

This is verified by: $\lambda_{\max}(\mathcal{L}^c_{A_1}(P_z)) \approx -0.9995 < 0$, $\lambda_{\max}(\mathcal{L}^c_{A_2}(P_z)) \approx -1.1056 < 0$, $\lambda_{\max}(\mathcal{L}^c_{A_3}(P_z)) \approx -0.1101 < 0$, $\lambda_{\max}(\mathcal{L}^c_{A_4}(P_z)) \approx -0.8657 < 0$, $\lambda_{\max}(\mathcal{L}^c_{A_5}(P_z)) \approx +12.621 \geq 0$, $\lambda_{\max}(\mathcal{L}^c_{A_6}(P_z)) \approx +0.2085 \geq 0$.

The graphical criterion is now applied in the discrete-time setting to the following matrices:

$$\begin{aligned} A_1 &= \begin{bmatrix} -0.5 & 0 \\ -0.7 & 0.5 \end{bmatrix} & A_2 &= \begin{bmatrix} 0.8 & 0.4 \\ -0.4 & 0.2 \end{bmatrix} \\ A_3 &= \begin{bmatrix} -0.4 & 1 \\ 0.2 & 0.3 \end{bmatrix} & A_4 &= \begin{bmatrix} 0.5 & -0.4 \\ 0.5 & 0.5 \end{bmatrix} \end{aligned} \tag{6}$$

The resulting plot is presented in Figure 3. This time, there exists $z = (-0.1; -0.05) \in \mathcal{R}_{A_1} \cap \mathcal{R}_{A_2} \cap \mathcal{R}_{A_3} \cap \mathcal{R}_{A_4} \cap \mathcal{D}$, hence Theorem 1 guarantees that the positive-definite matrix P_z given by

$$P_z = \begin{bmatrix} 0.9 & -0.05 \\ -0.05 & 1.1 \end{bmatrix}$$

is such that $\mathcal{L}^d_{A_i}(P_z) \in \mathbb{S}_2^{--}$ for $i = 1, \ldots, 4$.

This is also verified by: $\lambda_{\max}(\mathcal{L}^d_{A_1}(P_z)) \approx -0.0387 < 0$, $\lambda_{\max}(\mathcal{L}^d_{A_1}(P_z)) \approx -0.0446 < 0$, $\lambda_{\max}(\mathcal{L}^d_{A_1}(P_z)) \approx -0.0386 < 0$, $\lambda_{\max}(\mathcal{L}^d_{A_1}(P_z)) \approx -0.3580 < 0$.

IV. Conclusion and Perspectives

This paper has introduced a straightforward graphical criterion to the existence of a CQLF to a set of second-order LTI systems, simply by identifying the intersection of several ellipses. The approach can be applied to several classes of nonlinear systems, including switched linear systems, linear time-varying systems with a bounded state matrix, polytopic (quasi-)linear parameter-varying systems, Takagi-Sugeno (fuzzy) systems, etc. This graphical criterion is illustrated in both continuous- and discrete-time settings. The simplicity and graphical nature of the method makes it very intuitive, and will perhaps motivate further algebraic approaches, including for systems of higher order. It is also relatively easy to adapt the proposed criterion to similar LMIs with a minimum decay rate guarantee. These considerations remain to be investigated in the future.

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Appendix: Matlab code

Figure 2 is obtained with:

```
A = {[-10 0;2 -0.5],[-1 -2;2 -1],[0 0.5;-1 -1],[-3 5;-7 -2],[1 2;3 4],[0 -0.5;0.2 -0.3]};
CQLF_criterion(A,'c','colorList',{'b','m','g','c','r','y'},'lineStyle','-');
```

Figure 3 is obtained with:

```
A = {[-0.5 0;-0.7 0.5],[0.8 0.4;-0.4 0.2],[-0.4 1;0.2 0.3],[0.5 -0.4;0.5 0.5]};
CQLF_criterion(A,'d','opacity',0.3,'lineStyle','-');
```

The function "CQLF_criterion" is given thereafter:

```
function CQLF_criterion(A,time,varg)
arguments
A % A cell array of 2x2 real matrices
time='c'; % 'c' for continuous, 'd' for discrete
varg.resolution = 1500; % plot resolution
varg.colorList = arrayfun(@(x)rand(1,3),1:numel(A),'UniformOutput',false); % cell array specifying a color for each system
varg.opacity = 0.5; % patches opacity
varg.lineStyle = 'none'; % outline of the patches
end
z1 = linspace(-2,2,varg.resolution);
z2 = linspace(-2,2,varg.resolution);
[Z1,Z2] = meshgrid(z1,z2);
hold on;
for i=1:numel(A)
[a11,a12,a21,a22] = deal(A{i}(1,1),A{i}(1,2),A{i}(2,1),A{i}(2,2));
f = @(x,y)(x*(a11-a22)+y*(a12+a21)+a11+a22)+sqrt((x*(a12-a21)+y*(a11+a22)+a12+a21).^2+(x*(a11+a22)+y*(a21-a12)+a11-a22).^2)<0;
g = @(x,y)(x*(a11^2+a12^2-a21^2-a22^2)/2+y*(a11*a21+a12*a22)+(a11^2+a12^2+a21^2+a22^2)/2-1+sqrt((x*(a21^2+a12^2-a11^2-a22^2+2)/2+y*(a12*a22-a11*a21)+(a22^2+a12^2-a21^2-a11^2)/2).^2+(x*(a11*a12-a21*a22)+y*(a11*a22+a12*a21-1)+a11*a12+a21*a22).^2))<0;
val = ((time=='c')*f(Z1,Z2)+(time=='d')*g(Z1,Z2));
C = contourc(z1,z2,val,[1 1]);
l = sprintf('$$\\mathcal{%s}_{A_%d}$$',(time=='c')*'Q'+(time=='d')*'R',i);
patch(C(1,2:end),C(2,2:end),varg.colorList{i},'FaceAlpha',varg.opacity,'LineStyle',varg.lineStyle,'DisplayName',l);
end
contour(Z1,Z2,(Z1.^2+Z2.^2<1),[1 1],'--','LineColor','k','HandleVisibility','off');
legend('Location','southwest','Interpreter','latex');
axis([[-1.05,1.05,-1.05,1.05]]);
pbaspect([1 1 1]);
end
```